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Structurally Unstable Quadratic Vector Fields of Codimension Two: Families Possessing One Finite Saddle-Node and a Separatrix Connection

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Abstract
This paper is part of a series of works whose ultimate goal is the complete classification of phase portraits of quadratic differential systems in the plane modulo limit cycles. It is estimated that the total number may be around 2000, so the work to find them all must be split in different papers in a systematic way so to assure the completeness of the study and also the non intersection among them. In this paper we classify the family of phase portraits possessing one finite saddle-node and a separatrix connection and determine that there are a minimum of 77 topologically different phase portraits plus at most 16 other phase portraits which we conjecture to be impossible. Along this paper we also deploy a mistake in the book (Artés et al. in Structurally unstable quadratic vector fields of codimension one, Birkhäuser/Springer, Cham, 2018) linked to a mistake in Reyn and Huang (Separatrix configuration of quadratic systems with finite multiplicity three and a $M^0_{1,1}$ type of critical point at infinity. Report Technische Universiteit Delft, pp 95–115, 1995).

Keywords Quadratic differential systems · Structural stability · Codimension two · Phase portrait · Saddle-node

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1 Introduction

In this paper we study the simplest non-linear polynomial differential equations, the planar quadratic differential systems. A *polynomial differential system* on the plane is

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a system of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in the variables x and y over $\mathbb{R}$. We call *degree* of a system (1) the integer $n = \max(\deg(p), \deg(q))$. In particular we call *quadratic* a differential system (1) with degree $n = 2$.

This paper is part of a series of papers already published (and some more that will come) whose ultimate goal is the total classification of phase portraits of quadratic systems, done for several members of a group of researchers. So some parts of the introduction and the techniques used may be common with them.

The linear differential equations were completely solved by Laplace in 1812 for every dimension, not just planar. After the resolution of linear differential systems, it seemed natural to address the classification of quadratic differential systems. However, it was found that the problem would not have an easy and fast solution. Unlike the linear systems that can be solved analytically, quadratic systems (not even, therefore, those of higher degree) do not generically admit a solution of that kind, at least, with a finite number of terms.

Therefore, for the resolution of non-linear differential systems, another strategy was chosen and it allowed the creation of a new area of knowledge in Mathematics: the Qualitative Theory of Differential Equations [37]. The idea is quite simple: since we are not able to give a concrete mathematical expression to the solution of a system of differential equations, this theory intends to express by means of a complete and precise drawing, the behavior of any particle located in a vector field governed by such a differential equation, i.e. its phase portrait.

Even with all the reductions made to the problem until now, there are still difficulties. The most expressive difficulty is that the phase portraits of differential systems may have invariant sets that are not punctual, as the limit cycles. A linear system cannot generate limit cycles; at most they can present a completely circular phase portrait where all the orbits are periodic. But a differential system in the plane, polynomial or not, and starting with the quadratic ones, may present several of these limit cycles. It is trivial to verify that there can be an infinite number of these cycles in non-polynomial problems, but the intuition seems to indicate that a polynomial system should not have an infinite number of limit cycles in a similar way that it cannot have an infinite number of isolated singular points. And because the number of singular points is linked to the degree of the polynomial system, it also seems logical to think that the number of limit cycles could also have a similar link, either directly as the number of singular points, or even in an indirect way from the number of the parameters of such systems. In fact, it is already proved that quadratic systems have a finite number of limit cycles [21] and there are two independent proofs that any given polynomial system has a finite number of limit cycles [28, 33]. However, it is worth mentioning that none of both has been yet fully understood by the mathematical community.

In 1900, David Hilbert [30, 31] proposed a set of 23 problems to be solved in the 20th century, and among them his well-known 16th problem asks for the maximum number of limit cycles $H(n)$ a polynomial differential system in the plane with degree

n may have. More than one hundred years after, we do not yet have an uniform upper bound for this generic problem, only for specific families of such a system.

Therefore, the complete classification of quadratic systems is a very difficult task at the moment and it depends enormously of the culmination of Hilbert's 16th problem, even at least partially for $H(2)$. At this moment we simply know that $H(2) \geq 4$ but no example with 5 limit cycles has been found. The first example with 4 limit cycles was found by Shi Song Ling in [45]. In fact, only three phase portraits have been found up to now with such number of limit cycles and all three derive directly from phase portraits with a weak focus of third order which have a limit cycle along a strong focus [3].

Even so, a lot of problems have been appearing related to quadratic systems and for which it has been possible to give an answer. In fact, there are more than one thousand articles published directly related to quadratic systems. John Reyn, from Delft University (Netherlands), was committed in preparing a bibliography that was published several times until his retirement [38]. It is worth mentioning that in the last three decades many other articles related to quadratic systems have appeared, what figures that the mentioned amount of one thousand papers in that bibliography has already been widely exceeded. It is worth mentioning that he estimated the total number of different phase portraits (modulo limit cycles) to be around 2000.

In those more than one thousand papers mentioned, many families have been studied, partially or completely, but the collection of all the works is not helpful to provide a complete classification since there are many intersections among the papers (same phase portrait may belong to several families), or even worse, there may be phase portraits that have never appeared in any of them.

So, we need to obtain a systematic procedure which studies independent families producing always different phase portraits with the assurance that after a finite number of families, all of them will have appeared. With this goal in mind, Reyn tried to study families according to the number of finite singularities that have escaped to infinity. He was successful in the cases with two or more singularities escaping to infinity [39, 40, 44]. In the case when just one singularity has escaped to infinity, they published [43] with the case when one of the infinite singularities is nilpotent or degenerate, but their work where this singularity is just semi-elemental remained just as a report [42] since several mistakes were detected. Some missed phase portraits were already reported in [6] and here we will report an impossible phase portrait which induced a mistake in [6]. Finally Reyn in his book [41] recognized the impossibility to deal with the case where no finite singularity escapes to infinity using his tools.

Given the difficulty of solving the 16th Hilbert's problem, if we want to obtain a global classification of quadratic systems before this problem is solved, this will have to be done modulo limit cycles. We propose to carry out a systematic global classification and, for this, we cannot be attained only to the study of families of systems that do not give more than extremely local visions of the global parameter space. Even applying to our quadratic system a linear change of coordinates plus a translation and a time rescaling, which supposes a reduction from the initial 12 parameters to a limited set of systems with 5 parameters, $\mathbb{R}^5$ is still a very large space. And moreover, there is not just a single family with 5 parameters that contains all quadratic systems. One needs several such families. The study of families has been very useful to provide examples, but not for the systematic classification.

The other systematic way to try to obtain the complete classification of phase portraits of quadratic systems was started with the study of the structurally stable quadratic systems, modulo limit cycles. That is, the goal was to determine how many and which phase portraits of a quadratic system cannot be modified by small perturbations in their coefficients. To obtain a structurally stable system modulo limit cycles we need very few conditions: we do not allow the existence of multiple singular points and the existence of connections of separatrices. Centers, weak foci and semi-stable cycles are submerged in the quotient modulo limit cycles. This systematic analysis [2] showed that the structurally stable quadratic systems modulo limit cycles produce a total of 44 topologically distinct phase portraits.

The natural problem to be studied after was the structurally unstable quadratic differential systems of codimension one. This study [6] was done in approximately 20 years and finally we obtained at least 204 (and at most 211) topologically phase portraits of codimension one modulo limit cycles.

The pattern of work in these two papers (and the ones continuing after) is quite similar. First we need to produce by combination of singularities and separatrices, all *potential* (see definition below) phase portraits of a given codimension and after one must either find a concrete example of every phase portrait, or produce a proof which shows its impossibility.

Definition 1 By a potential phase portrait we understand a phase portrait which is compatible with the number and type of singularities with what can be obtained in a fixed class of systems.

So, a potential phase portrait may still be not realizable by other deeper reasons.

In several previous papers these phase portraits were called simply as “possible” but the interpretation of this word could make people uncomfortable when something called “possible” finally becomes impossible or non-realizable. So, we have decided for a different word. The candidates of phase portraits that we first obtain have the potential of being finally realizable, but maybe they are not at the end.

The types of proofs that work to show impossibilities of phase portraits use to deal with the number of contact points that the flow can have with a straight line. Some newer proofs deal with geometric concepts like the position and tangencies of orbits and characteristic directions. Also, the non realizable phase portraits of a certain codimension become a key tool to prove the impossibility of related phase portraits of lower and higher codimension.

The way to obtain examples of the phase portraits comes mainly from already studied families of the same or higher codimensions. If they are of the same codimension, they must directly appear in those studies. In case of using higher codimension examples, then by perturbing one or more of the unstable elements of that phase portrait, one obtains the desired phase portrait. In this way the study of structurally stable quadratic systems is complete, that is, from 72 initially potential phase portraits, we obtained examples of 44 and proved the impossibility of the remaining 28. And up to now, no result has contradicted this statement.

In 1998, just after ending [2] and starting the production of topologically potential phase portraits of codimension one, the number, and particularly the size of the already studied families, was not large. But new techniques created by the school of Sibirskii [9, Chapter 5] about invariant polynomials, allowed a growth in the dimension of the bifurcation diagrams that could be studied, and it became a gorgeous source of phase portraits which helped to complete [6].

Anyway, in [6] the work could not be completely ended since after studying more than 500 potential different phase portraits, finding examples for 204 of them, and impossibilities of more than 300, there remained seven phase portraits for which we were unable to provide neither an example nor a proof of impossibility. And all seven cases are related with the existence of a graphic and the behavior of the focus inside. The tools of contact points are useless in these cases. The proofs of impossibilities might be related to the impossibility of certain phase portraits with limit cycles. The fact that we were not able to prove such impossibility, together with the fact that we have not found such phase phase portraits in none of the papers previously published, made us conjecture their impossibility.

This fact will produce a cascade effect in higher codimensions since conjectured impossibility of some codimension one phase portraits will extend into some more codimension two phase portraits, plus some new ones which will appear.

The next step is now the study of codimension two phase portraits and this was already initiated in [14, 15]. In the first paper, the scheme of work for codimension two was introduced. Since the number of cases in codimension two will exceed by large those of codimension one, it was proposed to split it in several classes and [15] already studied the first of them, concretely the phase portraits containing exactly two finite saddle-nodes, or one cusp as the only unstable elements. In [14] we find a continuation of the work where phase portraits having exactly one finite and one infinite saddle-node (this includes two classes) as the only unstable elements, are studied.

In what follows, we recall some definitions and notation used in those papers, and then we explain all the cases of structurally unstable quadratic systems of codimension two, one by one, and present the completion of the fourth class.

Let X be a vector field. A point $p \in \mathbb{R}^2$ such that $X(p) = 0$ (respectively $X(p) \neq 0$) is called a *singular point* (respectively *regular point*) of the vector field X.

Let $P_n(\mathbb{R}^2)$ be the set of all polynomial vector fields on $\mathbb{R}^2$ of the form $X(x, y) = (P(x, y), Q(x, y))$, with P and Q polynomials in the variables x and y of degree at most n (with $n \in \mathbb{N}$). In this set we consider the *coefficient topology* by identifying each vector field $X \in P_n(\mathbb{R}^2)$ with a point of $\mathbb{R}^{(n+1)(n+2)}$ (see more details in [6]).

For $X \in P_n(\mathbb{R}^2)$, we consider the *Poincaré compactified vector field* $p(X)$ *corresponding to* X as the vector field induced on $\mathbb{S}^2$ as described in [1, 6, 26, 29, 46]). Concerning this, a singular point q of $X \in P_n(\mathbb{R}^2)$ is called *infinite* (respectively *finite*) if it is a singular point of $p(X)$ in $\mathbb{S}^1$ (respectively in $\mathbb{S}^2 \backslash \mathbb{S}^1$).

Now, we present the local classification of the singular points of $p(X)$. Let q be a singular point of $p(X)$.

The classical definitions are:

- q is *non-degenerate* if $\det(Dp(X)(q)) \neq 0$, i.e. the determinant of the linear part of $p(X)$ at the singular point q is nonzero;

- q is *hyperbolic* if the two eigenvalues of $Dp(X)(q)$ have real part different from 0;
- q is *semi-hyperbolic* if exactly one eigenvalue of $Dp(X)(q)$ is equal to 0.

However, we will also use new notation introduced in [9] directly related to the Jacobian matrix of the singularity. We have:

- q is *elemental* if both of its eigenvalues are non-zero;
- q is *semi-elemental* if exactly one of its eigenvalues equals to zero;
- q is *nilpotent* if both of its eigenvalues are zero, but its Jacobian matrix at this point is non-identically zero;
- q is *intricate* if its Jacobian matrix is identically zero;
- q is an *elemental saddle* if $\det(Dp(X)(q)) < 0$, i.e. the product of the eigenvalues of $Dp(X)(q)$ is negative;
- q is an *elemental anti-saddle* if $\det(Dp(X)(q)) > 0$ and the neighborhood of q is not formed by periodic orbits (in which case we would call it a *center*), i.e., it is either a node or a focus.

Nodes and foci can be algebraically distinguished by means of the sign of the discriminant of the Jacobian matrix, but from the topological point of view, this distinction is useless.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to summarize in a single word the rather complicated behavior of phase curves around such a singularity. We prefer to avoid the use of the word "degenerate". The word "degenerate" has been so widely used for so many different things that the reader may misinterpret its meaning easily. In [9] the word "degenerate" is used only to indicate systems with an infinite number of finite singularities (even if they are complex). We have seen in some papers an elementary node with identical eigenvalues being called "degenerate", or a weak focus, and also any multiple singularity.

Remark 1 Saddles have always (topological) index -1 and anti-saddles have index $+1$ (see [26, 32] for the definition of index of a singular point).

We encourage the reader to recall the definition of characteristic directions and finite sectorial decomposition of vector fields $p(X) \in P_n(\mathbb{S}^2)$ (or $X \in P_n(\mathbb{R}^2)$) (for instance, see [26]).

Let $p(X) \in P_n(\mathbb{S}^2)$ (respectively $X \in P_n(\mathbb{R}^2)$). A *separatrix* of $p(X)$ (respectively X) is an orbit which is either a singular point (respectively a finite singular point), or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point (respectively a finite singular point). Neumann [35] proved that the set formed by all separatrices of $p(X)$, denoted by $S(p(X))$, is closed. The open connected components of $\mathbb{S}^2 \setminus S(p(X))$ are called *canonical regions* of $p(X)$. We define a *separatrix configuration* as the union of $S(p(X))$ plus one representative solution chosen from each canonical region. Two separatrix configurations S_1 and S_2 of vector fields of $P_n(\mathbb{S}^2)$ (respectively $P_n(\mathbb{R}^2)$) are said to be *topologically equivalent* if there is an orientation-preserving homeomorphism of $\mathbb{S}^2$ (respectively $\mathbb{R}^2$) which maps the trajectories of S_1 onto the trajectories of S_2. However, in order to reduce the number of different phase portraits to half, normally the condition of orientation-preserving is skipped.

Definition 2 We define *skeleton of separatrices* as the union of $S(p(X))$ without the representative solution of each canonical region.

Some canonical regions accept only one representative orbit but other regions whose border is a graphic (see definition below) accept two different representatives and thus, a skeleton of separatrices can still produce different separatrix configurations.

We call a *heteroclinic orbit* a separatrix which starts and ends on different points (being a separatrix of both) and a *homoclinic orbit* as a separatrix which starts and ends at the same point. A *loop* is formed by a homoclinic orbit and its associated singular point. These orbits are also called separatrix connections or saddle connections.

A *(non-degenerate) graphic* as defined in [27] is formed by a finite sequence of singular points $r_1, r_2, \ldots, r_n$ (with possible repetitions) and non-trivial connecting orbits γ_i for $i = 1, \ldots, n$ such that γ_i has r_i as α-limit set and r_{i+1} as ω-limit set for $i < n$ and γ_n has r_n as α-limit set and r_1 as ω-limit set. Also normal orientations n_j of the non-trivial orbits must be coherent in the sense that if γ_{j-1} has left-hand orientation then so does γ_j. A *polycycle* is a graphic which has a Poincaré return map.

A *degenerate graphic* is formed by a finite sequence of singular points $r_1, r_2, \ldots, r_n$ (with possible repetitions) and non-trivial connecting orbits and/or segments of curves of singular points γ_i for $i = 1, \ldots, n$ such that γ_i has r_i as α-limit set and r_{i+1} as ω-limit set for $i < n$ and γ_n has r_n as α-limit set and r_1 as ω-limit set. Also normal orientations n_j of the non-trivial orbits must be coherent in the sense that if γ_{j-1} has left-hand orientation then so does γ_j. For more details, see [27].

A vector field $p(X) \in P_n(\mathbb{S}^2)$ is said to be *structurally stable with respect to perturbations in* $P_n(\mathbb{S}^2)$ if there exists a neighborhood V of $p(X)$ in $P_n(\mathbb{S}^2)$ such that $p(Y) \in V$ implies that $p(X)$ and $p(Y)$ are topologically equivalent; that is, there exists a homeomorphism of $\mathbb{S}^2$, which preserves $\mathbb{S}^1$, carrying orbits of the flow induced by $p(X)$ onto orbits of the flow induced by $p(Y)$, preserving sense but not necessarily parameterization.

Since in this paper we are interested in the classification of the structurally unstable quadratic vector fields of codimension two, we recall the concept of quadratic vector fields of lower codimension in structurally stability.

Recalling the works of Peixoto [36], restricted to the set of the quadratic vector fields, we have the following result:

Theorem 1 *Consider* $p(X) \in P_n(\mathbb{S}^2)$ *(or* $X \in P_n(\mathbb{R}^2)$*). This system is structurally stable if and only if*

(i) *the finite and infinite singular points are hyperbolic;*
(ii) *the limit cycles are hyperbolic;*
(iii) *there are no saddle connections.*

Moreover, the structurally stable systems form an open and dense subset of $P_n(\mathbb{S}^2)$ *(or* $P_n(\mathbb{R}^2)$*).*

The studies done up to now on structurally stable systems and codimension one systems are modulo limit cycles, so it is sufficient to consider only conditions *(i)* and *(iii)* of Theorem 1. We refer to these conditions as *stable objects*.

According to [2] there are 44 topologically distinct structurally stable quadratic vector fields. Concerning the codimension one quadratic vector fields, we allow the break of only one stable object. In other words, a quadratic vector field X is *structurally unstable of codimension one* if and only if

(I) It has one and only one structurally unstable object of codimension one, i.e. one of the following types:

(I.1) a saddle-node q of multiplicity two with $\rho_0 = (\partial P/\partial x + \partial Q/\partial y)_q \neq 0$;
(I.2) a separatrix from one saddle point to another;
(I.3) a separatrix forming a loop for a saddle point with $\rho_0 \neq 0$ evaluated at the saddle;
(I.4) It has one unstructurally unstable limit cycle of multiplicity 2, that is, which under perturbation may produce at most two hyperbolic limit cycles;
(I.5) It has a weak focus of order 1.

(II) If the vector field has a saddle-node, none of its separatrices may go to a saddle point and no two separatrices of the saddle-node are continuation one of the other.

For the structurally unstable phase portraits of codimension one modulo limit cycles, we may tear apart the points (I.4) and (I.5). Also the point (I.3) requires no dedication: a phase portrait having a separatrix forming a loop for a saddle point with $\rho_0 = 0$ evaluated at the saddle as its only stability is in fact a codimension two phase portrait which modulo limit cycles is topologically equivalent to another of codimension one. In what follows, instead of talking about codimension one modulo limit cycles, we will simply say *codimension* one*.

As described in [6, Chapter 5], the codimension one* quadratic vector fields can be allocated in four classes, according to the coincidences that may occur with singular points or separatrices of structurally stable quadratic vector fields X.

(A) When a finite saddle and a finite node of X coalesce and disappear.
(B) When an infinite saddle and an infinite node of X coalesce and disappear.
(C) When a finite saddle (respectively node) and an infinite node (respectively saddle) of X coalesce and then they exchange positions.
(D) When we have a saddle-to-saddle connection. This class is split into five subclasses according to the type of the connection: (a) finite-finite (heteroclinic orbit), (b) loop (homoclinic orbit), (c) finite-infinite, (d) infinite-infinite between symmetric points and (e) infinite-infinite between adjacent points.

Recalling the main result in [6], the phase portraits in all these four classes sum up 211 topological distinct ones, where 204 of these total are proved to be realizable and the remaining 7 are conjectured to be impossible. However, when we started the study of codimension two phase portraits, we needed to rely on the codimension one* realizable ones and also on the non realizable ones. And some tricky situations lead us to discover some mistakes in [6] which make that the number of realizable phase portraits has been reduced to 202 (maintaining the 7 conjectured impossible). One mistake was found in [15] and another in this paper. In [15] it was proven that phase portrait from [6] named as $\mathbb{U}^1_{A,49}$ is not realizable and must be renamed as $\mathbb{U}^{1,I}_{A,49}$. And here we will prove that $\mathbb{U}^1_{D,62}$ is also not realizable and must be renamed as $\mathbb{U}^{1,I}_{D,62}$

The current step is to classify, modulo limit cycles, the codimension two quadratic vector fields.

Up to now, we have mentioned many times the word "codimension" and this is a clear concept in geometry. However, in this classification we want to obtain topologically distinct phase portraits, and we want to group them according to their level of genericity. So, what was clear for structurally stable phase portraits and for codimension one* phase portraits may become a little weird if we continue in this same way. We do not want to classify phase portraits in a simple euclidean space, but on the moduli space of phase portraits under the topological equivalence and the modulo limit cycles condition. Thus, some phase portraits which are geometrically different and which have different geometrical codimension may be topologically equivalent, and it must be given a unique topological codimension in this moduli space. The works done up to now in quadratic systems of topological codimension zero, one and two have had no problem to determine what conditions were required, but starting at codimension three and higher, the conditions may become less clear. In paper [10] the authors make a complete description of the concept of codimension related to polynomial systems and specially to quadratic systems and give a global definition of codimension which here is adapted to phase portraits:

Definition 3 We say that a phase portrait of a quadratic vector field is structurally stable (has topological codimension zero) if any sufficiently small perturbation in the parameter space leaves the phase portrait topologically equivalent the previous one.

Definition 4 We say that a phase portrait of a quadratic vector field is structurally unstable of topological codimension $k \in \mathbb{N}$ if any sufficiently small perturbation in the parameter space either leaves the phase portrait topologically equivalent to the previous one or it moves it to a lower codimension one, and there exists at least one such as perturbation which perturbs the phase portrait into one of codimension $k - 1$, or there exists at least one couple of chained perturbations which perturbs the phase portrait into one of codimension $k - 2$.

Remark 2 1. When applying these definitions, modulo limit cycles, to phase portraits with centers, it would say that some phase portraits with centers would be of codimension as low as two, while geometrically they occupy a much smaller region in $\mathbb{R}^{12}$. So, the best way to avoid inconsistencies in the definitions is to tear apart the phase portraits with centers, that we know they are in number 31 [47], and just work with systems without centers.

2. The last part of the definition mentioning the possibility of a chain of two perturbations, refers to some special cases of high codimension which are explained in [10] but has no effect in codimension two.
3. Starting in cubic systems, the definition of topologically equivalence, modulo limit cycles, becomes more complicated since we can have limit cycles having only one singularity in its interior or more than one. There is even a proof of existence of up to 13 limit cycles which are nested in a tricky way with one limit cycle surrounding all nine singularities of a cubic system [22]. So we cannot collapse the limit cycle because its interior is also relevant for the phase portrait.
4. Moreover, our definition of codimension also needs more precision starting with cubic systems due to new phenomena that may happen there.

Table 1 Classes of structurally unstable quadratic vector fields of codimension two* considered from combinations of the classes of codimension one*: (A), (B), (C) and (D) (which in turn is split into a, b, c, d and e)

	(A)	(B)	(C)	(D)
(A)	(AA)	–	–	–
(B)	(AB)	(BB)	–	–
(C)	(AC)	(BC)	(CC)	–
(D)	(AD) (5 cases)	(BD) (5 cases)	(CD) (5 cases)	See Table 2

Table 2 Sub-classes of structurally unstable quadratic vector fields of codimension two* in the class (DD) (see Table 1)

	a	b	c	d	e
a	(aa)				
b	(ab)	(bb)			
c	(ac)	(bc)	(cc)		
d	(ad)	(bd)	(cd)	(dd)	
e	(ae)	(be)	(ce)	(de)	(ee)

5. As we have already been doing along this introduction, when we talk about "codimension", we will refer to the topological codimension as defined in Definitions 3 and 4.

Then, according to this definition concerning codimension two, and the previously known results of codimension one*, we have the result:

Theorem 2 *A polynomial vector field in $P_2(\mathbb{R}^2)$ is structurally unstable of codimension two modulo limit cycles if and only if all its objects are stable except for the break of exactly two stable objects. In other words, we allow the presence of two unstable objects of codimension one or one of codimension two.*

Combining the classes of codimension one* quadratic vector fields one to each other, we obtain 10 new classes, where one of them is split into 15 sub-classes, according to Tables 1 and 2.

Analogously, instead of talking about codimension two modulo limit cycles, we will simply say *codimension* two*.

Geometrically, the codimension two* classes can be described as follows. Let X be a codimension one* quadratic vector field. We have the following classes:

(AA) When X already has a finite saddle-node and either a finite saddle (respectively a finite node) of X coalesces with the finite saddle-node, giving birth to a semi-elemental triple saddle: $\overline{s}_{(3)}$ (respectively a triple node: $\overline{n}_{(3)}$), or when both separatrices of the saddle-node limiting its parabolic sector coalesce, giving birth to a cusp of multiplicity two: $\widehat{cp}_{(2)}$, or when another finite saddle-node is formed, having then two finite saddle-nodes: $\overline{sn}_{(2)}+\overline{sn}_{(2)}$. Since the phase portraits with $\overline{s}_{(3)}$ and with $\overline{n}_{(3)}$ would be topologically equivalent to structurally stable phase portraits and we are mainly interested in new phase portraits, we will skip them in this classification. Anyway, we may find them in the papers [16] and [19].

(AB) When X already has a finite saddle-node and an infinite saddle and an infinite node of X coalesce: $\overline{sn}_{(2)}+\overline{\binom{0}{2}}SN$.

(AC) When X already has a finite saddle-node and a finite saddle (respectively a finite node) and an infinite node (respectively an infinite saddle) of X coalesce: $\overline{sn}_{(2)}+\overline{\binom{1}{1}}SN$.

