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Plethysms and operads

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Abstract

We introduce the $\mathcal{T}$-construction, an endofunctor on the category of generalized operads, as a general mechanism by which various notions of plethystic substitution arise from more ordinary notions of substitution. In the special case of one-object unary operads, i.e. monoids, we recover the T-construction of Giraudo. We realize several kinds of plethysm as convolution products arising from the homotopy cardinality of the incidence bialgebra of the bar construction of various operads obtained from the $\mathcal{T}$-construction. The bar constructions are simplicial groupoids, and in the special case of the terminal reduced operad Sym, we recover the simplicial groupoid of Cebrian (Algebraic Geom Topol 21(1):421–446, 2021), a combinatorial model for ordinary plethysm in the sense of Pólya, given in the spirit of Waldhausen S and Quillen Q constructions. In some of the cases of the $\mathcal{T}$-construction, an analogous interpretation is possible.

Mathematics Subject Classification 05A19 · 18B40 · 18N50 · 16T10 · 18M65 · 18M80 · 13J05

List of symbols

Grpd	Category of groupoids and groupoid morphisms
Set	Category of sets and set maps
(P, μ, η)	Generic strong cartesian monad (1.1.1 and 1.3.1)
Id	Identity monad (1.3.3)
M	Free monoid monad (page 8)
M^{r}	Free semigroup monad (1.3.5)
S	Free symmetric monoidal category monad (1.3.7)
S^{r}	Free symmetric semimonoidal category monad (1.3.8)
L	Monad $A \mapsto \mathsf{P}1 \times A$ (page 22)
Y	Generic (locally finite) monoid (1.3.6)
Y	Monad given by $A \mapsto Y \times A$, for Y a monoid (1.3.6)
TY	Giraudo T-construction of Y (4.1.3)
$D_{A,B}$	Strength natural transformation (1.3.1)

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D_B	Strength for $A = 1$ (page 22)
R_A	Projection $\mathsf{P}1 \times A \mapsto A$ (page 22)
$\mathcal{B}$	Two-sided bar construction (page 16)
$\mathcal{B}^{\mathsf{P}}$	Two-sided bar construction relative to a monad P (page 19)
$\mathcal{B}_n$	n-Simplices of the two-sided bar construction $\mathcal{B}$ (page 16)
$\mathcal{B}_n^{\circ}$	Subgroupoid of connected objects of $\mathcal{B}_n$ (page 49)
Sym	The reduced symmetric operad (1.3.8)
Ass	The reduced associative operad (1.3.5)
$\mathcal{E}$	Generic ambient cartesian category, mainly **Set** or **Grpd**) (1.1)
C	Category internal to $\mathcal{E}$ (page 8)
Q	P-operad internal to $\mathcal{E}$ (page 8)
Q_0, Q_1	Objects and operations of Q (page 8)
$(\mathsf{Q}, \mu^Q, \eta^Q)$	Monad on $\mathcal{E}/Q_0$ defined by the P-operad Q (page 14)
$\mathcal{T}_{\mathsf{P}}C$	$\mathcal{T}$-construction from C to a P-operad (page 22)
$\mathcal{T}^{\mathsf{P}}Q$	$\mathcal{T}$-construction from a P-operad to a category C (page 27)
$\mathcal{T}_{\mathsf{P}}Q$	$\mathcal{T}$-construction from a P′-operad to a P-operad (page 32)
Λ	Set of infinite vectors $\lambda = (\lambda_1, \lambda_2, \dots)$ of natural numbers with $\lambda_i = 0$ for all i large enough (page 40)
$\mathbb{1}$	Simplex category (page 16)
$T\mathbf{S}$	Simplicial groupoid of [9] (page 55)

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Introduction

Plethysm is a substitution law in the ring of power series in infinitely many variables. It was introduced by Pólya [38] in unlabelled enumeration theory in combinatorics, motivated as a series analogue of the wreath product of permutation groups. Another notion of plethysm was defined by Littlewood [27] in the context of symmetric functions and representation theory of the general linear groups [30]. It appears also in algebraic topology, in connection with λ-rings [5] and power operations in cohomology [2]. The two notions of plethysm are closely related, as described in [4, 41].

The plethysms we deal with emerge from Pólya's notion, which we proceed to recall. Let $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ be the ring of power series in the infinite set of variables $\mathbf{x} = (x_1, x_2, \dots)$ without constant term. Given $F, G \in \mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$, their *plethystic substitution* is defined as

$$(G \circledast F)(x_1, x_2, \dots) = G(F_1, F_2, \dots), \text{ where } F_k(x_1, x_2, \dots) = F(x_k, x_{2k}, \dots).$$

The formal power series may be expressed as

$$F(\mathbf{x}) = \sum_{\lambda} F_{\lambda} \frac{\mathbf{x}^{\lambda}}{\mathrm{aut}(\lambda)},$$

where $\mathrm{aut}(\lambda)$ are certain symmetry factors (see Section 5 below).

It is well appreciated in combinatorics that bijective proofs give a deeper understanding than algebraic manipulation, as well as a better ground for further development. The so-called objective method, pioneered by Lawvere [25], Joyal [22], and Baez and Dolan [1], provides a systematic approach to bijective proofs. One of the starting points of objective combinatorics is the theory of species, developed by Joyal [22] as a combinatorial theory of formal series. Within this context, Pólya enumeration theory of unlabelled structures, including cycle index series and their plethysm, was entirely renewed.

However, a full combinatorial model of plethysm was only given a few years later by Nava and Rota [37]. They developed the notion of partitional, a functor from the groupoid of partitions to the category of finite sets, and showed that a suitable notion of composition of partitionals yields plethystic substitution of their generating functions, in analogy with composition of species and composition of their exponential generating functions. A variation of this combinatorial interpretation was given shortly after by Bergeron [3], who instead of partitionals considered permutationals, functors from the groupoid of permutations to the category of finite sets. This approach is nicely related to the theory of species and their cycle index series through an adjunction. Later on, Nava [36] studied both partitionals and permutationals from the point of view of incidence coalgebras, and added a third class of functors called linear partitionals.

The bialgebras arising from the various plethystic substitutions are called plethystic bialgebras in the present article. Here is a general definition, as given in [9]: the *plethystic bialgebra* is the free polynomial algebra on the linear functionals $A_\lambda(F) = F_\lambda$ with comultiplication dual to plethystic substitution,

$$\Delta(A_\lambda)(F, G) = A_\lambda(G \circledast F).$$

In this definition, the difference between the three bialgebras of Nava depends on the definition of the linear functionals A_λ, which in turn depend on the definition of the symmetry factors $\text{aut}(\lambda)$.

The present work introduces a construction on operads, called the $\mathcal{T}$-construction, which establishes a relationship between ordinary substitutions and plethystic substitutions. In particular, this construction produces combinatorial models for the partitional and the linear partitional (also called exponential) cases, but also for other kinds of plethysm: plethysm of power series with variables indexed by a (locally finite) monoid, introduced by Méndez and Nava [34] in the course of generalizing Joyal's theory of colored species to an arbitrary set of colors; plethysm in two variables $\mathbf{x}, \mathbf{y}$; plethysm of series with coefficients in a noncommutative ring, in the style of [6], and plethysm of series with noncommuting variables. All these plethysms and their bialgebras are explained in Section 5. The $\mathcal{T}$-construction relies on operads and the theory of decomposition spaces and their incidence bialgebras.

The theory of operads has long been a standard tool in topology and algebra [28, 31], and in category theory [26], and it is getting increasingly important also in combinatorics [21, 33]. In the present work, for maximal flexibility, we work with operads in the form of generalized multicategories [26]. This allows us to cover simultaneously notions such as monoids, categories, nonsymmetric operads and symmetric operads.

On the other hand, decomposition spaces (certain simplicial spaces) provide a general machinery to objectify the notion of incidence algebra in algebraic combinatorics. They were introduced by Gálvez, Kock and Tonks [16–18] in this framework, and they are the same as 2-Segal spaces introduced by Dyckerhoff and Kapranov [13] in the context of homological algebra and representation theory. To recover the algebraic incidence coalgebra from the categorified incidence coalgebra one takes homotopy cardinality, a cardinality functor defined from groupoids to the rationals.

It was shown in [24] that the two-sided bar construction [32, 46] of an operad is a Segal groupoid, a particular type of decomposition space, and classical constructions of bialgebras arising from operads factor through this construction (see [11, 42, 43] for related constructions). Next we give two relevant examples of bialgebras that arise as incidence bialgebras of operads.

Example Let $x\mathbb{Q}[\![x]\!]$ be the ring of formal power series in x without constant term, and let $F, G \in x\mathbb{Q}[\![x]\!]$. The *Faà di Bruno bialgebra* $\mathcal{F}$ is the free algebra $\mathbb{Q}[A_1, A_2, \dots]$, where $A_n \in x\mathbb{Q}[\![x]\!]^*$ is the linear map defined by

$$A_n(F) = \frac{d^n F}{dx^n}.$$

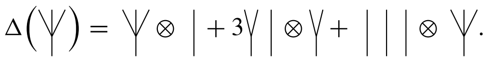
Its comultiplication is defined to be dual to substitution of power series. That is

$$\Delta(A_n)(F, G) = A_n(G \circ F).$$

It is a result of Joyal [22, §7.4] that this bialgebra can be objectified by using the category of finite sets and surjections $\mathbf{S}$. In the context of Segal spaces and incidence bialgebras the result reads as follows: *the Faà di Bruno bialgebra $\mathcal{F}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of the fat nerve $N\mathbf{S}$ of the category* $\mathbf{S}$. The comultiplication here is given by summing over factorizations of surjections.

Example In earlier work [9] the author found a simplicial groupoid $T\mathbf{S}$ (like $N\mathbf{S}$ arising from the category of surjections cf. 6 below), which plays the same role for plethystic substitution: *the homotopy cardinality of the incidence bialgebra of $T\mathbf{S}$ is isomorphic to the (partitional)* *plethystic bialgebra.* The comultiplication extracted from this simplicial groupoid can be interpreted as summing over certain transversals of partitions, as in the work of Nava and Rota [37].

Now, it is well-known that $N\mathbf{S}$ is equivalent to the two-sided bar construction of Sym, the terminal reduced symmetric operad. This equivalence takes the surjection $n \twoheadrightarrow 1$ to the unique n-ary operation, and the comultiplication of an operation runs through all possible 2-step factorizations. For example



The starting point of the present work is the observation that also $T\mathbf{S}$ is equivalent to the two-sided bar construction of an operad. As we shall see, this operad can be obtained from Sym by the aforementioned $\mathcal{T}$-construction, which makes sense for any (nice enough) operad. As stated above, this construction leads to many other flavors of plethysm, some of which had already been studied in various contexts. For instance, from Ass, the reduced associative operad, we obtain the exponential plethystic bialgebra, and from n-colored Sym or Ass, we obtain the n-variables plethystic bialgebra. The results relating the bialgebras to these operads are explained in Section 5.

Let us give a brief introduction to the $\mathcal{T}$-construction. The word $\mathcal{T}$-construction comes from the simplicial T-construction [9], where T stands for transversal (in the sense of Nava–Rota [37]), and which is analogous to Waldhausen S and Quillen Q constructions. By coincidence Giraudo [20] had used the same letter T for a functor from monoids to nonsymmetric operads. The $\mathcal{T}$-construction of the present work encompasses both these constructions, and the letter T has been maintained, but now in a fancier font.

Let us first describe Giraudo's T-construction [20]. Let $(Y, \cdot, 1)$ be a monoid. Then

$$TY := \bigsqcup_{n \geq 0} TY(n),$$

where for all $n \geq 1$,

$$TY(n) := \{(x_1, \ldots, x_n) \mid x_i \in Y \text{ for all } i = 1, \ldots, n\},$$

so that the n-ary operations are n-tuples of elements of Y. The substitution law in TY,

$$\circ_i : TY(n) \times TY(m) \longrightarrow TY(n + m - 1),$$

is defined as follows: for all $x \in TY(n)$, $y \in TY(m)$, and $i = 1, \ldots, n$,

$$x \circ_i y := (x_1, \ldots, x_{i-1}, x_i \cdot y_1, \ldots, x_i \cdot y_m, x_{i+1}, \ldots, x_n).$$

Our $\mathcal{T}$-construction is developed in the context of P-operads (generalized multicategories in Leinster [26] terminology), for P a cartesian monad on a cartesian category $\mathcal{E}$. This level of generality allows us to work with symmetric, nonsymmetric, colored and noncolored operads on the same footing. This includes also monoids and categories. A P-operad is represented by a span and two arrows

$$\begin{array}{ccc} & Q_1 & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ \mathsf{P}Q_0 & & Q_0 \end{array} \qquad \begin{array}{c} \mathsf{P}Q_1 \times_{\mathsf{P}Q_0} Q_1 \xrightarrow{\ m\ } Q_1 \\ \\ Q_0 \xrightarrow{\ \ e\ \ } Q_1, \end{array}$$

where Q_0 is thought of as the object of colors, Q_1 is thought of as the object of operations, s returns the P-configuration of input colors, t returns the output color, e is the unit and m is composition. All these arrows have to satisfy associativity and unit axioms.

For instance, if Id is the identity monad, then an Id-operad is a category internal to $\mathcal{E}$. The $\mathcal{T}$-construction is in fact a composition of two constructions, one from P-operads to (internal) categories and one from categories to P-operads. The latter contains the Giraudo T-construction for the case $\mathcal{E} = \mathbf{Set}$ if we consider monoids as categories with one object.

However, we will mainly be interested in $\mathcal{E} = \mathbf{Grpd}$. In particular, nonsymmetric operads will be considered as M^{r}-operads, where M^{r} is the free semimonoidal category monad in **Grpd**, and symmetric operads as S^{r}-operads, where S^{r} is the free symmetric semimonoidal category monad in **Grpd**. There are two main reasons for working over **Grpd**: on the one hand, note that unlike nonsymmetric operads, symmetric operads cannot be portrayed as P-operads in **Set**, because the free commutative monoid monad is not cartesian; on the other hand, working in **Grpd** adapts better with the theory of decomposition spaces and incidence coalgebras. This theory uses weak notions of simplicial groupoids, slice categories, and pullbacks, but by keeping track of fibrancy we can stay within strict notions and strict monads in the style of [45].

In order for the $\mathcal{T}$-construction to work, it is necessary to assume that the monads come equipped with a strength. This notion goes back to work of A. Kock [23] in enriched category theory, but it has turned out to be fundamental for the role monads play in functional programming [35, 44]. In Section 1 we recall the theory of generalized operads, the notion of strong monad, and the two-sided bar construction in this context.

In Section 2 we briefly explain Segal groupoids, incidence coalgebras and homotopy cardinality. Section 3 is devoted to the $\mathcal{T}$-construction, and Section 4 to some examples. Next, in Section 5 we introduce the bialgebras and state and prove the main results: the equivalence between the homotopy cardinality of the incidence bialgebras of the two-sided bar constructions of operads obtained from the $\mathcal{T}$-construction and the plethystic bialgebras, as well as the Faà di Bruno bialgebra and some of its variations. Finally, in Section 6 we prove the equivalence between $T\mathbf{S}$ and $\mathcal{B}\widetilde{\mathsf{Sym}}$, and we characterize some of the two-sided bar constructions as simplicial groupoids similar to $T\mathbf{S}$.

1 Monads and operads

As mentioned in the introduction, the $\mathcal{T}$-construction fits neatly within the context of generalized operads and strong monads. The following discussion of generalized operads is taken from [26]. Let us start by expressing the notions of category and of plain operad in this setting.

A small category C can be described by sets and functions

$$C_0 \xleftarrow{s} C_1 \xrightarrow{t} C_0 \qquad C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \qquad C_0 \xrightarrow{e} C_1$$

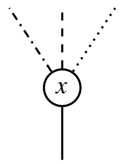
where the pullback is taken along $C_1 \xrightarrow{s} C_0 \xleftarrow{t} C_1$, satisfying associativity and identity axioms, which can be expressed with commutative diagrams in **Set** (see "Appendix A.1"). The set C_0 is the set of objects and C_1 is the set of arrows of C. The map s returns the source of an arrow and t returns its target. The maps m and e represent composition and identities.

A nonsymmetric operad can be defined in a similar way. Let $\mathsf{M}\colon \mathbf{Set} \to \mathbf{Set}$ be the free monoid monad: it sends a set A to $\coprod_{n\in\mathbb{N}} A^n$ (see Example 1.3.4 below). Then a nonsymmetric colored set operad can be described as consisting of sets and functions

$$
\begin{array}{ccc}
& Q_1 & \\
{}^{s}\swarrow & & \searrow^{t} \\
\mathsf{M}Q_0 & & Q_0
\end{array}
\qquad
\begin{array}{c}
\mathsf{M}Q_1 \times_{\mathsf{M}Q_0} Q_1 \xrightarrow{m} Q_1 \\
\\
Q_0 \xrightarrow{\quad e \quad} Q_1
\end{array}
\tag{1.0.1}
$$

satisfying associativity and identity axioms, which can be expressed with commutative diagrams in **Set** (see "Appendix 1") and involve the monad structure on M. The set Q_0 is the set of objects (or colors) and Q_1 is the set of operations of Q. The map s assigns to an operation the sequence of objects constituting its source, and t returns its target. The maps m and e represent composition and identities.

Operations of classical operads, such as nonsymmetric operads or symmetric operads are pictured as



,

where the different types of lines represent objects or colors of Q_0.

1.1 P-operads

The above characterization of nonsymmetric **Set** operads can be generalized to any ambient category and any monad P as long as they are cartesian. The classical case is **Set**; we shall be concerned also with **Grpd**.

Definition 1.1.1 A category is *cartesian* if it has all pullbacks. A functor is *cartesian* if it preserves pullbacks. A natural transformation is *cartesian* if all its naturality squares are pullbacks. A monad (P, μ, η) is *cartesian* if P is cartesian as a functor and μ and η are cartesian natural transformations.

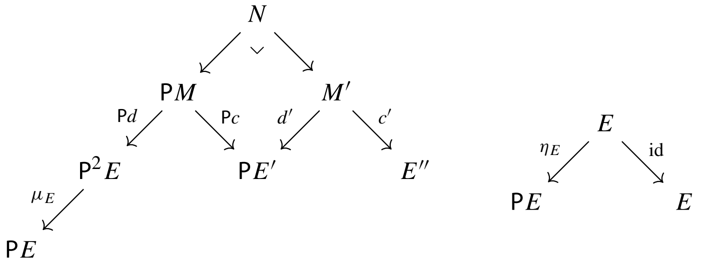
Given a cartesian category $\mathcal{E}$ and a cartesian monad (P, μ, η), we define $\mathcal{E}_{(\mathsf{P})}$ as the bicategory whose 0-cells are the objects E of $\mathcal{E}$, whose 1-cells $E \to E'$ are spans $\mathsf{P}E \leftarrow M \to E'$, and 2-cells are the usual morphisms $M \to N$ between spans:

$$
\begin{array}{ccc}
& M & \\
{}^{d}\swarrow & \downarrow & \searrow^{c} \\
\mathsf{P}E & & E' \\
{}_{q}\nwarrow & & \nearrow_{p} \\
& N. &
\end{array}
$$

Given two 1-cells

$$
\begin{array}{ccc}
& M & \\
{}^{d}\swarrow & & \searrow^{c} \\
\mathsf{P}E & & E'
\end{array}
\qquad
\begin{array}{ccc}
& M' & \\
{}^{d'}\swarrow & & \searrow^{c'} \\
\mathsf{P}E' & & E''
\end{array}
$$

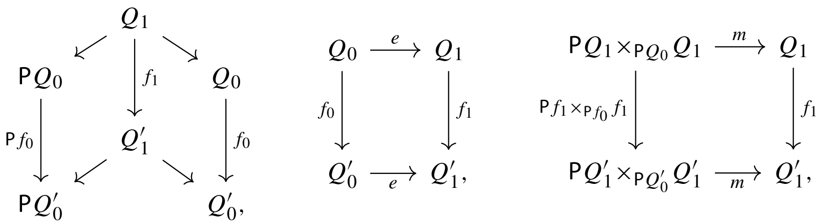
the composite is given by taking a pullback and using the multiplication μ of P, and the 1-cell identity is given by η and id. They are shown in the following diagram:



Composition and identity of 2-cells are obvious. Since composition assumes a global choice of pullbacks, and since the pasting of two chosen pullbacks is not generally a chosen pullback, composition is associative up to coherent isomorphism. The coherence 2-cells are defined using the universal property of the pullback.

Definition 1.1.2 (Burroni [7]) Let P be a cartesian monad in a cartesian category $\mathcal{E}$. A P-*operad* is a monad in the bicategory $\mathcal{E}_{(\mathsf{P})}$.

This means that a P-operad Q consists precisely of objects Q_0 and Q_1 of $\mathcal{E}$ together with maps s, t, composition m and identities e as in Diagram (1.0.1) satisfying associativity and identity axioms ("Appendix 1"). A *morphism* $Q \to Q'$ of P-operads is defined as a pair of arrows $Q_0 \xrightarrow{f_0} Q'_0$, $Q_1 \xrightarrow{f_1} Q'_1$, satisfying the following diagrams,

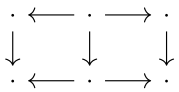


(1.1.1)

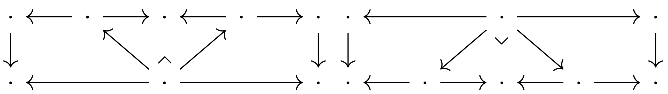
regarding compatibility with the spans, identities and composition maps. Notice that this is not an arrow in $\mathcal{E}_{(\mathsf{P})}$. The category of P-operads is denoted **P-Operad**.

1.2 Morphisms of spans

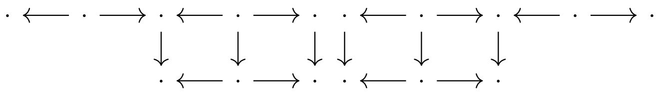
In Section 3 we will deal with morphisms between long horizontal composites of spans. It is thus worth to set up a framework for such morphisms: consider the following diagrams, named blocks, made of maps in $\mathcal{E}$,



(1.2.1)



(1.2.2)

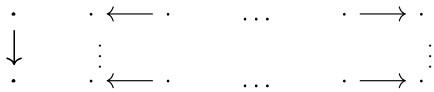


(1.2.3)

Notice that (1.2.2) induce isomorphisms of spans if the vertical maps are isomorphisms, since in this case they represent horizontal composition of spans. Diagram (1.2.1) is an isomorphism when all the vertical arrows are isomorphisms, and (1.2.3) are isomorphisms when all the vertical arrows and the span projected away are isomorphisms. Besides, the blocks can be horizontally and vertically attached in the obvious way to get morphisms of longer spans, with the only restriction that the diagrams (1.2.3) can be attached to the right and to the left respectively.

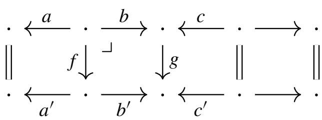
Lemma 1.2.1 *Any pasting of blocks defines a morphism between the limit of the top row and the limit of the bottom row. Moreover, such a morphism is an isomorphism if it can be constructed from blocks that are isomorphisms.*

The morphisms between long spans are pictured with diagrams



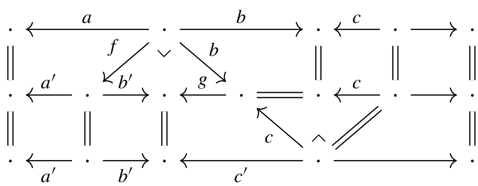
where the left bold part is the limit of the diagram: the upper dot is the limit of the upper row, and same for the bottom row. Observe that the decomposition of a morphism into blocks is not unique, and there may be decompositions of isomorphisms whose blocks are not necessarily isomorphisms. Here is an example that will be used later on.