(AD) When X has already a finite saddle-node and a separatrix connection is formed, considering all five types of class (D).

(BB) When an infinite saddle (respectively an infinite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a triple saddle: $\overline{\binom{0}{3}}S$ (respectively a triple node: $\overline{\binom{0}{3}}N$). This case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to an structurally stable one.

(BC) When a finite anti-saddle (respectively finite saddle) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a nilpotent elliptic-saddle $\widehat{\binom{1}{2}}E-H$ (respectively nilpotent saddle $\widehat{\binom{1}{2}}HHH-H$). Or it may also happen that a finite saddle (respectively a finite node) coalesces with an elemental infinite node (respectively an infinite saddle) in a phase portrait having already an $\overline{\binom{0}{2}}SN$, having then in total $\overline{\binom{1}{1}}SN+\overline{\binom{0}{2}}SN$.

(BD) When we have an infinite saddle-node $\overline{\binom{0}{2}}SN$ plus a separatrix connection, considering all five types of class (D).

(CC) This case has two possibilities:

  (i) a finite saddle (respectively finite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to a semi-elemental triple saddle $\overline{\binom{2}{1}}S$ (respectively a semi-elemental triple node $\overline{\binom{2}{1}}N$),

  (ii) a finite saddle (respectively finite node) and an infinite node (respectively an infinite saddle) of X coalesce plus another existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to two infinite saddle-nodes $\overline{\binom{1}{1}}SN+\overline{\binom{1}{1}}SN$.

The first case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to a structurally stable one.

One could think also in the possibility of two finite singularities coalescing with an infinite node (respectively saddle) leading to a nilpotent or intricate singularity. However, it is proved in [9] that such a possibility cannot involve a unique infinite singularity, but at least two, and then the codimension is higher. If several finite singularities coalesce with a single infinite singularity, they all do along the same affine direction and we just get semi-elemental singularities.

(CD) When we have an infinite saddle-node $\overline{\binom{1}{1}}SN$ plus a saddle to saddle connection, considering all five types of class (D).

(DD) When we have two saddle to saddle connections, which are grouped as follows:

- (aa) two finite-finite heteroclinic connections;
- (ab) a finite-finite heteroclinic connection and a loop;
- (ac) a finite-finite heteroclinic connection and a finite-infinite connection;
- (ad) a finite-finite heteroclinic connection and an infinite-infinite connection between symmetric points;
- (ae) a finite-finite heteroclinic connection and an infinite-infinite connection between adjacent points;
- (bb) two loops;
- (bc) a loop and a finite-infinite connection;
- (bd) a loop and an infinite-infinite connection between symmetric points;
- (be) a loop and an infinite-infinite connection between adjacent points;
- (cc) two finite-infinite connections;
- (cd) a finite-infinite connection and an infinite-infinite connection between symmetric points;
- (ce) a finite-infinite connection and an infinite-infinite connection between adjacent points;
- (dd) two infinite-infinite connections between symmetric points;
- (de) an infinite-infinite connection between symmetric points and an infinite-infinite connection between adjacent points;
- (ee) two infinite-infinite connections between adjacent points.

Some of these cases have been proved to be empty in an on course paper [11].

The class (AA) with a cusp or two finite saddle-nodes has already been studied in [15] and the classes (AB) and (AC) with a finite saddle-node and both types of infinite saddle-nodes have also been completed in [14].

The main goal of this paper is to present the global phase portraits of the vector fields $X \in P_2(\mathbb{R}^2)$ belonging to the class (AD) and make sure that they are realizable.

Let $\sum_0^2$ denote the set of all planar structurally stable vector fields and $\sum_i^2(S)$ denote the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of codimension i, modulo limit cycles belonging to the set S, where S is a set of vector fields with the same type of instability. For instance, $X \in \sum_2^2(AD)$ denote the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of codimension two* belonging to the class (AD).

With all of these we can formulate the next theorem.

Theorem 3 *If $X \in \sum_2^2(AD)$ then there are at least* 77 *topologically different phase portraits (given in Figs.* 1, 2, 3*) modulo orientation and modulo limit cycles and at most* 93.

In several papers where the phase portraits of a family of quadratic systems were classified starting from a given normal form [7, 12, 13] and which split the parameter region in several hundreds of sets, a classification technique using topological invariants was needed in order to detect topologically equivalent phase portraits which may occur in different parts. Even though this same technique could be used here, we consider it is not necessary since the phase portraits (class (D)) from which we start producing the potential phase portraits of class (AD) are already different, and thus, we cannot obtain the same phase portrait from two different sources. We can only obtain two equivalent phase portraits by colliding two different antisaddles with a saddle starting from the same phase portrait of class (D), and these cases are easily detected along the proof of this theorem and the repetitions are conveniently teared appart. For example in Fig. 14 we will see how phase portrait $\mathbb{U}^1_{D,3}$ which has two antisaddles which may coalesce with the saddle, produce only one phase portrait $\mathbb{U}^2_{AD,3}$. Other similar cases appear.

In [6] we already detected seven potential phase portraits of codimension one* in class (D) for which we were not able to find an example, neither to produce a proof of impossibility. All these cases were related with the existence of graphics for which (once fixed every other direction of the flow) the stability or instability of the focus inside the graphic could mean the difference between having an example or not. For several reasons developed in [6], these phase portraits were conjectured to be impossible. From these seven phase portraits, one can develop easily eleven more phase portraits of class (AD). However, if the conjecture is true, these phase portraits will be also impossible. On the contrary, if we had found an example of one of these eleven phase portraits in class (AD), we could easily bifurcate the corresponding phase portrait of codimension one*. But this has not happened as it was expected by the conjecture. Moreover, when developing all the topological possibilities of phase portraits in class (AD), we meet again with the same problem we had in [6] and we detect some skeletons of separatrices for which there are two potential phase portraits, and we are only able to find example of one of them. That is, from a certain realizable codimension one* phase portrait of class (D) having a graphic one can produce the coalescence of a finite saddle and a finite node. If the focus inside the graphic has a certain stability (relative to other stabilities in the phase portrait) we are able to find an example, but it seems that such coalescence is not possible in the opposite case. This phenomena produces that the number of conjectured impossible cases increases from codimension one* to codimension two*. Some more cases will be added when the classes (BD), (CD) and (DD) are completed. And this will even increase higher when codimension three is studied.

Conjecture 1 *The 11 phase portraits of codimension two* that can be developed from the codimension one* portraits (by coalescing a finite saddle and a finite anti-saddle) shown in Fig. 4, plus the 5 codimension two* phase portraits shown in Fig. 5 are non realizable.*

Note that the five phase portraits shown in Fig. 5 are very similar to other five realizable cases. The only difference is the stability of the focus inside the graphic. Consequently, we have named them with a number related with the realizable case.

During the study of this class we have found of a second mistake in [6]. In that book we claimed to have at least 204 different realizable phase portraits (and 211 at most). In [15] we already proved that $\mathbb{U}^1_{A,49}$ was impossible (and renamed it to $\mathbb{U}^{1,I}_{A,49}$). Now, we have found another impossibility which comes from next proposition:

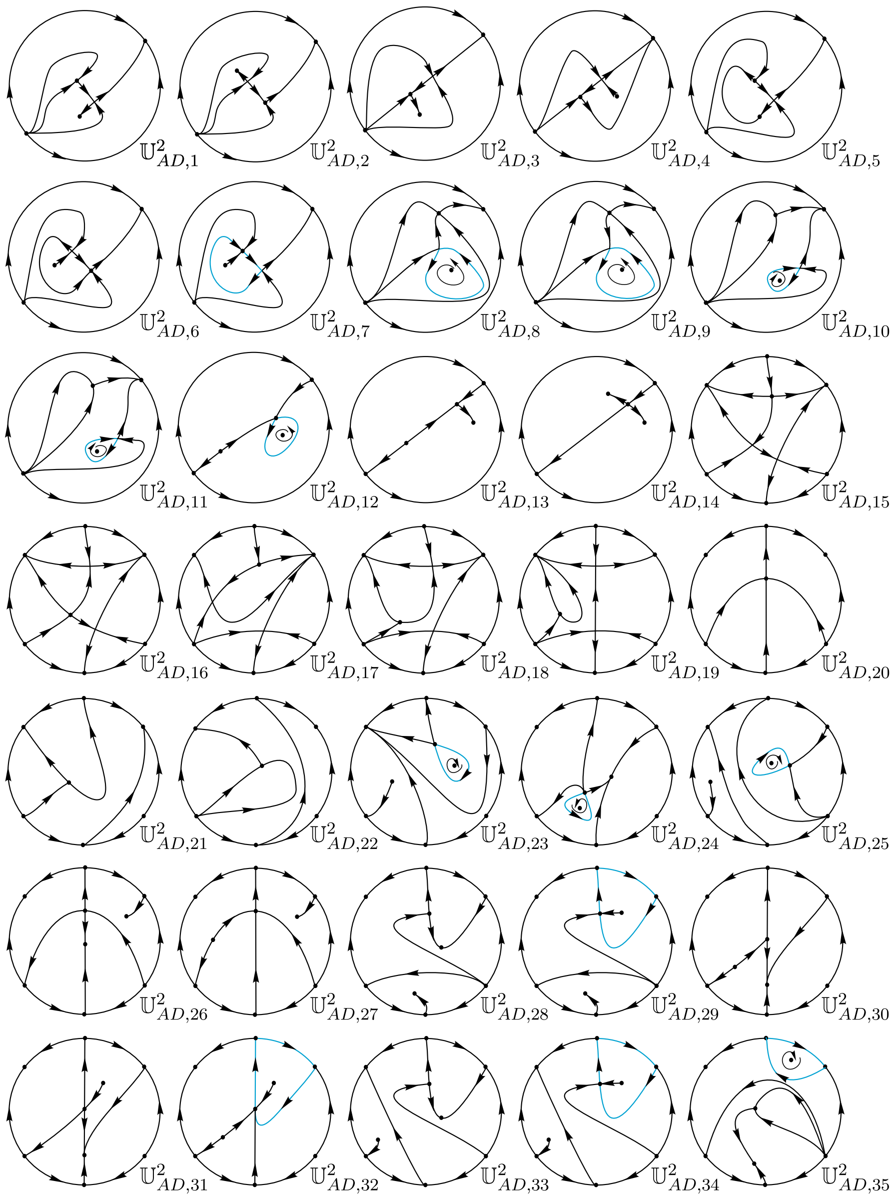


Fig. 1 Structurally unstable quadratic phase portraits of codimension two* of class (AD)

Proposition 1 *Phase portrait* $\mathbb{U}^1_{D,62}$ *from* [6] *is impossible (and must be renamed to* $\mathbb{U}^{1,I}_{D,62}$*). Thus, the realizable cases of structurally unstable phase portraits of quadratic systems of codimension one** *is at least 202 and at most 209.*

The mistake we did in [6] regarding this phase portrait was due because we trusted the report [42] and derived an example of realization of $\mathbb{U}^1_{D,62}$ from phase portrait

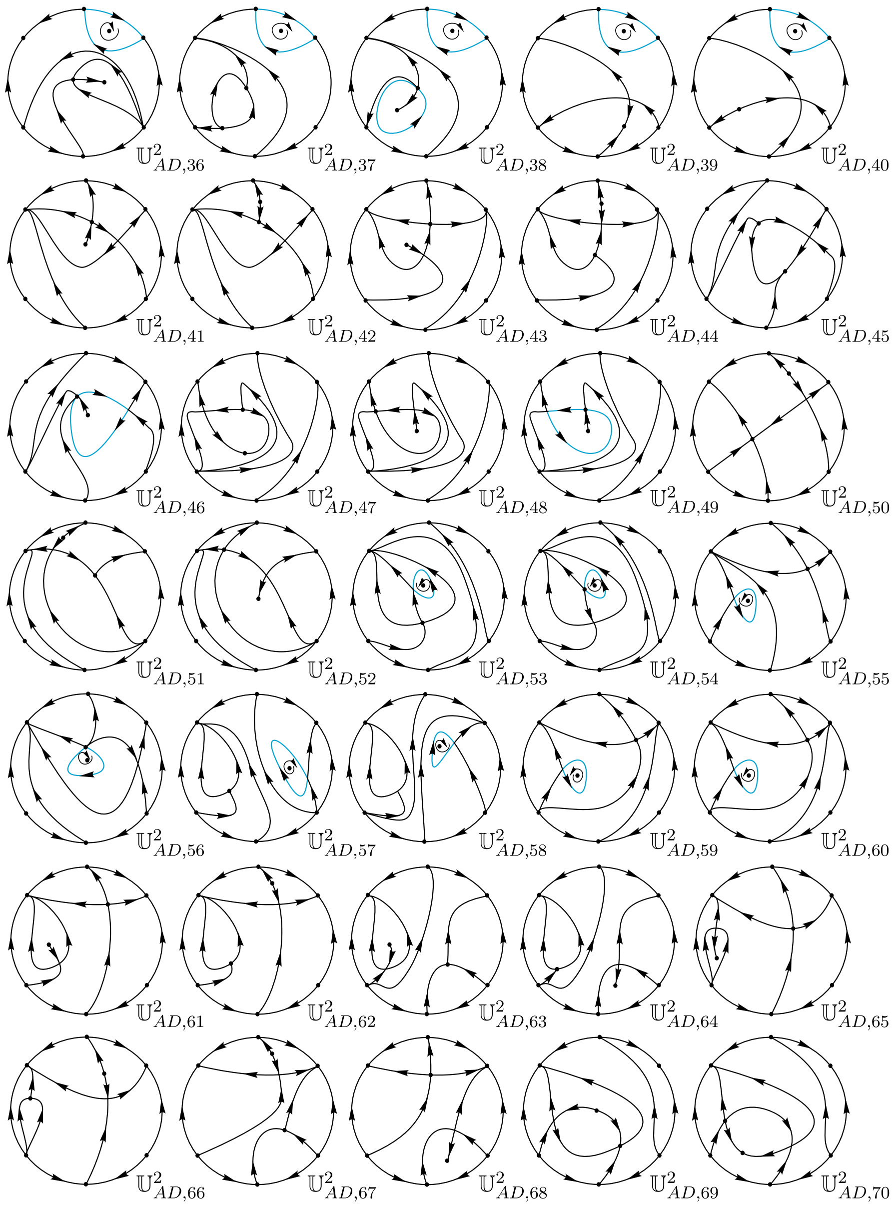


Fig. 2 (Cont.) Structurally unstable quadratic phase portraits of codimension two* of class (AD)

*an*18 in [42]. However, now that we have tried to derive codimension two* phase portraits from $\mathbb{U}^1_{D,62}$ and we have checked that none seems to appear in already done classifications which could contain them, we have rechecked the arguments given in [42]. We concluded that they were not strong and finally worked out a proof of its impossibility.

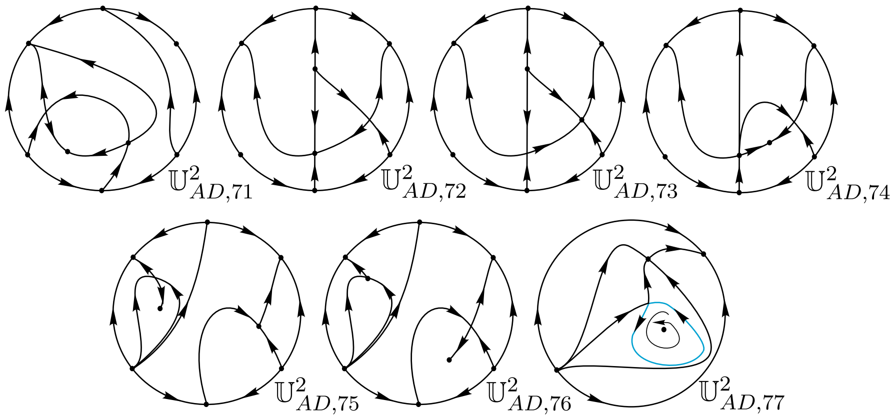


Fig. 3 (Cont.) Structurally unstable quadratic phase portraits of codimension two* of class (AD)

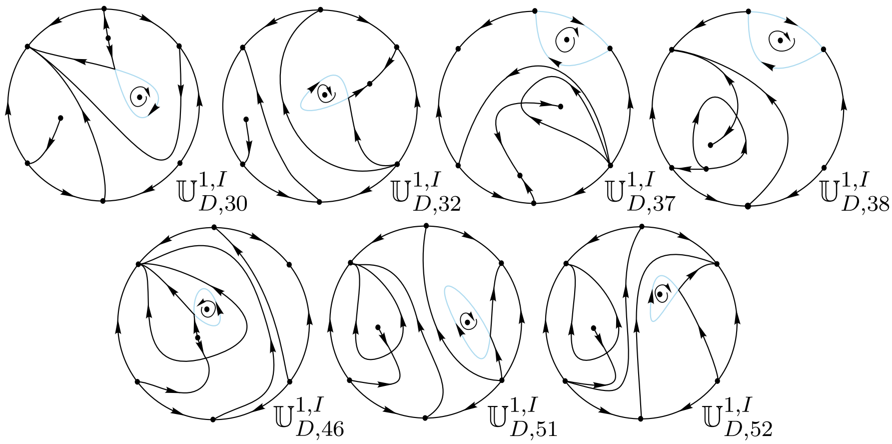


Fig. 4 Conjectured impossible structurally unstable quadratic phase portraits of codimension one*

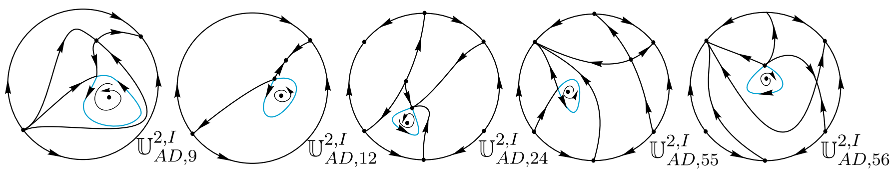


Fig. 5 Conjectured impossible structurally unstable quadratic phase portraits of codimension two* of class (AD)

In Sect. 2 we give a short description of the graphics that have been found in this paper (or some previous papers of this research line) linking them with the classification given in [27]. And we also explain a little about limit cycles even though they are out of the goal of this paper. In Sect. 3 we will prove Proposition 1. In Sect. 4 we make a brief description of phase portraits of codimensions zero* and one* that are needed in this paper. In Sect. 5, we make the list of topologically potential phase portraits of codimension two* in the class (AD) removing already some which can be proved to be impossible at that same moment. In Sect. 6, we prove the realization of 77 of them, and will justify the reasons why we conjecture the impossibility of the remaining 16.

2 Graphics and Limit Cycles

Even though the goal of this paper deals little with graphics and limit cycles, it is out of doubt that these are two of the most important elements in Qualitative Theory.

Limit cycles are the most elusive phenomena in phase portraits. They may appear either from bifurcation of a weak focus (Hopf-bifurcation), by bifurcation of a graphic, by bifurcation of a multiple singularity (finite or infinite), by bifurcation of a multiple limit cycle, by bifurcation of a period annulus, or by bifurcation of degenerate systems (with a common factor between p and q of (1)) and only the first case can be fully algebraically controlled. The other cases are generically non-algebraic. Examples of these bifurcations may be found in hundreds of papers, but in [13], by a simple control of neighbor regions, examples of all these bifurcations may be found.

Our goal to find all the topologically different phase portraits modulo limit cycles tears apart this big problem, but it is not an irrelevant goal. Whenever the mathematical community finally gets the complete set of phase portraits of quadratic systems (or whatever other family), the subset of the phase portraits modulo limit cycles will be the base for such classification.

It is expected to obtain more than one thousand (maybe even up to 2000) different phase portraits of quadratic systems modulo limit cycles. For quite many of them it will be trivial to determine that they will not have limit cycles (in the case they do not have a finite anti-saddle). And the phase portraits having an invariant straight line are known to be bounded to just one limit cycle [23, 25]. But for all the others, it will be needed to determine exactly how many different phase portraits can be obtained from that skeleton by adding limit cycles. Up to now and up to our knowledge, there is just one non trivial skeleton of phase portrait which could theoretically have limit cycles, and it has been proved the absence of limit cycles in it. Concretely structurally stable phase portrait $\mathbb{S}^2_{7,1}$ obtained in [2] was conjectured by statistical tools to be incompatible with limit cycles in [4] and proved in [5]. For all other non-trivial skeletons of phase portraits found up to now, there is not a single proof determining which is the maximum number of limit cycles it may have. There are many papers related to maximum number of limit cycles, but they are always linked to a certain normal form. Most of them simply prove that a concrete normal form may have just one limit cycle. But this does not imply that the skeletons of phase portraits obtained in other normal forms, may not have more limit cycles in the whole classification.

Up to now, it is known that there are examples of phase portraits of quadratics systems with four limit cycles distributed in two nests around two foci, three aronud one and one around the other. And even though it is conjectured that four and this distribution is the effective maximum, there is not yet any conclusive global proof. The phase portraits for which there are examples with 4 limit cycles belong to just three skeletons of phase portraits, concretely the structurally stable $\mathbb{S}^2_{4,1}$ and $\mathbb{S}^2_{11,2}$ from

[2], and the codimension one* $\mathbb{U}^1_{B,31}$ from [6], but this (3, 1) distribution is compatible with many more skeletons. The proof that they may have at least 4 limit cycles was given in several papers since they appear in classifications with a weak focus of order 3 already having a limit cycle around a strong focus [3].

But not even if the maximum bound were four (and the maximum distribution (3, 1)), we would be close to obtain all the phase portraits of quadratic systems. Any of the three above mentioned skeletons of phase portraits may have the topologically different configurations (0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (2, 1) and (3, 1). That is 7 different configurations. But even this is not a simple criteria to obtain a simple upper bound of the total number of phase portraits. There are phase portraits like $\mathbb{S}^2_{5,1}$ from [2] which has up to three finite anti-saddles. One of them receives (or emits) a single separatrix, a second anti-saddle receives (or emits) exactly two separatrices, and a third anti-saddle receives (or emits) exactly three separatrices. So, the fact that a limit cycle could be surrounding any of the three anti-saddles would generate a topologically different phase portrait. And in case there were two nests of limit cycles, and assuming that they could have up to 4 limits cycles, the number of cases would increase up to 25 possibilities. But from these 25 possibilities, up to now only six have been confirmed to exist. In fact, a very recent paper [48, Theorem 5.4] reduces these 25 possibilities to just 13 (assuming that 3 is the maximum of limit cycles around each singularity) when proving that a quadratic system with 4 real finite singularities can only have distributions $(n, 0)$ or (1, 1).

We are collecting a large database and recording the maximum number of limit cycles found in each one of the skeletons classified up to now.

With all this we want to remark that the topological classification of phase portraits modulo limit cycles is important since it produces a complete set of skeletons from which the complete set of phase portraits must be located. For each particular skeleton, it must be studied if it contains none, one, two or up to three anti-saddles around which limit cycles may be located (it is easy to prove that at most two of them may be foci). If there is a complete collection of phase portraits modulo limit cycles, and an upper bound of limit cycles is found, this will give a quite rough upper bound for the number of different phase portraits. But the real number will need a deeper study case by case. Nowadays it seems still quite far the moment to obtain the final complete classification, but the classification modulo limit cycles is achievable with the current techniques and affordable with some effort (better said, quite a lot of effort), so we think it is worth trying for it.

Now we talk a little about graphics. Graphics are also very important because they can become the bifurcation edge which leads to the formation of limit cycles. There has been a lot of literature related to graphics in the past, and one of the most relevant papers is [27] where the authors list the complete set of 121 different graphics that may appear in quadratic systems. The graphics in this list can be of different types. Many of them imply the connection of one (or more) couple of separatrices, finite or infinite. Other graphics are formed simply because a separatrix arrives to the nodal part of a saddle-node (finite or infinite) or an even more degenerated singularity in concomitance with other properties of the phase portrait. Unfortunately, most of these graphics cannot be detected by means of algebraic tools. In many studies of families of systems where a complete bifurcation is given of the parameter space, after all the algebraic bifurcations are given, the use of continuity and coherence arguments allows the detection of some other non-algebraic bifurcations where these graphics appear.

Our methodical study of phase portraits of quadratic systems modulo limit cycles started with codimension zero (structurally stable) [2] and of course these phase portraits cannot have any graphic at all. The second step was the classification of codimension one* phase portraits, and there we could start finding some graphics, but not too many. Concretely we could find graphic (F_2^1) from [27] in $\mathbb{U}^1_{A,10}$, $\mathbb{U}^1_{A,13}$, $\mathbb{U}^1_{A,37}$, $\mathbb{U}^1_{A,43}$, $\mathbb{U}^1_{A,59}$, $\mathbb{U}^1_{A,64}$, and $\mathbb{U}^1_{A,70}$. This graphic consists simply in one finite saddle-node which sends its center manifold (separatrix of zero eigenvalue) to the nodal part of itself. We also have graphic (I^2_{19}) from [27] in $\mathbb{U}^1_{B,29}$, $\mathbb{U}^1_{B,30}$ (twice), $\mathbb{U}^1_{B,33}$, $\mathbb{U}^1_{B,36}$ and $\mathbb{U}^1_{B,38}$. This graphic consists on one elemental infinite saddle which sends one of its separatrices to the nodal part of an infinite adjacent saddle-node formed by the coalescence of two infinite singularities. There are no graphics in the class (C) of codimension one* phase portraits. Finally, in class (D) we find the graphics (F_1^1), (H_1^1) and (I_1^2) from [27]. The first one is just a loop of a finite elemental saddle, the second one is a separatrix connection between opposite infinite elemental saddles, and the third one is a separatrix connection between adjacent infinite elemental saddles. The loop is present in $\mathbb{U}^1_{D,1}$, $\mathbb{U}^1_{D,6}$, $\mathbb{U}^1_{D,7}$, $\mathbb{U}^1_{D,8}$, $\mathbb{U}^1_{D,9}$, $\mathbb{U}^1_{D,12}$, $\mathbb{U}^1_{D,19}$, $\mathbb{U}^1_{D,20}$, $\mathbb{U}^1_{D,22}$, $\mathbb{U}^1_{D,23}$, $\mathbb{U}^1_{D,30}$, $\mathbb{U}^1_{D,31}$, $\mathbb{U}^1_{D,32}$, $\mathbb{U}^1_{D,46}$, $\mathbb{U}^1_{D,47}$, $\mathbb{U}^1_{D,48}$, $\mathbb{U}^1_{D,49}$, $\mathbb{U}^1_{D,50}$, $\mathbb{U}^1_{D,51}$, $\mathbb{U}^1_{D,52}$, $\mathbb{U}^1_{D,53}$ and $\mathbb{U}^1_{D,54}$. The second graphic appears in $\mathbb{U}^1_{D,10}$ and $\mathbb{U}^1_{D,11}$. And the third one can be seen in $\mathbb{U}^1_{D,28}$, $\mathbb{U}^1_{D,29}$, $\mathbb{U}^1_{D,37}$, $\mathbb{U}^1_{D,38}$ and $\mathbb{U}^1_{D,39}$. No other graphic from these five types may appear since all the remaining 116 imply higher codimension.