Example 1.2.2 The following diagram represents an isomorphism of composites of spans:



(1.2.4)

Indeed, it can be expressed by pasting isomorphism blocks:



(1.2.5)

1.3 Strong monads

We now recall the notion of strong monad [23], which is central in the $\mathcal{T}$-construction. From now on the ambient category $\mathcal{E}$ is required to have a terminal object, hence all finite limits.

Definition 1.3.1 Let (P, μ, η) be a monad on $\mathcal{E}$. A *strength* for P is a natural transformation with components $D_{A,B} : A \times \mathsf{P}B \to \mathsf{P}(A \times B)$, satisfying the following two axioms concerning tensoring with 1 and consecutive applications of D,

$$\begin{array}{ccc} 1\times \mathsf{P}A & \xrightarrow{D_{1,A}} & \mathsf{P}(1\times A) \\ & \searrow^{p_2} & \downarrow{\scriptstyle \mathsf{P}p_2} \\ & & \mathsf{P}A \end{array} \tag{1.3.1a}$$

$$\begin{array}{ccccc} (A\times B)\times \mathsf{P}C & \longrightarrow & A\times(B\times \mathsf{P}C) & \xrightarrow{A\times D_{B,C}} & A\times \mathsf{P}(B\times C) \\ {\scriptstyle D_{A\times B,C}}\downarrow & & & & \downarrow{\scriptstyle D_{A,B\times C}} \\ \mathsf{P}((A\times B)\times C) & & \longrightarrow & & \mathsf{P}(A\times(B\times C)) \end{array} \tag{1.3.1.b}$$

and two axioms concerning compatibility with monad unit and multiplication,

$$\begin{array}{ccc} A\times B & \xrightarrow{A\times\eta_B} & A\times \mathsf{P}B \\ & \searrow^{\eta_{A\times B}} & \downarrow{\scriptstyle D_{A,B}} \\ & & \mathsf{P}(A\times B) \end{array} \tag{1.3.2a}$$

$$\begin{array}{ccccc} A\times \mathsf{P}^2B & \xrightarrow{D_{A,\mathsf{P}B}} & \mathsf{P}(A\times \mathsf{P}B) & \xrightarrow{\mathsf{P}D_{A,B}} & \mathsf{P}^2(A\times B) \\ {\scriptstyle A\times\mu_B}\downarrow & & & & \downarrow{\scriptstyle \mu_{A\times B}} \\ A\times \mathsf{P}B & & \xrightarrow[D_{A,B}]{} & & \mathsf{P}(A\times B) \end{array} \tag{1.3.2b}$$

Before seeing some examples of P-operads and strong monads, we prove the following lemma, which will be useful in Section 3.

Lemma 1.3.2 *Let u be the unique morphism $u\colon P1 \to 1$. Then the square*

$$\begin{array}{ccc} A\times P^2 1 & \xrightarrow{A\times Pu} & A\times P1 \\ {\scriptstyle D_{A,P1}}\downarrow & \lrcorner & \downarrow{\scriptstyle D_{A,1}} \\ P(A\times P1) & \xrightarrow[P(A\times u)]{} & P(A\times 1) \end{array} \tag{1.3.3}$$

is a pullback.

Proof Observe that if we project the bottom rows of this square to the first component,

$$\begin{array}{ccc} A\times \mathsf{P}^2 1 & \xrightarrow{A\times \mathsf{P}u} & A\times \mathsf{P}1 \\ {\scriptstyle D_{A,\mathsf{P}1}}\downarrow & \lrcorner & \downarrow{\scriptstyle D_{A,1}} \\ \mathsf{P}(A\times \mathsf{P}1) & \xrightarrow{\mathsf{P}(A\times u)} & \mathsf{P}(A\times 1) \\ {\scriptstyle \mathsf{P}p_2}\downarrow & & \downarrow{\scriptstyle \mathsf{P}p_2} \\ \mathsf{P}^2 1 & \xrightarrow[\mathsf{P}u]{} & \mathsf{P}1 \end{array}$$

then the lower square is a pullback because P is cartesian, and the outer square is a pullback because it is a projection, by (1.3.1a). Therefore the upper square is a pullback too. □

Let us see some examples of strong monads.

Example 1.3.3 Obviously the identity monad is strong. If we take the identity monad Id on any cartesian cartesian category $\mathcal{E}$ then a Id-operad is the same as a category internal to $\mathcal{E}$, and a non colored Id-operad is a monoid in $\mathcal{E}$. In particular if $\mathcal{E} = \mathbf{Set}$ they are small categories and monoids, respectively.

Example 1.3.4 Let (M, μ, η) be the free monoid monad on the category $\mathcal{E}$=**Set**. As mentioned above, a M-operad is the same thing as a colored nonsymmetric set operad. Here is the full explicit description of M. Let A be a set and $a_0, \ldots, a_n \in A$, then

$$\mathsf{M}A = \coprod_{n\in N} A^n,$$
$$\eta_A(a_0) = (a_0),$$
$$\mu_A\big((a_1, \ldots, a_i), \ldots, (a_j, \ldots, a_n)\big) = (a_1, \ldots, a_n). \tag{1.3.4}$$

The free monoid monad is strong with the following strength:

$$D_{A,B}\colon A\times\mathsf{M}B \longrightarrow \mathsf{M}(A\times B)$$
$$\big(a, (b_1, \ldots, b_n)\big) \longmapsto \big((a, b_1), \ldots, (a, b_n)\big).$$

It is straightforward to check that the diagrams (1.3.2b) and (1.3.2a) are satisfied and clear that $D_{A,B}$ is injective. This last feature is relevant because to define the $\mathcal{T}$-construction, in Section 3, it will be necessary that D_{1,C_0} is a monomorphism.

Example 1.3.5 The free semigroup monad M^{r} on **Set** is defined in the same way as the free monoid monad, except that in this case $\mathsf{M}^{\mathrm{r}}A = \coprod_{n\geq 1} A^n$. This means that a M^{r}-operad is a nonsymmetric operad without nullary operations. The terminal M^{r}-operad is denoted Ass, which is of course the reduced associative operad. Notice that M^{r} is also a strong cartesian monad on **Grpd**. In this sense the operad Ass can also be considered as an M^{r}-operad in **Grpd**, with discrete groupoid of objects and discrete groupoid of operations. The context will suffice to distinguish between **Set** and **Grpd**, but in the main applications (Section 5) we work over **Grpd**.

Example 1.3.6 Let Y be a monoid. Denote by Y the monad on **Set** given by $\mathsf{Y}A = Y\times A$ with unit and multiplication given by those of Y. Then Y is strong with strength given by the associator of the cartesian product. Therefore in this case the strength is an isomorphism. The same holds if Y is a monoid in **Grpd** and Y is then a monad on **Grpd**.

Example 1.3.7 Let (S, μ, η) be the free symmetric monoidal category monad on **Grpd**. An S-operad is an operad internal to groupoids, so that it has a groupoid of colors and a groupoid of operations. Let A be a groupoid and $\mathfrak{S}_n$ the symmetric group on n elements. The monad S acts on A by

$$\mathsf{S}A = \coprod_{n\in N} A^n//\mathfrak{S}_n,$$

where // means homotopy quotient [1, 15]. Hence it is analogous to M, but we add an arrow

$$(a_1, \ldots, a_n) \xrightarrow{\sigma} (a_{\sigma 1}, \ldots, a_{\sigma n})$$

for every element $\sigma \in \mathfrak{S}_n$. The multiplication and unit natural transformations are defined as in (1.3.4) for both objects and operations. Notice that any symmetric operad Q is in particular an S-operad, where the groupoid of objects Q_0 is discrete and the groupoid Q_1 has only the arrows coming from the permutations of its source sequence. In other words, a symmetric operad is an S-operad

$$\mathsf{S}Q_0 \xleftarrow{s} Q_1 \xrightarrow{t} Q_0$$

such that Q_0 is discrete and s is a discrete fibration. The strength for S is defined the same way as for M,

$$\begin{aligned} D_{A,B}\colon A \times \mathsf{S}B &\longrightarrow \mathsf{S}(A \times B) \\ \big(a, (b_1, \ldots, b_n)\big) &\longrightarrow \big((a, b_1), \ldots, (a, b_n)\big), \end{aligned}$$

and it is again a monomorphism, since it is injective both on objects and morphisms.

Observe that symmetric operads cannot be expressed as P-operads in **Set**, since the actions of the symmetric groups have to be encoded necessarily as morphisms in Q_1. Also, the only monad P one could attempt to use to define them is the free commutative monoid monad, but it is not cartesian.

Example 1.3.8 As for M and M^{r}, we can remove the empty sequence from S to get a monad S^{r} on **Grpd** whose operads do not have nullary operations. We denote by Sym the terminal S^{r}-operad, which is the reduced commutative operad.

1.4 The two-sided bar construction for P-operads

The two-sided bar construction for operads is standard [32]. In this section we introduce the construction in the more general setting of P-operads by using induced monads. Any P-operad Q defines a monad $(\mathsf{Q}, \mu^{\mathsf{Q}}, \eta^{\mathsf{Q}})$ on the slice category of $\mathcal{E}$ over Q_0

$$\mathsf{Q} : \mathcal{E}/Q_0 \longrightarrow \mathcal{E}/Q_0,$$

given by pullback and composition, as shown in the following diagram for an element $X \xrightarrow{f} Q_0$ of $\mathcal{E}/Q_0$

$$\begin{array}{ccccc} & & \mathsf{Q}X & & \\ & \swarrow & \lrcorner & \searrow & \\ \mathsf{P}X & & & & Q_1 \\ & \searrow^{\mathsf{P}f} & & \swarrow_{s} & & \searrow^{t} \\ & & \mathsf{P}Q_0 & & & & Q_0. \end{array} \tag{1.4.1}$$

The image of f is thus the dashed composite. The multiplication μ^{Q} and the unit η^{Q} are defined by the following morphisms

$$\begin{array}{ccccccccccc} \mathsf{Q}^2X & & \mathsf{P}^2X & \xrightarrow{\mathsf{P}^2 f} & \mathsf{P}^2Q_0 & \xleftarrow{\mathsf{P}s} & \mathsf{P}Q_1 & \xrightarrow{\mathsf{P}t} & \mathsf{P}Q_0 & \xleftarrow{s} & Q_1 & \xrightarrow{t} & Q_0 \\ \Big\downarrow \mu^{\mathsf{Q}}_X & & \| & & \| & & & \nwarrow^{s} \quad \lrcorner \quad \nearrow^{t} & & & & & \| \\ & & \mathsf{P}^2X & \xrightarrow{\mathsf{P}^2 f} & \mathsf{P}^2Q_0 & \longleftarrow & & \mathsf{P}Q_1 \times_{\mathsf{P}Q_0} Q_1 & & \longrightarrow & & & Q_0 \\ & & \Big\downarrow \mu_X & & \Big\downarrow \mu_{Q_0} & & & \Big\downarrow m & & & & & \| \\ \mathsf{Q}X & & \mathsf{P}X & \xrightarrow{\mathsf{P}f} & \mathsf{P}Q_0 & & \xleftarrow{s} & Q_1 & & \xrightarrow{t} & & & Q_0, \end{array} \tag{1.4.2}$$

$$
\begin{array}{ccc}
\boldsymbol{X} & & X \xrightarrow{\ f\ } Q_0 = = = Q_0 = = = Q_0 \\
{\scriptstyle \eta^{\mathsf{Q}}_{\boldsymbol{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \eta_X} \qquad \big\downarrow{\scriptstyle \eta_{Q_0}} \qquad \big\downarrow{\scriptstyle e} \qquad \big\| \\
\mathsf{P}_{\boldsymbol{Q}}\boldsymbol{X} & & X \xrightarrow[\mathsf{P}f]{} \mathsf{P}Q_0 \xleftarrow[s]{} Q_1 \xrightarrow[t]{} Q_0.
\end{array}
\tag{1.4.3}
$$

Definition 1.4.1 An *algebra* over the P-operad Q is an algebra over the monad Q.

Notice that the category $\mathcal{E}/Q_0$ has a terminal object, $Q_0 \xrightarrow{1} Q_0$, so that there is an algebra over Q given by the unique arrow $q : \mathsf{Q}1 \to 1$. Moreover, since $\mathcal{E}$ has a terminal object, the P-operad $\mathsf{P} : \mathcal{E}/1 \to \mathcal{E}/1$ itself can be represented by the span

$$\mathsf{P}1 \longleftarrow \mathsf{P}1 \longrightarrow 1,$$

and is the terminal P-operad. Now, the terminal arrow $u : Q_0 \to 1$ induces, by postcomposition, a functor $u_! : \mathcal{E}/Q_0 \to \mathcal{E}/1$. The diagram

$$
\begin{array}{ccccc}
\mathsf{P}Q_0 & \longleftarrow & Q_1 & \longrightarrow & Q_0 \\
{\scriptstyle \mathsf{P}u}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle u} \\
\mathsf{P}1 & \longleftarrow & \mathsf{P}1 & \longrightarrow & 1.
\end{array}
\tag{1.4.4}
$$

represents a natural transformation $u_!\mathsf{Q} \overset{\phi}{\Rightarrow} \mathsf{P}u_!$ which is compatible with the comultiplications and units of Q and P, meaning that

$$
\begin{aligned}
u_!\mathsf{Q}^2 \overset{\phi^2}{\Longrightarrow} \mathsf{P}^2u_! \overset{\mu u_!}{\Longrightarrow} \mathsf{P}u_! &= u_!\mathsf{Q}^2 \overset{u_!\mu^{\mathsf{Q}}}{\Longrightarrow} u_!\mathsf{Q} \overset{\phi}{\Rightarrow} \mathsf{P}u_! \qquad \text{and} \\
u_! \overset{u_!\eta^{\mathsf{Q}}}{\Longrightarrow} u_!\mathsf{Q} \overset{\phi}{\Rightarrow} \mathsf{P}u_! &= u_! \overset{\eta u_!}{\Longrightarrow} \mathsf{P}u_!
\end{aligned}
$$

or, equivalently,

$$
\begin{array}{ccccccccccc}
\mathsf{P}Q_0 & \xleftarrow{\mu_{Q_0}} & \mathsf{P}^2Q_0 & \leftarrow & \mathsf{P}Q_1 & \rightarrow & \mathsf{P}Q_0 & \leftarrow & Q_1 & \rightarrow & Q_0 \\
{\scriptstyle \mathsf{P}u}\downarrow & & \downarrow{\scriptstyle \mathsf{P}^2u} & & \downarrow & & {\scriptstyle \mathsf{P}u}\downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\
\mathsf{P}1 & \xleftarrow[\mu_1]{} & \mathsf{P}^21 & \longleftarrow & \mathsf{P}^21 & \xrightarrow{\mathsf{P}u} & \mathsf{P}1 & \longleftarrow & \mathsf{P}1 & \longrightarrow & 1 \\
\| & & & & & \nwarrow & & \nearrow{\scriptstyle \mathsf{P}u} & & & \| \\
\| & & & & & & \mathsf{P}^21 & & & & \| \\
\| & & & & & & \downarrow{\scriptstyle \mu_1} & & & & \| \\
\mathsf{P}1 & & & \longleftarrow & & & \mathsf{P}1 & & \longrightarrow & & 1
\end{array}
=
\begin{array}{ccccccccccc}
\mathsf{P}Q_0 & \xleftarrow{\mu_{Q_0}} & \mathsf{P}^2Q_0 & \leftarrow & \mathsf{P}Q_1 & \rightarrow & \mathsf{P}Q_0 & \leftarrow & Q_1 & \rightarrow & Q_0 \\
\| & & & & & \nwarrow & & \nearrow & & & \| \\
\| & & & & & & Q_2 & & & & \| \\
\| & & & & & & \downarrow & & & & \| \\
\mathsf{P}Q_0 & & & \longleftarrow & & & Q_1 & & \longrightarrow & & Q_0 \\
{\scriptstyle \mathsf{P}u}\downarrow & & & & & & \downarrow & & & & \downarrow{\scriptstyle u} \\
\mathsf{P}1 & & & \longleftarrow & & & \mathsf{P}1 & & \longrightarrow & & 1,
\end{array}
\tag{1.4.5}
$$

$$
\begin{array}{ccccc}
Q_0 & = = = & Q_0 & = = = & Q_0 \\
{\scriptstyle \eta_{Q_0}}\big\downarrow & & \big\downarrow{\scriptstyle e} & & \big\| \\
\mathsf{P}Q_0 & \longleftarrow & Q_1 & \longrightarrow & Q_0 \\
{\scriptstyle \mathsf{P}u}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle u} \\
\mathsf{P}1 & \longleftarrow & \mathsf{P}1 & \longrightarrow & 1.
\end{array}
=
\begin{array}{ccccc}
Q_0 & = = = & Q_0 & = = = & Q_0 \\
{\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u} \\
1 & \longleftarrow & 1 & \longrightarrow & 1 \\
{\scriptstyle \eta_1}\big\downarrow & & \big\downarrow{\scriptstyle \eta_1} & & \big\| \\
\mathsf{P}1 & \longleftarrow & \mathsf{P}1 & \longrightarrow & 1.
\end{array}
\tag{1.4.6}
$$

Lemma 1.4.2 *The natural transformation ϕ is cartesian.*

Proof Let us describe the naturality squares of ϕ. Let H be a map in $\mathcal{E}/Q_0$, that is, a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Q_0. & \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} \mathsf{P}X \underset{\mathsf{P}Q_0}{\times} Q_1 & \xrightarrow{u_!\mathsf{Q}H} & \mathsf{P}Y \underset{\mathsf{P}Q_0}{\times} Q_1 & \longrightarrow & Q_1 \\ {\scriptstyle \phi_X}\downarrow & & {\scriptstyle \phi_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \mathsf{P}X & \xrightarrow[\mathsf{P}u_!H]{} & \mathsf{P}Y & \xrightarrow[\mathsf{P}g]{} & \mathsf{P}Q_0. \end{array}$$

From (1.4.1) it is clear that the pullback square on the right is precisely the definition of $u!\mathsf{Q}g$. From (1.4.1) and (1.4.4) we have that the square on the left is the naturality square for ϕ at H, and moreover that ϕ_X and ϕ_Y are projections. But $\mathsf{P}u_!H = \mathsf{P}h$ and $\mathsf{P}g \circ \mathsf{P}h = \mathsf{P}f$, so that the composite square is precisely the definition of $u!\mathsf{Q}f$, which is a pullback. As a consequence, the naturality square is a pullback too. □

Given a P-operad Q, we define its *two-sided bar construction* [24, 32, 46]

$$\mathcal{B}Q : \mathbb{1}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

as the two-sided bar construction of Q, ϕ and the terminal algebra 1. This means that the space of n-simplicies $\mathcal{B}_n Q$ is given by

$$\mathsf{P}u_!\mathsf{Q}^n 1,$$

the inner face maps are given by the monad multiplication μ^{Q}, the bottom face map is given by $c : \mathsf{Q}1 \to 1$ and the top face maps are given by ϕ and μ. Similarly, the degeneracy maps are given by η^{Q}. Diagrams (1.4.5) and (1.4.6) and the monad axioms for P and Q guarantee that the simplicial identities are satisfied.

In practice, the bar construction of Q is simply

$$\mathsf{P}Q_0 \Lleftarrow \mathsf{P}Q_1 \Lleftarrow \mathsf{P}Q_2 \Lleftarrow \mathsf{P}Q_3 \Lleftarrow \cdots, \tag{1.4.7}$$

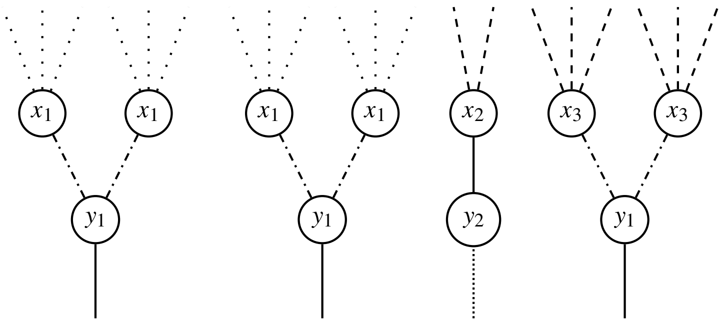
where

(i) $Q_2 := \mathsf{P}Q_1 \underset{\mathsf{P}Q_0}{\times} Q_1$ and $Q_3 := \mathsf{P}^2 Q_1 \underset{\mathsf{P}^2 Q_0}{\times} \mathsf{P}Q_1 \underset{\mathsf{P}Q_0}{\times} Q_1$, etc.;
(ii) the bottom face maps d_0 are induced by t;
(iii) the top face maps d_n are induced by s and μ;
(iv) the inner face maps are induced by m and μ, and
(v) the degeneracy maps are induced by e and η.

Henceforth we may indiscriminately use this simplicial notation. Let us see some examples.

Example 1.4.3 Let C be a small category. Hence C is a Id-operad in **Set**. Then $\mathcal{B}C$ is the nerve of C. Moreover, we can consider C as a category internal to **Grpd** whose groupoid of objects has as morphisms the isomorphisms of C, and whose groupoid of arrows has as morphisms the isomorphisms of the arrow category of C. In this case $\mathcal{B}C$ is the fat nerve of C, whose groupoid of n-simplices is the groupoid $\mathrm{Map}(\Delta[n], C)$. In the theory of incidence coalgebras, this is often more interesting than the ordinary nerve, cf. [9, 17, 18]

Example 1.4.4 If Q is a symmetric operad, as in Example 1.3.7, then $\mathcal{B}Q$ is the usual operadic two-sided bar construction. Its n-simplices have as objects forests of n-level Q-trees, and as morphisms permutations at each level. For example, the following picture



is an object of $\mathsf{P}Q_2$ with $(2! \cdot 2!^2 \cdot 3!^4) \cdot (2!) \cdot (2! \cdot 2!^2)$ automorphisms.

The following result is a reformulation of [46, Proposition 4.4.1] and [24, Proposition 3.3] in the context of P-operads.

Proposition 1.4.5 *The simplicial object $\mathcal{B}Q$ is a strict category object.*

Proof We have to check that the squares

$$\begin{array}{ccc} \mathcal{B}_{n+2}Q & \xrightarrow{d_0} & \mathcal{B}_{n+1}Q \\ {\scriptstyle d_{n+2}}\downarrow & & \downarrow{\scriptstyle d_{n+1}} \\ \mathcal{B}_{n+1}Q & \xrightarrow[d_0]{} & \mathcal{B}_nQ. \end{array} \tag{1.4.8}$$

are pullbacks for $n \geq 0$. We show the case $n = 0$, the rest are similar. The square is given by

$$\begin{array}{ccc} \mathsf{P}u_!\mathsf{Q}\mathsf{Q}1 & \xrightarrow{\mathsf{P}u_!\mathsf{Q}c} & \mathsf{P}u_!\mathsf{Q}1 \\ {\scriptstyle \mathsf{P}(\phi_{\mathsf{Q}1})}\downarrow & & \downarrow{\scriptstyle \mathsf{P}(\phi_1)} \\ \mathsf{P}\mathsf{P}u_!\mathsf{Q}1 & \xrightarrow{\mathsf{P}\mathsf{P}u_!c} & \mathsf{P}\mathsf{P}u_!1 \\ {\scriptstyle \mu^{\mathsf{P}}_{u!\mathsf{Q}1}}\downarrow & & \downarrow{\scriptstyle \mu^{\mathsf{P}}_{u!1}} \\ \mathsf{P}u_!\mathsf{Q}1 & \xrightarrow[\mathsf{P}u_!c]{} & \mathsf{P}u_!1. \end{array} \tag{1.4.9}$$

The bottom square is cartesian because it is a naturality square for μ^{P}, and P is a cartesian monad. The top square is P applied to a naturality square of ϕ, which is cartesian, by Lemma 1.4.2. Since P preserves pullbacks, the square is cartesian. □

This allows to obtain the following result, in the special case where $\mathcal{E} = \mathbf{Grpd}$.