In the studies of the classes (AA) [15] and (AB) and (AC) [14], the only graphics we see, are those which are inherited from the respective phase portraits of codimension one* having already a graphic. In the studies of the classes (AD), (BD) and (CD) we will start incorporating more graphics from [27] since we will see for example loops having a saddle-node instead of a saddle. Also the class (DD) will provide graphics with two separatrix connections. Anyway, the graphics will appear in bigger numbers when codimension three is studied.

Concretely in (AD) we have already known graphic (F_1^1) (loop) in $\mathbb{U}^2_{AD,8}$, $\mathbb{U}^2_{AD,10}$, $\mathbb{U}^2_{AD,11}$, $\mathbb{U}^2_{AD,53}$, $\mathbb{U}^2_{AD,55}$, $\mathbb{U}^2_{AD,57}$, $\mathbb{U}^2_{AD,58}$, $\mathbb{U}^2_{AD,59}$, $\mathbb{U}^2_{AD,60}$ and $\mathbb{U}^2_{AD,77}$; graphic (I_1^2) in $\mathbb{U}^2_{AD,35}$, $\mathbb{U}^2_{AD,36}$, $\mathbb{U}^2_{AD,37}$, $\mathbb{U}^2_{AD,38}$, $\mathbb{U}^2_{AD,39}$ and $\mathbb{U}^2_{AD,40}$; and graphic (F_2^1) in $\mathbb{U}^2_{AD,38}$.

The new graphics we find here are (F_3^1) (loop with a finite saddle-node) in $\mathbb{U}^2_{AD,9}$, $\mathbb{U}^2_{AD,12}$, $\mathbb{U}^2_{AD,23}$, $\mathbb{U}^2_{AD,24}$, $\mathbb{U}^2_{AD,25}$, $\mathbb{U}^2_{AD,54}$ and $\mathbb{U}^2_{AD,56}$; graphic (F_3^2) (heteroclinic orbit involving a finite saddle and a finite saddle-node having only one separatrix connection) in $\mathbb{U}^2_{AD,7}$, $\mathbb{U}^2_{AD,46}$ and $\mathbb{U}^2_{AD,49}$; graphic (H_3^1) (heteroclinic orbit involving a pair of infinite opposite saddles and a finite saddle-node having only one separatrix connection) in $\mathbb{U}^2_{AD,14}$, and graphic (I_8^2) (heteroclinic orbit involving two infinite saddles and a finite saddle-node having only one separatrix connection) in $\mathbb{U}^2_{AD,29}$, $\mathbb{U}^2_{AD,32}$ and $\mathbb{U}^2_{AD,34}$.

3 Impossibility of $\mathbb{U}^{1,I}_{D,62}$

In order to prove this, we need a couple of technical lemmas.

Lemma 1 *There are no contact points of the flow of quadratic systems with straight lines which are characteristic directions of isolated infinite singularities unless these straight lines are invariant.*

Proof Take an infinite singularity of a quadratic system. By means of a rotation we may put it at the infinite singular point $[1:0:0]$, that is, at the end of the x-axis. Assume there is a contact point of the flow with the x-axis. By means of a translation we can put that contact point on the affine origin.

Then the system must be

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2, \\ \dot{y} &= ex + fy + 2mxy + ny^2. \end{aligned} \tag{2}$$

We use the notation and normal forms from [9] which are the most effective. The coefficient of x^2 in the second equation must be zero in order to fix the infinite singularity, and the constant term of the second equation must be zero in order to force the contact point of the flow with the x-axis at the origin of the affine plane.

Then the system has a singular point at the origin of the local chart U_1 and looks like

$$\begin{aligned} \dot{w} &= (2m - g)w + ez + H.O.T., \\ \dot{z} &= -gz + H.O.T., \end{aligned} \tag{3}$$

and the polynomial of characteristic directions is $PCD(w, z) = -z(2mw + ez)$. In order to have the direction of the x-axis characteristic in the affine plane, the direction on the z-axis must be characteristic on the U_1 chart, and this implies that the variable w must be a common factor of the $PCD(w, z)$. So we need $e = 0$. In the case of non intricate singularities, this is equivalent to compute the eigenvectors of the Jacobian matrix of the system at the singularity, and we clearly see that in order to have the vector $(0, 1)$ as eigenvector, we need also $e = 0$.

And clearly, if $e = 0$ then $y = 0$ is an invariant straight line. □

If there cannot be a contact point in the straight line defined by a characteristic direction of an infinite singular point, even less we can have a finite singularity on it, and the flow must be transversal all along the line.

Lemma 2 *Consider an elemental or semi-elemental infinite singularity of a quadratic system having an affine characteristic direction. The orbits which are tangent to an affine characteristic direction (which is not an invariant straight line) of such an infinite singularity stay locally on the opposite affine semi-plane than those which are tangent with the same line on the opposite infinite singular point.*

Proof A restricted version of this lemma was already proved in [15] in the case the infinite singular point was a saddle. Now, with the help of Lemma 1 and the geometrical classification of singularities done in [9], we can extend the result. We suspect that this lemma may be true for every infinite singularity of a quadratic system (having affine characteristic directions), even for the intricate ones. But then the proof would have to consider many more cases, and since we only need for elemental and semi-elemental ones, we restrict the statement to what it is.

For the proof of the lemma, we will need to work at an intermediate point between the classical topological classification of singularities and the geometrical classification since the way the orbits reach the singularities will be relevant. This is the qualitative equivalence defined by Jiang and Llibre in [34].

In [9] (diagrams from 6.5 to 6.8) one can find all the configurations of infinite singularities that a quadratic system can have according to the geometrical equivalence. From them, it is easy to count that there are 60 geometrically different real isolated infinite singularities, from the simplest saddle, to some intricate multiplicity seven singularities. The possibilities for elemental and semi-elemental are 14.

If we extract from them those which are qualitatively different and which have affine characteristic directions, one obtains the following points (using the notation from [9]): N^f, N^∞, N^*, S, $\overline{\binom{0}{2}}SN$ and $\overline{\binom{1}{1}}SN$.

It is important to notice the qualitative difference between N^f and N^∞. In the first case, all the affine orbits arrive at the singular point tangent to the affine characteristic direction. In the second case, they arrive tangent to the infinity direction (except just one). This is related to the biggest eigenvalue of the Jacobian matrix. In the case of N^* we have a star node. It is also important to notice that the nodal part of the saddle-node $\overline{\binom{1}{1}}SN$ behaves as an N^f, while the nodal part (in both local charts) of a saddle-node $\overline{\binom{0}{2}}SN$ behaves as a N^∞ (see Fig. 6). Even though someone may find a bit weird the shape we have given to some nodal orbits, this is exactly the way they behave. Notice that every regular point close enough to the infinite node must have an orbit connecting to it, and the orbit must arrive to the infinite singularity in the required characteristic direction. So, for example, in the case of a node N^f (see Fig. 6), a regular point close to infinity and with $y < 0$ must belong to an orbit which must turn and move into the upper half-plane so to arrive to the infinite node tangent to the x-axis. The case of the star node N^* may also seem strange since one may imagine the star node formed by straight lines. If that were the case, then the x-axis would be an invariant straight line but we are assuming precisely that this is not the case. There are quadratic systems having a star node at infinity (as well as finite ones) and no invariant straight line. So, the orbit that arrives to the star node tangent to the x-axis leaves a region in the upper half-plane whose regular points must belong to orbits that must enter in the lower half-plane in order to arrive to the star node.

Now, for each one of these singularities we will prove with a picture that the orbits tangent to the characteristic direction of the infinite singularity cannot be all in the same semi-plane. Assume the contrary and we obtain Fig. 6.

We have also assumed that the infinite singular point is the point $[1 : 0 : 0]$ on local chart U_1 (and their opposite in chart V_1), that the characteristic direction is the x-axis

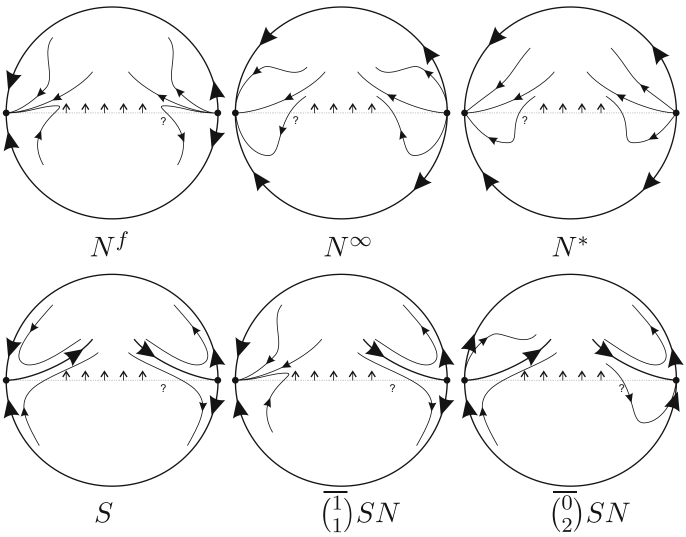


Fig. 6 Proof of Lemma 2

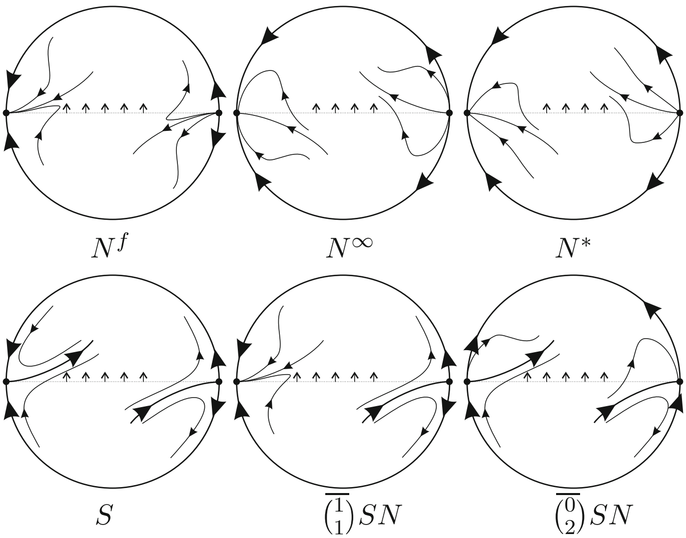


Fig. 7 Proof of Lemma 2 (cont.)

and that the flow on that axis goes upwards. This can always be done by means of rotations, translations, symmetries and time changes.

In each one of the cases (see Fig. 6) there is always a contradiction since it is needed that some orbits cut the x-axis in the opposite direction. Otherwise, if the tangencies in one of the local charts take place in the lower half-plane (see Fig. 7) all cases are compatible with the flow on the x-axis. □

The next corollary follows immediately from Fig. 6.

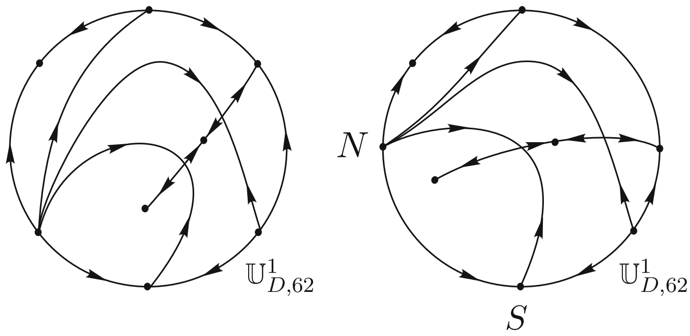


Fig. 8 Phase portrait $\mathbb{U}^1_{D,62}$

Corollary 1 *A separatrix of an infinite elemental or semi-elemental singularity of a quadratic system cannot cross the straight line defined by its characteristic direction.*

Now we are able to prove Proposition 1 and state clearly that phase portrait $\mathbb{U}^1_{D,62}$ from [6] is impossible.

Proof We start by bringing here the image of this phase portrait. We have drawn it in the left of Fig. 8, exactly as it is given in [6] and in the right we plot an equivalent one, with a saddle at the origin of the affine plane and an infinite node at the end of the x-axis. The third infinite singularity may be moved to $[1 : 1 : 0]$ by means of an affine change. This is a codimension one* phase portrait since there is a separatrix connection of type (c) that joins the finite saddle at the origin with the infinite saddle at the end of the y-axis.

Now one must realize that the separatrices of the infinite saddle do not seem to satisfy Lemma 2. But this may be due that we simply have not plotted them well. One needs to prove that in a stronger way.

We start by plotting just the infinite singularities at $N[1 : 0 : 0]$, $S[0 : 1 : 0]$ and $N[1 : -1 : 0]$, plus the saddle which will have the connection at the origin, plus both axes which are not invariant, so the characteristic directions arriving to the infinite singular points $N[1 : 0 : 0]$ and $S[0 : 1 : 0]$ are not the axes. We also assume that the infinite saddle which will produce the separatrix connection is on the local chart V_2. All this can be done by means of affine linear changes. Assume that the characteristic directions are situated in the most natural way as appear in Fig. 9a. Since the separatrix from the saddle $S[0 : 1 : 0]$ on the negative y semi-plane must connect with a separatrix of the saddle at the origin, it must be on the left of the characteristic direction. Thus, by Lemma 2, the separatrix of the saddle $S[0 : 1 : 0]$ on the positive y semi-plane must be on the right, and then it is impossible to arrive to the node $N[1 : 0 : 0]$ on the negative x semi-plane.

So, let us put the characteristic direction in the less natural way, that is, in the left of the origin (see Fig. 9b). Now it seems that the separatrices of the infinite saddle fit correctly. However, we still need to plot the opposite stable separatrix of the saddle at the origin so that it comes from the infinite node $N[1 : 0 : 0]$ on the negative x semi-plane. Well, it seems compatible with the flow (see Fig. 9c) on the positive y

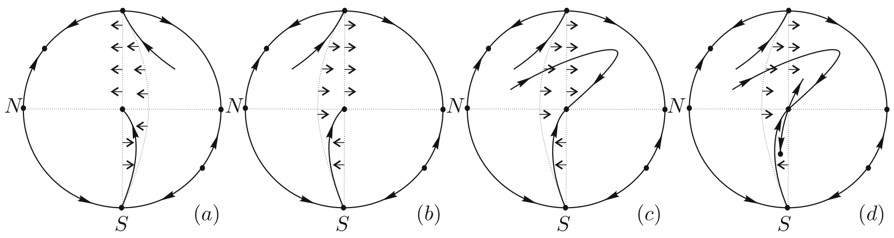


Fig. 9 Impossibility of phase portrait $\mathbb{U}^{1,I}_{D,62}$

semi-plane, but the problem stays in the negative y semi-axis. The flow there moves to the left, and since the origin is a saddle, the only way this may happen is like we see in Fig. 9d. This forces that the flow on the x-axis must go down when $x > 0$ and up in $x < 0$.

We must situate the second finite saddle in the fourth quadrant because this saddle must receive one separatrix from the infinite node in the fourth quadrant and send one to the finite anti-saddle we have in third quadrant. But then, this saddle cannot receive its remaining stable separatrix from $N[1:0:0]$ in local chart V_1. Thus, we see that this is incompatible with phase portrait $\mathbb{U}^1_{D,62}$. □

Even though two errors have been found in [6], one in class (A) and one in class (D), which would force a renumbering of phase portraits, we have preferred to keep the gaps unfilled in order to avoid incompatibilities between papers which would increase confusion. Anyway, in the error for class (D) we have been lucky since it is the last numbered case of (D) the one which has turned to be impossible. Within time, a new complete and consistent notation will be created based on the classification of singularities given in [11].

Since two errors have been already found in [6], we have wanted to convince ourselves and the scientific community that the remaining 202 phase portraits are realizable, we have reproduced all the examples given in the book. All of them have been tested numerically with the program P4 [26] and even though many of them are infinitesimal perturbations of codimension two systems, all of them can be checked to be what they represent. We offer the complete collection of P4 files in a zip file that is free for downloading at "http://mat.uab.cat/~artes/articles/SU2AD/P4su1.zip". Please, be careful since several of the examples are on the limit of what can be computed numerically with P4. For some of them it is needed to adapt the integration parameters in order to obtain the desired phase portrait. The arguments and techniques to modify those parameters are explained in the last chapter of [26]. One needs to reproduce the examples of that chapter which grow in increasing difficulty in order to understand the use of P4. There are pairs of several examples in [6] which show exactly the same coefficients. This is perfectly normal since we forgot to add the parameter ε which is positive in one of them and negative in the other, making a saddle-node to split in a saddle and a node, or to disappear. Anyway, the value of ε is not the same for every example. In some of them, it can be relatively large so to allow a better view of the phase portraits and in other cases it must be very small since a bigger one may imply more bifurcations than wished. In the zipped file we offer, the values of ε are already given and they have been checked to work. Also, there are some of the examples that show limit cycles. Since the classification was done modulo limit cycles, they are perfectly acceptable as representatives of their class.

So, the conclusion is that structurally unstable quadratic systems of codimension one* can have at least 202 different phase portraits and at most 209, and this gap of seven remains as a conjecture to be impossible. Moreover we confirm the goodness of these 202 phase portraits, it is also true that there are some typos in [6] that need to be corrected. We make here a list of them:

1. In equation (6.9), the coefficient of xy in the second equation must be $((2 + 2h - n)(1 - \varepsilon) - 2l)$.
2. In page 198, Table 6.4, the example for $\mathbb{U}^1_{A,66}$ must be $(h, l, n) = (4/10, -194/10, -1)$.
3. In page 198, Table 6.4, the example for $\mathbb{U}^1_{A,67}$ must be $(h, l, n) = (-99/1000, 1/10, 81/100)$. Moreover, it is not a bifurcation of V_{113} from [18], but of V_{122}.
4. In page 211, Table 6.5, the example for $\mathbb{U}^1_{B,5}$ must be $(h, l, n) = (1 + \sqrt{7} + 10^{-6}, n - 2h - 42/100, 7)$ (the same as $\mathbb{U}^1_{B,23}$ but with different sign of ε).
5. In page 211, Table 6.5, the example for $\mathbb{U}^1_{B,7}$ must be $(h, l, n) = (-5, 10, 10)$ (the same as $\mathbb{U}^1_{B,10}$ but with different sign of ε).
6. In page 211, Table 6.5, the example for $\mathbb{U}^1_{B,24}$ must be $(h, l, n) = (40001/1000, 206/100, 25)$.
7. In page 212, Table 6.6, the example for $\mathbb{U}^1_{B,38}$ must be $(h, l, n) = (96/100, 1/10, 81/100)$.
8. In page 212, Table 6.6, the example for $\mathbb{U}^1_{B,39}$ must be $(h, l, n) = (98/100, 1/10, 81/100)$.
9. In equations (6.101) and (6.102), the coefficient of y^2 in the second equation must be $3321/400$.
10. In equation (6.103), the coefficient of y^2 in the second equation must be $5229/100$.

Since the mistake (in [6]) detected in this paper came from a mistake in [42], we have checked with special care all other examples which came also from that paper. There was no problem at all in the examples of class (C) since the normal form was given, the parameters were fixed and all are right. And with respect to class (D), the paper [42] was used twice in page 239, Section 6.5.5. The first use was of phase portrait $an18$ which we have proved impossible here. The second use was of phase portrait $en09$ without giving it explicitly. Anyway, $en09$ does really exist, and from it, we can really obtain $\mathbb{U}^1_{D,40}$. Concretely system $\dot{x} = 2x/5 - 3y/10 - y^2 - 3xy - x^2/10$, $\dot{y} = xy + x/5$ is a representative of $\mathbb{U}^1_{D,40}$ and $\dot{x} = 2x/5 - 3y/10 - y^2 - 3xy$, $\dot{y} = xy + x/5$ is a representative of $en09$.

4 Quadratic Vector Fields of Codimension Zero and One

In this section we summarize all the needed results from the book of Artés, Llibre and Rezende [6]. The following result is a restriction of Theorem 1.1 of [6] to the class

(D). We denote by $\sum_1^2(D)$ the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of codimension one* belonging to the class (D).

Theorem 4 *If $X \in \sum_1^2(D)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 61 *phase portraits of Figs.* 10 *and* 11*, and all of them are realizable.*

Here we have already corrected the error detected in this paper and have reduced the number of phase portraits of class (D) from 62 to 61.

In [6], quite many topologically potential phase portraits of codimension one* were discarded because they were not realizable. From all of them, we just need one phase portrait which appeared in page 77 of [6] but did not receive a formal name until [14, Figure 13].

Proposition 2 *Phase portrait $\mathbb{U}^{1,I}_{A,3}$ given in Fig.* 12 *is not realizable.*

An important result to study the impossibility of some phase portraits is [6, Corollary 3.29].

Corollary 2 *If one of the structurally stable vector fields that bifurcates from a potential structurally unstable vector field of codimension one* is not realizable, then this unstable system is also not realizable.*

It would be nice if this theorem could be adapted to higher codimensions, but unfortunately this is not so clear. The idea is that if a phase portrait shows several unstabilities, one can produce potential phase portraits bifurcating from it just breaking one of such unstabilities. For example, a codimension three phase portrait with a finite saddle-node $\overline{sn}_{(2)}$ and two infinite saddle-nodes $\overline{\binom{1}{1}}SN$ and $\overline{\binom{0}{2}}SN$ could be susceptible to be bifurcated in up to six possibilities of codimension two. But maybe there are some linked unstabilities which cannot be broken independently. One clear case is a phase portrait with a graphic and a center inside. One can break the center, while respecting the graphic, but not otherwise: If one breaks the graphic, the center must also disappear. Also some very intricate singularities in some phase portraits force the existence of invariant straight lines which are separatrix connections. We can break the intricate singularity while respecting the invariant line, but we cannot break the invariant line and produce new phase portraits with the same intricate singularity. This is a result that must be considered for every particular class of systems. For the class we are involved now (AD), we can prove it. In fact, the main problem deals with the concept of "codimension" which may be thought from a geometrical or a topological point of view, and which for lower codimensions up to 2 has been easy to deal with, but that starting at codimension 3 has turned much more difficult. We are working in the preprint [10] which deeply affords the concept of codimension in polynomial differential systems, and particularly for quadratic systems. In the paper, it is explained that at some level of degeneracy of the system, the codimension of the configuration of singularities, or the codimension of the phase portrait is not a simple direct sum of the individual codimensions of the different unstable objects it may have. The paper also determines the topological codimension of every topological configuration of singularities from [11]. These topological configurations of singularities were extracted in [8] from the geometrical configurations given in the book [9]. The topological codimension of every topological configuration of singularities will be the skeleton upon which we will be able to study the phase portraits of codimension 3 and higher.

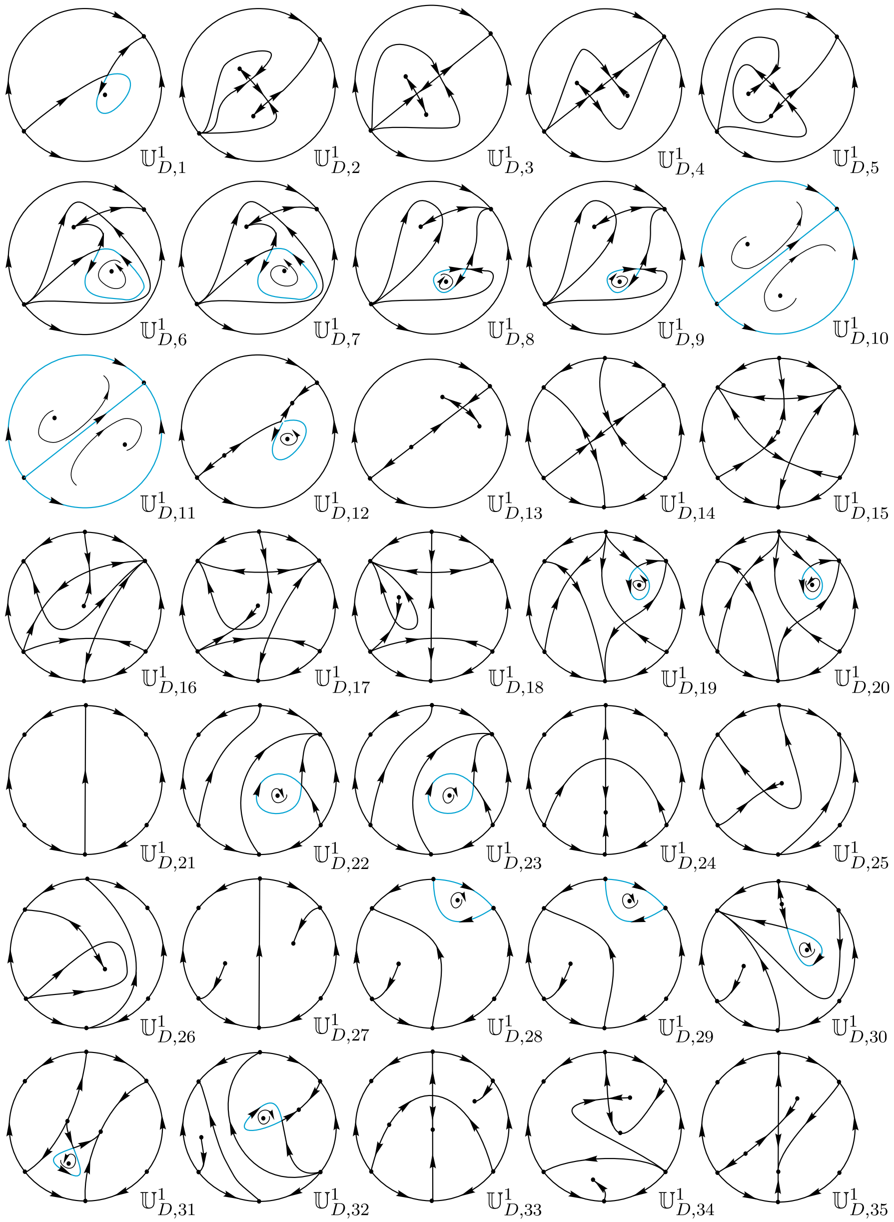


Fig. 10 Unstable quadratic systems of codimension one* (cases with a separatrix connection)

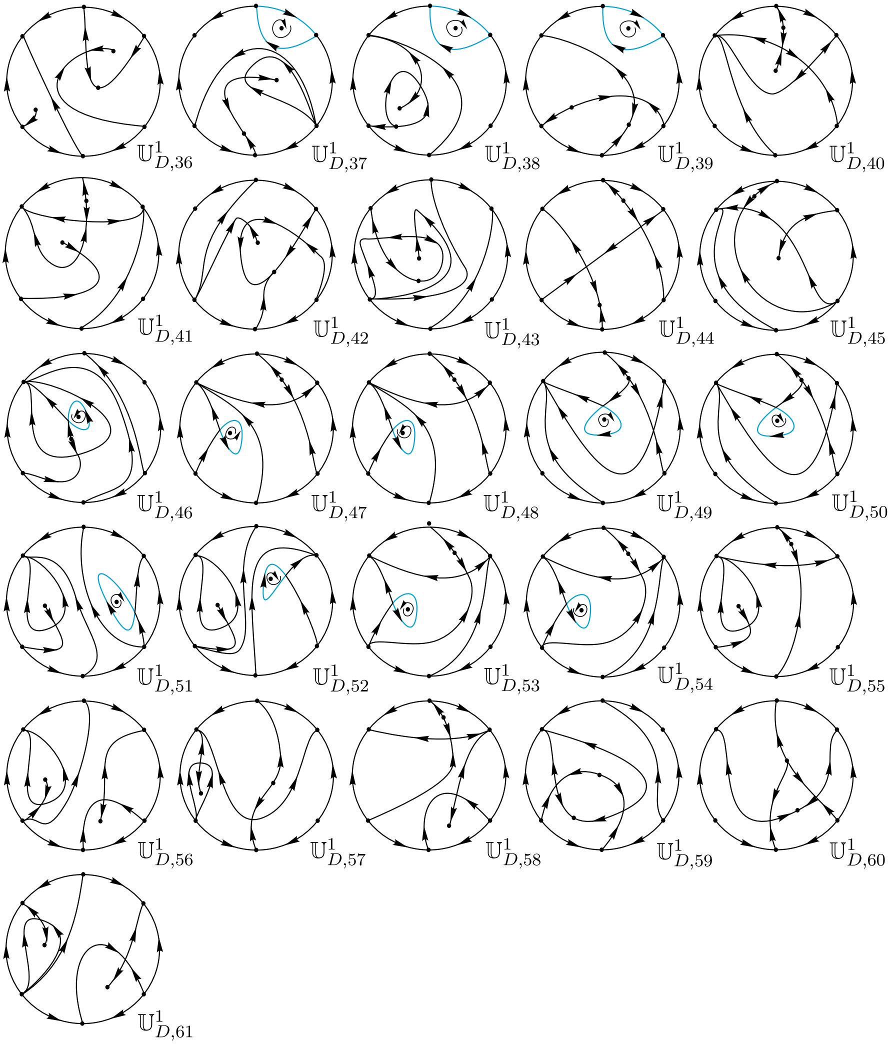
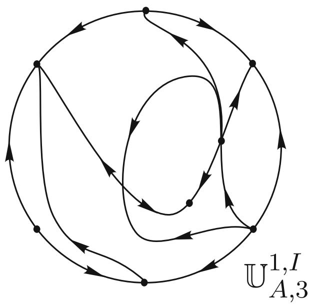


Fig. 11 (Cont.) Unstable quadratic systems of codimension one* (cases with a separatrix connection)

Fig. 12 Non realizable unstable quadratic system of codimension one*



In this sense, we must put in stand-by our own result [15, Theorem 6] until these higher codimension systems are studied.

Theorem 5 *If one of the phase portraits of codimension one* that bifurcates from a potential codimension two* class (AD) phase portrait is not realizable, then this latter phase portrait is also not realizable.*

Proof In our case, we have a finite semi-elemental saddle-node and a separatrix connection which is not necessarily algebraic. We can always break the separatrix connection by means of a rotated vector field, and this respects the number of finite singularities. It is true that a rotated vector field may turn a semi-elemental saddle-node into a cusp, but in order to break the connection we just need an infinitesimal perturbation, and under these conditions, the saddle-node remains unaffected.

If we want to break the saddle-node, we must simply do first a perturbation of the system as we did in [6, Lemma 3.24] in order to make it disappear or split it into two singularities. Of course, this perturbation may (almost surely) break also the separatrix connection. But by means of another rotated vector field which preserves all the finite singularities, we may recover the separatrix connection. □

This same theorem will be also be true in the case of the class (CD) with a very similar proof. However, it is not clear in the class (BD) because after splitting an infinite saddle-node $\overline{\binom{0}{2}}SN$, the required rotated vector field needed to recover the separatrix connection may affect the infinite singularities. One would like to think that the main parameters needed to produce both perturbations are of different types, for example, one being the parameters of the quadratic part of the equation and other of the linear or constant part. Thus the effect of each parameter would be stronger for the bifurcation for which it is required and weaker for the collateral effect it produces. Anyway, this is something which will be considered at its proper time.

We will have a similar problem when looking for examples in Sect. 6 that can be derived from systems having a $\overline{\binom{0}{2}}SN$ at infinity since we will need to perturb it and reproduce a separatrix connection. We will do it so that all our examples will be certain, but this is not enough to turn this fact into a general theorem as we have done for the class (AD).

5 Proof of Theorem 3: The Topologically Potential Phase Portraits

Here we consider all 61 realizable structurally unstable quadratic vector fields of codimension one* from class (D).

This paper leads more with the topology of the space than with a bifurcation diagram. We do not work with normal forms, neither parameters. So, we think it is better to talk on more topological terms.

Let H_0 be the set of all the quadratic systems. That is H_0 can be assimilated with $\mathbb{R}^{12}$ (thus including also the linear systems and even the constant and null systems). Or if preferred to work in a compact set, H_0 can be assimilated to $\mathbb{S}^{11}$ (removing just the null system). Since each differential system has a unique phase portrait, H_0 can be considered to contain all the phase portraits of quadratic systems (including also lower degree ones).

The *equivalence relation modulo limit cycles* is defined in H_0 as follows:

Definition 5 Two systems S_1, S_2 *are equivalent modulo limit cycles*, $S_1 \sim_{LC} S_2$, if and only if by identifying the unique focus inside each eye of limit cycles with each one of the points inside the closed region bounded by the largest one of the limit cycles of an eye of limit cycles, the two phase portraits become topologically equivalent with the resulting quotient topology on the plane.

So we obtain the space of all the different phase portraits modulo limit cycles of quadratic systems as $\hat{H}_0 = H_0/\sim$. The structurally stable quadratic systems occupy a generic space in $\hat{H}_0$ while the non stable ones occupy a space of measure zero.

Let H_1 be the complementary space of the structurally stable systems. That is, H_1 is the space of all the non stable phase portraits. Equivalently, we can define the space of all the different (modulo limit cycles) non stable phase portraits of quadratic systems as $\hat{H}_1 = H_1/\sim$. So $\hat{H}_0\setminus\hat{H}_1$ is exactly the space of the structurally stable ones. $\hat{H}_0\setminus\hat{H}_1$ is divided into disconnected pieces. All these pieces are open sets bordered by parts of $\hat{H}_1$. So, each piece of $\hat{H}_0\setminus\hat{H}_1$ must have a common border with at least another piece of $\hat{H}_0\setminus\hat{H}_1$.

In a similar way, the set of the codimension one* phase portraits occupy a generic part of $\hat{H}_1$ and the set of higher codimension phase portraits (modulo limit cycles) that we may call $\hat{H}_2$ is an hyper-surface of $\hat{H}_1$.

Thus, the set of codimension one* phase portraits is $\hat{H}_1\setminus\hat{H}_2$.

We will say that two structurally stable systems modulo limit cycles (equivalently two pieces of $\hat{H}_0\setminus\hat{H}_1$) are adjacent if they share a border which is a piece of $\hat{H}_1\setminus\hat{H}_2$. That is, we ask them to share a border of non-null measure in $\hat{H}_1$.

We can extend these definition to higher codimensions naturally.

It is not known the number of disconnected pieces of $\hat{H}_0\setminus\hat{H}_1$ but we know for sure that there are at least 44 since this is the number of topologically different structurally stable quadratic phase portraits modulo limit cycles that exist. It could happen that different pieces of $\hat{H}_0\setminus\hat{H}_1$ would share the same phase portrait. It is convenient to define another equivalence relation between pieces of $\hat{H}_0\setminus\hat{H}_1$ and say that two pieces are equivalent if they produce the same topological phase portrait. In this sense, we may say that $\tilde{H}_0 = (\hat{H}_0\setminus\hat{H}_1)/\sim$ has exactly 44 pieces. We may extend the same definition to higher codimensions.

In the same way, we know that $\tilde{H}_1 = (\hat{H}_1\setminus\hat{H}_2)/\sim$ has between 202 and 209 (7 conjectured impossible) pieces. In this paper we are looking for the number of pieces of $\tilde{H}_2 = (\hat{H}_2\setminus\hat{H}_3)/\sim$ that contain a finite saddle-node and a separatrix connection.

With the list of potential phase portraits of class (AD) we have the list of all the potential borders that we may have between some pieces of $\tilde{H}_1$ (those having the required conditions of unstability), but maybe some of them are not realizable. In order to exist, the bifurcations of the potential phase portrait must really exist.

In order to check this, we need to consider all the phase portraits of codimension one* (of the required subfamilies), and study which are the potential borders of the pieces in $\tilde{H}_1$

Definition 6 We will call evolution of a phase portrait in $\tilde{H}_k$ (with $k = 0, 1, \ldots$) to the tree of phase portraits in $\tilde{H}_{k+1}$ which border it, and complemented with the phase portraits in $\tilde{H}_k$ that can be found beyond that border by a small perturbation.

We use the term *evolution* because we want to avoid the use of the term "perturbation". A "perturbation" is classically a small modification in the parameter space so to break and bifurcate some degeneracy. An "evolution" is a trip to the borders of the region to look what can be found in their borders and what is beyond them. Notice that we only consider the border of exactly one dimension less of the starting region. The pieces of border of lower dimension will be considered when studying the evolution of the phase portraits of the previous borders. In order to use a simpler language, we will say that a phase portrait *produces by evolution* the corresponding tree of phase portraits, or simply *produces* the corresponding tree of phase portraits. Moreover we will describe the tree first mentioning the borders, and later the phase portraits beyond the borders.

Remark 3 It is important to mention that phase portraits corresponding to some low codimension may also appear in some higher codimensions. The simple reason is that two geometrically distinct singularities may be topologically equivalent. The simplest cases are the triple semi-elemental nodes and saddles which topologically behave as if they were elemental ones. Then for example, some codimension two* phase portraits may be topologically equivalent to structurally stable ones. In this sense, one may have a piece of $\tilde{H}_0$ which is bordered by a piece of $\tilde{H}_1$ and this last one has a border in $\tilde{H}_2$ which is topologically equivalent to what we had in the original piece of $\tilde{H}_0$.

Remark 4 Even more curious is a situation that we will see several times in the paper. We have a phase portrait with a finite saddle-node and a separatrix connection which uses one (or two) separatrix of the saddle-node. Then, if the saddle-node disappears, obviously the connection must also disappear. So, we have an object of codimension two* and while looking for one of their first level bifurcations which should be in codimension one*, it is forced to be in codimension 0. We have already found several of these cases in different bifurcation diagrams (see for example Figures 40, 41 and 42 in [7]), and the simplest explanation is that the separatrix connection simply "persists" in the complex space. Anyway, the phase portrait allows other perturbations in codimension one which confirms its codimension two by Definition 4.

In order to find all the topologically potential phase portraits belonging to the class (AD) of codimension two*, it is necessary to consider all possible ways of coalescing two finite singular points (in a phase portrait that already contains a separatrix connection) producing a finite saddle-node and maintaining the connection. It is not necessary to study the already known non-realizable phase portraits of class (D). Technically it could also be possible to start from phase portraits in class (A) and check all the possible separatrix connections than can be produced. However, this alternative algorithm does not guarantee that we do not miss some possibilities.

So we will consider one by one all the 61 realizable phase portraits from class (D) and determine the set of potential phase portraits of class (AD). In other studies where the determination of impossible phase portraits needed long proofs, we were forced to use a temporary notation for the set of candidates. But in this case, the proofs are much easier, and we can provide the definitive notation. Anyway, this notation cannot be considered definitive until we provide the example which proves the realization in Sect. 6.

After the coalescence of a finite saddle and a finite node, we will obtain a codimension two* phase portrait. We can bifurcate this phase portrait in several ways, either producing the disappearance of the saddle-node or by breaking the separatrix connection (and normally this last option produces two possibilities). If the separatrix connection is related with the saddle-node, its disappearance may produce also the disappearance of the connection and we will have a structurally stable system. But in some tricky case, a (or even two) new separatrix connection may appear after the disappearance of the saddle-node. Then we will obtain an unstable phase portrait of class (D) and several related structurally stable ones. In case we break the separatrix connection, we will always obtain phase portraits of class (A).

Phase portrait $\mathbb{U}^1_{D,1}$ cannot produce by evolution any portrait of class (AD) since its only anti-saddle (more concretely a focus) is already confined inside the graphic. If that focus coalesces with the only available finite saddle, it will produce a cusp, and the connection disappears. In this case, the border of the region is phase portrait $\mathbb{U}^2_{AA,1}$ from [6]. This is not its unique border. Other borders may be phase portraits where another saddle-node (finite or infinite) appears. This will happen in some other cases so we will not comment any more these possibilities.

Phase portrait $\mathbb{U}^1_{D,2}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,1}$ and $\mathbb{U}^2_{AD,2}$ (see Fig. 13). After bifurcation by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{2,1}$ (see Remark 4). A perturbation breaking just the connection produces codimension one* phase portraits $\mathbb{U}^1_{A,5}$ or $\mathbb{U}^1_{A,2}$ from $\mathbb{U}^2_{AD,1}$ and phase portraits $\mathbb{U}^1_{A,4}$ or $\mathbb{U}^1_{A,5}$ from $\mathbb{U}^2_{AD,2}$. In Fig. 13 we have drawn the complete bifurcation diagram.

Phase portrait $\mathbb{U}^1_{D,3}$ may produce phase portrait $\mathbb{U}^2_{AD,3}$ (see Fig. 14) and after bifurcation we get phase portraits $\mathbb{S}^2_{2,1}$, $\mathbb{U}^1_{A,2}$ or $\mathbb{U}^1_{A,3}$. By symmetry, the other anti-saddle coalescing with the saddle leads to same conclusion.

Phase portrait $\mathbb{U}^1_{D,4}$ may produce phase portrait $\mathbb{U}^2_{AD,4}$ (see Fig. 15) and after bifurcation we get phase portraits $\mathbb{S}^2_{2,1}$, $\mathbb{U}^1_{A,7}$ or $\mathbb{U}^1_{A,6}$. By symmetry, the other anti-saddle coalescing with the saddle leads to same conclusion.

Phase portrait $\mathbb{U}^1_{D,5}$ may produce phase portraits $\mathbb{U}^2_{AD,5}$, $\mathbb{U}^2_{AD,6}$ and $\mathbb{U}^2_{AD,7}$ (see Fig. 16). After bifurcation by disappearance of the saddle-node of the first two we get phase portraits $\mathbb{S}^2_{2,1}$. However, in the third case, the separatrix connection may remain and we get phase portrait $U^1_{D,1}$. If it does not remain, we get phase portrait $\mathbb{S}^2_{2,1}$ without limit cycle, or with limit cycle. Of course, generically the connection will not survive.

If bifurcation breaks the separatrix connection we get phase portraits $\mathbb{U}^1_{A,7}$ from all of them in one of the possible breaks, or respectively phase portraits $\mathbb{U}^1_{A,9}$, $\mathbb{U}^1_{A,8}$ and $\mathbb{U}^1_{A,10}$ with the other break.

For the sake of the argumentation we will describe the evolution of phase portrait $\mathbb{U}^1_{D,7}$ before $\mathbb{U}^1_{D,6}$.

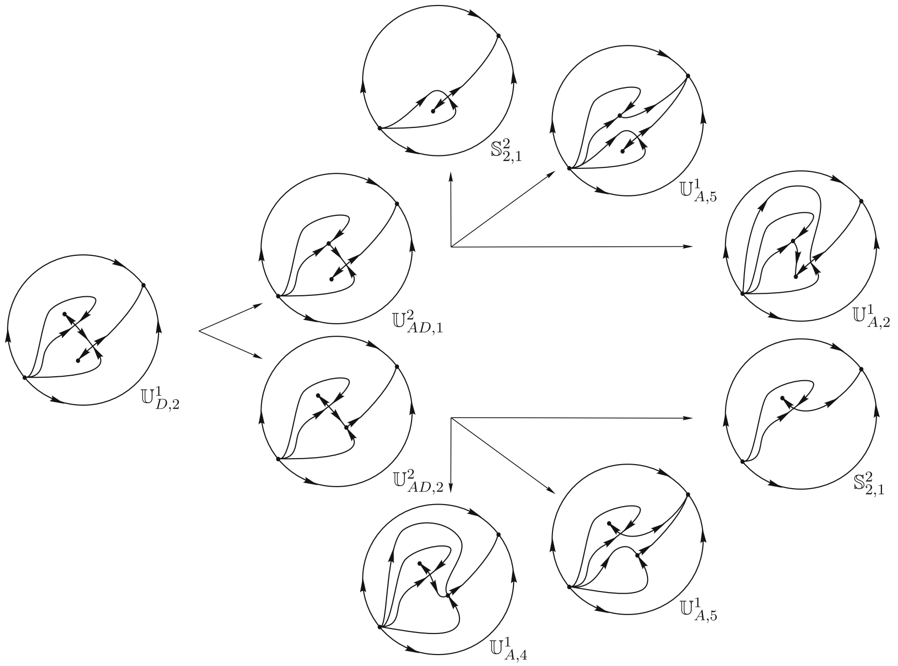


Fig. 13 Unstable phase portraits $\mathbb{U}^2_{AD,1}$ and $\mathbb{U}^2_{AD,2}$

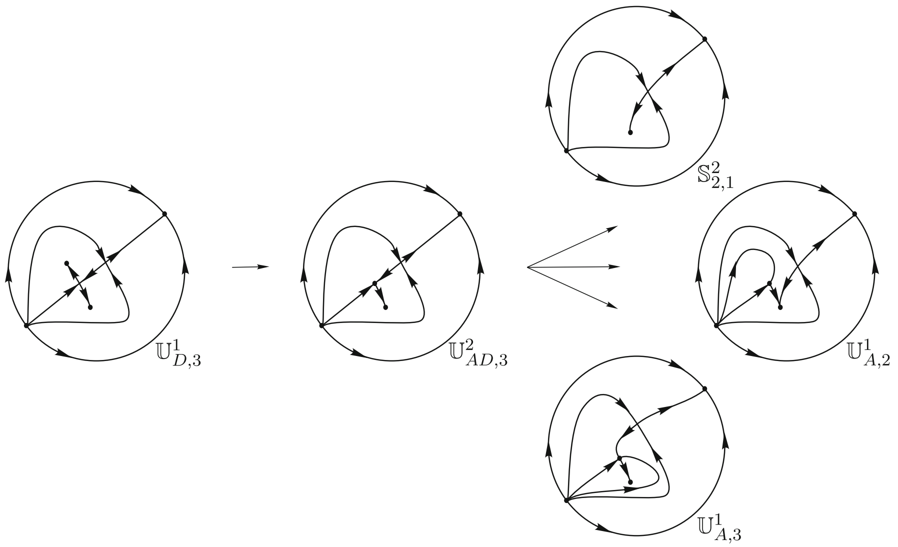


Fig. 14 Phase portrait $\mathbb{U}^2_{AD,3}$

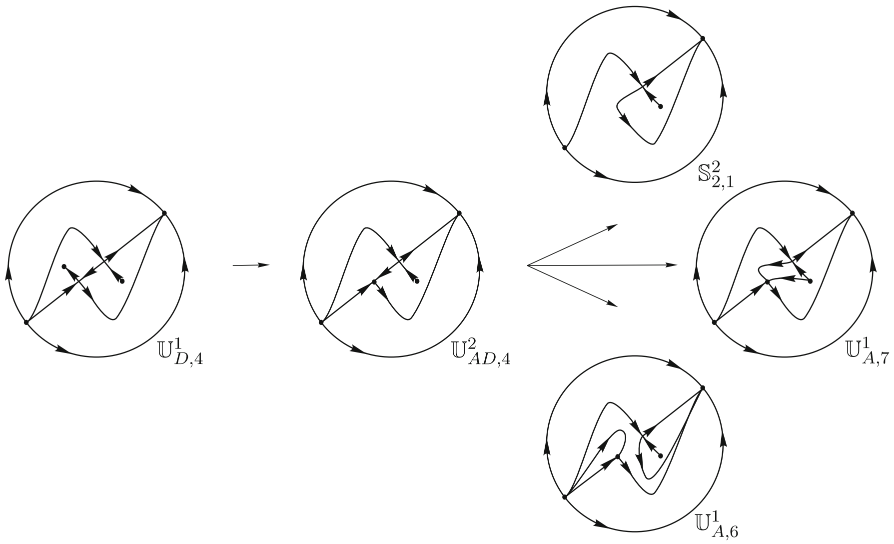


Fig. 15 Phase portrait $\mathbb{U}^2_{AD,4}$

Phase portrait $\mathbb{U}^1_{D,7}$ may produce phase portraits $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,9}$ (see Fig. 17). In the first phase portrait the separatrix connection has no relation with the saddle-node and may remain. So, if the saddle-node disappears we get a phase portrait of class (D), concretely $\mathbb{U}^1_{D,1}$. From $\mathbb{U}^2_{AD,9}$, if the saddle-node disappears, the loop also vanishes and we get $\mathbb{S}^2_{2,1}$ (with limit cycle). If we break the loop, we obtain $\mathbb{U}^1_{A,4}$ (with limit cycle) or $\mathbb{U}^1_{A,8}$ from the first, and $\mathbb{U}^1_{A,3}$ (with limit cycle) or $\mathbb{U}^1_{A,10}$ from the second.

We want just to remark the subtle difference between $\mathbb{U}^2_{AD,9}$ and $\mathbb{U}^2_{AD,7}$. In the case of $\mathbb{U}^2_{AD,9}$ the separatrix connection involves only the saddle-node and if the singularity disappears, the connection also. But in the case of $\mathbb{U}^2_{AD,7}$ the separatrix connection involved the saddle-node and a saddle, and there exists the possibility that even after the disappearance of the saddle-node, a new separatrix connection may be formed with separatrices of the saddle. We will also see later (see $\mathbb{U}^2_{AD,47}$) a case in which after the disappearance of the saddle-node, two different separatrix connections (and 3 generic possibilities) may appear.

Phase portrait $\mathbb{U}^1_{D,6}$ may produce phase portraits $\mathbb{U}^2_{AD,77}$ and $\mathbb{U}^{2,I}_{AD,9}$ (see Fig. 18). The reason why we jump numeration from expected 10 to 77 is that for a long time we have thought that $\mathbb{U}^2_{AD,77}$ would be impossible and thus was given the name $\mathbb{U}^{2,I}_{AD,8}$. However we have delayed the completion of this paper until the complete study of the family of quadratic systems with a finite saddle-node and a weak focus (**QSwf1sn**) of first order which was on process [20]. Finally from this paper we have obtained a clear example of $\mathbb{U}^2_{AD,77}$ proving it is realizable. Since the numeration in this paper was already done, and a renumbering could lead to mistakes, we have preferred to call it as $\mathbb{U}^2_{AD,77}$.

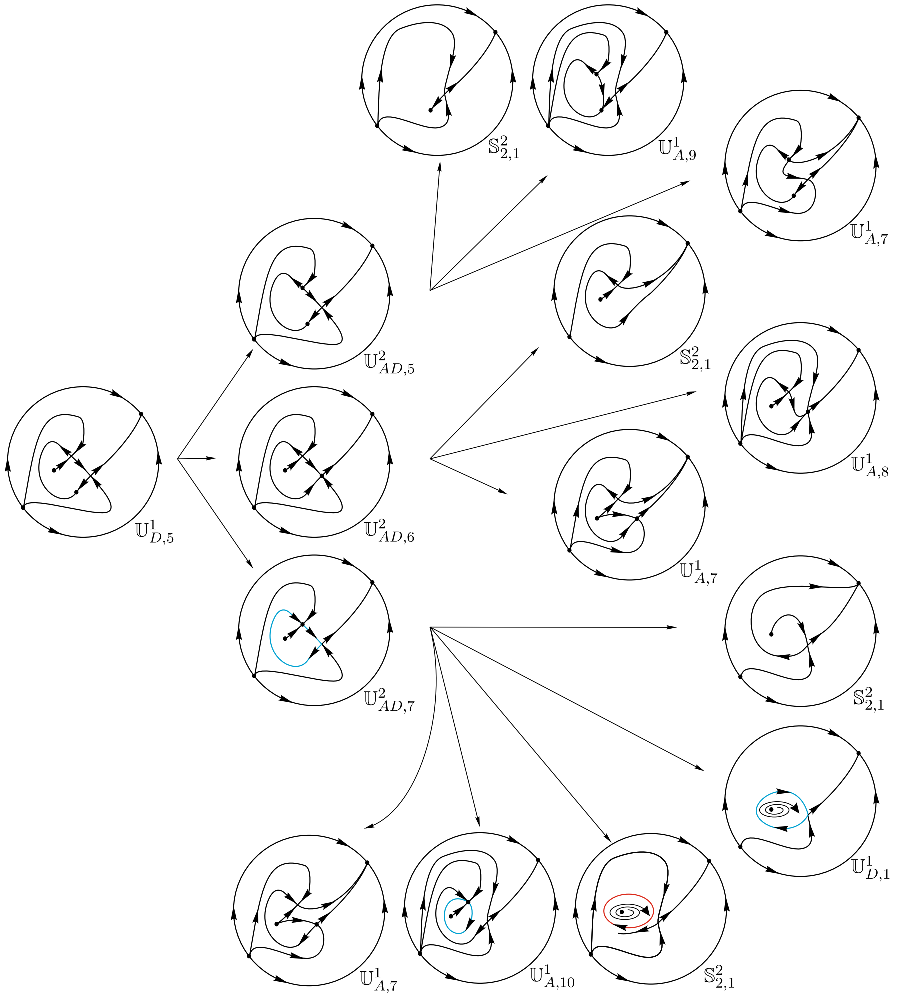


Fig. 16 Phase portraits $\mathbb{U}^2_{AD,5}$, $\mathbb{U}^2_{AD,6}$ and $\mathbb{U}^2_{AD,7}$

Phase portraits $\mathbb{U}^2_{AD,77}$ and $\mathbb{U}^{2,I}_{AD,9}$ have the same skeleton of separatrices, as the phase portraits $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,9}$ (respectively), but now the focus inside the loop has opposite stability. As we will show in Sect. 6, both $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,77}$ can be obtained from class (AB) and **QSwf1sn** (respectively) but we have only been able to found $\mathbb{U}^2_{AD,9}$ and $\mathbb{U}^{2,I}_{AD,9}$ remains not found.