Proposition 1.4.6 *Let* $\mathsf{P} : \mathbf{Grpd} \to \mathbf{Grpd}$ *be a cartesian monad that preserves fibrations. Let* Q *be a* P*-operad such that* Q_0 *is a discrete groupoid. Then the simplicial groupoid* $\mathcal{B}Q$ *is a Segal groupoid.*

Proof It is enough to see that the strict pullbacks 1.4.8 are also homotopy pullbacks. For $n = 0$, notice that $\mathsf{P}u_!\mathsf{Q}1 \xrightarrow{\mathsf{P}u_!c} \mathsf{P}u_!1$ is precisely the map $\mathsf{P}Q_1 \xrightarrow{\mathsf{P}m} \mathsf{P}Q_0$. But since Q_0 is discrete m is a fibration, which means that $\mathsf{P}m$ is a fibration, because P preserves fibrations. This implies that the square is also a homotopy pullback. Moreover, since pullbacks preserve fibrations the map $\mathsf{P}u_!\mathsf{QQ}1 \xrightarrow{\mathsf{P}u_!\mathsf{Q}c} \mathsf{P}u_!\mathsf{Q}1$ is again a fibration. The same argument then implies that the square for $n = 1$ is also a homotopy pullback, and so on. □

Suppose now that $\mathsf{R} : \mathcal{E} \to \mathcal{E}$ is another cartesian monad and that there is a cartesian monad map $\mathsf{P} \overset{\psi}{\Rightarrow} \mathsf{R}$. Then we can take the bar construction over R

$$\mathcal{B}^{\mathsf{R}}Q : \mathbb{1}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

whose n-simplices are given by

$$\mathsf{R}u_!\mathsf{Q}^n 1 \ \text{(or } \mathsf{R}Q_n).$$

In this case all the face maps coincide with the previous ones except the top face map, which is given by

$$\mathsf{R}u_!\mathsf{Q}^{n+1}1 \xrightarrow{\mathsf{R}(\phi_{Q1})} \mathsf{RP}u_!\mathsf{Q}^n 1 \xrightarrow{\mathsf{R}(\psi_{u_!\mathsf{Q}^n 1})} \mathsf{RR}u_!\mathsf{Q}^n 1 \xrightarrow{\mu^{\mathsf{R}}_{u_!\mathsf{Q}^n 1}} \mathsf{R}u_!\mathsf{Q}^n 1.$$

Since ψ is cartesian the simplicial object $\mathcal{B}^{\mathsf{R}}$ is also a strict category object. Moreover, if R preserves fibrations, it is a Segal groupoid, for the same reason as $\mathcal{B}Q$ in Proposition 1.4.6. The main examples of this bar construction that we use come from the natural transformations $\mathsf{M}^{\mathsf{r}} \Rightarrow \mathsf{S}$, $\mathsf{S}^{\mathsf{r}} \Rightarrow \mathsf{S}$ and $\mathsf{M}^{\mathsf{r}} \Rightarrow \mathsf{M}$, as in [24].

2 Segal groupoids and incidence coalgebras

Throughout this section, pullbacks and fibers of groupoids refer to homotopy pullbacks and homotopy fibers. A brief introduction to the homotopy approach to groupoids in combinatorics can be found in [15, §3].

2.1 Segal groupoids

A simplicial groupoid $X\colon \mathbb{1}^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$ is a *Segal space* [17, §2.9, Lemma 2.10] if the following square is a pullback for all $n > 0$:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_0} & X_n \\ {\scriptstyle d_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_n} \\ X_n & \xrightarrow[d_0]{} & X_{n-1}. \end{array} \tag{2.1.1}$$

Segal spaces arise prominently through the fat nerve construction: the fat nerve of a category $\mathcal{C}$ is the simplicial groupoid $X = N\mathcal{C}$ with $X_n = \mathrm{Fun}([n], \mathcal{C})^{\simeq}$, the groupoid of functors $[n] \to \mathcal{C}$. In this case the pullbacks above are strict, so that all the simplices are strictly determined by X_0 and X_1, respectively the objects and arrows of $\mathcal{C}$, and the inner face maps are given by composition of arrows in $\mathcal{C}$. In the general case, X_n is determined from X_0 and X_1 only up to equivalence, but one may still think of it as a "category" object whose composition is defined only up to equivalence.

Remark 2.1.1 Despite the Segal conditions (2.1.1) require the squares to be homotopy pullbacks, if the top or bottom face maps are fibrations, the ordinary pullbacks are also homotopy pullbacks. In the present work, homotopy pullbacks mostly arise in this way.

2.2 Incidence coalgebras

Let X be a simplicial groupoid. The spans

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1, \qquad X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1,$$

define two functors

$$\begin{array}{rcl} \Delta\colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd}_{/X_1\times X_1} \\ S \xrightarrow{s} X_1 & \longmapsto & (d_2,d_0)_! \circ d_1^*(s), \end{array} \qquad \begin{array}{rcl} \epsilon\colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd} \\ S \xrightarrow{s} X_1 & \longmapsto & t_! \circ s_0^*(s). \end{array}$$

Recall that upperstar is homotopy pullback and lowershriek is postcomposition. This is the general way in which spans interpret homotopy linear algebra [16].

Segal spaces are a particular case of decomposition spaces [17, Proposition 3.7], simplicial groupoids with the property that the functor Δ is coassociative with the functor ϵ as counit (up to homotopy). In this case Δ and ϵ endow $\mathbf{Grpd}_{/X_1}$ with a coalgebra structure [17, §5] called the *incidence coalgebra* of X. Note that in the special case where X is the nerve of a poset, this construction becomes the classical incidence coalgebra [39, 40] construction after taking cardinality, as we shall do shortly.

The morphisms of decomposition spaces that induce coalgebra homomorphisms are the so-called *CULF* functors [17, §4], standing for conservative and unique-lifting-of-factorisations. A Segal space X is *CULF monoidal* if it is a monoid object in the monoidal category $(\mathbf{Dcmp}^{\mathrm{CULF}}, \times, 1)$ of decomposition spaces and CULF functors [17, §9]. More concretely, it is CULF monoidal if there is a product $X_n \times X_n \to X_n$ for each n, compatible with the degeneracy and face maps, and such that for all n the squares

$$\begin{array}{ccc} X_n \times X_n & \xrightarrow{g\times g} & X_1 \times X_1 \\ \downarrow & \lrcorner & \downarrow \\ X_n & \xrightarrow[g]{} & X_1 \end{array} \tag{2.2.1}$$

are pullbacks [17, §4]. Here g is induced by the unique endpoint-preserving map $[1] \to [n]$. For example the fat nerve of a monoidal extensive category is a CULF monoidal Segal space. Recall that a category $\mathcal{C}$ is monoidal extensive if it is monoidal $(\mathcal{C}, +, 0)$ and the natural functors $\mathcal{C}_{/A} \times \mathcal{C}_{/B} \to \mathcal{C}_{/A+B}$ and $\mathcal{C}_{/0} \to 1$ are equivalences.

If X is CULF monoidal then the resulting coalgebra is in fact a bialgebra [17, §9], with product given by

$$\begin{array}{ccccc} \odot\colon \mathbf{Grpd}_{/X_1} \otimes \mathbf{Grpd}_{/X_1} & \xrightarrow{\sim} & \mathbf{Grpd}_{/X_1\times X_1} & \xrightarrow{+_!} & \mathbf{Grpd}_{/X_1} \\ (G \to X_1) \otimes (H \to X_1) & \longmapsto & G \times H \to X_1 \times X_1 & \longmapsto & G \times H \to X_1. \end{array}$$

Briefly, a product in X_n compatible with the simplicial structure endows X with a product, but in order to be compatible with the coproduct it has to satisfy the diagram (2.2.1) (i.e. it has to be a CULF functor).

2.3 Homotopy cardinality

A groupoid X is *finite* if $\pi_0(X)$ is a finite set and $\pi_1(x) = \mathrm{Aut}(x)$ is a finite group for every point x. If only the latter is satisfied then it is called *locally finite*. A morphism of groupoids is called finite when all its fibers are finite. The *homotopy cardinality* [1], [16, §3] of a finite goupoid X is defined as

$$|X| := \sum_{x\in\pi_0 X} \frac{1}{|\,\mathrm{Aut}(x)|} \in \mathbb{Q},$$

and the homotopy cardinality of a finite map of groupoids $A \xrightarrow{p} B$ is

$$|p| := \sum_{b\in\pi_0 B} \frac{|A_b|}{|\,\mathrm{Aut}(b)|}\delta_b,$$

in $\mathbb{Q}_{\pi_0 B}$, the vector space spanned by $\pi_0 B$. In this sum, A_b is the homotopy fiber and δ_b is a formal symbol representing the isomorphism class of b. A simple computation shows that $|1 \xrightarrow{\ulcorner b \urcorner} B| = \delta_b$.

A Segal space X is *locally finite* [18, §7] if X_1 is a locally finite groupoid and both $s_0\colon X_0 \to X_1$ and $d_1\colon X_2 \to X_1$ are finite maps. In this case one can take homotopy cardinality to get a comultiplication

$$\begin{aligned} \Delta\colon \mathbb{Q}_{\pi_0 X_1} &\longrightarrow \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ |S \xrightarrow{s} X_1| &\longmapsto |(d_2, d_0)_! \circ d_1^*(s)| \end{aligned}$$

and similarly for ϵ (cf. [18, §7]). Moreover, if X is CULF monoidal then $\mathbb{Q}_{\pi_0 X_1}$ acquires a bialgebra structure with the product $\cdot = |\odot|$. In particular, if we denote by $+$ the monoidal product in X, then $\delta_a \cdot \delta_b = \delta_{a+b}$ for any $|1 \xrightarrow{\ulcorner a \urcorner} X_1|$ and $|1 \xrightarrow{\ulcorner b \urcorner} X_1|$. The following result gives a closed formula for the computation of the comultiplication when X is a Segal space.

Lemma 2.3.1 *([9, 19]) Let X be a Segal space. Then for f in X_1 we have*

$$\Delta(\delta_f) = \sum_{b\in\pi_0 X_1} \sum_{a\in\pi_0 X_1} \frac{|\,\mathrm{Iso}(d_0 a, d_1 b)_f|}{|\,\mathrm{Aut}(b)||\,\mathrm{Aut}(a)|}\delta_a \otimes \delta_b,$$

where $\mathrm{Iso}(d_0 a, d_1 b)$ *is the set of morphisms from $d_0 a$ to $d_1 b$ and* $\mathrm{Iso}(d_0 a, d_1 b)_f$ *is its homotopy fiber along d_1.*

3 The $\mathcal{T}$-construction

Throughout this section (P, μ, η) is a cartesian strong monad on a cartesian category $\mathcal{E}$, and category means a category internal to $\mathcal{E}$. As mentioned in the introduction, the $\mathcal{T}$-construction consists of two constructions, one from internal categories to P-operads and another from P-operads to categories. With the purpose of reducing the diagrams and fiber products, we use the following notation for the endofunctors and natural transformations featuring in this section,

$$\begin{array}{lcl} \mathsf{L} : \mathcal{E} \longrightarrow \mathcal{E} & \qquad & F : \mathsf{Id} \Longrightarrow \mathsf{L} \\ A \longmapsto A \times \mathsf{P}1, & & F_A : A \times 1 \xrightarrow{\mathrm{id}\times\eta_1} \mathsf{L}A, \\ & & \\ D : \mathsf{L} \Longrightarrow \mathsf{P} & & R : \mathsf{L} \Longrightarrow \mathsf{Id} \\ D_A : A \times \mathsf{P}1 \longrightarrow \mathsf{P}A, & & R_A : A \times \mathsf{P}1 \xrightarrow{p_1} A. \end{array}$$

Observe that L is cartesian as a functor. Also, notice that R and F are cartesian natural transformations. Finally, by monomorphism we refer to the 1-categorical notion. In the case of most interest where $\mathcal{E}$ is **Set** or **Grpd**, this means injective on objects and injective on arrows.

The material of this section is highly technical. The casual or application-oriented reader might wish to regard it as a black box and take on faith the well-definedness of the constructions, and still be able to appreciate the examples worked out in Chapters 4 and 5.

3.1 From categories to P-operads

Let C be a category such that $D_{C_0} : \mathsf{P}1 \times C_0 \to \mathsf{P}C_0$ is a monomorphism. It is convenient in this section to adopt a simplicial nomenclature. Hence C is represented by the span

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle d_1}\swarrow & & \searrow{\scriptstyle d_0} \\ C_0 & & C_0 \end{array} \qquad \begin{array}{l} C_1 \times_{C_0} C_1 =: C_2 \xrightarrow{d_1} C_1 \\ \\ C_0 \xrightarrow{\quad e \quad} C_1, \end{array}$$

with the only inconvenience that some of the face maps share their names. Notice that we still denote by e the degeneracy map s_0. We now construct a P-operad $\mathcal{T}_{\mathsf{P}}C$ from the category C. To keep notation short, the simplicial nomenclature for $\mathcal{T}_{\mathsf{P}}C$ is $\widetilde{C}_i$ for the simplices and $\widetilde{d}_i$ for the face maps. The span defining the objects and operations of $\mathcal{T}_{\mathsf{P}}C$ is given by the pullback

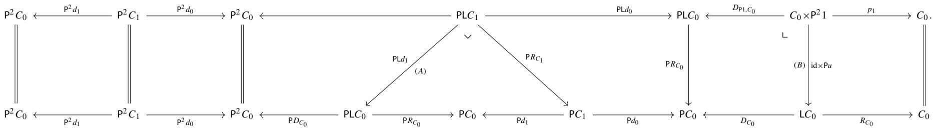
$$\begin{array}{ccccc} & & \widetilde{C}_1 & & \\ & {\scriptstyle i_1}\swarrow & \lrcorner & \searrow & \\ & \mathsf{P}C_1 & & \mathsf{L}C_0 & \\ {\scriptstyle \mathsf{P}d_1}\swarrow & & \searrow{\scriptstyle \mathsf{P}d_0} \quad {\scriptstyle D_{C_0}}\swarrow & & \searrow{\scriptstyle R_{C_0}} \\ \mathsf{P}C_0 & & \mathsf{P}C_0 & & C_0. \end{array} \qquad \widetilde{d}_1 : \widetilde{C}_1 \to \mathsf{P}C_0, \quad \widetilde{d}_0 : \widetilde{C}_1 \to C_0 \tag{3.1.1}$$

Observe that $\widetilde{C}_0 = C_0$, so that $\mathcal{T}_{\mathsf{P}}C$ has the same objects as C. Besides, the morphism i_1 is a monomorphism, since monomorphisms are preserved by pullbacks and D_{C_0} is a monomorphism.

To define composition we need to specify a map $\widetilde{C}_2 \xrightarrow{\widetilde{d}_1} \widetilde{C}_1$, where $\widetilde{C}_2 := \mathsf{P}\widetilde{C}_1 \underset{\mathsf{P}C_0}{\times} \widetilde{C}_1$, satisfying the axioms of "Appendix A.1".

However, to describe it we have to express $\widetilde{C}_2$ in a way we can naturally use composition in the original category $C_2 \xrightarrow{d_1} C_1$. The following diagram represents an isomorphism

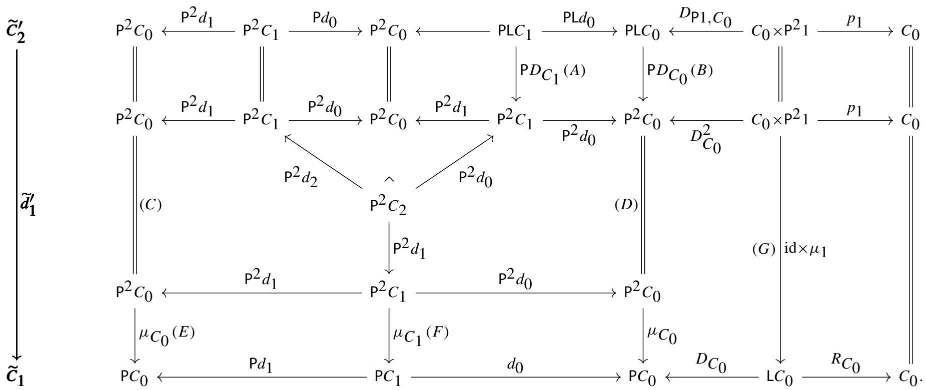
$$\widetilde{C}_2 \cong \mathsf{P}^2 C_1 \underset{\mathsf{P}^2 C_0}{\times} \mathsf{PL}C_1 \underset{\mathsf{PL}C_0}{\times} (C_0 \times \mathsf{P}^2 1) =: \widetilde{C}_2',$$



(3.1.2)

It is clear that all the squares in (3.1.2) commute. Moreover, the square (A) is cartesian because R and P are cartesian, and the square (B) is the same as (1.3.3) of Lemma 1.3.2.

Definition 3.1.1 The composition of $\mathcal{T}_{\mathsf{P}}C$ is given by the following arrow $\widetilde{C}'_2 \xrightarrow{\widetilde{d}'_1} \widetilde{C}_1$,



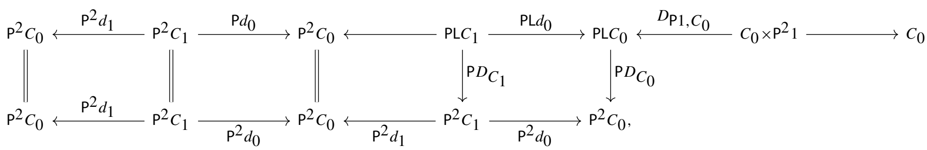
(3.1.3)

It is clear that the diagram commutes: (A) is P applied to a naturality square of D; (B) is the definition of D^2; (C) and (D) are P^2 applied to axioms (A.1.1a) and (A.1.1b) for composition in C; (E) and (F) are naturality squares of μ, and (G) is again axiom (1.3.2b) for strong monads. The remaining squares are trivial.

Notice that from this definition it is clear that $\widetilde{d}_1$ satisfies axioms (A.2.1a) and (A.2.1b). Furthermore, there is a map

$$\widetilde{C}'_2 \xrightarrow{i'_2} \mathsf{P}^2 C_2,$$

given by the diagram



which clearly makes the square

$$\begin{array}{ccc} \widetilde{C}'_2 & \xrightarrow{i'_2} & \mathsf{P}^2C_2 \\ & & \downarrow{\scriptstyle \mathsf{P}^2 d_1} \\ \widetilde{d}'_1 \downarrow & & \mathsf{P}^2C_1 \\ & & \downarrow{\scriptstyle \mu_{C_1}} \\ \widetilde{C}_1 & \underset{i_1}{\rightarrowtail} & \mathsf{P}C_1, \end{array} \quad \text{and therefore also the square} \quad \begin{array}{ccc} \widetilde{C}_2 & \xrightarrow{i_2} & \mathsf{P}^2C_2 \\ & & \downarrow{\scriptstyle \mathsf{P}^2 d_1} \\ \widetilde{d}_1 \downarrow & & \mathsf{P}^2C_1 \\ & & \downarrow{\scriptstyle \mu_{C_1}} \\ \widetilde{C}_1 & \underset{i_1}{\rightarrowtail} & \mathsf{P}C_1, \end{array} \tag{3.1.4}$$

commute, for the corresponding arrow i_2. This says, roughly speaking, that composition in $\mathcal{T}_{\mathsf{P}}C$ is "the same" as composition in P^2C, as it is also clear in most of the examples.

We have to check that composition is associative (A.2.3). We state first the following lemma. We omit its proof since it is long and unenlightening; it can be found in [10].

Lemma 3.1.2 *There is a map $\widetilde{C}_3 \xrightarrow{i_3} P^3C_3$ such that the following diagrams commute*

$$\begin{array}{ccc} \widetilde{C}_3 & \xrightarrow{i_3} & P^3C_3 \\ & & \downarrow{\scriptstyle P^3 d_1} \\ \widetilde{d}_1 \downarrow & & P^3C_2 \\ & & \downarrow{\scriptstyle P\mu_{C_2}} \\ \widetilde{C}_2 & \underset{i_2}{\longrightarrow} & P^2C_2, \end{array} \quad \begin{array}{ccc} \widetilde{C}_3 & \xrightarrow{i_3} & P^3C_3 \\ & & \downarrow{\scriptstyle P^3 d_2} \\ \widetilde{d}_2 \downarrow & & P^3C_2 \\ & & \downarrow{\scriptstyle \mu_{PC_2}} \\ \widetilde{C}_2 & \underset{i_2}{\longrightarrow} & P^2C_2. \end{array} \tag{3.1.5}$$

Proposition 3.1.3 *Composition is associative.*

Proof In view of Lemma 3.1.2 there is a diagram

$$\begin{array}{ccccccc} \widetilde{C}_3 & & & \xrightarrow{\widetilde{d}_1} & & & \widetilde{C}_2 \\ & \searrow^{i_3} & & & & \swarrow^{i_2} & \\ & & \mathsf{P}^3C_3 & \xrightarrow{\mathsf{P}^3d_1} \mathsf{P}^3C_2 \xrightarrow{\mathsf{P}\mu_{C_2}} & \mathsf{P}^2C_2 & & \\ & & {\scriptstyle \mathsf{P}^3d_2}\downarrow(A) & \downarrow{\scriptstyle \mathsf{P}^3d_1}\,(B) & \downarrow{\scriptstyle \mathsf{P}^2d_1} & & \\ \widetilde{d}_2 \downarrow & & \mathsf{P}^3C_2 & \xrightarrow{\mathsf{P}^3d_1} \mathsf{P}^3C_1 \xrightarrow{\mathsf{P}\mu_{C_1}} & \mathsf{P}^2C_1 & & \downarrow \widetilde{d}_1 \\ & & {\scriptstyle \mu_{\mathsf{P}C_2}}\downarrow(C) & \downarrow{\scriptstyle \mu_{\mathsf{P}C_1}}\,(D) & \downarrow{\scriptstyle \mu_{C_1}} & & \\ & & \mathsf{P}^2C_2 & \xrightarrow{\mathsf{P}^2d_1} \mathsf{P}^2C_1 \xrightarrow{\mu_{C_1}} & \mathsf{P}C_1 & & \\ & \nearrow^{i_2} & & & & \nwarrow^{i_1} & \\ \widetilde{C}_2 & & & \xrightarrow{\widetilde{d}_1} & & & \widetilde{C}_1, \end{array}$$

where the four trapeziums are diagrams (3.1.4) and (3.1.5) of Lemma 3.1.2. The inner squares are the following: (A) is P^3 applied to associativity of C; (B) is P applied to naturality of μ at d_1; (C) is naturality of μ at $\mathsf{P}d_1$ and (D) is the associativity law of μ. Since i_1 is a monomorphism (3.1.1) and all the inner diagrams commute, so does the outer square, as we wanted to see. □

The unit morphism of $\mathcal{T}_{\mathsf{P}}C$ is easier to obtain than composition. Recall that the unit is a morphism $\widetilde{e}\colon C_0 \to \widetilde{C}_1$ such that the following diagram (1) commutes,

$$
\begin{array}{ccccc}
 & & C_0 & & \\
 & \swarrow^{\eta_{C_0}} & \downarrow{\scriptstyle \widetilde{e}} & \searrow^{\mathrm{id}} & \\
\mathsf{P}C_0 & \xleftarrow[d_1]{} & \widetilde{C}_1 & \xrightarrow[d_0]{} & C_0.
\end{array}
$$

Definition 3.1.4 The unit of $\mathcal{T}_{\mathsf{P}}C$ is given by the following arrow:

$$
\begin{array}{c}
\boldsymbol{C_0} \\
\Big\downarrow{\scriptstyle \widetilde{e}} \\
\boldsymbol{\widetilde{C}_1}
\end{array}
\qquad
\begin{array}{ccccccccc}
\mathsf{P}C_0 & \xleftarrow[\eta_{C_0}]{} & C_0 & = & C_0 & = & C_0 & = & C_0 \\
\Big\| & (A) & \big\downarrow{\scriptstyle \eta_{C_0}} & (B) & \Big\downarrow{\scriptstyle \eta_{C_0}} & (C) & \Big\downarrow{\scriptstyle F_{C_0}} & (D) & \Big\| \\
 & & \mathsf{P}C_0 & & & & & & \\
 & & \big\downarrow{\scriptstyle \mathsf{P}e} & & & & & & \\
\mathsf{P}C_0 & \xleftarrow[\mathsf{P}d_1]{} & \mathsf{P}C_1 & \xrightarrow[\mathsf{P}d_0]{} & \mathsf{P}C_0 & \xleftarrow[D_{C_0}]{} & \mathsf{L}C_0 & \xrightarrow[R_{C_0}]{} & C_0.
\end{array}
\tag{3.1.6}
$$

It is clear that all the diagrams commute: (A) and (B) come from P applied to (A.1.2a) and (A.1.2b), this is $d_1 \circ e = \mathrm{id} = d_0 \circ e$; (C) is the compatibility between D and η (1.3.2a), and (D) is obvious from the definitions of R and F.

We have to verify now that composition with the unit morphism is the identity (1). To prove it we follow the same strategy as for associativity. That is, we project the diagrams into diagrams in the original category C containing the corresponding unit axioms. Again, the proof of the following lemma can be found in [10]. Recall first that

$$
C_2 := C_1 \times_{C_0} C_1 \quad \text{and} \quad \widetilde{C}_2 := \mathsf{P}\widetilde{C}_1 \underset{\mathsf{P}C_0}{\times} \widetilde{C}_1.
$$

Lemma 3.1.5 *We have commutative squares*

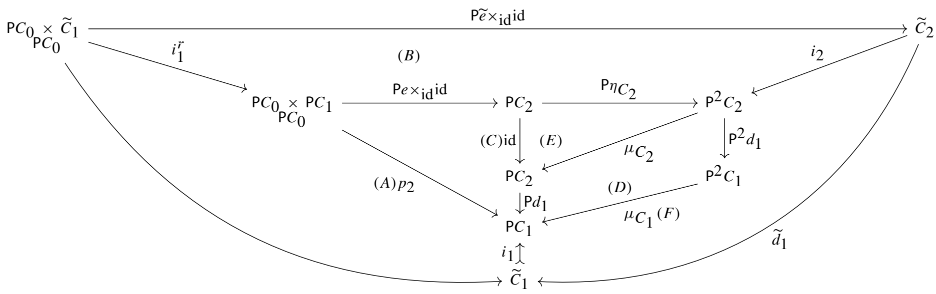
$$
\begin{array}{ccc}
PC_0 \underset{PC_0}{\times} \widetilde{C}_1 & \xrightarrow{i_1^l} & PC_0 \underset{PC_0}{\times} PC_1 \\
 & & \big\downarrow{\scriptstyle Pe \times_{id} id} \\
\Big\downarrow{\scriptstyle P\widetilde{e} \times_{id} id} & & PC_2 \\
 & & \big\downarrow{\scriptstyle P\eta_{C_2}} \\
\widetilde{C}_2 & \xrightarrow[i_2]{} & P^2 C_2,
\end{array}
\tag{3.1.7a}
$$

$$
\begin{array}{ccc}
\widetilde{C}_1 \underset{C_0}{\times} C_0 & \xrightarrow{i_1^r} & PC_1 \underset{PC_0}{\times} PC_0 \\
 & & \big\downarrow{\scriptstyle id \times_{id} Pe} \\
\Big\downarrow{\scriptstyle \eta_{\widetilde{C}_1} \times_{\eta_{C_0}} \widetilde{e}} & & PC_2 \\
 & & \big\downarrow{\scriptstyle \eta_{PC_2}} \\
\widetilde{C}_2 & \xrightarrow[i_2]{} & P^2 C_2,
\end{array}
\tag{3.1.7b}
$$

where i_1^l and i_1^r are the morphisms corresponding to i_1.

Proposition 3.1.6 *The unit morphism $\widetilde{e}$ of $\mathcal{T}_P C$ satisfies the left and right composition axioms (1).*

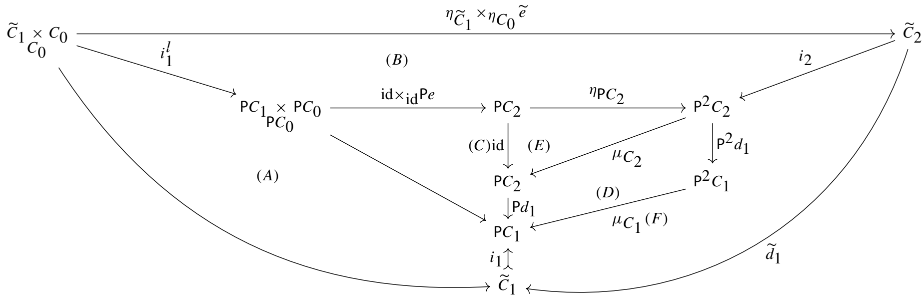
Proof For the left composition (A.2.4a), the required commutative triangle is the outline of the diagram



(3.1.8)

We have that Diagram (A) commutes by definition of i_1^l; (B) is precisely (3.1.7a) of Lemma 3.1.5; (C) is P applied to the left composition with the unit axiom in the category C (A.1.4a); (D) is naturality of μ at d_1; (E) is P of the unit axiom of P applied to C_2, and (F) is the same as (3.1.4). Since i_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see.

For the right composition (A.2.4b), the required commutative triangle is the outline of the diagram



(3.1.9)

We have that Diagram (A) commutes by definition of i_1^r, (B) is precisely (3.1.7a) of Lemma 3.1.5, (C) is P applied to the right composition with the unit axiom in the category C (A.1.4b); (D) is again naturality of μ at d_1, (E) is the unit axiom of P applied to $\mathsf{P}C_2$ and (F) is the same as (3.1.4), as before. Since i_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see. □

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let C and C' be two categories and $C \xrightarrow{f} B$ a functor, that is a diagram

$$\begin{array}{ccccc} C_0 & \xleftarrow{d_1} & C_1 & \xrightarrow{d_0} & C_0 \\ f_0\downarrow & & \downarrow f_1 & & \downarrow f_0 \\ B_0 & \xleftarrow{d_1} & B_1 & \xrightarrow{d_0} & B_0 \end{array}$$

satisfying the commutative squares of 1.1.1. Then $\mathcal{T}_{\mathsf{P}} f$ is the morphism given by

$$
\begin{array}{ccccccccccc}
\mathcal{T}_{\mathsf{P}} C & & \mathsf{P}C_0 & \xleftarrow{\mathsf{P}d_1} & \mathsf{P}C_1 & \xrightarrow{\mathsf{P}d_0} & \mathsf{P}C_0 & \xleftarrow{D_{C_0}} & \mathsf{L}C_0 & \xrightarrow{R_{C_0}} & C_0 \\
\downarrow{\scriptstyle \mathcal{T}_{\mathsf{P}} f} & & \downarrow{\scriptstyle \mathsf{P}f_0} & & \downarrow{\scriptstyle \mathsf{P}f_1} & & \downarrow{\scriptstyle \mathsf{P}f_0} & & \downarrow{\scriptstyle \mathsf{L}f_0} & & \downarrow{\scriptstyle f_0} \\
\mathcal{T}_{\mathsf{P}} B & & \mathsf{P}B_0 & \xleftarrow{\mathsf{P}d_1} & \mathsf{P}B_1 & \xrightarrow{\mathsf{P}d_0} & \mathsf{P}B_0 & \xleftarrow{D_{B_0}} & \mathsf{L}B_0 & \xrightarrow{R_{B_0}} & B_0 .
\end{array}
$$

It is a bit tedious but not difficult to see that $\widetilde{f}$ satisfies again the commutative squares of 1.1.1 [10]. Moreover, given another morphism $B \xrightarrow{g} A$ it is clear that $\mathcal{T}_{\mathsf{P}}(g \circ f) = \mathcal{T}_{\mathsf{P}} g \circ \mathcal{T}_{\mathsf{P}} f$, just because of the functoriality of P and L.

Since the construction is functorial, if the strength D_A is a monomorphism for every object $A \in \mathcal{E}$ then $\mathcal{T}^{\mathsf{P}}$ is in fact a functor from P-operads to categories internal to $\mathcal{E}$.

3.2 From P-operads to categories

This construction has a similar structure as the construction above, so we follow the same steps. Let Q be a P-operad,

$$
\mathsf{P}Q_0 \xleftarrow{d_1} Q_1 \xrightarrow{d_0} Q_0 \qquad \mathsf{P}Q_1 \underset{\mathsf{P}Q_0}{\times} Q_1 := Q_2 \xrightarrow{d_1} Q_1 \qquad Q_0 \xrightarrow{e} Q_1,
$$

and assume that $D_{Q_0} : \mathsf{P}1 \times Q \to \mathsf{P}Q$ is a monomorphism. We construct a category $\mathcal{T}^{\mathsf{P}} Q$ from the P-operad Q. In this case, the simplicial nomenclature for $\mathcal{T}^{\mathsf{P}} Q$ is $\overline{Q}_i$ for the simplices and $\overline{d}_i$ for the face maps. The following pullback defines the objects and arrows of $\mathcal{T}^{\mathsf{P}} Q$:

$$
\begin{array}{ccccc}
 & & \overline{C}_1 & & \\
 & \swarrow & \lrcorner & \searrow{\scriptstyle j_1} & \\
 & \mathsf{L}C_0 & & C_1 & \\
\swarrow{\scriptstyle R_{C_0}} & & \searrow{\scriptstyle D_{C_0}} \quad \swarrow{\scriptstyle d_1} & & \searrow{\scriptstyle d_0} \\
C_0 & & \mathsf{P}C_0 & & C_0 .
\end{array}
\qquad (\overline{d}_1 : \overline{C}_1 \to C_0,\ \overline{d}_0 : \overline{C}_1 \to C_0) \tag{3.2.1}
$$

Observe that $\overline{Q}_0 = Q_0$, so that again $\mathcal{T}^{\mathsf{P}} Q$ has the same objects as Q. Besides, the morphism j_1 is a monomorphism, since monomorphisms are preserved by pullbacks and D_{Q_0} is a monomorphism.

To define composition we need to define a map $\overline{Q}_2 \xrightarrow{\overline{d}_1} \overline{Q}_1$, where $\overline{Q}_2 := \overline{Q}_1 \times_{Q_0} \overline{Q}_1$, satisfying the axioms of "Appendix 1". However, to specify this map we need to express it in a way we can naturally use composition in the original P-operad $Q_2 \xrightarrow{d_1} Q_1$. The following diagram represents an isomorphism

$$
\overline{Q}_2 \cong \mathsf{L}^2 Q_0 \underset{\mathsf{PL}Q_0}{\times} \mathsf{L}Q_1 \underset{\mathsf{P}Q_0}{\times} Q_1 =: \overline{Q}'_2,
$$

$$
\begin{array}{ccccccccccccc}
Q_0 & \longleftarrow & \mathsf{L}^2 Q_0 & \xrightarrow{\mathsf{L}D_{Q_0}} & \mathsf{LP}Q_0 & \xleftarrow{\mathsf{L}d_1} & & \mathsf{L}Q_1 & & \longrightarrow & \mathsf{P}Q_0 & \xleftarrow{d_1} Q_1 \xrightarrow{d_0} & Q_0 . \\
\Vert & & \downarrow{\scriptstyle R_{\mathsf{L}Q_0}} & & (A)\downarrow{\scriptstyle R_{\mathsf{P}Q_0}} & & \swarrow{\scriptstyle R_{Q_1}} & (B) & \searrow{\scriptstyle \mathsf{L}d_0} & & \Vert & \Vert & \Vert \\
Q_0 & \xleftarrow[R_{Q_0}]{} & \mathsf{L}Q_0 & \xrightarrow[D_{Q_0}]{} & \mathsf{P}Q_0 & \xleftarrow[d_1]{} Q_1 \xrightarrow[d_0]{} & Q_0 & \xleftarrow[R_{Q_0}]{} & \mathsf{L}Q_0 & \xrightarrow[D_{Q_0}]{} & \mathsf{P}Q_0 & \xleftarrow[d_1]{} Q_1 \xrightarrow[d_0]{} & Q_0
\end{array}
\tag{3.2.2}
$$

It is clear that all the squares in (3.2.2) commute. Moreover, the squares (A) and (B) are cartesian because so is R.

Definition 3.2.1 The composition of $\overline{Q}$ is given by the following arrow $\overline{Q}'_2 \xrightarrow{\overline{d}'_1} \overline{Q}_1$,

$$
\begin{array}{c|ccccccccccccc}
\overline{Q}'_2 & Q_0 & \longleftarrow & \mathsf{L}^2 Q_0 & \xrightarrow{\mathsf{L}D_{Q_0}} & \mathsf{LP}Q_0 & \xleftarrow{\mathsf{L}d_1} & \mathsf{L}Q_1 & \longrightarrow & \mathsf{P}Q_0 & \xleftarrow{d_1} & Q_1 & \xrightarrow{d_0} & Q_0 \\
\downarrow & \Vert & & \downarrow{\scriptstyle \mathrm{id}\times D_{\mathsf{P}1}} & (A) & \downarrow{\scriptstyle D_{\mathsf{P}Q_0}} & (B) & \downarrow{\scriptstyle D_{Q_1}} & & \Vert & & \Vert & & \Vert \\
 & \Vert & & Q_0\times\mathsf{P}^2 1 & \xrightarrow[D^2_{Q_0}]{} & \mathsf{P}^2 Q_0 & \xleftarrow[\mathsf{P}d_1]{} & \mathsf{P}Q_1 & \xrightarrow{\mathsf{P}d_0} & \mathsf{P}Q_0 & \xleftarrow{d_1} & Q_1 & \xrightarrow[d_0]{} & Q_0 \\
 & \Vert & & \downarrow{\scriptstyle \mathrm{id}\times\mu_1}\ (C) & & \downarrow{\scriptstyle \mu_{Q_0}} & (D) & \nwarrow{\scriptstyle d_2} & Q_2 & \nearrow{\scriptstyle d_0} & (E) & & & \Vert \\
 & \Vert & & & & & & & \downarrow{\scriptstyle d_1} & & & & & \Vert \\
\overline{Q}_1 & Q_0 & \xleftarrow[R_{Q_0}]{} & \mathsf{P}Q_0 & \xrightarrow[D_{Q_0}]{} & \mathsf{P}Q_0 & \xleftarrow[d_1]{} & & Q_1 & & \xrightarrow[d_0]{} & & & \mathsf{P}Q_0 .
\end{array}
\tag{3.2.3}
$$

Let us see that all the diagrams commute: (A) is a combination of natuarlity of D applied to D_{Q_0} and axiom (1.3.1.b) concerning consecutive applications of the strength, (B) is naturality of D at d_1, (C) is axiom (1.3.2b) for strong monads and (D) and (E) are respectively axioms (A.2.1a) and (A.2.1b) for composition in Q. The remaining diagrams are clear.

Notice that from this definition it is clear that $\overline{d}_1$ satisfies axioms (A.1.1a) and (A.1.1b). Furthermore, there is a morphism

$$
\overline{Q}'_2 \xrightarrow{j'_2} Q_2,
$$

given by the diagram

$$
\begin{array}{ccccccccccccc}
Q_0 & \longleftarrow & \mathsf{L}^2 Q_0 & \xrightarrow{\mathsf{L}D_{Q_0}} & \mathsf{LP}Q_0 & \xleftarrow{\mathsf{L}d_1} & \mathsf{L}Q_1 & \longrightarrow & \mathsf{P}Q_0 & \xleftarrow{d_1} & Q_1 & \xrightarrow{d_0} & Q_0 \\
 & & & & \downarrow{\scriptstyle D_{\mathsf{P}Q_0}} & & \downarrow{\scriptstyle D_{Q_1}} & & \Vert & & \Vert & & \Vert \\
 & & & & \mathsf{P}^2 Q_0 & \xleftarrow[\mathsf{P}d_1]{} & \mathsf{P}Q_1 & \xrightarrow{\mathsf{P}d_0} & \mathsf{P}Q_0 & \xleftarrow{d_1} & Q_1 & \xrightarrow[d_0]{} & Q_0 ,
\end{array}
$$

which clearly makes the square

$$
\begin{array}{ccc}
\overline{Q}'_2 & \xrightarrow{j'_2} & Q_2 \\
{\scriptstyle \overline{d}'_1}\downarrow & & \downarrow{\scriptstyle d_1} \\
\overline{Q}_1 & \underset{j_1}{\rightarrowtail} & Q_1 ,
\end{array}
\quad \text{and therefore also the square} \quad
\begin{array}{ccc}
\overline{Q}_2 & \xrightarrow{j_2} & Q_2 \\
{\scriptstyle \overline{d}_1}\downarrow & & \downarrow{\scriptstyle d_1} \\
\overline{Q}_1 & \underset{j_1}{\rightarrowtail} & Q_1 ,
\end{array}
\tag{3.2.4}
$$

commute, for the corresponding j_2. This says, roughly speaking, that composition in $\overline{Q}$ is "the same" as composition in Q, as is also clear in most of the examples.

We have to check that composition is associative (A.1.3). As in the previous subsection, the proof of the following lemma is omitted; it can be read in [10].

Lemma 3.2.2 *There is a morphism $\overline{Q}_3 \xrightarrow{j_3} Q_3$ such that the following diagrams commute*

$$\begin{array}{ccc} \overline{Q}_3 & \xrightarrow{j_3} & Q_3 \\ {\scriptstyle \overline{d}_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ \overline{Q}_2 & \xrightarrow[j_2]{} & Q_2, \end{array} \qquad \begin{array}{ccc} \overline{Q}_3 & \xrightarrow{j_3} & Q_3 \\ {\scriptstyle \overline{d}_2}\downarrow & & \downarrow{\scriptstyle d_2} \\ \overline{Q}_2 & \xrightarrow[j_2]{} & Q_2. \end{array} \tag{3.2.5}$$

Proposition 3.2.3 *Composition is associative.*

Proof In view of Lemma 3.2.2 there is a diagram

$$\begin{array}{ccccccc} \overline{C}_3 & & & \xrightarrow{\overline{d}_2} & & & \overline{C}_2 \\ & \searrow^{j_3} & & & & \swarrow_{j_2} & \\ & & C_3 & \xrightarrow{d_2} & C_2 & & \\ {\scriptstyle \overline{d}_1}\downarrow & & {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle \overline{d}_1} \\ & & C_2 & \xrightarrow[d_1]{} & C_1 & & \\ & \nearrow^{j_2} & & & & \nwarrow_{j_1} & \\ \overline{C}_2 & & & \xrightarrow[\overline{d}_1]{} & & & \overline{C}_1. \end{array}$$

The four trapeziums are the commutative diagrams (3.2.4) and (3.2.5) of Lemma 3.2.2 respectively, and the inner square is associativity of composition in C (A.2.3). Since j_1 is a monomorphism and all the inner diagrams commute, so does the outer square, as we wanted to see. □

The unit morphism of the new category is easier to obtain than composition. Recall that the unit is a morphism $e\colon Q_0 \to \overline{Q}_1$ such that this diagram (1) commutes

$$\begin{array}{ccccc} & & Q_0 & & \\ & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle e} & \searrow^{\mathrm{id}} & \\ Q_0 & \xleftarrow[\overline{d}_1]{} & \overline{Q}_1 & \xrightarrow[\overline{d}_0]{} & Q_0. \end{array}$$

Definition 3.2.4 The unit of $\overline{Q}$ is given by the following arrow:

$$\begin{array}{ccccccccc} Q_0 & = & Q_0 & = & Q_0 & = & Q_0 & = & Q_0 \\ {\scriptstyle \mathrm{id}}\downarrow{\scriptstyle (A)} & & \downarrow{\scriptstyle F_{Q_0}} & & {\scriptstyle (B)}\downarrow{\scriptstyle \eta_{Q_0}\,(C)} & & \downarrow{\scriptstyle e} & & {\scriptstyle (D)}\downarrow{\scriptstyle \mathrm{id}} \\ Q_0 & \xleftarrow[R_{Q_0}]{} & \mathsf{L}Q_0 & \xrightarrow[D_{Q_0}]{} & \mathsf{P}Q_0 & \xleftarrow[d_1]{} & Q_1 & \xrightarrow[d_0]{} & Q_0. \end{array} \tag{3.2.6}$$

It is clear that all the diagrams commute: (A) is obvious from the definitions of R and F, (B) is axiom (1.3.2a) for strong monads, and (C) and (D) are respectively the unit axioms (A.2.2b) and (A.2.2a) of Q.

We have to check that composition with the unit morphism is the identity (1). To prove it we follow the same strategy as for associativity. That is, we project the diagrams into diagrams in the original P-operad Q containing the corresponding unit axioms. The proof of the following lemma can be found in [10]. Recall first that

$$Q_2 := \mathsf{P}Q_1 \underset{\mathsf{P}Q_0}{\times} Q_1 \quad \text{and} \quad \overline{Q}_2 := \overline{Q}_1 \times_{Q_0} \overline{Q}_1.$$

Lemma 3.2.5 *We have commutative squares*

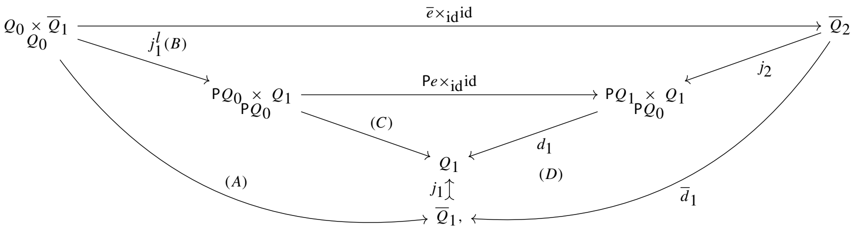
$$\begin{array}{ccc} Q_0 \underset{Q_0}{\times} \overline{Q}_1 & \xrightarrow{j_1^l} & \mathsf{P}Q_0 \underset{\mathsf{P}Q_0}{\times} Q_1 \\ {\scriptstyle \overline{e}\times_{id} id}\downarrow & & \downarrow{\scriptstyle \mathsf{P}e\times_{id} id} \\ \overline{Q}_2 & \xrightarrow[j_2]{} & Q_2, \end{array} \qquad (3.2.7a)$$

$$\begin{array}{ccc} \overline{Q}_1 \underset{Q_0}{\times} Q_0 & \xrightarrow{j_1^r} & Q_1 \underset{Q_0}{\times} Q_0 \\ {\scriptstyle id\times_{id}\overline{e}}\downarrow & & \downarrow{\scriptstyle \eta_{Q_1}\times_{\eta_{Q_0}} e} \\ \overline{Q}_2 & \xrightarrow[j_2]{} & Q_2, \end{array} \qquad (3.2.7b)$$

where j_1^l and j_1^r are the morphisms corresponding to j_1.

Proposition 3.2.6 *The unit morphism $\overline{e}$ of $\overline{Q}$ satisfies the left and right composition axioms (1).*

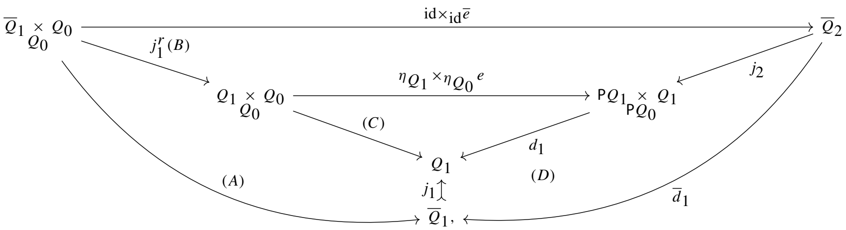
Proof For the left composition (A.1.4a), the required commutative triangle is the outline of the diagram



(3.2.8)

We have that Diagram (A) commutes by definition of j_1^l, (B) is precisely (3.2.7a) of Lemma 3.2.5, (C) is the left composition with unit axiom in the P-operad C (A.2.4a) and (D) is the same as (3.2.4). Since j_1 is a monomorphism and all the inner diagrams commute, so does the outer triangle, as we wanted to see.