Remark 5 Notice that if phase portrait $\mathbb{U}^{2,I}_{AD,9}$ would exist, then a perturbation of it would produce $\mathbb{U}^1_{A,10}$ with a limit cycle. We have not found any example in all the bibliography we have checked. If it could be proved to be impossible, then by Theorem 5 we would obtain the impossibility of $\mathbb{U}^{2,I}_{AD,9}$. The opposite is not true. The existence of $\mathbb{U}^1_{A,10}$ with a limit cycle is not yet a proof of the existence $\mathbb{U}^{2,I}_{AD,9}$ but it

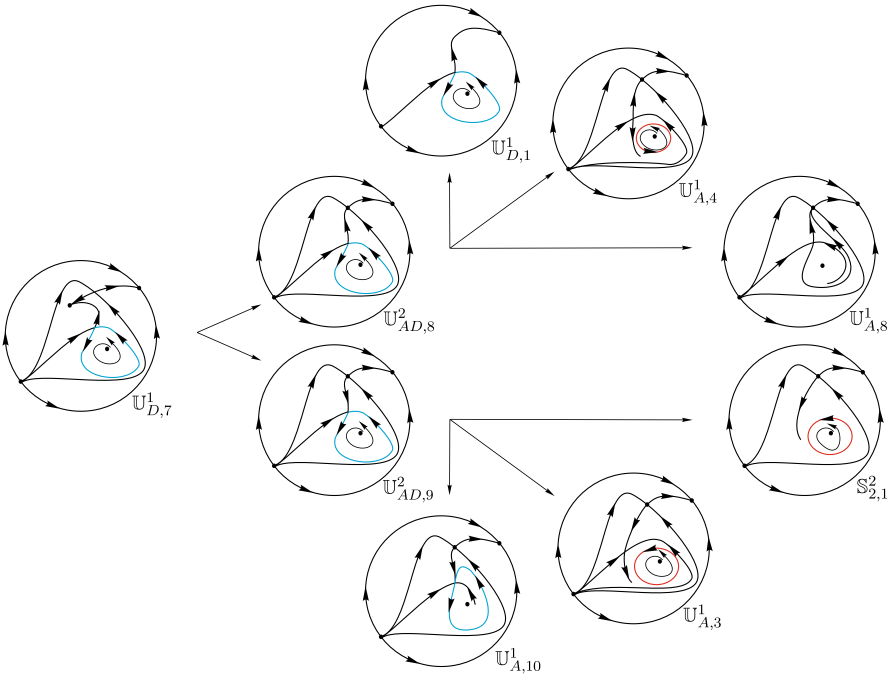


Fig. 17 Phase portraits $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,9}$

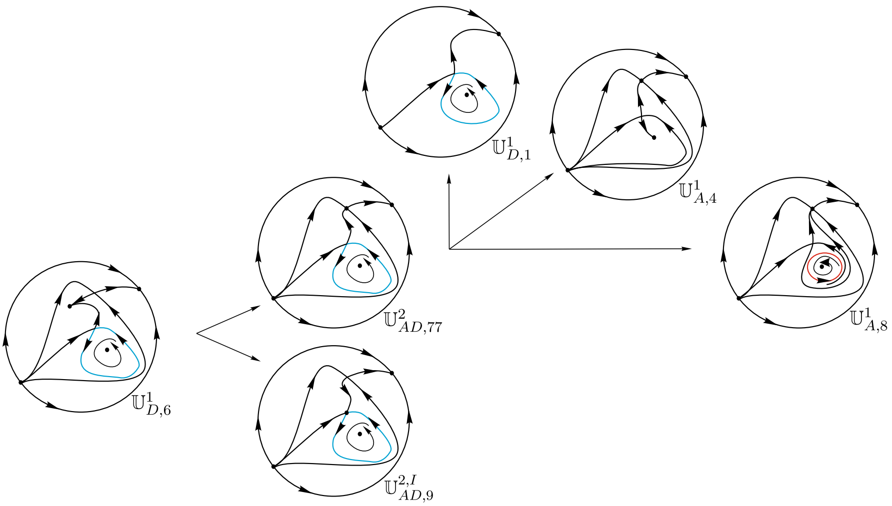


Fig. 18 Phase portrait $\mathbb{U}^2_{AD,77}$ and conjectured impossible $\mathbb{U}^{2,I}_{AD,9}$

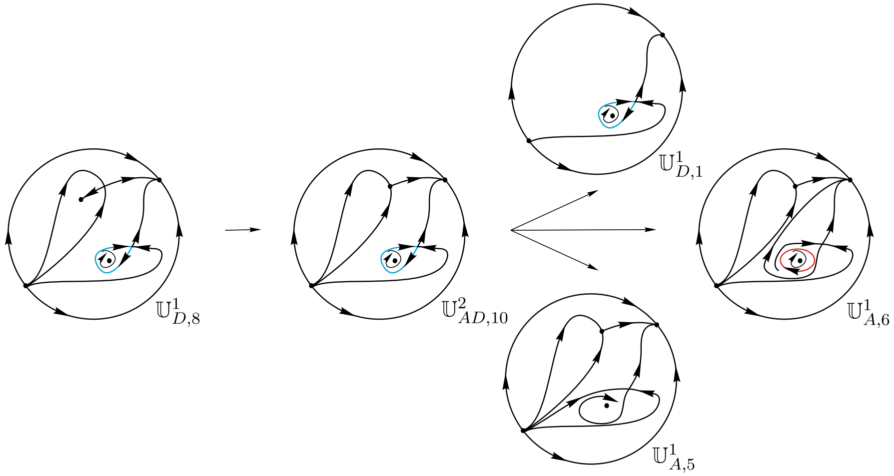


Fig. 19 Unstable phase portrait $\mathbb{U}^2_{AD,10}$

is a nice starting point from which trying to produce it. This is an argument why we have conjectured $\mathbb{U}^{2,I}_{AD,9}$ to be non-realizable. Similar arguments apply to the rest of conjectured impossible cases.

Before considering phase portrait $\mathbb{U}^1_{D,8}$ we must point out a small drawing typo in this phase portrait (and the same typo in $\mathbb{U}^1_{D,9}$, $\mathbb{U}^1_{D,20}$, $\mathbb{U}^1_{D,31}$, $\mathbb{U}^1_{D,48}$, $\mathbb{U}^1_{D,53}$ and $\mathbb{U}^1_{D,54}$) in [6]. In all of them the orbits inside the loop must turn in the same sense as the loop.

Phase portrait $\mathbb{U}^1_{D,8}$ may produce phase portrait $\mathbb{U}^2_{AD,10}$ (see Fig. 19) and after bifurcation we get phase portraits $\mathbb{U}^1_{D,1}$ (plus $\mathbb{S}^2_{2,1}$ with or without limit cycle), $\mathbb{U}^1_{A,5}$ or $\mathbb{U}^1_{A,6}$ (with limit cycle).

Phase portrait $\mathbb{U}^1_{D,9}$ may produce phase portrait $\mathbb{U}^2_{AD,11}$ (see Fig. 20) and after bifurcation we get phase portraits $\mathbb{U}^1_{D,1}$ (plus $\mathbb{S}^2_{2,1}$ with or without limit cycle), $\mathbb{U}^1_{A,6}$ or $\mathbb{U}^1_{A,5}$ (with limit cycle).

Phase portraits $\mathbb{U}^1_{D,10}$ and $\mathbb{U}^1_{D,11}$ may not produce phase portraits of class (AD) since they do not have finite saddles.

Phase portrait $\mathbb{U}^1_{D,12}$ may produce phase portraits $\mathbb{U}^2_{AD,12}$ and $\mathbb{U}^{2,I}_{AD,12}$ (see Fig. 21). Even though this may seem a symmetrical situation, it is not and both resulting phase portraits are different regardless they share the same skeleton. Moreover, we have found an example for one of them and the other is conjectured impossible. After bifurcation by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{4,1}$ (with limit cycle). By breaking just the loop can produce codimension one* phase portraits $\mathbb{U}^1_{A,12}$ (with limit cycle) or $\mathbb{U}^1_{A,13}$. We must mention here that the phase portrait $\mathbb{U}^{2,I}_{AD,12}$ it is drawn in [18] as phase portrait $5.7L_9$ in Fig. 10. However there is a typo there since the focus inside the graphic must have the opposite stability, and then in fact corresponds to $\mathbb{U}^2_{AD,12}$. In order to confirm the typo, one must look at Figs. 63 and 64 where the region $5.7L_5$ (which is in fact $5.7L_9$

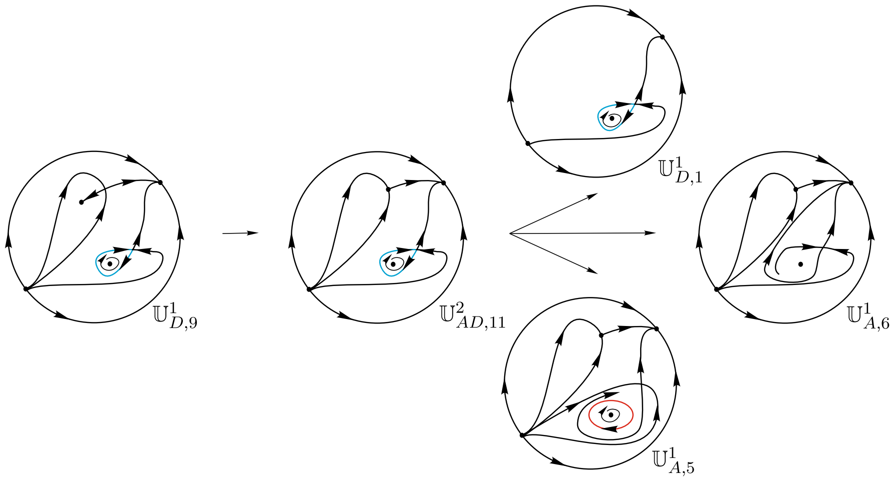


Fig. 20 Unstable phase portrait $\mathbb{U}^2_{AD,11}$

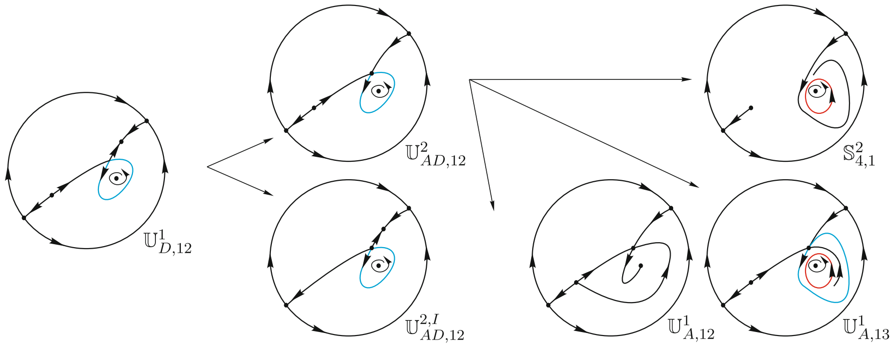


Fig. 21 Unstable phase portrait $\mathbb{U}^2_{AD,12}$ and conjectured impossible $\mathbb{U}^{2,I}_{AD,12}$

due to another typo) is surrounded by regions V_{53}, V_{54}, $V_{87} \equiv V_{94}$ and $V_{95} \equiv V_{107}$ and all them have that focus as attractor.

Phase portrait $\mathbb{U}^1_{D,13}$ may produce phase portraits $\mathbb{U}^2_{AD,13}$ and $\mathbb{U}^2_{AD,14}$ (see Fig. 22). After bifurcation of $\mathbb{U}^2_{AD,13}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{4,1}$. By breaking just the connection one can produce codimension one* phase portraits $\mathbb{U}^1_{A,12}$ or $\mathbb{U}^1_{A,11}$. After bifurcation of $\mathbb{U}^2_{AD,14}$ by disappearance of the saddle-node, the separatrix connection may persist and we get phase portrait $\mathbb{U}^1_{D,10}$ or $\mathbb{S}^2_{4,1}$ with limit cycle on one or other anti-saddle (which must be a focus). By breaking just the connection one can produce codimension one* phase portrait $\mathbb{U}^1_{A,13}$ (just one because of symmetry).

Phase portrait $\mathbb{U}^1_{D,14}$ may not produce phase portraits of class (AD) since it does not have finite anti-saddles.

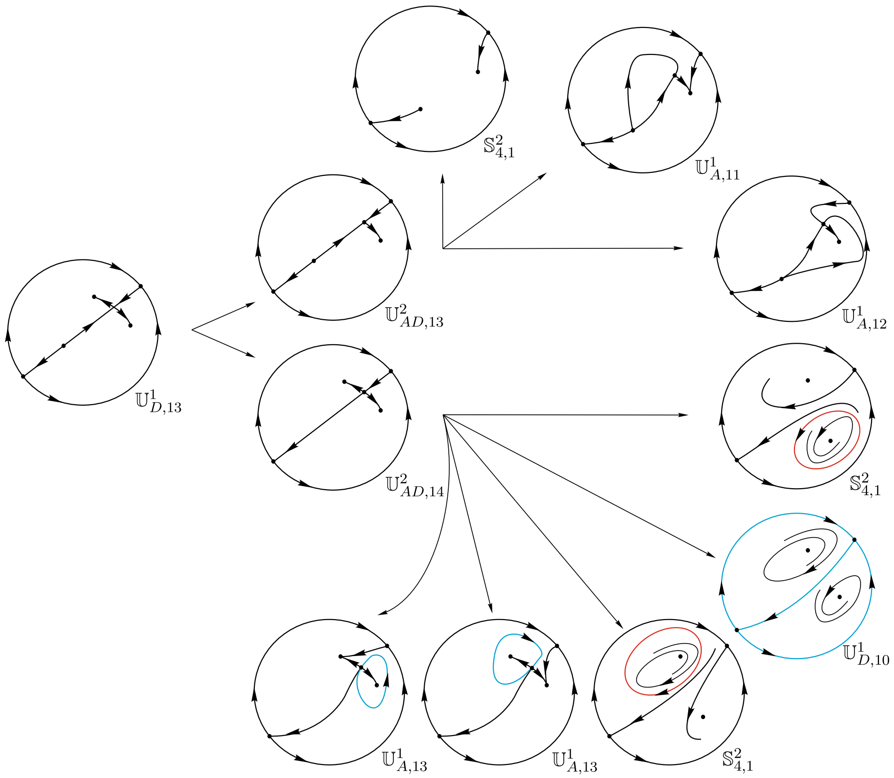


Fig. 22 Unstable phase portraits $\mathbb{U}^2_{AD,13}$ and $\mathbb{U}^2_{AD,14}$

Phase portrait $\mathbb{U}^1_{D,15}$ may produce phase portraits $\mathbb{U}^2_{AD,15}$ and $\mathbb{U}^2_{AD,16}$ (see Fig. 23). After bifurcation of $\mathbb{U}^2_{AD,15}$ by disappearance of the saddle-node, the separatrix connection may remain and we get phase portrait $\mathbb{U}^1_{D,14}$ (plus $\mathbb{S}^2_{6,1}$ just one by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,14}$ or $\mathbb{U}^1_{A,15}$. After bifurcation of $\mathbb{U}^2_{AD,16}$ by disappearance of the saddle-node, the separatrix connection is lost and we get phase portrait $\mathbb{S}^2_{6,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,16}$ and $\mathbb{U}^1_{A,14}$.

Phase portrait $\mathbb{U}^1_{D,16}$ may produce phase portrait $\mathbb{U}^2_{AD,17}$ (see Fig. 24). After bifurcation by disappearance of the saddle-node, the separatrix connection is lost and we get phase portrait $\mathbb{S}^2_{6,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,18}$ and $\mathbb{U}^1_{A,15}$.

Phase portrait $\mathbb{U}^1_{D,17}$ may produce phase portrait $\mathbb{U}^2_{AD,18}$ (see Fig. 25). After bifurcation by disappearance of the saddle-node, the separatrix connection is lost and we get phase portrait $\mathbb{S}^2_{6,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,17}$ and $\mathbb{U}^1_{A,16}$.

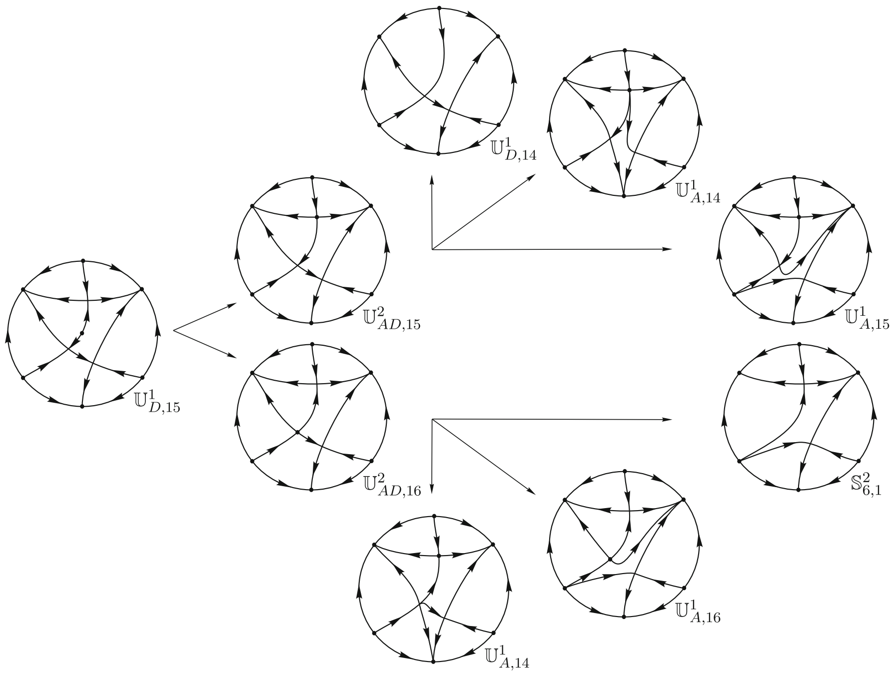


Fig. 23 Unstable phase portraits $\mathbb{U}^2_{AD,15}$ and $\mathbb{U}^2_{AD,16}$

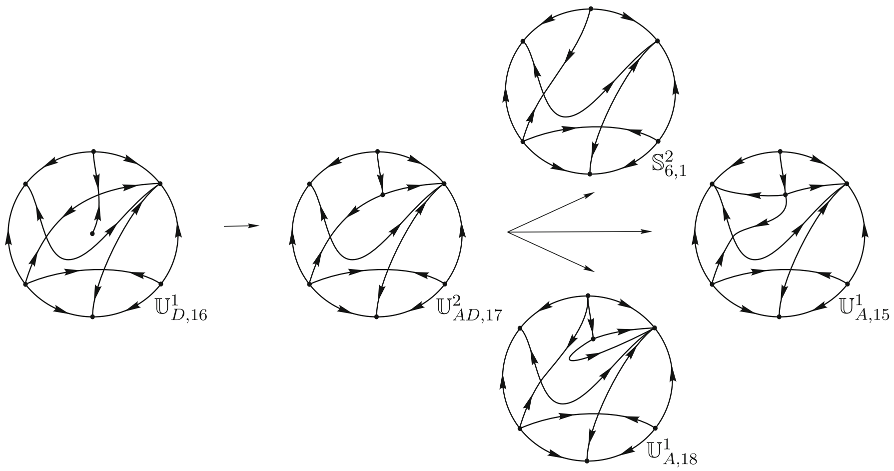


Fig. 24 Unstable phase portrait $\mathbb{U}^2_{AD,17}$

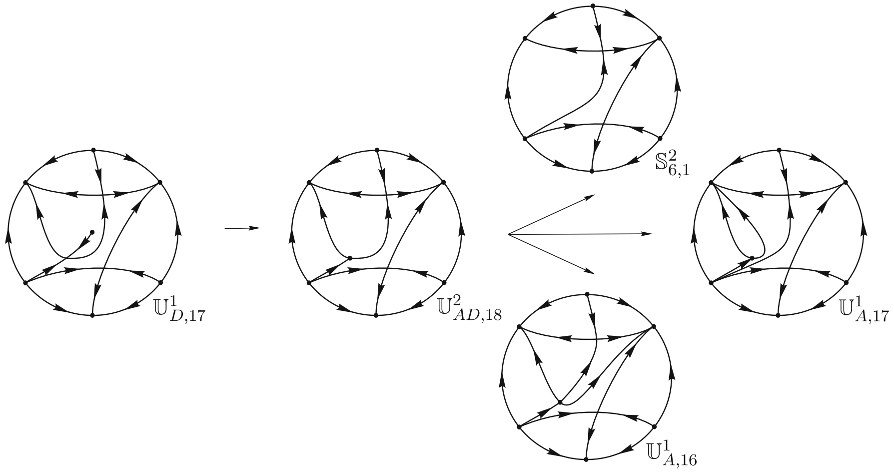


Fig. 25 Unstable phase portrait $\mathbb{U}^2_{AD,18}$

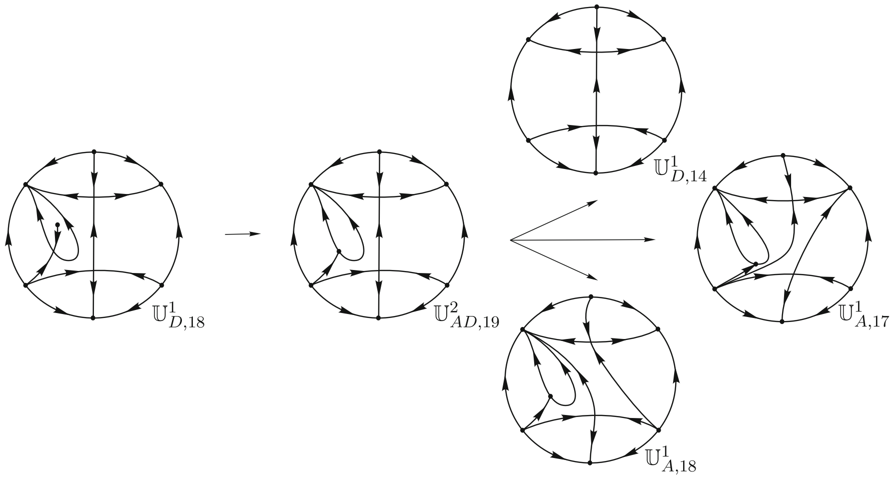


Fig. 26 Unstable phase portrait $\mathbb{U}^2_{AD,19}$

Phase portrait $\mathbb{U}^1_{D,18}$ may produce phase portrait $\mathbb{U}^2_{AD,19}$ (see Fig. 26). After bifurcation by disappearance of the saddle-node, the separatrix connection may remain and we get phase portrait $\mathbb{U}^1_{D,14}$ (plus $\mathbb{S}^2_{6,1}$ just one by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,17}$ and $\mathbb{U}^1_{A,18}$.

Phase portraits $\mathbb{U}^1_{D,19}$, $\mathbb{U}^1_{D,20}$, $\mathbb{U}^1_{D,22}$ and $\mathbb{U}^1_{D,23}$ may not produce phase portraits of class (AD) for the same reason as $\mathbb{U}^1_{D,1}$. And obviously $\mathbb{U}^1_{D,21}$ neither may by lack of finite singularities.

Phase portrait $\mathbb{U}^1_{D,24}$ may produce phase portrait $\mathbb{U}^2_{AD,20}$ (see Fig. 27). After bifurcation by disappearance of the saddle-node, the separatrix connection may remain and

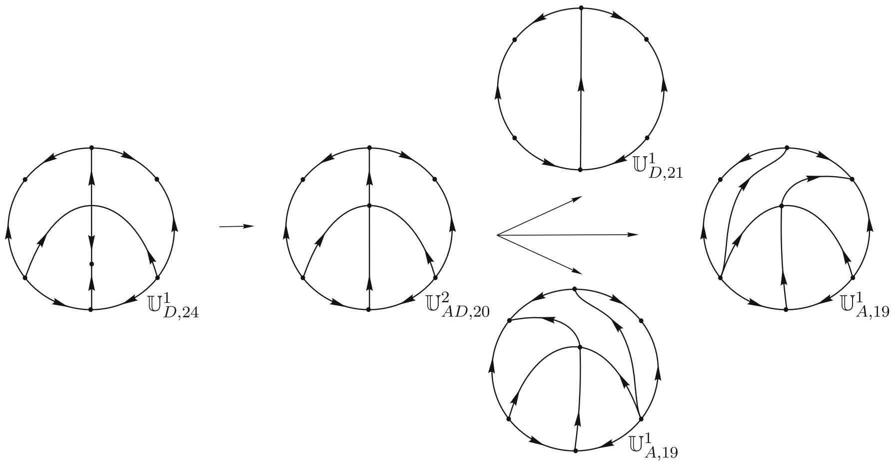


Fig. 27 Unstable phase portrait $\mathbb{U}^2_{AD,20}$

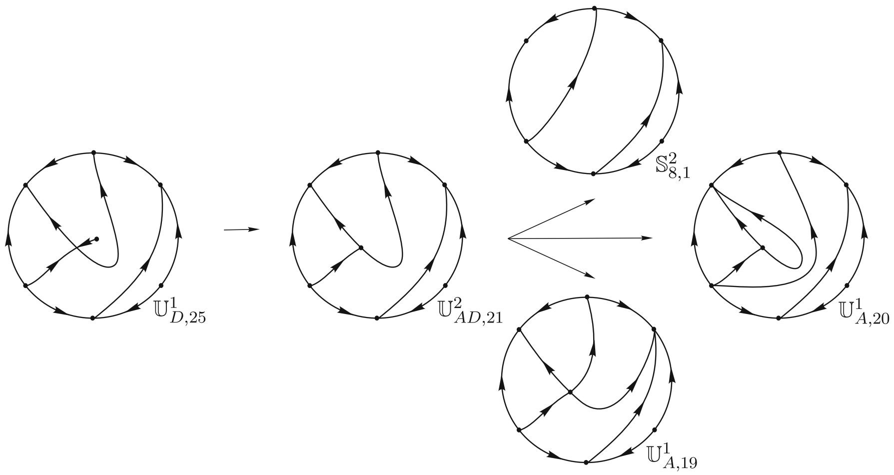


Fig. 28 Unstable phase portrait $\mathbb{U}^2_{AD,21}$

we get phase portrait $\mathbb{U}^1_{D,21}$ (plus $\mathbb{S}^2_{8,1}$ just one by symmetry). By breaking just the connection one can produce only phase portrait $\mathbb{U}^1_{A,19}$ (because of symmetry).

Phase portrait $\mathbb{U}^1_{D,25}$ may produce phase portrait $\mathbb{U}^2_{AD,21}$ (see Fig. 28). After bifurcation by disappearance of the saddle-node, the separatrix connection vanishes and we get phase portrait $\mathbb{S}^2_{8,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,19}$ and $\mathbb{U}^1_{A,20}$.

Phase portrait $\mathbb{U}^1_{D,26}$ may produce phase portrait $\mathbb{U}^2_{AD,22}$ (see Fig. 29). After bifurcation by disappearance of the saddle-node, the separatrix connection may remain and we get phase portrait $\mathbb{U}^1_{D,21}$ (plus $\mathbb{S}^2_{8,1}$ just one by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,20}$ and $\mathbb{U}^1_{A,21}$.

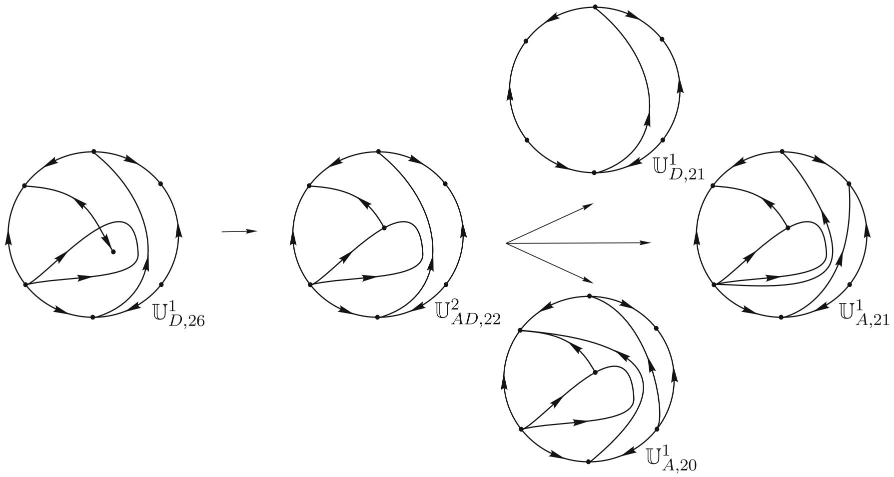


Fig. 29 Unstable phase portrait $\mathbb{U}^2_{AD,22}$

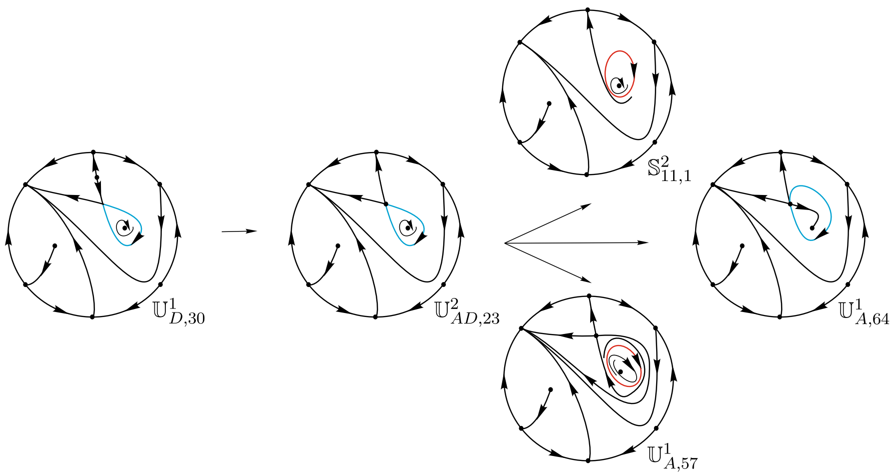


Fig. 30 Unstable phase portrait $\mathbb{U}^2_{AD,23}$

Phase portraits $\mathbb{U}^1_{D,27}$, $\mathbb{U}^1_{D,28}$ and $\mathbb{U}^1_{D,29}$ may not produce phase portraits of class (AD) since they do not have finite saddles.

Phase portrait $\mathbb{U}^1_{D,30}$ may produce phase portrait $\mathbb{U}^2_{AD,23}$ (see Fig. 30). After bifurcation by disappearance of the saddle-node, the separatrix connection vanishes and we get phase portrait $\mathbb{S}^2_{11,1}$ (with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,64}$ and $\mathbb{U}^1_{A,57}$ (with limit cycle).

Phase portrait $\mathbb{U}^1_{D,31}$ may produce phase portrait $\mathbb{U}^2_{AD,24}$ and $\mathbb{U}^{2,I}_{AD,24}$ which we conjecture as impossible (see Fig. 31). After bifurcation of $\mathbb{U}^2_{AD,24}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable

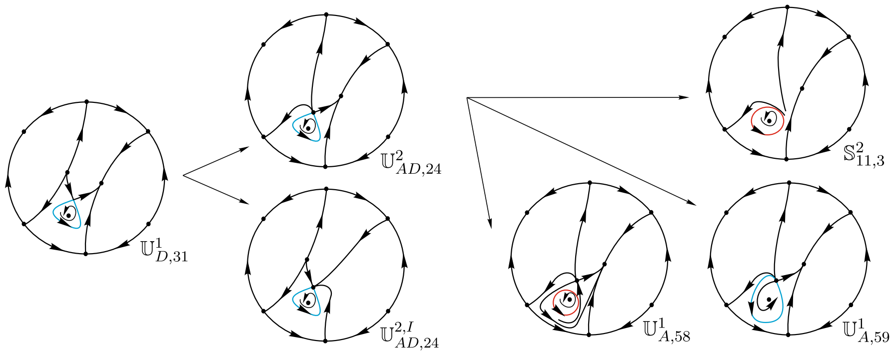


Fig. 31 Unstable phase portrait $\mathbb{U}^2_{AD,24}$ and conjectured impossible $\mathbb{U}^{2,I}_{AD,24}$

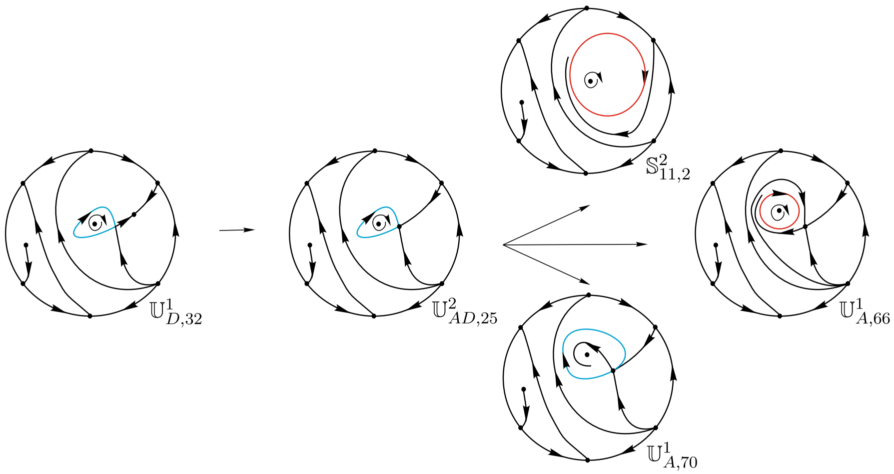


Fig. 32 Unstable phase portrait $\mathbb{U}^2_{AD,25}$

phase portrait $\mathbb{S}^2_{11,3}$ (with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,58}$ (with limit cycle) or $\mathbb{U}^1_{A,59}$.

Phase portrait $\mathbb{U}^1_{D,32}$ may produce phase portrait $\mathbb{U}^2_{AD,25}$ (see Fig. 32). After bifurcation by disappearance of the saddle-node, the separatrix connection vanishes and we get phase portrait $\mathbb{S}^2_{11,2}$ (with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,70}$ and $\mathbb{U}^1_{A,66}$ (with limit cycle).

Phase portrait $\mathbb{U}^1_{D,33}$ may produce phase portraits $\mathbb{U}^2_{AD,26}$ and $\mathbb{U}^2_{AD,27}$ (see Fig. 33). After bifurcation of $\mathbb{U}^2_{AD,26}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{11,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,62}$ or $\mathbb{U}^1_{A,68}$. After bifurcation of $\mathbb{U}^2_{AD,27}$ by disappearance of the saddle-node, the separatrix connection may remain and we get phase portrait $\mathbb{U}^1_{D,27}$ (plus $\mathbb{S}^2_{11,2}$ and $\mathbb{S}^2_{11,3}$). By breaking just the connection one produces phase portraits $\mathbb{U}^1_{A,61}$ or $\mathbb{U}^1_{A,67}$.

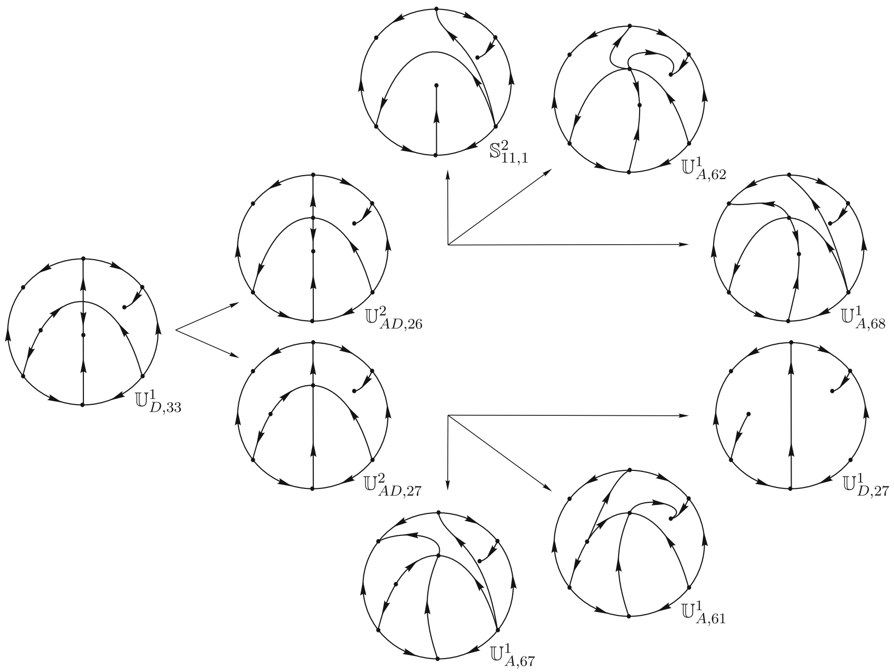


Fig. 33 Unstable phase portraits $\mathbb{U}^2_{AD,26}$ and $\mathbb{U}^2_{AD,27}$

Phase portrait $\mathbb{U}^1_{D,34}$ may produce phase portraits $\mathbb{U}^2_{AD,28}$ and $\mathbb{U}^2_{AD,29}$ (see Fig. 34). After bifurcation of $\mathbb{U}^2_{AD,28}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{11,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,68}$ or $\mathbb{U}^1_{A,63}$. After bifurcation of $\mathbb{U}^2_{AD,29}$ by disappearance of the saddle-node, the separatrix connection may remain and we get phase portrait $\mathbb{U}^1_{D,29}$ (from which $\mathbb{S}^2_{11,1}$ with limit cycle or $\mathbb{S}^2_{11,2}$ without may bifurcate). By breaking just the connection one produces phase portraits $\mathbb{U}^1_{A,67}$ or $\mathbb{U}^1_{A,64}$.

Phase portrait $\mathbb{U}^1_{D,35}$ may produce phase portraits $\mathbb{U}^2_{AD,30}$, $\mathbb{U}^2_{AD,31}$ and $\mathbb{U}^2_{AD,32}$ (see Fig. 35). After bifurcation of $\mathbb{U}^2_{AD,30}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{11,3}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,60}$ or $\mathbb{U}^1_{A,61}$. After bifurcation of $\mathbb{U}^2_{AD,31}$ by disappearance of the saddle-node, the separatrix connection is lost and we get phase portrait $\mathbb{S}^2_{11,3}$. By breaking just the connection one produces phase portraits $\mathbb{U}^1_{A,61}$ or $\mathbb{U}^1_{A,58}$. After bifurcation of $\mathbb{U}^2_{AD,32}$ by disappearance of the saddle-node, the separatrix connection may persist and we get phase portrait $\mathbb{U}^1_{D,28}$ (or its respective bifurcations $\mathbb{S}^2_{11,1}$ and $\mathbb{S}^2_{11,2}$ with limit cycle if the connection does

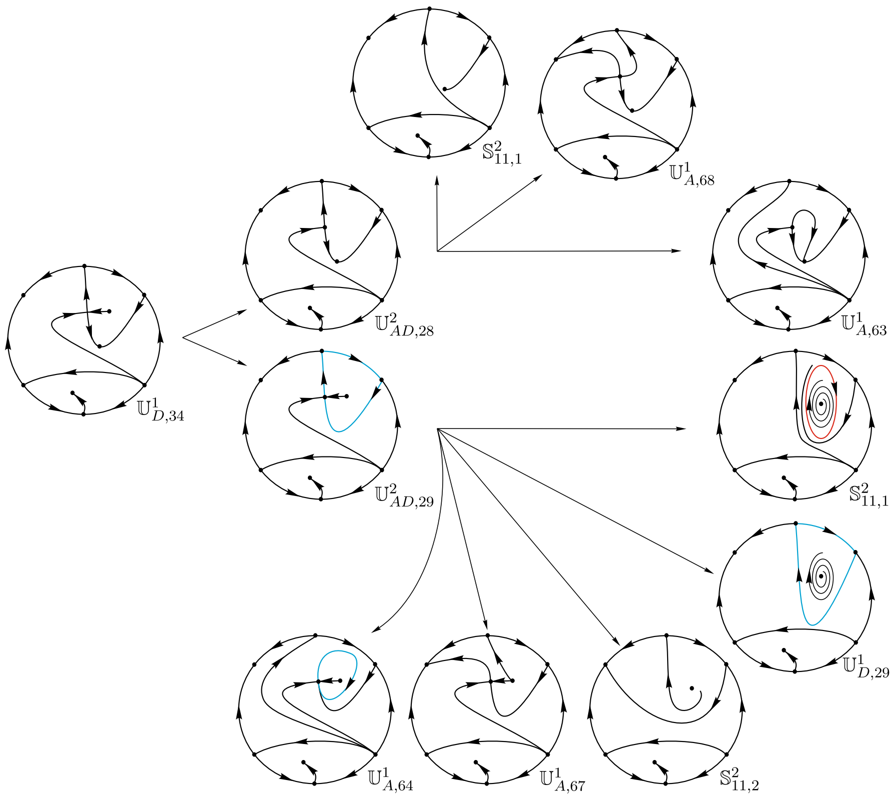


Fig. 34 Unstable phase portraits $\mathbb{U}^2_{AD,28}$ and $\mathbb{U}^2_{AD,29}$

not persist). By breaking just the connection one produces phase portraits $\mathbb{U}^1_{A,59}$ or $\mathbb{U}^1_{A,62}$.

Phase portrait $\mathbb{U}^1_{D,36}$ may produce phase portraits $\mathbb{U}^2_{AD,33}$ and $\mathbb{U}^2_{AD,34}$ (see Fig. 36). After bifurcation of $\mathbb{U}^2_{AD,33}$ by disappearance of the saddle-node, the separatrix connection is lost and we get the structurally stable phase portrait $\mathbb{S}^2_{11,2}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,69}$ or $\mathbb{U}^1_{A,67}$. After bifurcation of $\mathbb{U}^2_{AD,34}$ by disappearance of the saddle-node, the separatrix connection may persist and we get phase portrait $\mathbb{U}^1_{D,28}$ (or its respective bifurcations $\mathbb{S}^2_{11,1}$ and $\mathbb{S}^2_{11,2}$ with limit cycle if the connection does not persist). By breaking just the connection one produces phase portraits $\mathbb{U}^1_{A,70}$ or $\mathbb{U}^1_{A,68}$.

Phase portrait $\mathbb{U}^1_{D,37}$ may produce phase portraits $\mathbb{U}^2_{AD,35}$ and $\mathbb{U}^2_{AD,36}$ (see Fig. 37). After bifurcation of any of them by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,28}$ (and also its possible unfoldings $\mathbb{S}^2_{11,1}$ and $\mathbb{S}^2_{11,2}$ with limit cycle). By breaking just the connection in $\mathbb{U}^2_{AD,35}$ one can produce phase portraits $\mathbb{U}^1_{A,56}$ or $\mathbb{U}^1_{A,65}$ (with limit cycle). By breaking just the connection in $\mathbb{U}^2_{AD,36}$ one produces phase portraits $\mathbb{U}^1_{A,57}$ or $\mathbb{U}^1_{A,66}$ (with limit cycle).

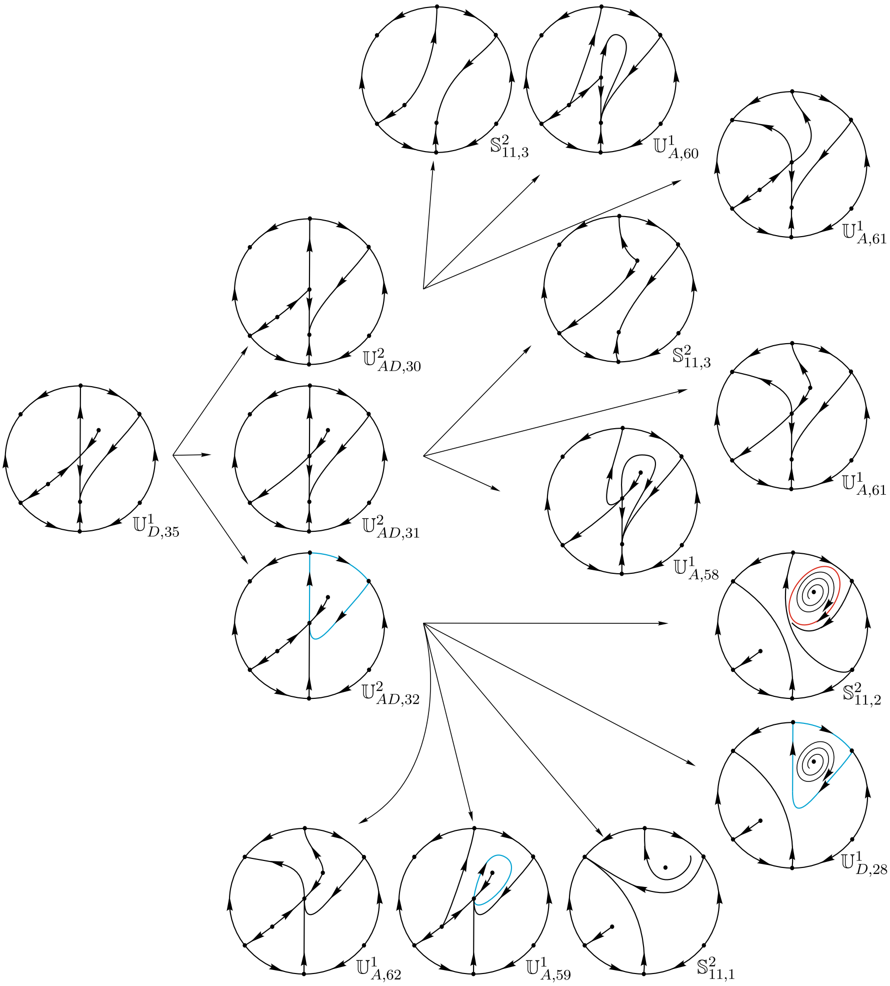


Fig. 35 Unstable phase portraits $\mathbb{U}^2_{AD,30}$, $\mathbb{U}^2_{AD,31}$ and $\mathbb{U}^2_{AD,32}$

For the next example we need a classical result [24, Theorem 3]:

Lemma 3 *Assume a phase portrait of a quadratic system with two foci (or centers). The flow around these foci must always rotate in opposite clockwise sense. The same happens with two graphics which are not nested.*

Proof The proof of this lemma follows easily from the fact that a straight line passing through two singular points cannot have more contact points. □

Phase portrait $\mathbb{U}^1_{D,38}$ may produce phase portraits $\mathbb{U}^2_{AD,37}$ and $\mathbb{U}^2_{AD,38}$. Topologically, it could also produce phase portrait $\mathbb{U}^{2,I}_{AD,38}$ (see Fig. 38) but this case is impossible by Lemma 3. After bifurcation of $\mathbb{U}^2_{AD,37}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,28}$ (and

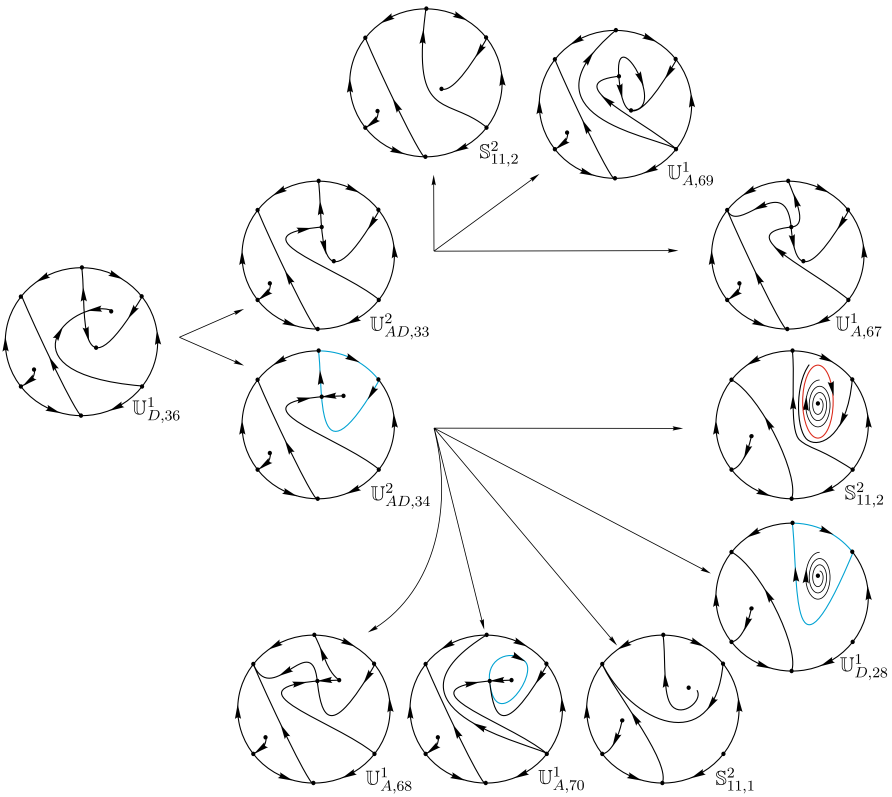


Fig. 36 Unstable phase portraits $\mathbb{U}^2_{AD,33}$ and $\mathbb{U}^2_{AD,34}$

also its possible unfoldings $\mathbb{S}^2_{11,1}$ with limit cycle and $\mathbb{S}^2_{11,2}$). By breaking just the connection in $\mathbb{U}^2_{AD,37}$ one can produce phase portraits $\mathbb{U}^1_{A,63}$ or $\mathbb{U}^1_{A,69}$ (with limit cycle). After bifurcation of $\mathbb{U}^2_{AD,38}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,29}$ (with limit cycle and also its possible unfoldings). By breaking just the connection in $\mathbb{U}^2_{AD,38}$ one produces phase portraits $\mathbb{U}^1_{A,64}$ or $\mathbb{U}^1_{A,70}$ (with limit cycle).

Phase portrait $\mathbb{U}^1_{D,39}$ may produce phase portraits $\mathbb{U}^2_{AD,39}$ and $\mathbb{U}^2_{AD,40}$ (see Fig. 39). They are very similar but not identical, and their different bifurcations will corroborate it. After bifurcation of any of them by disappearance of the saddle-node, the separatrix connection persists and we get the phase portrait $\mathbb{U}^1_{D,29}$ in the first case and $\mathbb{U}^1_{D,28}$ in the second, both without limit cycle. By breaking just the connection in $\mathbb{U}^2_{AD,39}$ one can produce phase portraits $\mathbb{U}^1_{A,67}$ or $\mathbb{U}^1_{A,68}$ (with limit cycle). By breaking just the connection in $\mathbb{U}^2_{AD,40}$ one produces the same phase portraits, but now the limit cycle is in $\mathbb{U}^1_{A,67}$.