For the right composition (A.1.4b), the required commutative triangle is the outline of the diagram



(3.2.9)

We have that Diagram (A) commutes by definition of j_1^r, (B) is precisely (3.2.7b) of Lemma 3.2.5, (C) is the right composition with unit axiom in the P-operad Q (A.2.4b), and (D) is the same as (3.2.4), as before. Since j_1 is a monomorphism and all the inner diagrams commute so does the outer triangle, as we wanted to see. □

The last thing to check is that the construction is functorial. First of all we have to specify how the construction acts on morphisms. Let Q and Q' be two P-operads and $Q \xrightarrow{f} B$ a morphism, that is a diagram

$$\begin{array}{ccccc} \mathsf{P}Q_0 & \xleftarrow{d_1} & Q_1 & \xrightarrow{d_0} & Q_0 \\ {\scriptstyle \mathsf{P}f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \mathsf{P}B_0 & \xleftarrow{d_1} & B_1 & \xrightarrow{d_0} & B_0 \end{array}$$

satisfying the commutative squares of 1.1.1. Then $\mathcal{T}^{\mathsf{P}} f$ is the functor given by

$$\begin{array}{ccccccccccc} \mathcal{T}^{\mathsf{P}} Q & & Q_0 & \xleftarrow[R_{Q_0}]{} & \mathsf{L}Q_0 & \xrightarrow[D_{Q_0}]{} & \mathsf{P}Q_0 & \xleftarrow[d_1]{} & Q_1 & \xrightarrow[d_0]{} & Q_0 \\ \downarrow{\scriptstyle \mathcal{T}^{\mathsf{P}} f} & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \mathsf{L}f_0} & & \downarrow{\scriptstyle \mathsf{P}f_0} & & \downarrow{\scriptstyle \mathsf{P}f_1} & & \downarrow{\scriptstyle f_0} \\ \mathcal{T}^{\mathsf{P}} B & & B_0 & \xleftarrow[R_{B_0}]{} & \mathsf{L}B_0 & \xrightarrow[D_{B_0}]{} & \mathsf{P}B_0 & \xleftarrow[d_1]{} & B_1 & \xrightarrow[d_0]{} & B_0. \end{array}$$

It is a bit tedious but not difficult to see that $\mathcal{T}^{\mathsf{P}} f$ satisfies again the commutative squares of 1.1.1 [10]. Moreover, given another morphism $B \xrightarrow{g} A$ it is clear that $\mathcal{T}^{\mathsf{P}}(g \circ f) = \mathcal{T}^{\mathsf{P}} g \circ \mathcal{T}^{\mathsf{P}} f$, just because of the functoriality of P and L.

Since the construction is functorial, if the strength D_A is a monomorphism for every object $A \in \mathcal{E}$ then $\mathcal{T}^{\mathsf{P}}$ is in fact a functor from P-operads to categories internal to $\mathcal{E}$.

3.3 The composite construction

Since we have defined a construction from P-operads to categories and a construction from categories to P-operads, we obtain a composite construction from P′-operads to P-operads, for P′ and P not necessarily the same monad. In particular, since a category is the same as an Id-operad, the composite construction for $\mathsf{P}' = \mathsf{Id}$ is the same as the functor from categories to P-operads. From now on we call $\mathcal{T}$-construction any of the three constructions, the context will suffice to distinguish, but we are mainly interested in landing on a P-operad, rather than a category. To keep notation short, we denote by

$$\mathcal{T}_{\mathsf{P}} Q := \mathcal{T}_{\mathsf{P}} \mathcal{T}^{\mathsf{P}'} Q$$

the composite construction that produces a P-operad from the P′-operad Q. The monad P′ will be always clear from the context.

3.4 Finiteness conditions

In Section 5 we will be interested in computing the incidence bialgebra of the bar construction of several P-operads in $\mathcal{E} = \mathbf{Grpd}$. Recall that to be able to take the homotopy cardinality, the bar construction has to be locally finite as a simplicial groupoid (in the sense of [18]). We now define the notion of locally finite operad (in the sense of ([24]) in the setting of P-operads, which is the sufficient condition for its bar construction to be locally finite, and we give sufficient conditions on the $\mathcal{T}$-construction to preserve locally finiteness.

Definition 3.4.1 A natural transformation is *finite* if all its components are finite. A monad (P, μ, η) on **Grpd** is *locally finite* if μ and η are finite natural transformations. A P-operad Q is locally finite if Q_1 is locally finite, and the maps d_1 and e are finite.

In the special case of $\mathsf{P} = \mathsf{Id}$, P-operads are just categories, and the notion of locally finite agrees with the standard notion. Notice that Q can be locally finite even if P is not. The condition of P being locally finite appears in the $\mathcal{T}$-construction.

Example 3.4.2 For a classical symmetric or nonsymmetric operad, the locally finiteness condition amounts to saying that every operation can be expressed as a composition of operations in a finite number of ways. For instance, the operads Ass and Sym are locally finite. For this it is important that nullary operations are excluded. The non-reduced versions, where there is a nullary operation, are *not* locally finite.

The bar construction of Q is locally finite if Q is locally finite and P preserves locally finite groupoids and finite maps (see Section 1). Also, given another locally finite monad R on $\mathcal{E}$ that preserves locally finite groupoids and finite maps, if there is a cartesian monad map $\mathsf{P} \overset{\psi}{\Rightarrow} \mathsf{R}$ with ψ finite then the bar construction $\mathcal{B}^{\mathsf{R}}$ is also locally finite. Let us see that the $\mathcal{T}$-construction interacts well with finiteness, as long as some simple conditions are satisfied.

Lemma 3.4.3 *Let $\mathsf{P}\colon \boldsymbol{Grpd} \to \boldsymbol{Grpd}$ be a locally finite strong monad that preserves locally finite groupoids, finite maps and fibrations. Assume moreover that the strength D is finite. Consider a locally finite category C in $\boldsymbol{Grpd}$ such that C_0 is discrete and D_{C_0} is a monomorphism. Then the P-operad $\mathcal{T}_{\mathsf{P}}C$ is locally finite.*

Proof Recall from Diagram (3.1.1) that $\widetilde{C}_1$ is defined as the pullback

$$\begin{array}{ccc} \widetilde{C}_1 & \longrightarrow & \mathsf{L}C_0 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle D_{C_0}} \\ \mathsf{P}C_1 & \xrightarrow[\mathsf{P}d_0]{} & \mathsf{P}C_0. \end{array}$$

Notice that the pullback and the monomorphism refer to the 1-categorical notions, while the finite map condition is a homotopy notion.

Let us see first that $\widetilde{C}_1$ is locally finite. Since C_1 is locally finite and P preserves locally finite groupoids, $\mathsf{P}C_1$ is locally finite. Now, an automorphism in $\widetilde{C}_1$ is a pair of automorphisms $(f, g) \in \mathsf{P}C_1 \times \mathsf{L}C_0$ coinciding at $\mathsf{P}C_0$, but there is only a finite number of f's, since $\mathsf{P}C_1$ is locally finite, and for each f at most one g, since D_{C_0} is a monomorphism.

We have to prove also that $\widetilde{d}_1$ and $\widetilde{e}$ are finite maps. This follows from their definitions, 3.1.1 and 3.1.4: since C_0 is discrete, we have that d_0 is a fibration, and because P (and also L) preserves fibrations, all the right arrows in diagrams (3.1.3) and (3.1.6) are fibrations. As a consequence their limit is equivalent to their homotopy limit. Finally, notice that all the vertical maps involved in these two diagrams are finite. This implies that their homotopy limit, and hence their limit, is also finite. □

Lemma 3.4.4 *Let $\mathsf{P}\colon \boldsymbol{Grpd} \to \boldsymbol{Grpd}$ be a locally finite strong monad that preserves locally finite groupoids, finite maps and fibrations. Assume moreover that the strength D is finite. Consider a locally finite P-operad Q such that Q_0 is discrete and D_{Q_0} is a monomorphism. Then the P-operad $\mathcal{T}_{\mathsf{P}}C$ is locally finite.*

Proof The proof is analogous to the proof of Lemma 3.4.3. □

In particular these results imply of course that if P and P' are monads satisfying the conditions of Lemmas 3.4.3 and 3.4.4 and Q is a locally finite P'-operad then $\mathcal{T}_{\mathsf{P}}Q$ is locally finite.

Remark 3.4.5 In the sequel, we deal with the free semigroup monad M^{r} and the free symmetric semimonoidal category monad S^{r}, which preserve locally finite groupoids, finite maps and fibrations, as required by Lemmas 3.4.3 and 3.4.4. Moreover, their strength is finite, as can be easily seen from its definition (see Examples 1.3.4 and 1.3.7). Also, we use the reduced operads Ass and Sym, as well as their colored versions. They are all locally finite and have discrete groupoid of colors.

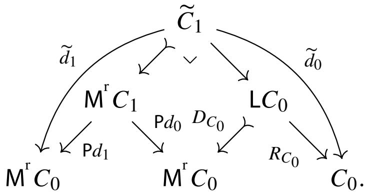
4 $\mathcal{T}$-construction for M^{r} and S^{r}-operads

In this section we unravel the $\mathcal{T}$-construction with some of the main examples. We begin discussing the construction from categories to M^{r}-operads and S^{r}-operads. When the category is just a monoid we get the Giraudo T-construction, which we recall next. Lastly we treat symmetric and nonsymmetric operads.

The choice of working with the reduced version of the operads (excluding nullary operations), is irrelevant for the sake of the $\mathcal{T}$-construction itself, which is abstract enough to work with any operad. The reason for preferring the reduced version is to stay within the realm of locally finite operads, as mentioned in Example 3.4.2 and Remark 3.4.5. Moreover, it is also easy to see that the cartesian monad maps $\mathsf{M}^{\mathrm{r}} \Rightarrow \mathsf{S}$, $\mathsf{S}^{\mathrm{r}} \Rightarrow \mathsf{S}$ and $\mathsf{M}^{\mathrm{r}} \Rightarrow \mathsf{M}$ are finite.

4.1 The $\mathcal{T}$-construction for categories

Let C be a category internal to **Set**, represented by the span $C_0 \leftarrow C_1 \rightarrow C_0$, and take the free semigroup monad M^{r}. The set of objects of $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$ is again C_0, while $\widetilde{C}_1$ is given by



(4.1.1)

Recall from Example 1.3.7 that the strength is given by

$$\begin{array}{rcl} D_{C_0} \colon \mathsf{L} C_0 & \longrightarrow & \mathsf{M}^{\mathrm{r}} C_0 \\ \big(c, (1, \ldots, 1)\big) & \longmapsto & \big((c, 1), \ldots, (c, 1)\big). \end{array} \tag{4.1.2}$$

Therefore, the pullback condition means that the elements in $\widetilde{C}_1$ that have input $c_1, \ldots, c_n$ and output c are the sequences of n arrows in C whose sources are $c_1, \ldots, c_n$ and whose targets are all c. Hence

$$\widetilde{C}_1 = \sum_{(c_1, \ldots, c_n; c)} \prod_{i=1}^{n} \mathrm{Hom}(c_i, c).$$

Substitution in $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$,

$$\circ\colon \prod_{i=1}^{k} \operatorname{Hom}(c_i, c) \times \prod_{i=1}^{k} \prod_{j=1}^{n_i} \operatorname{Hom}(d_j^i, c_i) \longrightarrow \prod_{\substack{1 \le i \le k \\ 1 \le j \le n_i}} \operatorname{Hom}(d_j^i, c),$$

goes as follows: for an operation $x \in \prod_{i=1}^{k} \operatorname{Hom}(c_i, c)$ and a sequence of k operations $y^i \in \prod_{j=1}^{n_i} \operatorname{Hom}(d_j^i, c_i)$, with $i = 1, \dots, k$,

$$x \circ (y^1, \dots, y^k) = (x_1 \circ y_1^1, \dots, x_1 \circ y_{n_1}^1, \dots, x_k \circ y_1^k, \dots, x_k \circ y_{n_k}^k) \in \prod_{\substack{1 \le i \le k \\ 1 \le j \le n_i}} \operatorname{Hom}(d_j^i, c).$$

Note that now the composition inside the parenthesis is composition of morphisms of C, while the composition on the left-hand side of the equation is composition in $\mathsf{M}^{\mathrm{r}} C$. It is not difficult to see that the composition we get from 3.1.1 agrees with the one defined above: both use the fact that $\widetilde{C}_2$ is a subset of $(\mathsf{M}^{\mathrm{r}})^2 C$ together with $(\mathsf{M}^{\mathrm{r}})^2 \circ$ and the monad multiplication. The identity elements of this operad are given by the identity morphisms of C. If the category C has coproducts $(+)$ then

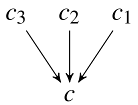
$$\prod_{i=1}^{n} \operatorname{Hom}(c_i, c) = \operatorname{Hom}(c_1 + \cdots + c_n, c),$$

so that the operations of $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$ are in fact arrows of C.

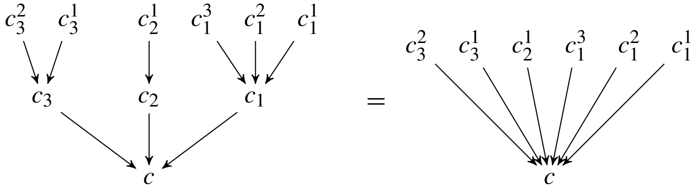
Since C can be considered as a category internal to **Grpd**, we can also compute $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ to get a symmetric operad. It is clear that $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C_1 = \mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C_1 // \mathfrak{S}$, where the action of the symmetric group $\mathfrak{S}_n$ is given by permutation of tuples, that is

$$\begin{array}{ccc} \mathfrak{S}_n \times \prod_{i=1}^{n} \operatorname{Hom}(c_i, c) & \longrightarrow & \prod_{i=1}^{n} \operatorname{Hom}(c_{\sigma(i)}; c) \\ (\sigma, (x_1, \dots, x_n)) & \longmapsto & (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{array}$$

It is useful to picture elements $(c_1, \dots, c_n; c)$ as (picturing $n = 3$)



Under this representation, composition in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ (or $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$) looks like



Example 4.1.1 Take $C = \{0 \rightleftarrows 1\}$. For any pair of objects of C there is exactly one morphism between them. Hence $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} C$ has one operation for each given sequence of inputs and output, so that it is the 2-colored associative operad Ass_2. In the same way $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$ is the 2-colored symmetric operad Sym_2. In fact it is straightforward to see that the $\mathcal{T}$-constructions of the discrete connected groupoid of n elements are Ass_n and Sym_n.

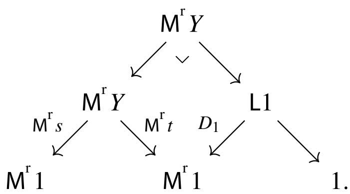
Example 4.1.2 Consider the category $C = \{0 \longrightarrow 1\}$. Note that in this case there is either one or no morphism between two objects of C. Thus clearly

$$\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C(c_1, \ldots, c_n; c) = \begin{cases} (c \to c_1, \ldots, c \to c_n) & \text{if } c = 0 \text{ or } c = c_1 = \cdots = c_n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Of course this operad is a suboperad of the previous one, since this category is a subcategory of the previous one. In particular composition is obvious.

Example 4.1.3 We now specialize to the case of categories with only one object, that is monoids, recovering the T-construction of Giraudo. This construction was introduced by Giraudo [20] as a generic method to build combinatorial operads from monoids.

Since a monoid is just a category with one object, it is represented by the span $1 \leftarrow Y \rightarrow 1$, and because the morphism $\mathsf{L}1 \xrightarrow{D_1} \mathsf{M}^{\mathrm{r}}1$ is an isomorphism, we have that $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} Y$ is given by



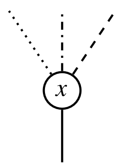
It is easy to see that this gives the same operad TY defined in the introduction, since TY is precisely $\mathsf{M}^{\mathrm{r}} Y$, and both compositions are defined by using composition in $(\mathsf{M}^{\mathrm{r}})^2 Y$ and the monad multiplitcation.

Example 4.1.4 If Y_1 is the singleton monoid, then $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}} Y_1 = \mathsf{Ass}$, the associative operad, and $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y_1 = \mathsf{Sym}$, the commutative operad.

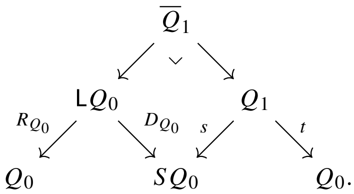
4.2 The $\mathcal{T}$-construction for operads

We now unravel the full $\mathcal{T}$-construction from nonsymmetric operads to S^{r}-operads. As we already know, the first ones are the same as M^{r}-operads in **Set**, but we view them as M^{r}-operads in **Grpd** with discrete groupoids of objects and arrows. At the end we will comment on other variations similar to this case, such as from symmetric operads to S^{r}-operads.

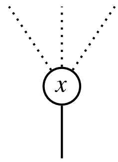
Let Q be an M^{r}-operad represented by the span $\mathsf{M}^{\mathrm{r}} Q_0 \leftarrow Q_1 \rightarrow Q_0$. Recall that elements of Q_1 are depicted as



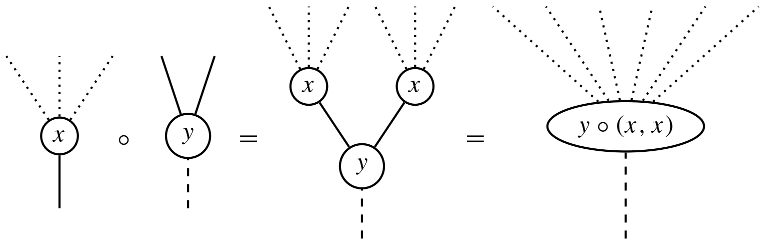
We apply first the $\mathcal{T}$-construction to get a category $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q$:



The strength morphism is the same as in (4.1.2). Therefore the elements of $\overline{Q}_1$ are the elements of Q_1 such that all the input objects coincide,



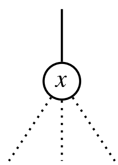
so that x is an arrow in the category $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q$. Notice that $\overline{Q}_2$ is a subset of Q_2. Therefore composition in $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q$ is the same as composition in Q. For example



(4.2.1)

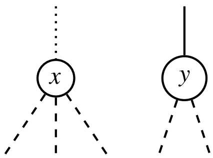
where $y \circ (x, x)$ is composition in Q. Hence the recipe is to repeat x for each input of y and use composition in Q.

It is not difficult to see that the categories $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} \mathsf{Ass}$ and $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} \mathsf{Sym}$ (as well as $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}} \mathsf{Ass}$ and $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}} \mathsf{Sym}$) are self dual (see Examples 4.2.1 and 4.2.2 below). An obvious consequence of this self duality is that we get equivalent operads by applying the $\mathcal{T}$-construction to their opposite categories. Nevertheless, when dealing with plethysm it is in fact more natural, from a combinatorial point of view, to apply the $\mathcal{T}$-construction to the opposite categories. This is particularly apparent when we interpret the simplicial groupoid $T\mathbf{S}$ [9] as an operad (see Examples 6.1.7 and 6.2.6). For this reason, before applying the $\mathcal{T}$-construction again we take the opposite category $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q^{\mathrm{op}}$. The arrows of this category are depicted as inverted arrows of $\mathcal{T}^{\mathsf{M}^{\mathrm{r}}} Q$:

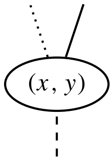


(4.2.2)

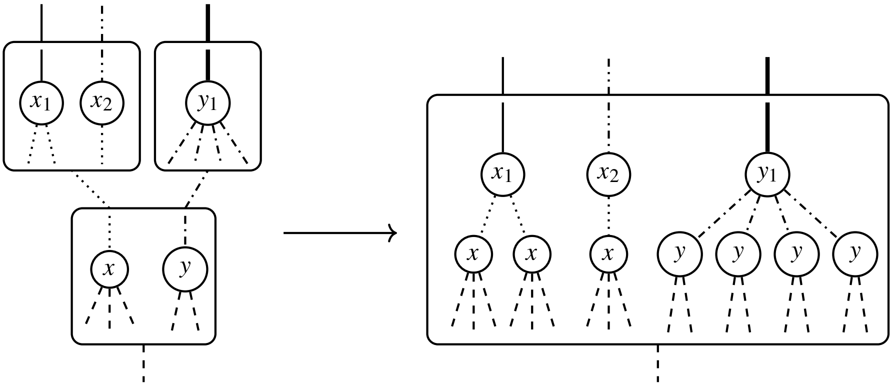
Now we finally apply the $\mathcal{T}$-construction to get a S^{r}-operad $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} Q^{\mathrm{op}}$ from the category $\mathcal{T}^{\mathsf{M}^{\mathsf{r}}} Q^{\mathrm{op}}$. This step was made above for any category: the objects of $\widetilde{Q}_1$ are sequences $(x_1, \dots, x_n)$ of arrows $x_i \in \overline{Q_1}$ with the same output. For instance the pair



is an operation



in $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} Q^{\mathrm{op}}$. Clearly $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} Q^{\mathrm{op}}$ is a symmetric operad, since the groupoid of objects is discrete and the morphisms in the groupoid $\widetilde{Q}_1$ are given by permutation of tuples. We show once and for all an example of composition in $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} Q^{\mathrm{op}}$:



(4.2.3)

and then of course we compose in Q, as in (4.2.1).

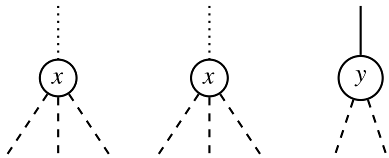
Since in all the examples we are interested in the step of taking the opposite category does not affect the result (by self duality), we will drop the op indication from now on, but be aware that all the representations of operations follow this convention.

Example 4.2.1 If the starting M^{r}-operad is Ass, which is a noncolored operad, then it is easy to see that the monoid $\mathcal{T}^{\mathsf{M}^{\mathsf{r}}} \mathsf{Ass}$ is isomorphic to $(\mathbb{N}^+, \times)$. Therefore the operations of $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} \mathsf{Ass}$ are sequences of natural numbers and composition is given by multiplication. For example

$$\big((2, 3), (4, 7)\big) \circ (5, 9) = (5 \cdot 2, 5 \cdot 3, 9 \cdot 4, 9 \cdot 7) = (10, 15, 36, 63).$$

If the starting M^{r}-operad is Ass_2 the 2-colored associative operad, then the category $\mathcal{T}^{\mathsf{M}^{\mathsf{r}}} \mathsf{Ass}_2$ has two objects and a morphism $\xrightarrow{n}$ for every pair of objects and positive natural number n. Composition is given by multiplication. The operations of $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}} \mathsf{Ass}_2$ are thus sequences of such arrows with the same output.

Suppose we start instead from a symmetric operad Q. Recall from Example 1.3.7 that a symmetric operad is an S-operad in **Grpd** such that Q_0 is discrete and $\mathsf{S}^{\mathsf{r}} Q_0 \xleftarrow{s} Q_1$ is discrete fibration. The $\mathcal{T}$-construction to get another S^{r}-operad is completely analogous to the previous case, but in this case the groupoid $\widetilde{Q}_1$ inherits morphisms from Q, so that for instance the element



has $2! \cdot 3!^2 \cdot 2!$ automorphisms, corresponding to $2!$ invariant permutations on (x, x, y) and permutations of the inputs. The latter contribution did not appear in the previous case, since Q was a planar operad. Notice that this means that $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Q^{\mathrm{op}}$ (also $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Q$) is not a symmetric operad, but just an S^{r}-operad in **Grpd**.