Phase portrait $\mathbb{U}^1_{D,40}$ may produce phase portraits $\mathbb{U}^2_{AD,41}$ and $\mathbb{U}^2_{AD,42}$ (see Fig. 40). After bifurcation of any of them by disappearance of the saddle-node, the separatrix

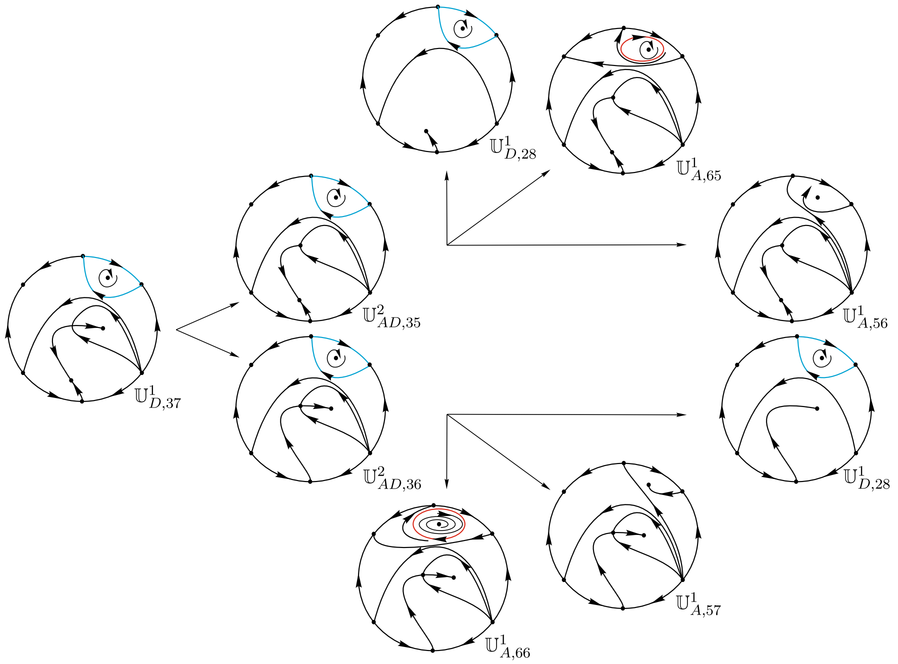


Fig. 37 Unstable phase portraits $\mathbb{U}^2_{AD,35}$ and $\mathbb{U}^2_{AD,36}$

connection vanishes and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection in $\mathbb{U}^2_{AD,41}$ one can produce phase portraits $\mathbb{U}^1_{A,31}$ or $\mathbb{U}^1_{A,28}$. By breaking just the connection in $\mathbb{U}^2_{AD,42}$ one can produce phase portraits $\mathbb{U}^1_{A,32}$ or $\mathbb{U}^1_{A,27}$.

Phase portrait $\mathbb{U}^1_{D,41}$ may produce phase portraits $\mathbb{U}^2_{AD,43}$ and $\mathbb{U}^2_{AD,44}$ (see Fig. 41). After bifurcation of $\mathbb{U}^2_{AD,43}$ by disappearance of the saddle-node, there is the possibility of the separatrix connection surviving and we get the phase portrait $\mathbb{U}^1_{D,25}$ (or its bifurcations in stable systems $\mathbb{S}^2_{9,1}$ and $\mathbb{S}^2_{9,2}$). In the case of $\mathbb{U}^2_{AD,44}$ the separatrix connection vanishes and we get $\mathbb{S}^2_{9,1}$. By breaking just the connection in $\mathbb{U}^2_{AD,43}$ one can produce phase portraits $\mathbb{U}^1_{A,31}$ or $\mathbb{U}^1_{A,34}$. By breaking just the connection in $\mathbb{U}^2_{AD,44}$ one can produce phase portraits $\mathbb{U}^1_{A,35}$ or $\mathbb{U}^1_{A,33}$.

Phase portrait $\mathbb{U}^1_{D,42}$ may produce phase portraits $\mathbb{U}^2_{AD,45}$ and $\mathbb{U}^2_{AD,46}$. Topologically, it could also produce phase portrait $\mathbb{U}^{2,I}_{AD,46}$ (see Fig. 42) but this case is impossible since the vanishing of the saddle-node would produce phase portrait $\mathbb{I}_{9,1}$. After bifurcation of $\mathbb{U}^2_{AD,45}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,52}$ or $\mathbb{U}^1_{A,42}$.

After bifurcation of $\mathbb{U}^2_{AD,46}$ by disappearance of the saddle-node, we get a tricky situation since two separatrices in $\mathbb{U}^2_{AD,46}$ arrive to the nodal part of the saddle-node, and the separatrix connection originally produced by the center manifold may persists in two different ways, to know $\mathbb{U}^1_{D,23}$ and $\mathbb{U}^1_{D,25}$ (with limit cycle). And if the connec-

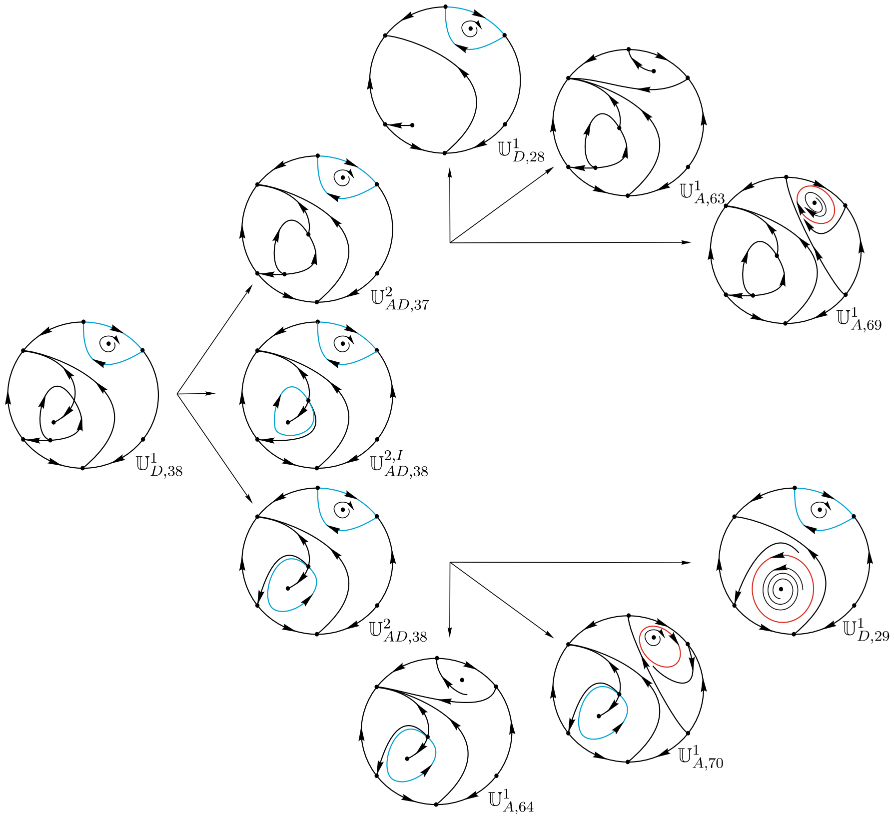


Fig. 38 Unstable phase portraits $\mathbb{U}^2_{AD,37}$, $\mathbb{U}^2_{AD,38}$ and impossible $\mathbb{U}^{2,I}_{AD,38}$

tion does not persist there are three possibilities, to know $\mathbb{S}^2_{9,2}$ (with limit cycle) if we are between the two unstable possibilities, $\mathbb{S}^2_{9,3}$ if we are beyond $\mathbb{U}^1_{D,23}$ and $\mathbb{S}^2_{9,1}$ (with limit cycle) if we are beyond $\mathbb{U}^1_{D,25}$.

By breaking just the connection in $\mathbb{U}^2_{AD,46}$ one produces phase portraits $\mathbb{U}^1_{A,51}$ or $\mathbb{U}^1_{A,43}$.

Phase portrait $\mathbb{U}^1_{D,43}$ may produce phase portraits $\mathbb{U}^2_{AD,47}$, $\mathbb{U}^2_{AD,48}$ and $\mathbb{U}^2_{AD,49}$ (see Fig. 43). After bifurcation of $\mathbb{U}^2_{AD,47}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,3}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,50}$ or $\mathbb{U}^1_{A,36}$.

After bifurcation of $\mathbb{U}^2_{AD,48}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,2}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,47}$ or $\mathbb{U}^1_{A,38}$.

After bifurcation of $\mathbb{U}^2_{AD,49}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,22}$ (and also its possible

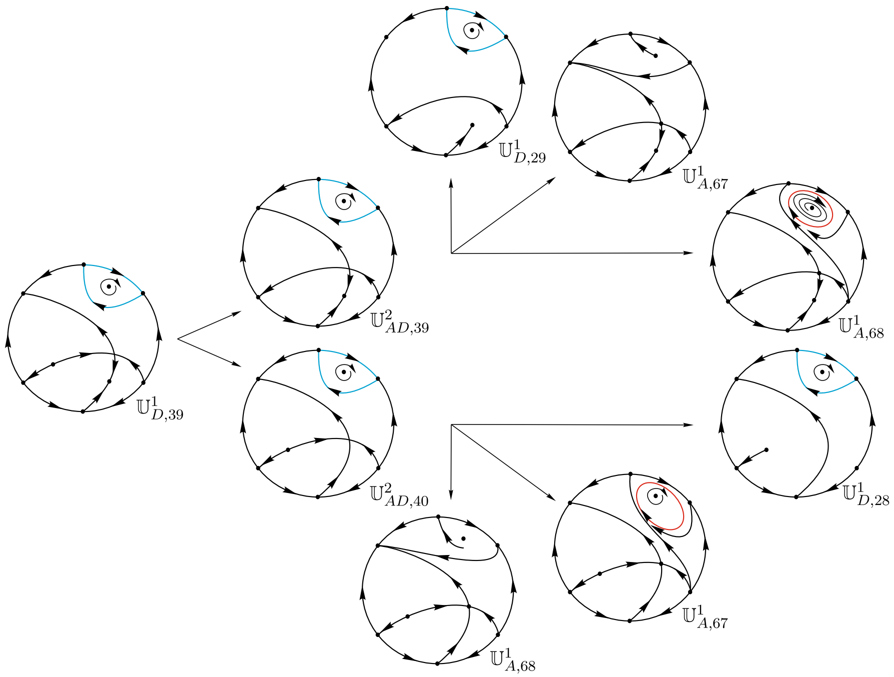


Fig. 39 Unstable phase portraits $\mathbb{U}^2_{AD,39}$ and $\mathbb{U}^2_{AD,40}$

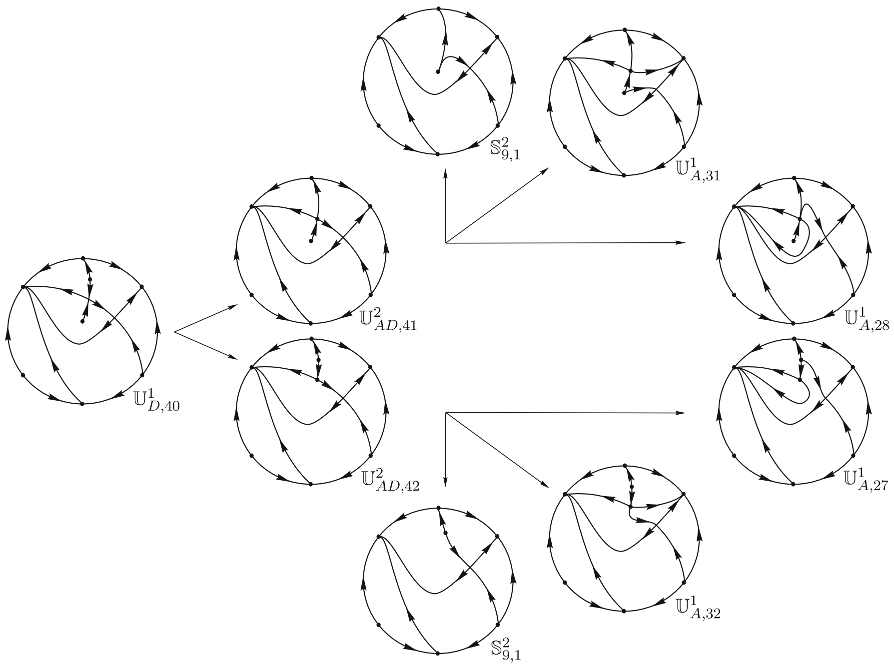


Fig. 40 Unstable phase portraits $\mathbb{U}^2_{AD,41}$ and $\mathbb{U}^2_{AD,42}$

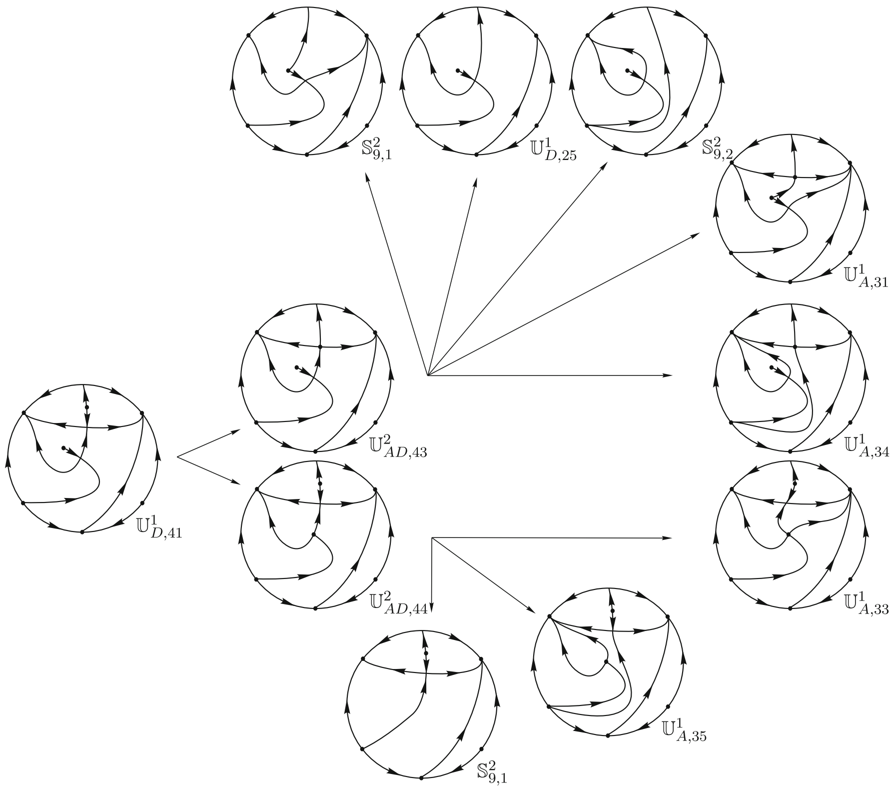


Fig. 41 Unstable phase portraits $\mathbb{U}^2_{AD,43}$ and $\mathbb{U}^2_{AD,44}$

unfoldings $\mathbb{S}^2_{9,3}$ with limit cycle or $\mathbb{S}^2_{9,2}$). By breaking just the connection in $\mathbb{U}^2_{AD,49}$ one produces phase portraits $\mathbb{U}^1_{A,37}$ or $\mathbb{U}^1_{A,48}$.

Phase portrait $\mathbb{U}^1_{D,44}$ may produce just phase portrait $\mathbb{U}^2_{AD,50}$ by symmetry (see Fig. 44). After bifurcation of $\mathbb{U}^2_{AD,50}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,54}$ or $\mathbb{U}^1_{A,55}$.

Phase portrait $\mathbb{U}^1_{D,45}$ may produce phase portraits $\mathbb{U}^2_{AD,51}$ and $\mathbb{U}^2_{AD,52}$ (see Fig. 45). After bifurcation of $\mathbb{U}^2_{AD,51}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,53}$ or $\mathbb{U}^1_{A,45}$.

After bifurcation of $\mathbb{U}^2_{AD,52}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,3}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,51}$ or $\mathbb{U}^1_{A,46}$.

Phase portrait $\mathbb{U}^1_{D,46}$ may produce phase portraits $\mathbb{U}^2_{AD,53}$ and $\mathbb{U}^2_{AD,54}$ (see Fig. 46). After bifurcation of $\mathbb{U}^2_{AD,53}$ by disappearance of the saddle-node, the separatrix con-

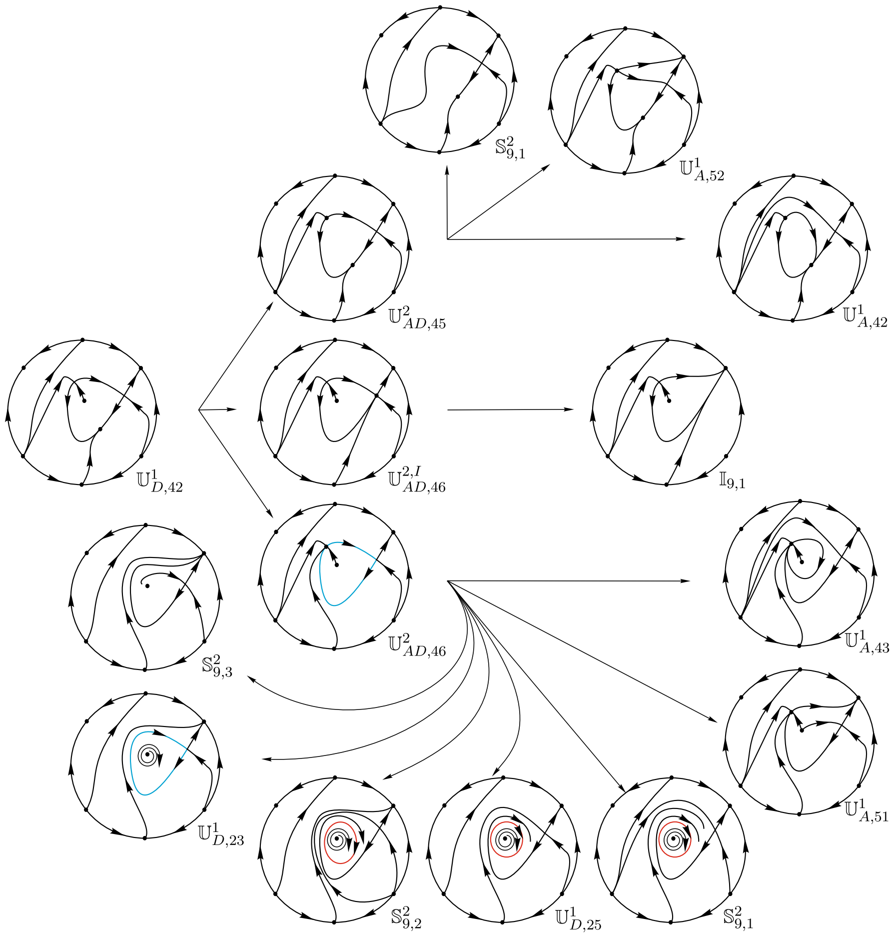


Fig. 42 Unstable phase portraits $\mathbb{U}^2_{AD,45}$, $\mathbb{U}^2_{AD,46}$ and impossible $\mathbb{U}^{2,I}_{AD,46}$

nection may remain and we get the phase portrait $\mathbb{U}^1_{D,22}$ (or its stable bifurcations $\mathbb{S}^2_{9,2}$ or $\mathbb{S}^2_{9,3}$ with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,38}$ or $\mathbb{U}^1_{A,24}$ with limit cycle.

After bifurcation of $\mathbb{U}^2_{AD,54}$ by disappearance of the saddle-node, the separatrix connection disappears and we get the phase portrait $\mathbb{S}^2_{9,3}$ with limit cycle. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,37}$ or $\mathbb{U}^1_{A,23}$ with limit cycle.

Phase portrait $\mathbb{U}^1_{D,47}$ may produce phase portrait $\mathbb{U}^{2,I}_{AD,55}$ but we conjecture it to be impossible (see Fig. 47). Even though we have not shown previously the possible bifurcation of conjectured impossible phase portraits, it is worth to do it in this case. After bifurcation of $\mathbb{U}^{2,I}_{AD,55}$ by disappearance of the saddle-node, the separatrix connection may remain and we get the phase portrait $\mathbb{U}^1_{D,23}$ (or its stable bifurcations

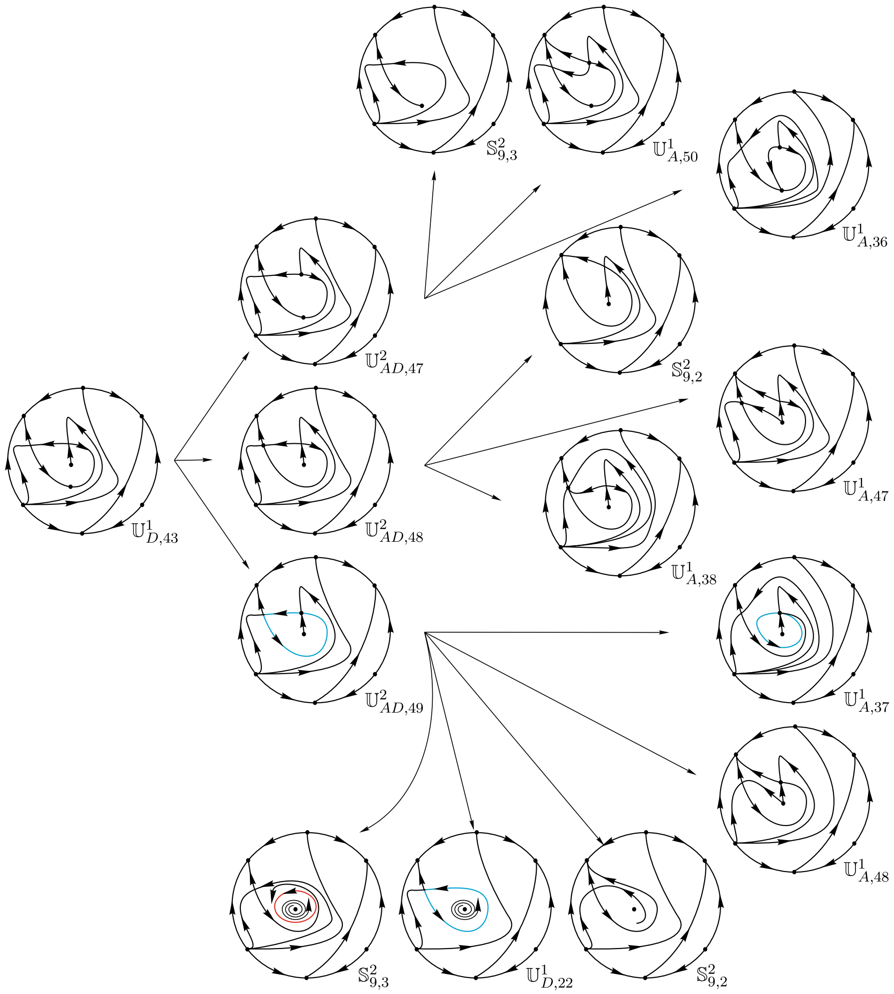


Fig. 43 Unstable phase portraits $\mathbb{U}^2_{AD,47}$, $\mathbb{U}^2_{AD,48}$ and $\mathbb{U}^2_{AD,49}$

$\mathbb{S}^2_{9,2}$ with limit cycle or $\mathbb{S}^2_{9,3}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,26}$ or $\mathbb{U}^1_{A,40}$ with limit cycle.

The fact that all the possible bifurcations from a codimension two* phase portrait may exist, it is not a proof that such portrait exist. However, if at least one of the possible topological bifurcations does not exist, it is a proof of its impossibility. Maybe in this case, it occurs that phase portrait $\mathbb{U}^1_{A,40}$ which is realizable, maybe cannot exist with limit cycle.