Example 4.2.2 If the starting S^{r}-operad is Sym, which is a noncolored symmetric operad, then it is easy to see that the monoid $\mathcal{T}^{\mathsf{S}^{\mathrm{r}}} \mathsf{Sym}$ is isomorphic to the monoid $(\mathbb{N}^+, \times)$ internal to groupoids where $\mathrm{Aut}(n) \cong \mathfrak{S}_n$. The objects of $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} \mathsf{Sym}$ are the same as the objects in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} \mathsf{Ass}$, and the morphisms are given by permutation of tuples (as in $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} \mathsf{Ass}$) plus the ones given by $\mathrm{Aut}(n)$ for each n. The colored case is analogous.

5 Plethysms and operads

Let us present the relation between the several plethystic bialgebras, operads and the $\mathcal{T}$-construction. Some proofs are omitted, since most of them are similar. The operads involved are the reduced symmetric operad Sym, the reduced associative operad Ass and their 2-colored versions. Also, playing the same role as these operads, we have a locally finite monoid Y. On the other hand, the $\mathcal{T}$-constructions are taken with respect to the monads S^{r} and M^{r}, as in Section 4, and everything is internal to $\mathcal{E} = \mathbf{Grpd}$.

Let us stress again that by Proposition 1.4.6, Lemmas 3.4.3 and 3.4.4 and the discussion of Section 4 all the bar constructions featuring in the present section are locally finite Segal groupoids, so that we can take cardinality to arrive at their incidence bialgebra in the classical sense of vector spaces.

It is appropriate to begin with the classical bialgebras, which are the main cases. The following standard notation is used:

- $\mathbf{x} = (x_1, x_2, \dots)$,
- Λ: set of infinite vectors of natural numbers with $\lambda_i = 0$ for all i large enough and $\lambda_j \neq 0$ for some j,
- $\Lambda \ni \lambda = (\lambda_1, \lambda_2, \dots)$,
- $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$,
- $\mathrm{aut}(\lambda) = 1!^{\lambda_1} \lambda_1! \cdot 2!^{\lambda_2} \lambda_2! \cdots$,
- $\lambda! = \lambda_1! \cdot \lambda_2! \cdots$,
- W: set of nonempty finite words of positive natural numbers,
- $W \ni \omega = \omega_1 \dots \omega_n$,
- $\mathbf{x}_\omega = x_{\omega_1} \cdots x_{\omega_n}$,
- $\omega! = \omega_1! \cdots \omega_n!$.

5.1 The classical case

The classical Faà di Bruno bialgebra $\mathcal{F}$ [12, 22] is obtained from the substitution of power series in one variable. Let $x\mathbb{Q}[\![x]\!]$ be the ring of formal power series with coefficients in $\mathbb{Q}$

without constant term. Elements of $x\mathbb{Q}[\![x]\!]$ are written

$$F(x) = \sum_{n\geq 1} \frac{F_n}{n!} x^n.$$

The set $x\mathbb{Q}[\![x]\!]$ forms a (noncommutative) monoid with substitution of power series. The *Faà di Bruno bialgebra* $\mathcal{F}$ is the free polynomial algebra $\mathbb{Q}[A_1, A_2, \dots]$ generated by the linear maps

$$\begin{aligned} A_i : x\mathbb{Q}[\![x]\!] &\longrightarrow \mathbb{Q} \\ F &\longmapsto F_i \end{aligned}$$

together with the comultiplication induced by substitution, meaning that

$$\Delta(A_n)(F, G) = A_n(G \circ F),$$

and counit given by $\epsilon(A_n) = A_n(x)$. The comultiplication of the generators can be explicitly described through the (exponential) Bell polynomials $B_{n,k}$, which count the number of partitions of an n-element set into k blocks:

$$\Delta(A_n) = \sum_{k=1}^{n} A_k \otimes B_{n,k}(A_1, A_2, \dots).$$

Theorem 5.1.1 *(Joyal, cf. modern reformulation in [15]) The Faà di Bruno bialgebra $\mathcal{F}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}$Sym.*

Note that Sym is of course the same as $\mathcal{T}_{\mathsf{Id}}$Sym and, as explained in Section 4, it is also $\mathcal{T}_{\mathsf{S}^{\mathsf{r}}}$ of the trivial monoid. This connects the Faà di Bruno bialgebra to the $\mathcal{T}$-construction in an analogous way as the plethystic bialgebras.

Let us recall how the classical plethystic substitution works [22, 37, 38]. Let $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ be the ring of power series in infinitely many variables without constant term and coefficients in $\mathbb{Q}$. Elements of $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ are written

$$F(\mathbf{x}) = \sum_{\lambda\in\Lambda} \frac{F_\lambda}{\mathrm{aut}(\lambda)} \mathbf{x}^\lambda.$$

Given two power series $F, G \in \mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$, their *plethystic substitution* is defined as

$$\begin{aligned} (G \circledast F)(x_1, x_2, \dots) &:= G(F_1, F_2, \dots), \quad \text{where} \\ F_k(x_1, x_2, \dots) &:= F(x_k, x_{2k}, \dots). \end{aligned} \tag{5.1.1}$$

The set $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ forms a (noncommutative) monoid with plethystic substitution. The *plethystic bialgebra* $\mathcal{P}$ [9, 36] is the free polynomial algebra $\mathcal{P} = \mathbb{Q}\big[\{A_\lambda\}_\lambda\big]$ generated by the set maps

$$\begin{aligned} A_\lambda : \mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!] &\longrightarrow \mathbb{Q} \\ F &\longmapsto F_\lambda \end{aligned}$$

together with the comultiplication induced by substitution, meaning that

$$\Delta(A_\lambda)(F, G) = A_\lambda(G \circledast F),$$

and counit given by $\epsilon(A_\lambda) = A_\lambda(x_1)$. The comultiplication of the generators can be explicitly described through the polynomials $P_{\sigma,\lambda}$, a plethystic version of the Bell polynomials which, in the terminology of Nava–Rota [37], count transversals of partitions.

$$\Delta(A_\sigma) = \sum_\lambda A_\lambda \otimes P_{\sigma,\lambda}\big(\{A_\mu\}_\mu\big).$$

Example 5.1.2

$$P_{(0,0,0,1,0,2),(1,2)}\big(\{A_\mu\}_\mu\big) = \\ = \frac{6!^2 2!4! \cdot 2!}{2!^2 2! \cdot 4!3!^2 2!} A_{(0,0,0,1)} A^2_{(0,0,1)} + \frac{6!^2 2!4! \cdot 2!}{2!^2 2! \cdot 6!3!2!} A_{(0,0,0,0,0,1)} A_{(0,0,1)} A_{(0,1)}. \tag{5.1.2}$$

We can give a heuristic explanation of this. Define the *k-th Verschiebung operator* $V^k : \Lambda \to \Lambda$ as

$$(V^k\lambda)_i = \begin{cases} \lambda_{i/k} & \text{if } k \mid i \\ 0 & \text{otherwise,} \end{cases} \tag{5.1.3}$$

Then Equation 5.1.2 indicates that the vector (0, 0, 0, 1, 0, 2) can be obtained in the following two ways,

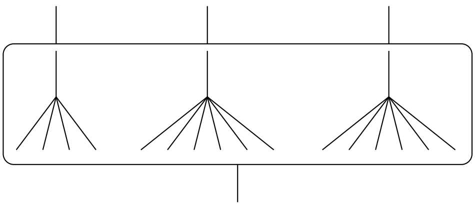
$$(0, 0, 0, 1, 0, 2) = V^1(0, 0, 0, 1) + V^2(0, 0, 1) + V^2(0, 0, 1),$$
$$(0, 0, 0, 1, 0, 2) = V^1(0, 0, 0, 0, 0, 1) + V^2(0, 0, 1) + V^2(0, 1),$$

where the Verschiebung operators used are determined by λ: one V^1 and two V^2. The coefficients are given by the automorphisms of the vectors involved. See [9] for a detailed explanation.

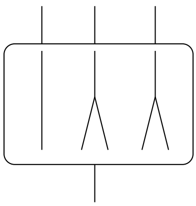
Theorem 5.1.3 *The plethystic bialgebra $\mathcal{P}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{\mathsf{S}^r}$* Sym.

Proof The comparison between these two incidence bialgebras was made in [9], where the simplicial interpretation of plethysm was established. In Section 6 we will see that indeed $\mathcal{B}\mathcal{T}_{\mathsf{S}^r}$ Sym is equivalent to $T\mathbf{S}$, the simplicial groupoid of [9]. □

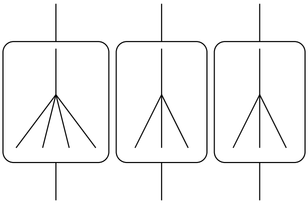
Example 5.1.4 Let us see the interpretation of $P_{(0,0,0,1,0,2),(1,2)}\big(\{A_\mu\}_\mu\big)$ (see Examples 5.1.2) from the point of view of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^r}$ Sym. The vectors $\sigma = (0, 0, 0, 1, 0, 2)$ and $\lambda = (1, 2)$ are represented by



and



respectively. What we want to count is, roughly speaking, the number of ways we can obtain σ as a composition of λ with three operations. It is straightforward to see that there are essentially two choices:



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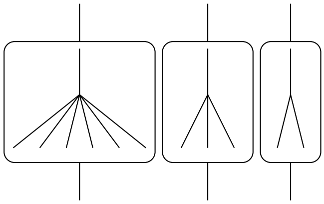
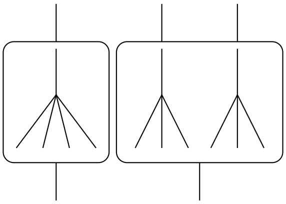
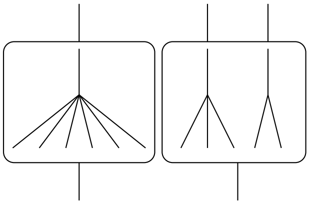


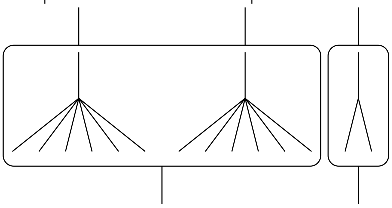
which clearly coincide with the ones of Example 5.1.2. This example could be misleading, in the sense that each operation above contains only one operation of Sym. This happens of course because $|\sigma| = |\lambda|$. For instance, if we take instead $\lambda = (1, 1)$ we obtain three possible choices:



,



and



5.2 Overview of variations

We proceed to introduce the variations of the plethystic bialgebra we explore. For the set of variables $(x_1, x_2, \dots)$, there are three sources of variations. At the level of power series they are the following:

(i) Commuting or noncommuting variables: of course in the classical case the variables commute. When the variables do not commute we will index them by $\omega \in W$, rather than $\lambda \in \Lambda$.

(ii) Commuting or noncommuting coefficients.

(iii) Two types of automorphisms: $\mathrm{aut}(\lambda)$ or $\lambda!$ for commuting variables, and $\omega!$ or 1 for noncommuting variables.

These variations are not independent: if the variables commute then the coefficients commute. Analogous variations can be obtained of the Faà di Bruno bialgebra, except in this case there is only one variable.

At the objective level, these three variations correspond (respectively) to the following choices:

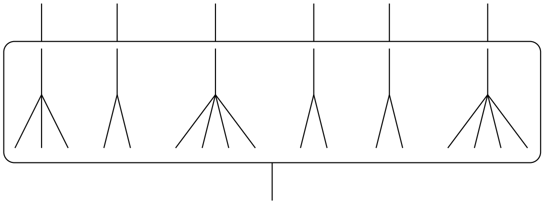
(i) $\mathcal{T}$-construction over S^{r} or over M^{r}.

(ii) Bar construction over S^{r} or over M^{r}.

(iii) Taking Sym or Ass as input operads.

The reason why they are not independent is clear here: there is a cartesian natural transformation $\mathsf{M}^{\mathrm{r}} \Rightarrow \mathsf{S}^{\mathrm{r}}$ that allows taking $\mathcal{B}^{\mathsf{S}^{\mathrm{r}}}$ of a M^{r}-operad (see Section 1), but no natural transformation in the opposite direction.

Let us give a brief justification of these correspondences. Consider the following sequence of operations:

$a =$ 

This could be either an operation in one of the following operads:

(i) $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$: in this case each operation has automorphisms, coming from the action of the symmetric group on Sym, and since the $\mathcal{T}$-construction is over S^{r} we can permute the operations. This means that the isomorphism class of a is given by $\lambda = (0, 3, 1, 2)$, since the order of the operations does not matter, and it has $\mathrm{aut}(\lambda) = 2!^3 3! \cdot 3!^1 1! \cdot 4!^2 2!$ automorphisms. The corresponding bialgebra is thus $\mathcal{P}$ and this particular operation corresponds to $A_{(0,3,1,2)}$, the linear map returning the coefficient of $x_2^3 x_3 x_4^2 / \mathrm{aut}(\lambda)$.

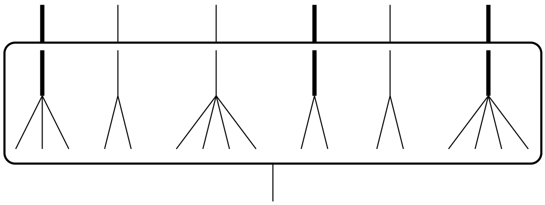
(ii) $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Sym}$: in this case the operations have automorphisms again, but since the $\mathcal{T}$-construction is over M^{r} we cannot permute them. This means that the isomorphism class of a is given by $\omega = (3, 2, 4, 2, 2, 4)$, so that it corresponds to noncommuting variables. Clearly it has $3!2!4!2!2!4!$ automorphisms. Now, depending on the bar construction it corresponds to commuting or noncommuting coefficients. This particular operation corresponds to $A_{(3,2,4,2,2,4)}$, the linear map returning the coefficient of $x_3 x_2 x_4 x_2 x_2 x_4 / \omega!$.

(iii) $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$: in this case the operations do not have automorphisms, and since the $\mathcal{T}$-construction is over M^{r} we cannot permute them. This means that the isomorphism class of a is given by $\omega = (3, 2, 4, 2, 2, 4)$, so that it corresponds to noncommuting variables, and it has no automorphisms. Now, depending on the bar construction it corresponds to commuting or noncommuting coefficients, as in the previous case. This particular operation corresponds to $a_{(3,2,4,2,2,4)}$, the linear map returning the coefficient of $x_3 x_2 x_4 x_2 x_2 x_4$.

(vi) $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$: in this case the operations do not have automorphisms, and since the $\mathcal{T}$-construction is over S^{r} we can permute them. This means that the isomorphism class of a is given by $\lambda = (0, 3, 1, 2)$, and it has $\lambda! = 3! \cdot 1! \cdot 2!$ automorphisms. Therefore it corresponds to commuting variables and coefficients. This particular operation corresponds to $a_{(0,3,1,2)}$, the linear map returning the coefficient of $x_2^3 x_3 x_4^2 / \lambda!$.

The cases of Sym and Ass are developed in Subsections 5.3 and 5.4 respectively. In Subsection 5.5 we generalize $\mathcal{P}_{\exp}$ to power series in the set of variables $(x_m \mid m \in Y)$ indexed over a locally finite monoid.

In Subsections 5.3 and 5.4 we also study the Faà di Bruno bialgebra in two variables and the plethystic bialgebra in the two sets of variables $(x_1, x_2, \dots), (y_1, y_2, \dots)$. For the plethystic case we only consider commuting variables and coefficients. Let us give a similar digression as above for the plethystic cases. Consider the following 2-colored operation:

$a =$ 

The isomorphism class of this operation is given by $(\lambda^1, \lambda^2) = \big((0, 2, 0, 2), (\mathbf{0{,}1{,}1{,}1})\big)$ (since everything commutes now), and it can either be an operation in $\mathcal{T}_{\mathsf{S}^r}\mathsf{Ass}_2$ or $\mathcal{T}_{\mathsf{S}^r}\mathsf{Ass}_2$. It thus corresponds to $A_{\left((0,2,0,2),(\mathbf{0,1,1,1})\right)} \in \mathcal{P}^2$ or to $a_{\left((0,2,0,2),(\mathbf{0,1,1,1})\right)} \in \mathcal{P}^2_{\mathrm{exp}}$, the linear maps returning the coefficients of $x_2^2 x_4^2 y_2 y_3 y_4 / \operatorname{aut}(\lambda^x) \operatorname{aut}(\lambda^y)$.

5.3 Bialgebras from Sym and Sym$_2$

We have already seen two bialgebras arising from Sym in Subsection 5.1, the Faà di Bruno bialgebra $\mathcal{F}$ and the plethystic bialgebra $\mathcal{P}$. Let us see the aforementioned variations.

Replace $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ by $\mathbf{x}\mathbb{Q}\langle\!\langle\mathbf{x}\rangle\!\rangle$, that is, noncommuting variables. Elements of $\mathbf{x}\mathbb{Q}\langle\!\langle\mathbf{x}\rangle\!\rangle$ are written

$$F(\mathbf{x}) = \sum_{\omega \in W} \frac{F_\omega}{\omega!}\mathbf{x}^\omega,$$

Substitution of power series in $\mathbf{x}\mathbb{Q}\langle\!\langle\mathbf{x}\rangle\!\rangle$ is defined in the same way as before (5.1.1). The *plethistic bialgebra with noncommuting variables* $\mathcal{P}^\diamond$ is defined as the free polynomial algebra $\mathbb{Q}\big[\{A_\omega\}_\omega\big]$ on the set maps A_ω and comultiplication and counit as usual.

Theorem 5.3.1 *The plethystic bialgebra with noncommuting variables $\mathcal{P}^\diamond$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}^r}\mathcal{T}_{\mathsf{M}^r}\mathsf{Sym}$.*

If we take $\mathbf{x}R\langle\!\langle\mathbf{x}\rangle\!\rangle$ with R a noncommutative unital ring, then we get the *noncommutative plethystic bialgebra with noncommuting variables* $\mathcal{P}^{\diamond,\mathrm{nc}}$, which is the free associative unital algebra $\mathbb{Q}\langle\{A_\omega\}_\omega\rangle$ together with the usual comultiplication and counit. In this case, substitution of power series is defined in the same way but it is not associative. However the comultiplication is still associative. A proof of this can be found in [6] for the one variable case, which is obtained below.

Theorem 5.3.2 *The noncommutative plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamond,\mathrm{nc}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{\mathsf{M}^r}\mathsf{Sym}$.*

Let us move forward to power series in two variables. All the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case. Also, for the bivariate plethystic bialgebras, we do not enter into noncommutativity of the variables or of the coefficients.

Let $x\mathbb{Q}[\![x, y]\!] + y\mathbb{Q}[\![x, y]\!]$ be the ring of formal power series in the variables x and y with coefficients in $\mathbb{Q}$ without constant term. We will denote this ring by Q in the following lines, for the sake of notation. Elements of Q are written

$$F(x, y) = \sum_{n+m \geq 1} \frac{F_{n,m}}{n!m!} x^n y^m.$$

The set $Q \times Q$ forms a (noncommutative) monoid with substitution of power series:

$$\begin{aligned} \big(Q \times Q\big) \times \big(Q \times Q\big) &\xrightarrow{\qquad\circ\qquad} Q \times Q \\ \big((F^1, F^2), (G^1, G^2)\big) &\longrightarrow \big(G^1(F^1, F^2), G^2(F^1, F^2)\big). \end{aligned}$$

We define the *Faà di Bruno bialgebra in two variables* $\mathcal{F}^2$ as the free polynomial algebra $\mathbb{Q}\big[\{A^i_{n,m}\}^{i=1,2}_{n+m\geq 1}\big]$ generated by the set maps

$$\begin{aligned} A^i_{n,m} : Q \times Q &\longrightarrow \mathbb{Q} \\ (F^1, F^2) &\longrightarrow F^i_{n,m} \end{aligned}$$

together with the comultiplication induced by substitution, meaning that

$$\Delta(A^i_{n,m})\big((F^1, F^2), (G^1, G^2)\big) = A^i_{n,m}\big((G^1, G^2) \circ (F^1, F^2)\big),$$

and counit given by $\epsilon(A^i_{n,m}) = A^i_{n,m}(x, y)$.

Theorem 5.3.3 *The Faà di Bruno bialgebra in two variables $\mathcal{F}^2$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathsf{Sym}_2$. The same holds for n variables and Sym_n.*

Notice that Sym_2 is the same as $\mathcal{T}_{\mathsf{Id}}\mathsf{Sym}$ and, as explained in Example 4.1.1, it is also $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} C$, where $C = \{0 \rightleftarrows 1\}$. This connects the Faà di Bruno bialgebra in two variables to the $\mathcal{T}$-construction in an analogous way as the plethystic bialgebras.

We can do the same with the power series ring in two sets of infinitely many variables $\mathbf{x}\mathbb{Q}[\![\mathbf{x}, \mathbf{y}]\!] + \mathbf{y}\mathbb{Q}[\![\mathbf{x}, \mathbf{y}]\!]$ with coefficients in $\mathbb{Q}$. We shall write

$$\mathbf{X} = (\mathbf{x}, \mathbf{y}),\ \ \lambda = (\lambda^1, \lambda^2) \in \Lambda^2,\ \ \mathrm{aut}(\lambda) = \mathrm{aut}(\lambda^1)\,\mathrm{aut}(\lambda^2),$$
$$\mathbf{X}^\lambda = \mathbf{x}^{\lambda^1}\mathbf{y}^{\lambda^2},\ \text{ and } \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] = \mathbf{x}\mathbb{Q}[\![\mathbf{x}, \mathbf{y}]\!] + \mathbf{y}\mathbb{Q}[\![\mathbf{x}, \mathbf{y}]\!],$$

so that elements of $\mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!]$ are written

$$F(\mathbf{X}) = \sum_\lambda \frac{F_\lambda}{\mathrm{aut}(\lambda)}\mathbf{X}^\lambda.$$

The set $\mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \times \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!]$ forms a (noncommutative) monoid with plethystic substitution of power series:

$$\begin{aligned} \big(\mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \times \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!]\big) \times \big(\mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \times \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!]\big) &\xrightarrow{\qquad\circledast\qquad} \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \times \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \\ \big((F^1, F^2), (G^1, G^2)\big) &\longmapsto \big(G^1(F^1, F^2), G^2(F^1, F^2)\big) \end{aligned}$$

The *plethystic bialgebra in two variables* $\mathcal{P}^2$ is defined as the free polynomial algebra $\mathbb{Q}\big[\{A^i_\lambda\}^{i=1,2}\big]$ generated by the set maps

$$\begin{aligned} A^i_\lambda : \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] \times \mathbf{X}\mathbb{Q}[\![\mathbf{X}]\!] &\longrightarrow \mathbb{Q} \\ (F^1, F^2) &\longmapsto F^i_\lambda \end{aligned}$$

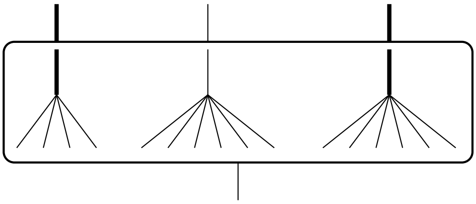
together with the comultiplication induced by substitution, meaning that

$$\Delta(A^i_\lambda)\big((F^1, F^2), (G^1, G^2)\big) = A^i_\lambda\big((G^1, G^2) \circ (F^1, F^2)\big),$$

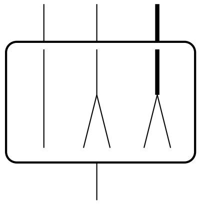
and counit given by $\epsilon(A^i_\lambda) = A^i_\lambda(x, y)$.

Theorem 5.3.4 *The plethystic bialgebra in two variables $\mathcal{P}^2$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}_2$.*

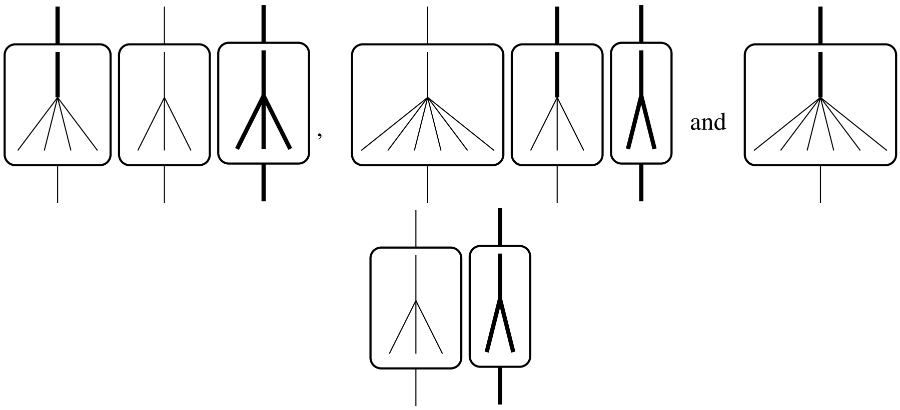
Example 5.3.5 We could define polynomials $P^2_{\sigma,\lambda}\big(\{A_\mu\}^i_\mu\big)$ to express the comultiplication of A^i_σ, in analogy to the univariate case. Let us see again the interpretation of $P^2_{\sigma,\lambda}\big(\{A_\mu\}^i_\mu\big)$, for $\sigma = \big((0,0,0,0,0,1), \mathbf{(0,0,0,1,0,1)}\big)$ and $\lambda = \big((1,1), \mathbf{(0,1)}\big)$, from the point of view of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}_2$. These two vectors are represented by



and



respectively. The output color depends on whether we are computing the comultiplication of A^1_σ or A^2_σ. We assume the former, without loss of generality. It is easy to see that there are essentially three options, which are the only possible colorings of the solutions for the analogous case of Example 5.3.5:



5.4 Bialgebras from Ass and Ass_2

Take again $x\mathbb{Q}[\![x]\!]$, but write now elements of $x\mathbb{Q}[\![x]\!]$ as

$$F(x) = \sum_{n\geq 1} f_n x^n.$$

The *ordinary Faà di Bruno bialgebra* $\mathcal{F}_{\text{ord}}$ is the free polynomial algebra $\mathbb{Q}[a_1, a_2, \dots]$ generated by the linear maps $a_i(F) = f_i$ together with the comultiplication induced by substitution and counit given by $\epsilon(a_n) = a_n(x)$, as before.

Theorem 5.4.1 *The ordinary Faà di Bruno bialgebra $\mathcal{F}_{\text{ord}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{S}^r}\mathsf{Ass}$.*

It is clear that $\mathcal{F}$ and $\mathcal{F}_{\text{ord}}$ are isomorphic bialgebras, since we have only changed the basis. However their combinatorial meaning is slightly different, and indeed $\mathcal{B}^{\mathsf{S}^r}\mathsf{Sym}$ and $\mathcal{B}^{\mathsf{S}^r}\mathsf{Ass}$

are not equivalent. Note that Ass is of course the same as $\mathcal{T}_{\mathsf{Id}}\mathsf{Ass}$ and, as explained in Section 4, it is also $\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}$ of the trivial monoid. This connects the ordinary Faà di Bruno bialgebra to the $\mathcal{T}$-construction.

If we replace above $\mathbb{Q}$ by R (a noncommutative unital ring), we obtain the *noncommutative Faà di Bruno bialgebra* $\mathcal{F}^{\mathrm{nc}}$ [6, 14, 29], the free associative unital algebra $\mathbb{Q}\langle a_1, a_2, \ldots\rangle$ generated by the set maps $a_i(F) = f_i$, together with the comultiplication induced by substitution and counit $\epsilon(a_n) = a_n(x)$, as before. In this case, substitution of power series is not associative, but the comultiplication is still coassociative [6]. It is clear that $\mathcal{F}$ and $\mathcal{F}_{\mathrm{ord}}$ are the abelianization of $\mathcal{F}^{\mathrm{nc}}$ [6].

Theorem 5.4.2 *The noncommutative Faà di Bruno bialgebra $\mathcal{F}^{\mathrm{nc}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{\mathsf{M}^r}\mathsf{Ass}$.*

We now move to the plethystic bialgebras. The *exponential plethystic bialgebra* $\mathcal{P}_{\exp}$ is the same bialgebra as $\mathcal{P}$, but in this case $\mathrm{aut}(\lambda) = \lambda! = \lambda_1!\lambda_2!\cdots$ [36]. The generators of this bialgebra are denoted by a_λ.

Theorem 5.4.3 *The exponential plethystic bialgebra $\mathcal{P}_{\exp}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{S^r}\mathsf{Ass}$.*

The *linear plethystic bialgebra with noncommuting variables* $\mathcal{P}^{\diamondsuit}_{\mathrm{lin}}$ is the same bialgebra as $\mathcal{P}^{\diamondsuit}$ but without automorphisms of ω. The generators for this bialgebra are denoted a_ω.

Theorem 5.4.4 *The linear plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit}_{\mathrm{lin}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{S^r}\mathcal{T}_{M^r}\mathsf{Ass}$.*

The *noncommutative linear plethystic bialgebra with non-commutating variables* $\mathcal{P}^{\diamondsuit,\mathrm{nc}}_{\mathrm{lin}}$ is the same as $\mathcal{P}^{\diamondsuit,\mathrm{nc}}$ but without automorphisms on ω. We write a_ω for its generators. Contrarily to what it may seem, the noncommutativity simplifies the explicit formula for the comultiplication of the generators. Denote by $|w|$ the length of a word. Let also W^W_n be the set of length n words of words of W. Finally, for $k \in \mathbb{N}$ and $\omega = \omega_1 \ldots \omega_n \in W$, define the kth Verschiebung operator as

$$k\omega = (k\omega_1)\ldots(k\omega_n).$$

Proposition 5.4.5 *The comultiplication of $\mathcal{P}^{\diamondsuit,\mathrm{nc}}$ is given by*

$$\Delta(a_\nu) = \sum_{\omega\in W}\sum_{\kappa\in W^W_{|\omega|}} T^{\kappa}_{\nu,\omega}\left(\prod_{i=1}^{|\omega|} a_{\kappa_i}\right)\otimes a_\omega,$$

where

$$T^{\boldsymbol{\kappa}}_{\nu,\omega} = \begin{cases} 1 & \text{if } \nu = \sum_{i=1}^{n}\omega_i\kappa_i \\ 0 & \text{otherwise.}\end{cases}$$

This proposition is analogous to [9, Proposition 3.3].

Theorem 5.4.6 *The noncommutative linear plethystic bialgebra with noncommuting variables $\mathcal{P}^{\diamondsuit,\mathrm{nc}}_{\mathrm{lin}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{M^r}\mathsf{Ass}$.*

Proof Notice that $\mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is discrete. Its elements are given by sequences of tuples

$$(m_1^1, \ldots, m_{n_1}^1), \ldots, (m_1^k, \ldots, m_{n_k}^k)$$

of positive natural numbers (see Example 4.2.1), but there is only the identity morphisms between them. Thus juxtaposition of sequences gives $\mathcal{B}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ a (nonsymmetric) monoidal structure. Sequences containing one tuple are called connected, and form an algebra basis of the incidence bialgebra. The subgroupoid of connected sequences is denoted $\mathcal{B}_1^{\circ}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$. It is clear that $\pi_0\mathcal{B}_1^{\circ}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass} = \mathcal{B}_1^{\circ}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is isomorphic to W, and that $\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass} = \mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is isomorphic to W^W. Although $\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass} = \mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ we keep using the notation δ_ω for the isomorphism class of $\omega \in \mathcal{B}_1\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$. It only remains to compute the comultiplication:

$$\Delta(\delta_\nu) = \sum_\omega \sum_\kappa |\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| \delta_\kappa \otimes \delta_\omega.$$

By the discussion above we only have to check that

$$|\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| = T^\kappa_{\nu,\omega},$$

but this is clear because there is only one morphism between $d_0\kappa$ and $d_1\omega$ and fibering over ν means taking the subset of those morphisms that give ν after composing, hence $|\operatorname{Iso}(d_0\kappa, d_1\omega)_\nu| = 1$ if $d_1(\kappa, \omega) = \nu$ and 0 otherwise, exactly as $T^\kappa_{\nu,\omega}$. □

Let us move forward to power series in two variables. Again, all the results are also valid for any number of variables, but for simplicity and notation we have chosen to show the two variables case.

Let $\mathbb{Q}\langle\langle x, y\rangle\rangle$ be the ring of formal power series in the noncommutative variables x and y with coefficients in $\mathbb{Q}$ without constant term. Elements of $\mathbb{Q}\langle\langle x, y\rangle\rangle$ are written

$$F(x, y) = \sum_\omega f_\omega \omega,$$

where ω is a nonempty word in x and y. The set $\mathbb{Q}\langle\langle x, y\rangle\rangle$ forms a noncommutative monoid with substitution of power series.

We define the *Faà di Bruno bialgebra in two noncommuting variables* $\mathcal{F}^{\langle 2\rangle}$ as the free polynomial algebra $\mathbb{Q}\big[\{a^i_\omega\}\big]$ generated by the set maps

$$\begin{aligned} a^i_\omega : \mathbb{Q}\langle\langle x, y\rangle\rangle \times \mathbb{Q}\langle\langle x, y\rangle\rangle &\longrightarrow \mathbb{Q} \\ (F^1, F^2) &\longrightarrow f^i_\omega \end{aligned}$$

together with the counit given by $\epsilon(a^i_\omega) = a^i_\omega(x, y)$ and the comultiplication induced by substitution.

Theorem 5.4.7 *The Faà di Bruno bialgebra in two noncommuting variables $\mathcal{F}^{\langle 2\rangle}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}^{S^r}\mathsf{Ass}_2$.*

We obtain the *noncommutative Faà di Bruno bialgebra in two noncommuting variables* $\mathcal{F}^{\langle 2\rangle,\mathrm{nc}}$ by taking above power series with coefficients in R.

Theorem 5.4.8 *The noncommutative Faà di Bruno bialgebra in two noncommuting variables $\mathcal{F}^{\langle 2\rangle,\mathrm{nc}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathsf{Ass}_2$.*

Finally, the *exponential plethystic bialgebra in two variables* $\mathcal{P}^2_{\text{exp}}$ is the same as $\mathcal{P}^2$ but with exponential automorphisms $\text{aut}(\lambda) = \lambda_1!\lambda_2!\cdots$. The generators of this bialgebra are denoted a^i_λ.

Theorem 5.4.9 *The exponential plethystic bialgebra in two variables $\mathcal{P}^2_{\text{exp}}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{BT}_{S^r}\mathsf{Ass}_2$.*

5.5 *Y*-plethysm and bialgebras from *Y*

In Subsection 5.4 we could have taken the locally finite monoid $(\mathbb{N}, \times)$ instead of Ass, since $\mathcal{T}^{\mathsf{M}^r}\mathsf{Ass} = (\mathbb{N}, \times)$ (Example 4.2.1). In fact, we have indirectly done so in the proof of Theorem 5.4.6. It is the case that the three plethystic bialgebras of Subsection 5.4 can be generalized to any locally finite monoid. In this section we explain the generalization of $\mathcal{P}_{\text{exp}}$, which arises from Y-plethysm, introduced by Méndez and Nava [34] in the context of colored species.

Let Y be a locally finite monoid; this means that any $m \in Y$ there has a finite number of two-step factorizations $m = nk$. This is the same as the finite decomposition property of Cartier–Foata [8]. Consider the set of variables $\mathbf{x} = \{x_m\}_{m\in Y}$, and the ring of formal power series $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$, following the same conventions as above. Elements of $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ are written

$$F(\mathbf{x}) = \sum_{\lambda\in\Lambda} \frac{f_\lambda}{\lambda!}\mathbf{x}^\lambda,$$

where now the sum is indexed by the subset $\Lambda \subseteq \text{Hom}_{\mathbf{Set}}(Y, \mathbb{N})$ of maps with finite support, and $\mathbf{x}^\lambda$ is the obvious monomial, for $\lambda \in \Lambda$. In this case $\lambda! = \prod \lambda_m!$.

The monoid structure of Y defines an operation $x_n \circledast x_m = x_{mn}$, which extends to a binary operation on $\mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!]$ as

$$\begin{aligned}(G \circledast F)(x_m|m \in Y) &:= G(F_m|m \in Y), \quad \text{where}\\ F_m(x_n|n \in Y) &:= F(x_{mn}|n \in Y).\end{aligned}$$

This substitution operation was introduced in [34] in the context of species colored over a monoid, although their conditions on the monoid are more restrictive. The main example comes from the monoid $(\mathbb{N}^+, \times)$, which gives ordinary plethysm. Another relevant example is $(\mathbb{N}, +)$, which gives $F_k(\mathbf{x}) = F(x_k, x_{k+1}, \dots)$, which appears in [33]. The power series F_m can be described by using the Verschiebung operators: for each $m \in Y$ we define the mth Verschiebung operator V^m on $\text{Hom}_{\mathbf{Set}}(Y, \mathbb{N})$ as follows: for each $\lambda \in \text{Hom}_{\mathbf{Set}}(Y, \mathbb{N})$ and $n \in Y$,

$$V^m\lambda(n) = \sum_{mk=n} \lambda_k.$$

Clearly if $Y = (\mathbb{N}^+, \times)$ this gives the usual Verschiebung operators [9, 36, 37]. The power series F_m can be expressed as

$$F_m(\mathbf{x}) = \sum_{\lambda} \frac{f_\lambda}{\text{aut}(\lambda)}\mathbf{x}^{V^m\lambda}.$$

As usual, we define the Y-plethystic bialgebra $\mathcal{P}^Y$ as the polynomial algebra $\mathbb{Q}\big[\{a_\lambda\}_\lambda\big]$ on the set maps $a_\lambda : \mathbf{x}\mathbb{Q}[\![\mathbf{x}]\!] \to \mathbb{Q}$ defined by $a_\lambda(F) = f_\lambda$, with comultiplication dual to plethystic substitution, that is

$$\Delta(a_\lambda)(F, G) = a_\lambda(G \circledast F),$$

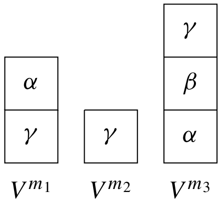
and counit given by $\epsilon(a_\lambda) = a_\lambda(x_1)$.

What follows is devoted to express the comultiplication of $\mathcal{P}^Y$. Consider a list $\boldsymbol{\mu} \in \Lambda^n$ of n infinite vectors, regarded as a representative element of a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$. We denote by $\mathrm{R}(\boldsymbol{\mu}) \subseteq \mathfrak{S}_n$ the set of automorphisms that maps the list $\boldsymbol{\mu}$ to itself. For example if $\boldsymbol{\mu} = \{\alpha, \alpha, \beta, \gamma, \gamma, \gamma\}$ then $\mathrm{R}(\boldsymbol{\mu})$ has $2! \cdot 1! \cdot 3!$ elements. Notice that if $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $\mathrm{R}(\boldsymbol{\mu}) \cong \mathrm{R}(\boldsymbol{\mu}')$. We may thus refer to $\mathrm{R}(\boldsymbol{\mu})$ for a multiset $\overline{\boldsymbol{\mu}} \in \Lambda^n/\mathfrak{S}_n$ by taking a representative, since we are only interested in its cardinality.

Fix two infinite vectors, $\sigma, \lambda \in \Lambda$, and a list of infinite vectors $\boldsymbol{\mu} \in \Lambda^n$, with $n = |\lambda|$. We define the set of $(\lambda, \boldsymbol{\mu})$*−decompositions* of σ as

$$T^{\boldsymbol{\mu}}_{\sigma,\lambda} := \left\{ p \colon \boldsymbol{\mu} \xrightarrow{\sim} \sum_{m \in Y} \{1, \dots, \lambda_m\} \,|\, \sigma = \sum_{\mu \in \boldsymbol{\mu}} V^{q(\mu)} \mu \right\},$$

where p is a bijection of n-element sets and q returns the index of $p(\mu)$ in the sum. A useful way to visualize an element of this set is as a placement of the elements of $\boldsymbol{\mu}$ over a grid with λ_m cells in the mth column such that if we apply V^m to the mth column and sum the cells the result is σ. For example, if $\lambda = (\lambda_{m_1}, \lambda_{m_2}, \lambda_{m_3}) = (2, 1, 3)$ and $\boldsymbol{\mu} = \{\alpha, \alpha, \beta, \gamma, \gamma, \gamma\}$ the placement



belongs to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ if $\sigma = V^{m_1}(\gamma+\alpha)+V^{m_2}(\gamma)+V^{m_3}(\alpha+\beta+\gamma)$, where the sum is a pointwise vector sum in Λ. Note that each such placement appears $|\,\mathrm{R}(\boldsymbol{\mu})|$ times in $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$. Observe also that if $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \Lambda^n$ are representatives of the same multiset then there is an induced bijection $T^{\boldsymbol{\mu}}_{\sigma,\lambda} \cong T^{\boldsymbol{\mu}'}_{\sigma,\lambda}$. We may thus refer to $T^{\boldsymbol{\mu}}_{\sigma,\lambda}$ for a class $\overline{\boldsymbol{\mu}} \in \Lambda^{|\lambda|}/\mathfrak{S}_{|\lambda|}$ by taking a representative, since we are only interested in its cardinality.

Proposition 5.5.1 *The comultiplication of $\mathcal{P}^Y$ is given by*

$$\Delta(\sigma) = \sum_{\lambda} \sum_{\overline{\boldsymbol{\mu}}} \frac{\mathrm{aut}(\sigma) \cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|}{\mathrm{aut}(\lambda) \cdot \mathrm{aut}(\boldsymbol{\mu})} \prod_{\mu \in \boldsymbol{\mu}} a_\mu. \tag{5.51}$$

This proposition is analogous to [9, Proposition 3.3].

Theorem 5.5.2 *The Y-plethystic bialgebra $\mathcal{P}^Y$ is isomorphic to the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y$.*

Proof of 5.5.2 Let us compute the homotopy cardinality of the incidence bialgebra of $\mathcal{B}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y$. First of all, notice that the elements of $\mathcal{B}_1\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y = \mathsf{S}^{\mathrm{r}}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y$ are sequences of tuples

$$(m^1_1, \dots, m^1_{n_1}), \dots, (m^k_1, \dots, m^k_{n_k})$$

of elements of Y. Juxtaposition of sequences gives $\mathcal{B}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}} Y$ a symmetric monoidal structure. Sequences containing only one tuple are called connected, and form an algebra basis of the incidence bialgebra. Since the morphisms between tuples are given by permutations, it is clear that the set of isomorphism classes of connected elements $\pi_0\mathcal{B}_1^{\circ}\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y$ is isomorphic to Λ, the subset of $\mathrm{Hom}_{\mathbf{Set}}(Y, \mathbb{N})$ consisting of maps with finite support. The isomorphism class δ_λ of a connected element λ is given by the map $Y \xrightarrow{\lambda} \mathbb{N}$ such that λ_m is the number of times m appears in λ. Be aware that the same notation is used for either the connected elements of $\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y$ and the maps representing their isomorphism class. Moreover,

$$\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y \cong \sum_n \Lambda^n//\mathfrak{S}_n,$$

so that an element $\tau \in \pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y$ may be identified with a multiset $\overline{\boldsymbol{\mu}}$ of maps. With these identifications we clearly have

$$|\operatorname{Aut}(\lambda)| = \lambda! \quad \text{and} \quad |\operatorname{Aut}(\tau)| = \operatorname{aut}(\boldsymbol{\mu}),$$

for λ connected and τ not necessarily connected. The left hand sides refer to the automorphisms groups in $\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y$, while the right hand sides were introduced above.

The assignment

$$\begin{aligned} \mathbb{Q}_{\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y} &\longrightarrow \mathcal{P}_{\exp} \\ \delta_\lambda &\longmapsto a_\lambda \\ \delta_{\lambda+\mu} = \delta_\lambda\delta_\mu &\longmapsto a_\lambda a_\mu, \end{aligned}$$

for λ and μ connected, defines an isomorphism of algebras. Notice that $\lambda+\mu$ is the monoidal sum in $\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y$, which does not correspond to the pointwise sum of their corresponding infinite vectors, since it has two connected components.

We have to compute the coproduct in $\mathbb{Q}_{\pi_0\mathcal{B}_1\mathcal{T}_{\mathsf{S}}^{\mathrm{r}}Y}$. It is enough to compute it for connected elements. From Lemma 2.3.1 we have, for σ connected,

$$\Delta(\delta_\sigma) = \sum_\lambda \sum_\tau \frac{|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma|}{|\operatorname{Aut}(\lambda)||\operatorname{Aut}(\tau)|}\delta_\tau \otimes \delta_\lambda. \tag{5.52}$$

In view of the discussion above, it only remains to show that

$$|\operatorname{Iso}(d_0\tau, d_1\lambda)_\sigma| = \operatorname{aut}(\sigma) \cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|.$$

Consider representatives for τ and λ,

$$\begin{aligned} \tau &= \big((m_1^1, \ldots, m_{n_1}^1), \ldots, (m_1^k, \ldots, m_{n_k}^k)\big) \\ \lambda &= (m_1, \ldots, m_k), \end{aligned}$$

then $d_0\tau = d_1\lambda = (1, \ldots, 1)$, k times. This means that

$$\operatorname{Iso}(d_0\tau, d_1\lambda) = \operatorname{Aut}(1, \ldots, 1) \cong \mathfrak{S}_k.$$

Any element $\phi \in \operatorname{Iso}(d_0\tau, d_1\lambda)$ induces a map between sequences

$$\big((m_1^1, \ldots, m_{n_1}^1), \ldots, (m_1^k, \ldots, m_{n_k}^k)\big) \xrightarrow{\phi} (m_1, \ldots, m_k).$$

We express it as a permutation on τ and write

$$\phi(\tau) = \big((m_1^{\phi(1)}, \ldots, m_{n_{\phi(1)}}^{\phi(1)}), \ldots, (m_1^{\phi(k)}, \ldots, m_{n_{\phi(k)}}^{\phi(k)})\big).$$

Now, consider the subset

$$\Big\{\phi \in \operatorname{Iso}(d_0\tau, d_1\lambda) \mid d_1\big((\phi(\tau), \lambda)\big) \simeq \sigma\Big\}.$$

It is straightforward to see that this subset is isomorphic to

$$T^{\boldsymbol{\mu}}_{\sigma,\lambda} := \left\{ p\colon \boldsymbol{\mu} \xrightarrow{\sim} \sum_{m\in M} \{1,\dots,\lambda_m\} \mid \sigma = \sum_{\mu\in\boldsymbol{\mu}} V^{q(\mu)}\mu \right\},$$

under the identifications $\tau \to \boldsymbol{\mu}$ and $\phi \to p$. The summation of the Verschiebung operators is precisely composition of $\phi(\tau)$ and λ. Finally, since $\mathrm{Iso}(d_0\tau, d_1\lambda)_\sigma$ is a homotopy fiber we have that

$$\mathrm{Iso}(d_0\tau, d_1\lambda)_\sigma \cong \mathrm{Aut}(\sigma) \times \left\{ \phi \in \mathrm{Iso}(d_0\tau, d_1\lambda) \mid d_1\big((\phi(\tau),\lambda)\big) \simeq \sigma \right\} \cong \mathrm{Aut}(\sigma) \times T^{\boldsymbol{\mu}}_{\sigma,\lambda}$$

and therefore

$$|\,\mathrm{Iso}(d_0\tau, d_1\lambda)_\sigma| = \mathrm{aut}(\sigma)\cdot |T^{\boldsymbol{\mu}}_{\sigma,\lambda}|,$$

as we wanted to see. □

This proves also Theorem 5.4.3 by taking the monoid $(\mathbb{N}^+, \times)$.

6 Relation with *T*S

We end this work by exploring the relations between the $\mathcal{T}$-construction and the simplicial groupoid $T\mathbf{S}$ of [9]. We first recall what this simplicial groupoid looks like. Then we prove that $T\mathbf{S}$ and $\mathcal{BT}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$ are equivalent simplicial groupoids. This proves in particular Theorem 5.1.3. Finally we show that the operads of Section 5 arising from Ass or Sym are also equivalent to similar simplicial groupoids.

6.1 The simplicial groupoid *T*S

It can be defined through a general construction [9], but we content ourselves with a brief description: objects in $T_1\mathbf{S}$ and $T_2\mathbf{S}$ (1 and 2-simplices of $T\mathbf{S}$) are, respectively, diagrams of finite sets and surjections

$$\begin{array}{ccc} & t_{01} & \\ \swarrow & & \searrow \\ t_{00} & \twoheadrightarrow & t_{11}, \end{array}\qquad\qquad \begin{array}{ccccc} & & t_{02} & & \\ & \swarrow & \lrcorner & \searrow & \\ & t_{01} & & t_{12} & \\ \swarrow & & \searrow\ \swarrow & & \searrow \\ t_{00} & \twoheadrightarrow & t_{11} & \twoheadrightarrow & t_{22}. \end{array}$$

Morphisms of such shapes are levelwise bijections $t_{ij} \xrightarrow{\sim} t'_{ij}$ compatible with the diagram. In general $T_n\mathbf{S}$ is an analogous pyramid, with t_{0n} in the peak, all of whose squares are pullbacks of sets. The face maps d_i remove all the sets containing an i index, and the degeneracy maps s_i repeat the ith diagonals. Diagrams whose last set is singleton are called *connected*. It is not difficult to see that $T\mathbf{S}$ is a Segal groupoid [9].

We now prove that $T\mathbf{S} \simeq \mathcal{BT}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$. We prove the equivalence by constructing an intermediate simplicial groupoid. More precisely, we find a subsimplicial groupoid of $T\mathbf{S}$ which is equivalent to $T\mathbf{S}$ and isomorphic to $\mathcal{BT}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$. First of all we need some notation and elementary results.

Definition 6.1.1 Consider the category of finite ordinals $[n] = \{1, \ldots, n\}$ and set maps. We say that a square

$$\begin{array}{ccc} [m] & \xrightarrow{q} & [n] \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ [l] & \xrightarrow[g]{} & [k] \end{array} \tag{6.1.1}$$

is monotone if it is a pullback of sets, p is monotone and q is monotone at each fiber over p, that is, $q_{|p^{-1}(i)|}$ is monotone for all $i \in [l]$.

Lemma 6.1.2 *Consider the category of finite ordinals and set maps.*

(i) The class of monotone pullback squares is closed under composition of squares.
(ii) Given a diagram $[l] \xrightarrow{g} [k] \xleftarrow{f} [n]$, there is a unique monotone square as 6.1.1.

Proof (i) is clear, and (ii) follows fom the fact that we can totally order the pullback,

$$P = \sum_{i \in [k]} [l]_i \times [n]_i,$$

by using the orders of $[l]$ and $[n]$. That is, given $a, b \in P$, then $a < b$ if $p(a) < p(b)$ or $p(a) = p(b)$ and $q(a) < q(b)$. □

Consider the full subsimplicial groupoid $\mathcal{V} \subseteq T\mathbf{S}$ containing only the simplices whose entries are the finite ordinals $[k]$, whose left-down-arrows and right-arrows are monotone surjections and whose left-down arrows are fiber-monotone in the sense of Definition 6.1.1, and whose pullback squares are monotone. Note that Lemma 6.1.2 ensures that $\mathcal{V}$ is well defined, meaning that the inclusion $\mathcal{V} \hookrightarrow T\mathbf{S}$ is a morphism of simplicial groupoids.

Lemma 6.1.3 *$\mathcal{V} \hookrightarrow T\mathbf{S}$ is an equivalence of simplicial groupoids.*

Proof Given an element of $T_n\mathbf{S}$,

$$\begin{array}{ccccccccc} & & \swarrow & & & & \searrow & & \\ & t_{01} & & & & & t_{n-1,n} & & \\ \swarrow & & \searrow & & & \swarrow & & \searrow & \\ t_{00} & \longrightarrow & t_{11} & \dashrightarrow & t_{n-1,n-1} & & \longrightarrow & & t_{nn}, \end{array}$$

it is clear we can choose an ordering of the t_{ii} and the $t_{i,i+1}$ such that all the arrows between them are monotone. Then by Lemma 6.1.2 there exists a unique ordering on the rest of the t_{ij}'s making the pullback squares monotone. Hence the inclusion is essentially surjective. Since we have taken the full inclusion, the automorphism group of any elment of $\mathcal{V}_n$ is equal to to its automorphism group as an element of $T_n\mathbf{S}$. Hence the inclusion is an equivalence. □

Note that in $\mathcal{V}$ the uniqueness of the monotone squares implies that the Segal maps are in fact isomorphisms,

$$\mathcal{V}_n \cong \mathcal{V}_1 \times_{\mathcal{V}_0} \cdots \times_{\mathcal{V}_0} \mathcal{V}_1.$$

In other words, there is a well-defined composition $d_1 \colon \mathcal{V}_1 \times_{\mathcal{V}_0} \mathcal{V}_1 \to \mathcal{V}_1$. In view of this we may drop the elements t_{ij} with $j \geq i + 2$ from the diagrams.

Lemma 6.1.4 *Let* V *be the operad whose* n*-ary operations are diagrams*

$$\begin{array}{ccc} & [m] & \\ \swarrow & & \searrow \\ [n] & \longrightarrow & 1 \end{array}$$

where $[m] \twoheadrightarrow [n]$ *is monotone, whose morphisms are entrywise bijections, and whose composition is given by monotone pullback squares. Then* $\mathcal{V} \cong \mathcal{B}\mathsf{V}$*.*

Proof The isomorphism is given by

$$\begin{array}{ccc} & [m_1] & \\ \swarrow & & \searrow \\ [n_1] & \longrightarrow & 1, \end{array} \cdot \, , \begin{array}{ccc} & [m_k] & \\ \swarrow & & \searrow \\ [n_2] & \longrightarrow & 1, \end{array} \quad \mapsto \quad \begin{array}{ccc} & [m_1 + \cdots + m_k] & \\ \swarrow & & \searrow \\ [n_1 + \cdots + n_k] & \rightarrowtail & [k] \end{array}$$

at the level of 1-simplices and similarly in general. □

Lemma 6.1.5 V *is isomorphic to* $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$*.*

Proof An operation of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ is a family of operations of Sym, which is equivalent to a monotone surjection $[m] \twoheadrightarrow [n]$. It is also clear that morphsims between operations of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ are the same as morphisms in V. Thus we only need to see that composition coincides. Let us denote by x the unique x-ary operation of Sym. Thus a general element of $\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ is a tuple $(x_1, \ldots, x_n)$. By definition of the $\mathcal{T}$-construction

$$\begin{aligned} &(x_1, \ldots, x_n) \circledast ((y_1^1, \ldots, y_{k_1}^1), \ldots, (y_1^n, \ldots, y_{k_n}^n)) \\ &\quad = (y_1^1 \cdot x_1, \ldots, y_{k_1}^1 \cdot x_1, \ldots, y_1^n \cdot x_n, \ldots, y_{k_n}^n \cdot x_n), \end{aligned}$$

which is nothing but the pullback

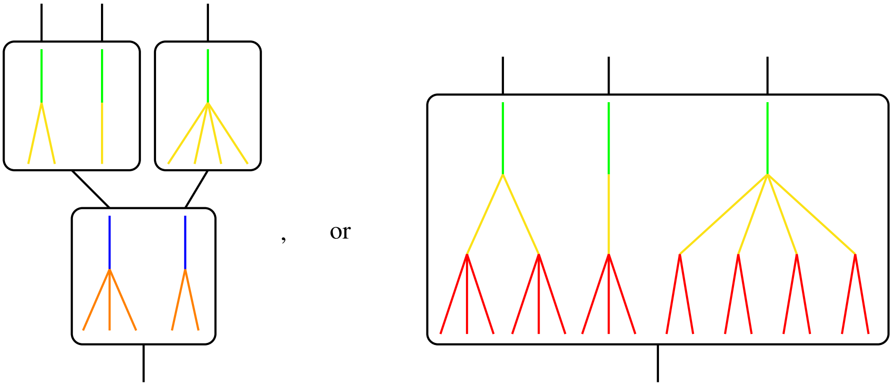
$$\begin{array}{ccccc} & & \left[\sum_{i,j} y_j^i x_i\right] & & \\ & \swarrow & \lrcorner & \searrow & \\ & \left[\sum_{i,j} y_j^i\right] & & \left[\sum_i x_i\right] & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ \left[\sum_i k_i^i\right] & \twoheadrightarrow & \cdot & \twoheadrightarrow & 1, \end{array}$$

the composition of their corresponding operations in V. □

Proposition 6.1.6 *The simplicial groupoids* $T\boldsymbol{S}$ *and* $\mathcal{B}\mathcal{T}_{\mathsf{S}^r}\mathsf{Sym}$ *are equivalent.*

Proof It is direct from Lemmas 6.1.2, 6.1.3 and 6.1.4. □

Example 6.1.7 Consider the following 2-simplex of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}$:



(6.1.2)

We use colors here only to make the comparison more pleasant, but of course this is not a colored operad. This 2-simplex corresponds, in $T\mathbf{S}$, to

$$\begin{array}{ccccc} & & 17 & & \\ & \swarrow & \downarrow & \searrow & \\ & 7 & & 5 & \\ \swarrow & & \searrow\ \swarrow & & \searrow \\ 3 & \twoheadrightarrow & 2 & \twoheadrightarrow & 1. \end{array} \tag{6.1.3}$$

It is opportune in this example to show that indeed the convention of drawing the operations upside down (or taking the opposite category before applying $\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}$, see Diagram (4.2.2) and the preceding digression) comes out more naturally in order to interpret $T\mathbf{S}$ as an operad. First of all, observe that at the level of finite sets and surjections the Verschiebung operators 5.1.3 can be regarded as a scalar multiplication,

$$V^S(X \to B) = S \times X \to X \to B,$$

in the sense that if λ represents the class of $X \to B$, then $V^{|S|}\lambda$ represents the class of $V^S(X \to B)$. Under this perspective we can write the information on (6.1.3) as

$$\begin{aligned} 17 \twoheadrightarrow 3 &= (9 \twoheadrightarrow 2) + (8 \twoheadrightarrow 1) = V^3(3 \twoheadrightarrow 2) + V^2(4 \twoheadrightarrow 1) \\ &= (3 \times 3 \twoheadrightarrow 2) + (2 \times 4 \twoheadrightarrow 1), \end{aligned} \tag{6.1.4}$$

We can clearly see this in (6.1.2). On the contrary, it is not difficult to check that without this convention Equation (6.1.4) would rather appear as

$$17 \twoheadrightarrow 3 = (9 \twoheadrightarrow 2) + (8 \twoheadrightarrow 1) = \big((2 \times 3 \twoheadrightarrow 1) + (1 \times 3 \twoheadrightarrow 1)\big) + (4 \times 2 \twoheadrightarrow 1).$$

6.2 Other *T*S-like simplicial groupoids

We now present other equivalences between variations of $T\mathbf{S}$ and some of the bar constructions treated before. First of all we introduce some notation: monotone surjections between ordered sets are denoted $a \overset{\bullet}{\twoheadrightarrow} b$. We call *linear surjection* $a \overset{\circ}{\twoheadrightarrow} b$ a surjection between finite sets $f : a \twoheadrightarrow b$ with an order on $f^{-1}(r)$ for each $r \in b$.

Notice that the composite of two monotone surjections is again a monotone surjection, and the composite of two linear surjections is again a linear surjection, with the obvious order. Moreover, given pullback squares

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & \cdot \\ p \downarrow^{\bullet} & \lrcorner & \downarrow^{\bullet} f \\ \cdot & \twoheadrightarrow & \cdot \end{array} \quad \text{and} \quad \begin{array}{ccc} \cdot & \twoheadrightarrow & \cdot \\ q \downarrow^{\circ} & \lrcorner & \downarrow^{\circ} g \\ \cdot & \twoheadrightarrow & \cdot \end{array},$$

we say that p and f are *compatible* if the order of p is induced by the order of f, in the sense of Lemma 6.1.2. Similarly, we say that q and g are compatible if the order of q is induced by the order of g.

The proofs of all the following results are similar to the one of Proposition 6.1.6. To avoid repetitiveness we give only intuitive explanations.

Example 6.2.1 The simplicial groupoid $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Ass}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that all the left-down surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{array}{ccc} & t_{01} & \\ \swarrow^{\circ} & & \searrow \\ t_{00} & \twoheadrightarrow & t_{11}. \end{array}$$

The isomorphism classes of connected diagrams are again infinite vectors $\lambda = (\lambda_1, \lambda_2, \dots)$ as in $T\mathbf{S}$, and the number of automorphisms of a connected element of class λ is precisely $\lambda_1! \cdot \lambda_2! \cdots$, since t_{01} is fixed.

Example 6.2.2 The simplicial groupoid $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is equivalent to the simplicial grou-poid constructed as $T\mathbf{S}$ but with the additional structure that the left-down surjections and the right surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{array}{ccc} & t_{01} & \\ \swarrow^{\circ} & & \searrow \\ t_{00} & \overset{\circ}{\twoheadrightarrow} & t_{11}. \end{array}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by words $\omega = \omega_1\omega_2 \dots \omega_n$ where ω_i is the size of the ith fiber. It does not have any automorphisms, since t_{01} and t_{00} are fixed.

Example 6.2.3 The simplicial groupoid $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Sym}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the right surjections are linear and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{array}{ccc} & t_{01} & \\ \swarrow & & \searrow \\ t_{00} & \overset{\circ}{\twoheadrightarrow} & t_{11}. \end{array}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by finite words $\omega = \omega_1\omega_2 \dots \omega_n$ where $\omega_i > 0$ is the size of the ith fiber. It has $\omega! := \omega_1!\omega_2! \cdots \omega_n!$ automorphisms, since t_{00} is fixed.

Example 6.2.4 The simplicial groupoid $\mathcal{B}^{\mathsf{M}}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Sym}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the right surjections are monotone. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{array}{ccc} & t_{01} & \\ \swarrow & & \searrow\!\!\!\searrow \\ t_{00} & \longrightarrow\!\!\!\!\!\bullet\!\!\twoheadrightarrow & t_{11}. \end{array}$$

Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by finite words $\omega = \omega_1\omega_2 \ldots \omega_n$ where $\omega_i > 0$ is the size of the ith fiber. It has $\omega! := \omega_1!\omega_2! \cdots \omega_n!$ automorphisms, since t_{00} is fixed. The difference between this simplicial groupoid and the one of Example 6.2.3 is that in this case t_{11} is also ordered. As a consequence the monoidal structure is not symmetric, so that the resulting incidence bialgebra is not commutative.

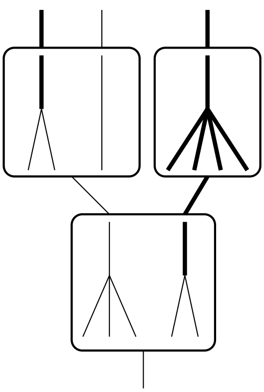
Example 6.2.5 The simplicial groupoid $\mathcal{B}^{\mathsf{M}}\mathcal{T}_{\mathsf{M}^{\mathrm{r}}}\mathsf{Ass}$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the left-down surjections and the right surjections are monotone and compatible. Morphisms are order-preserving levelwise bijections. Hence the 1-simplices are diagrams

$$\begin{array}{ccc} & t_{01} & \\ \swarrow\!\!\!\swarrow\!\!\bullet & & \searrow\!\!\!\searrow \\ t_{00} & \longrightarrow\!\!\!\!\!\bullet\!\!\twoheadrightarrow & t_{11}. \end{array}$$

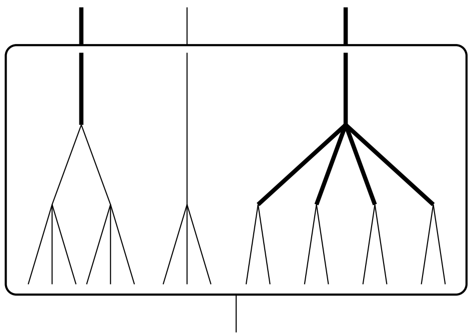
Observe that for a connected element, t_{00} is totally ordered. Thus the isomorphism classes of connected elements are given by words $\omega = \omega_1\omega_2 \ldots \omega_n$ where ω_i is the size of the ith fiber. It does not have any automorphisms, since t_{01} and t_{00} are fixed. Again, the difference between this simplicial groupoid and the one of Example 6.2.2 is that in this case t_{11} is ordered.

Example 6.2.6 Finally, the simplicial groupoid $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}_2$ is equivalent to the simplicial groupoid constructed as $T\mathbf{S}$ but with the additional structure that the objects are 2-colored and the right-down surjections are color preserving. Morphisms are color-preserving levelwise bijections.

For instance, the following 2-simplex of $\mathcal{B}^{\mathsf{S}}\mathcal{T}_{\mathsf{S}^{\mathrm{r}}}\mathsf{Sym}_2$,



, or



(6.2.1)

where now the colors do refer to the input and output colors, corresponds to the following 2-simplex:

$$\begin{array}{ccccc} & & 17 & & \\ & \swarrow & \vee & \searrow & \\ & 3+\mathbf{4} & & 5 & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ 1+\mathbf{2} & \twoheadrightarrow & 1+\mathbf{1} & \twoheadrightarrow & 1. \end{array} \tag{6.2.2}$$

Observe that indeed the right-down surjections are color-preserving.

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A Appendices

A.1 Axioms for internal category

Let $\mathcal{E}$ be a cartesian category. A category C internal to $\mathcal{E}$ can be described by objects and arrows of $\mathcal{E}$

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle s} \swarrow & & \searrow {\scriptstyle t} \\ C_0 & & C_0 \end{array} \qquad \begin{array}{c} C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \\[2ex] C_0 \xrightarrow{e} C_1 \end{array}$$

where the pullback is taken along $C_1 \xrightarrow{s} C_0 \xleftarrow{t} C_1$, satisfying the following commutative diagrams:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array} \tag{A.1.1a}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array} \tag{A.1.1b}$$

$$\begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle s} \\ & & C_0 \end{array} \tag{A.1.2a}$$

$$\begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle t} \\ & & C_0 \end{array} \tag{A.1.2b}$$

$$\begin{array}{ccc}
(C_1\times_{C_0}C_1)\times_{C_0}C_1 & \xrightarrow{m\times_{C_0}C_1} & C_1\times_{C_0}C_1 \\
\downarrow & & \\
C_1\times_{C_0}(C_1\times_{C_0}C_1) & & \downarrow{\scriptstyle m} \\
{\scriptstyle C_1\times_{C_0}m}\downarrow & & \\
C_1\times_{C_0}C_1 & \xrightarrow[m]{} & C_1
\end{array} \tag{A.1.3}$$

$$\begin{array}{ccc}
C_0\times_{C_0}C_1 & \xrightarrow{e\times_{C_0}C_1} & C_1\times_{C_0}C_1 \\
& {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle m} & \\
& C_1 &
\end{array} \tag{A.1.4a}$$

$$\begin{array}{ccc}
C_1\times_{C_0}C_0 & \xrightarrow{C_1\times_{C_0}e} & C_1\times_{C_0}C_1 \\
& {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle m} & \\
& C_1 &
\end{array} \tag{A.1.4b}$$

Axioms for P-operad

Let $\mathcal{E}$ be a cartesian category and (P, μ, η) a cartesian monad. A P-multicategory Q can be described by objects and arrows of $\mathcal{E}$

$$\begin{array}{ccc}
& Q_1 & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\
\mathsf{P}Q_0 & & Q_0
\end{array}
\qquad
\begin{array}{c}
\mathsf{P}Q_1\times_{\mathsf{P}Q_0}Q_1 \xrightarrow{m} Q_1 \\[2ex]
Q_0 \xrightarrow{e} Q_1
\end{array}$$

where the pullback is taken along $\mathsf{P}Q_1 \xrightarrow{t} \mathsf{P}Q_0 \xleftarrow{s} Q_1$, satisfying the following commutative diagrams:

$$\begin{array}{ccc}
\mathsf{P}Q_1\times_{\mathsf{P}Q_0}Q_1 & \xrightarrow{p_1} & \mathsf{P}Q_1 \\
& & \downarrow{\scriptstyle \mathsf{P}s} \\
{\scriptstyle m}\downarrow & & \mathsf{P}^2Q_0 \\
& & \downarrow{\scriptstyle \mu_{Q_0}} \\
Q_1 & \xrightarrow[s]{} & \mathsf{P}Q_0
\end{array} \tag{A.2.1a}$$

$$\begin{array}{ccc}
\mathsf{P}Q_1\times_{Q_0}Q_1 & \xrightarrow{m} & Q_1 \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle t} \\
Q_1 & \xrightarrow[t]{} & Q_0
\end{array} \tag{A.2.1b}$$

$$\begin{array}{ccc}
Q_0 & \xrightarrow{e} & Q_1 \\
& {\scriptstyle \eta_{Q_0}}\searrow & \downarrow{\scriptstyle s} \\
& & \mathsf{P}Q_0
\end{array} \tag{A.2.2a}$$

$$\begin{array}{ccc}
Q_0 & \xrightarrow{e} & Q_1 \\
& {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle t} \\
& & Q_0
\end{array} \tag{A.2.2b}$$

$$\begin{array}{ccc}
(\mathsf{P}^2 Q_1 \times_{\mathsf{P}^2 Q_0} \mathsf{P} Q_1) \times_{\mathsf{P} Q_0} Q_1 & \xrightarrow{\mathsf{P}m \times_{Q_0} Q_1} & \mathsf{P} Q_1 \times_{\mathsf{P} Q_0} Q_1 \\
\downarrow & & \\
\mathsf{P}^2 Q_1 \times_{\mathsf{P}^2 Q_0} (\mathsf{P} Q_1 \times_{\mathsf{P} Q_0} Q_1) & & \downarrow m \\
\downarrow \mu_{Q_1} \times_{\mu Q_0} m & & \\
\mathsf{P} Q_1 \times_{\mathsf{P} Q_0} Q_1 & \xrightarrow[m]{} & Q_1
\end{array} \tag{A.2.3}$$

$$\begin{array}{ccc}
\mathsf{P} Q_0 \times_{\mathsf{P} Q_0} Q_1 & \xrightarrow{\mathsf{P}e \times_{Q_0} Q_1} & \mathsf{P} Q_1 \times_{\mathsf{P} Q_0} Q_1 \\
{\scriptstyle p_2} \searrow & & \swarrow {\scriptstyle m} \\
& Q_1 &
\end{array} \tag{A.2.4a}$$

$$\begin{array}{ccc}
Q_1 \times_{Q_0} Q_0 & \xrightarrow{Q_1 \times_{Q_0} e} & \mathsf{P} Q_1 \times_{\mathsf{P} Q_0} Q_1 \\
{\scriptstyle p_1} \searrow & & \swarrow {\scriptstyle m} \\
& Q_1 &
\end{array} \tag{A.2.4b}$$

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