Phase portrait $\mathbb{U}^1_{D,48}$ (with the same skeleton as $\mathbb{U}^1_{D,47}$) may produce phase portrait $\mathbb{U}^2_{AD,55}$ (see Fig. 48). After bifurcation of $\mathbb{U}^2_{AD,55}$ by disappearance of the saddle-node, the separatrix connection may remain and we get the phase portrait $\mathbb{U}^1_{D,22}$ (or its stable bifurcations $\mathbb{S}^2_{9,2}$ or $\mathbb{S}^2_{9,3}$ with limit cycle). By breaking just the connection

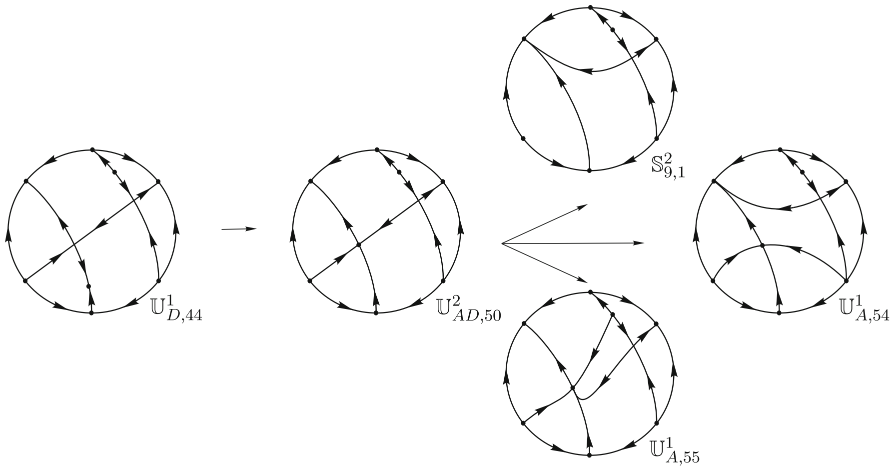


Fig. 44 Unstable phase portrait $\mathbb{U}^2_{AD,50}$

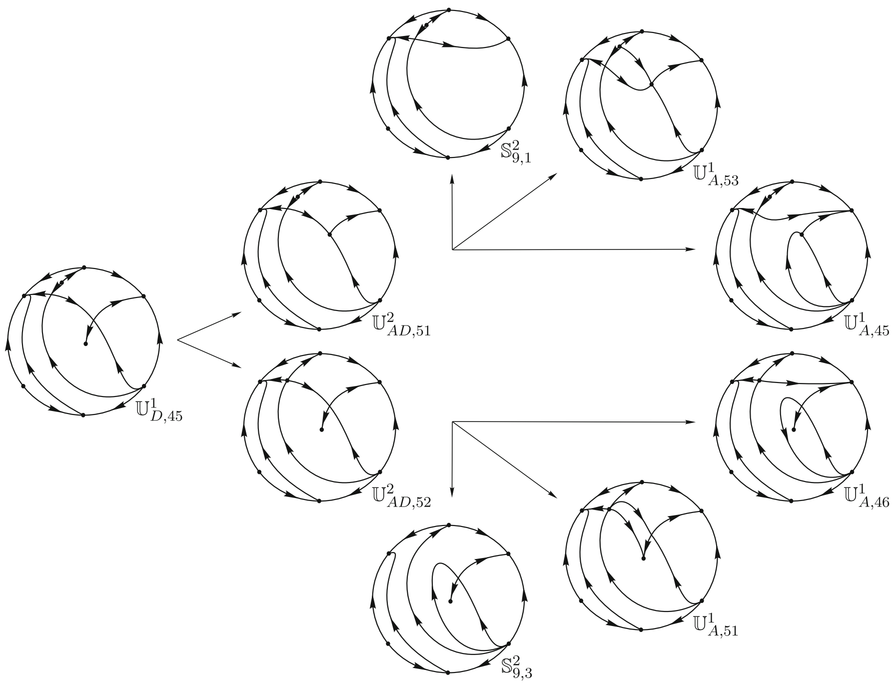


Fig. 45 Unstable phase portraits $\mathbb{U}^2_{AD,51}$ and $\mathbb{U}^2_{AD,52}$

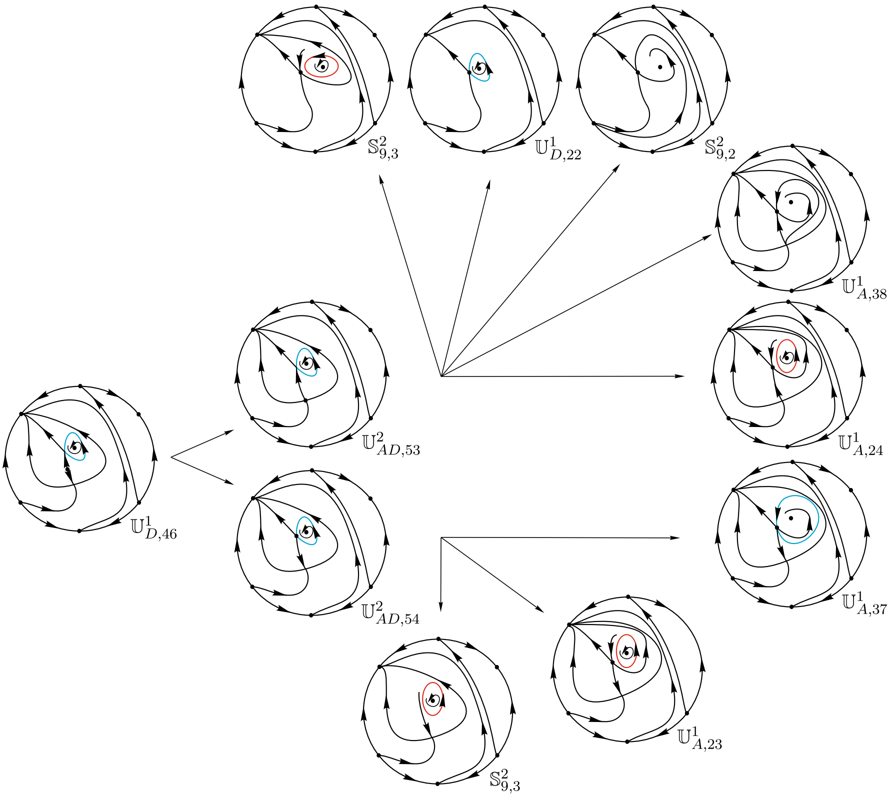


Fig. 46 Unstable phase portraits $\mathbb{U}^2_{AD,53}$ and $\mathbb{U}^2_{AD,54}$

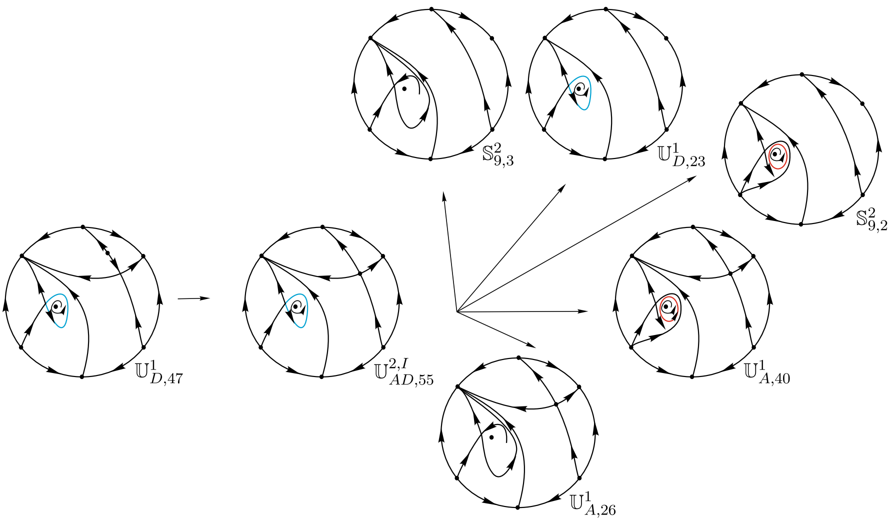


Fig. 47 Conjectured impossible phase portrait $\mathbb{U}^{2,I}_{AD,55}$

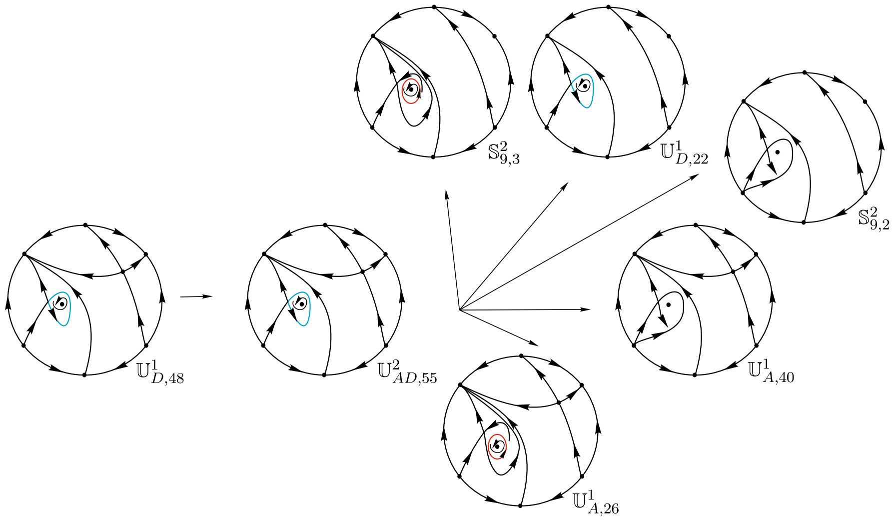


Fig. 48 Unstable phase portrait $\mathbb{U}^2_{AD,55}$

one can produce phase portraits $\mathbb{U}^1_{A,26}$ with limit cycle or $\mathbb{U}^1_{A,40}$. We may see that this case is very similar with the previous one, and only changes the stability of the focus inside the loop. And the bifurcations obtained are also very similar, with limit cycles in different cases. But when looking for examples of $\mathbb{U}^2_{AD,55}$ we have been successful while looking for $\mathbb{U}^{2,I}_{AD,55}$ not. That is, we have been able to produce $\mathbb{U}^1_{A,26}$ with limit cycle while nowhere appears $\mathbb{U}^1_{A,40}$ with it. This is clearly not a proof, but being already aware of the existences of these dual cases where one is possible, and another is not found, we believe that our conjecture is certain.

Phase portrait $\mathbb{U}^1_{D,49}$ may produce by evolution phase portrait $\mathbb{U}^{2,I}_{AD,56}$ (conjectured impossible) and impossible phase portrait $\mathbb{U}^{2,I}_{AD,56a}$ (see Fig. 49). Why $\mathbb{U}^{2,I}_{AD,56a}$ is impossible? Because after the disappearance of the saddle-node, we get a phase portrait with a loop which was not even named in [6] since it bifurcates in $\mathbb{I}_{9,1}$. However, we have not a proof of the impossibility of $\mathbb{U}^{2,I}_{AD,56}$. We have simply not found an example for it, and we have done for its dual case that we will see in the next example.

After bifurcation of $\mathbb{U}^{2,I}_{AD,56}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,43}$ with limit cycle or $\mathbb{U}^1_{A,28}$. Maybe $\mathbb{U}^1_{A,43}$ is not realizable with limit cycle.

Phase portrait $\mathbb{U}^1_{D,50}$ has the same skeleton as $\mathbb{U}^1_{D,49}$ and thus has a similar evolution. It may produce by evolution phase portrait $\mathbb{U}^2_{AD,56}$ and impossible phase portrait $\mathbb{U}^{2,I}_{AD,56b}$ (see Fig. 50). Phase portrait $\mathbb{U}^{2,I}_{AD,56b}$ is impossible for the same reason as $\mathbb{U}^{2,I}_{AD,56a}$. By the way, the names $\mathbb{U}^{2,I}_{AD,56a}$ and $\mathbb{U}^{2,I}_{AD,56b}$ are somehow artificial. The conjectured impossible phase portrait which has the same skeleton of separatrices as $\mathbb{U}^2_{AD,56}$ is $\mathbb{U}^{2,I}_{AD,56}$. Phase portraits $\mathbb{U}^{2,I}_{AD,56a}$ and $\mathbb{U}^{2,I}_{AD,56b}$ share a same skeleton

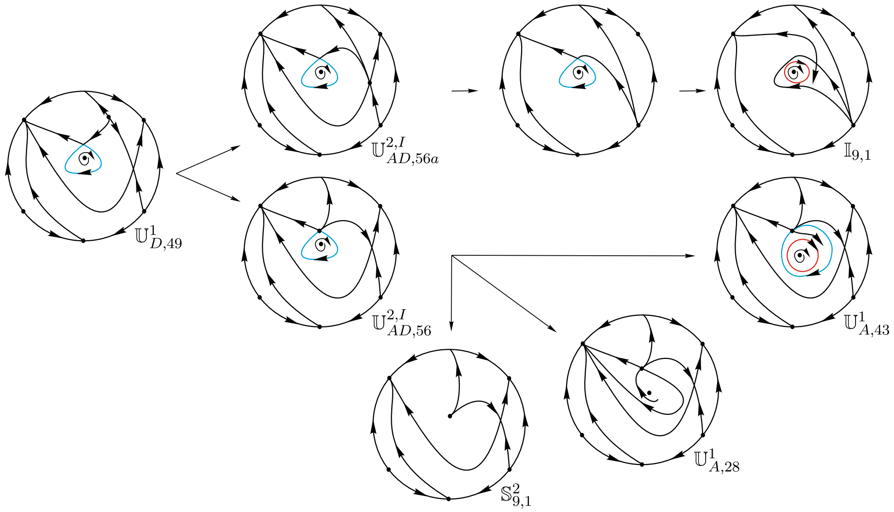


Fig. 49 Conjectured impossible phase portrait $\mathbb{U}^{2,I}_{AD,56}$ and impossible $\mathbb{U}^{2,I}_{AD,56a}$

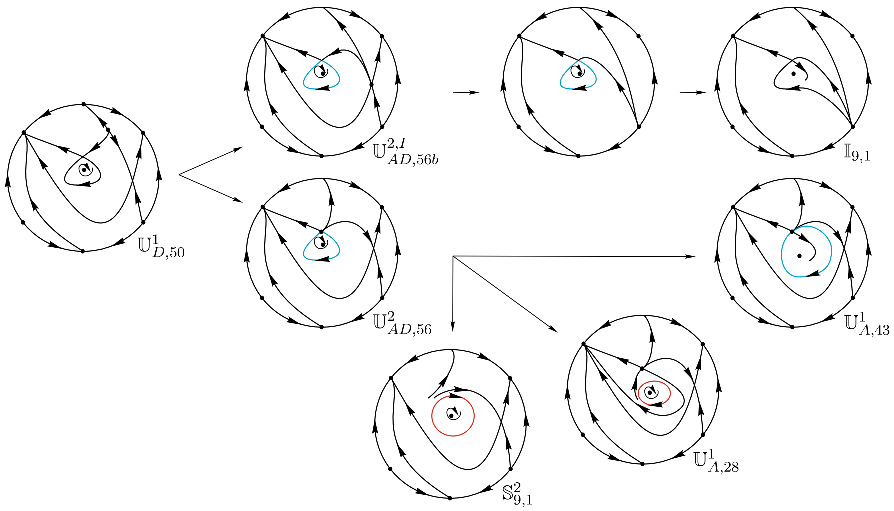


Fig. 50 Unstable phase portrait $\mathbb{U}^{2}_{AD,56}$ and impossible $\mathbb{U}^{2,I}_{AD,56b}$

(different from the one of $\mathbb{U}^{2}_{AD,56}$) and are proved both to be impossible. We just want to give a name to these impossible phase portraits for if ever we need to use them in a future paper in order to prove the impossibility of other phase portraits. And since the name $\mathbb{U}^{2,I}_{AD,56}$ is already needed to denote a conjectured impossible case, thus we give them a close name. We do not want to use a number at the end of the list like 78 for if ever a conjectured impossible phase portrait is finally found to be realizable and that number were needed.

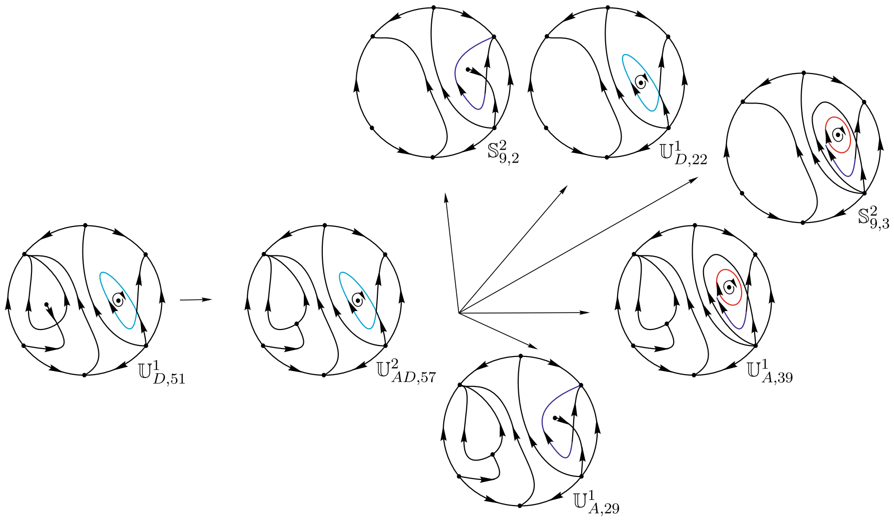


Fig. 51 Unstable phase portrait $\mathbb{U}^2_{AD,57}$

After bifurcation of $\mathbb{U}^2_{AD,56}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,1}$ with limit cycle. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,43}$ or $\mathbb{U}^1_{A,28}$ with limit cycle.

Phase portrait $\mathbb{U}^1_{D,51}$ may produce by evolution phase portrait $\mathbb{U}^2_{AD,57}$ (see Fig. 51). After bifurcation of $\mathbb{U}^2_{AD,57}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,22}$ (or its stable bifurcations $\mathbb{S}^2_{9,2}$ or $\mathbb{S}^2_{9,3}$ with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,39}$ with limit cycle or $\mathbb{U}^1_{A,29}$.

Phase portrait $\mathbb{U}^1_{D,52}$ may produce by evolution phase portrait $\mathbb{U}^2_{AD,58}$ (see Fig. 52). After bifurcation of $\mathbb{U}^2_{AD,58}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,22}$ (or its stable bifurcations $\mathbb{S}^2_{9,2}$ or $\mathbb{S}^2_{9,3}$ with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,30}$ with limit cycle or $\mathbb{U}^1_{A,44}$.

Phase portrait $\mathbb{U}^1_{D,53}$ may produce by evolution phase portrait $\mathbb{U}^2_{AD,59}$ (see Fig. 53). After bifurcation of $\mathbb{U}^2_{AD,59}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,22}$ (or its stable bifurcations $\mathbb{S}^2_{9,2}$ or $\mathbb{S}^2_{9,3}$ with limit cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,46}$ with limit cycle or $\mathbb{U}^1_{A,34}$.

Phase portrait $\mathbb{U}^1_{D,54}$ has the same skeleton as $\mathbb{U}^1_{D,53}$ and thus has a similar evolution. It may produce by evolution phase portrait $\mathbb{U}^2_{AD,60}$ (see Fig. 54). After bifurcation of $\mathbb{U}^2_{AD,60}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,23}$ (or its stable bifurcations $\mathbb{S}^2_{9,3}$ or $\mathbb{S}^2_{9,2}$ with limit

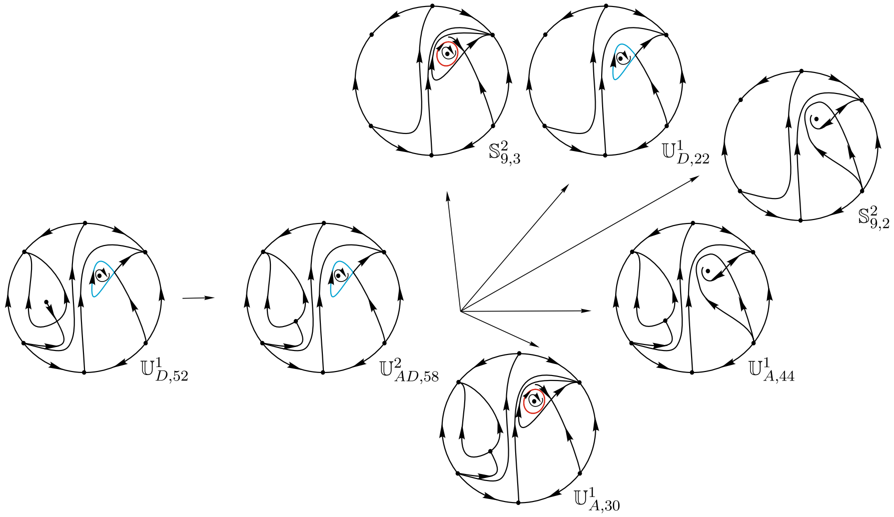


Fig. 52 Unstable phase portrait $\mathbb{U}^2_{AD,58}$

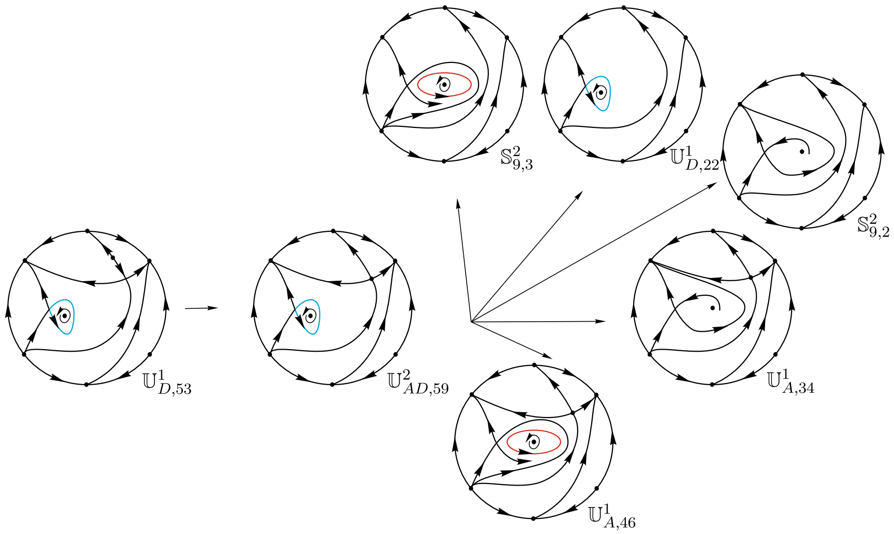


Fig. 53 Unstable phase portrait $\mathbb{U}^2_{AD,59}$

cycle). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,34}$ with limit cycle or $\mathbb{U}^1_{A,46}$.

Phase portrait $\mathbb{U}^1_{D,55}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,61}$ and $\mathbb{U}^2_{AD,62}$ (see Fig. 55). After bifurcation of $\mathbb{U}^2_{AD,61}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,26}$ (or its stable bifurcations $\mathbb{S}^2_{9,3}$ or $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,34}$ or $\mathbb{U}^1_{A,26}$.

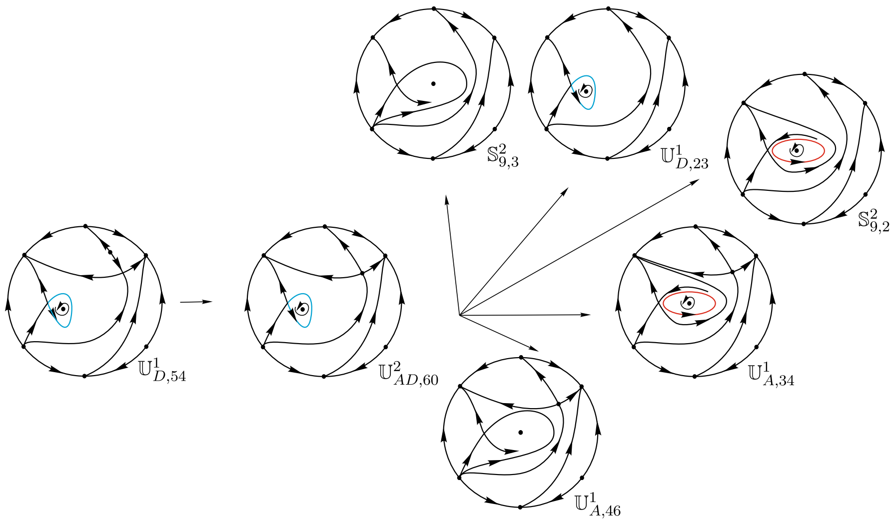


Fig. 54 Unstable phase portrait $\mathbb{U}^2_{AD,60}$

After bifurcation of $\mathbb{U}^2_{AD,62}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,24}$ (or its stable bifurcation $\mathbb{S}^2_{9,1}$ twice by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,35}$ or $\mathbb{U}^1_{A,25}$.

Phase portrait $\mathbb{U}^1_{D,56}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,63}$ and $\mathbb{U}^2_{AD,64}$ (see Fig. 56). After bifurcation of $\mathbb{U}^2_{AD,63}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,2}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,44}$ or $\mathbb{U}^1_{A,40}$.

After bifurcation of $\mathbb{U}^2_{AD,64}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,25}$ (or its stable bifurcation $\mathbb{S}^2_{9,1}$ or $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,44}$ or $\mathbb{U}^1_{A,41}$.

Phase portrait $\mathbb{U}^1_{D,57}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,65}$ and $\mathbb{U}^2_{AD,66}$ (see Fig. 57). After bifurcation of $\mathbb{U}^2_{AD,65}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,26}$ (or its stable bifurcation $\mathbb{S}^2_{9,3}$ or $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,46}$ or $\mathbb{U}^1_{A,40}$.

After bifurcation of $\mathbb{U}^2_{AD,66}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,24}$ (or its stable bifurcation $\mathbb{S}^2_{9,1}$ twice by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,41}$ or $\mathbb{U}^1_{A,45}$.

Phase portrait $\mathbb{U}^1_{D,58}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,67}$ and $\mathbb{U}^2_{AD,68}$ (see Fig. 58). After bifurcation of $\mathbb{U}^2_{AD,67}$ by disappearance of the saddle-node, the

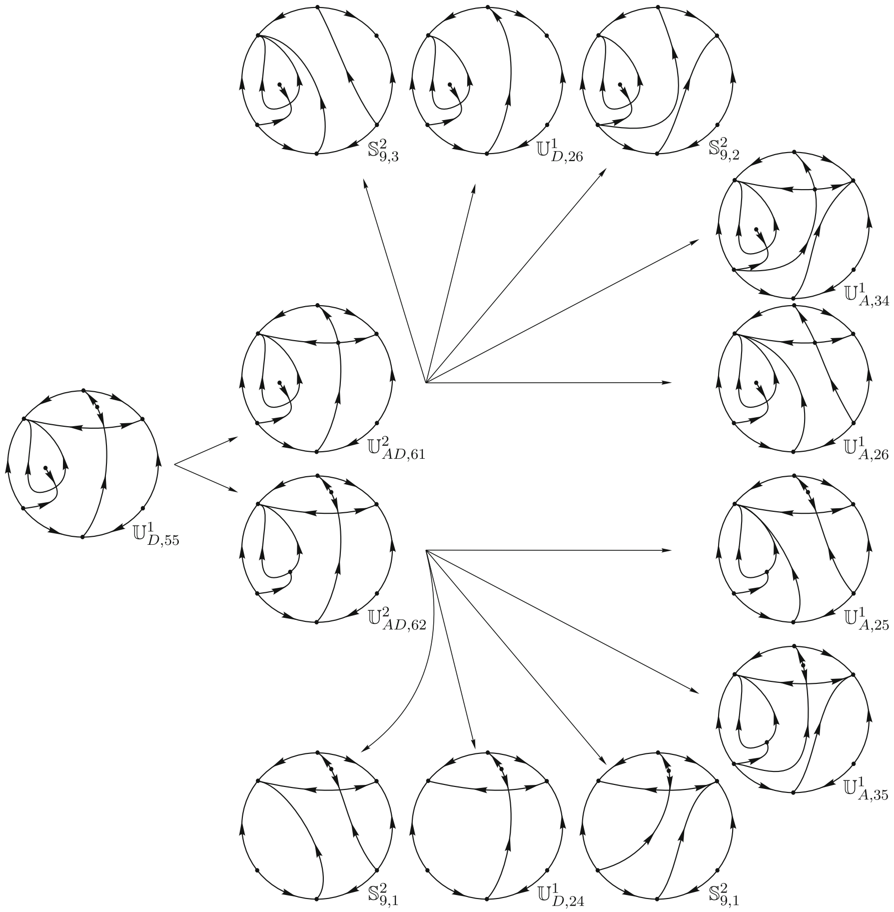


Fig. 55 Unstable phase portraits $\mathbb{U}^2_{AD,61}$ and $\mathbb{U}^2_{AD,62}$

separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,41}$ or $\mathbb{U}^1_{A,54}$.

After bifurcation of $\mathbb{U}^2_{AD,68}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,25}$ (or its stable bifurcations $\mathbb{S}^2_{9,1}$ and $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,40}$ or $\mathbb{U}^1_{A,54}$.

Phase portrait $\mathbb{U}^1_{D,59}$ has the trickiest evolution of all the cases in this study, as it usually happens with those portraits which can bifurcate into $\mathbb{S}^2_{10,13}$. Now we have two finite anti-saddles which may coalesce with two different finite saddles giving up to 4 possibilities.

Phase portrait $\mathbb{U}^1_{D,59}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,69}$, $\mathbb{U}^2_{AD,70}$ and $\mathbb{U}^2_{AD,71}$ (see Fig. 59). It could also produce $\mathbb{U}^{2,I}_{AD,69}$ but this phase portrait is not

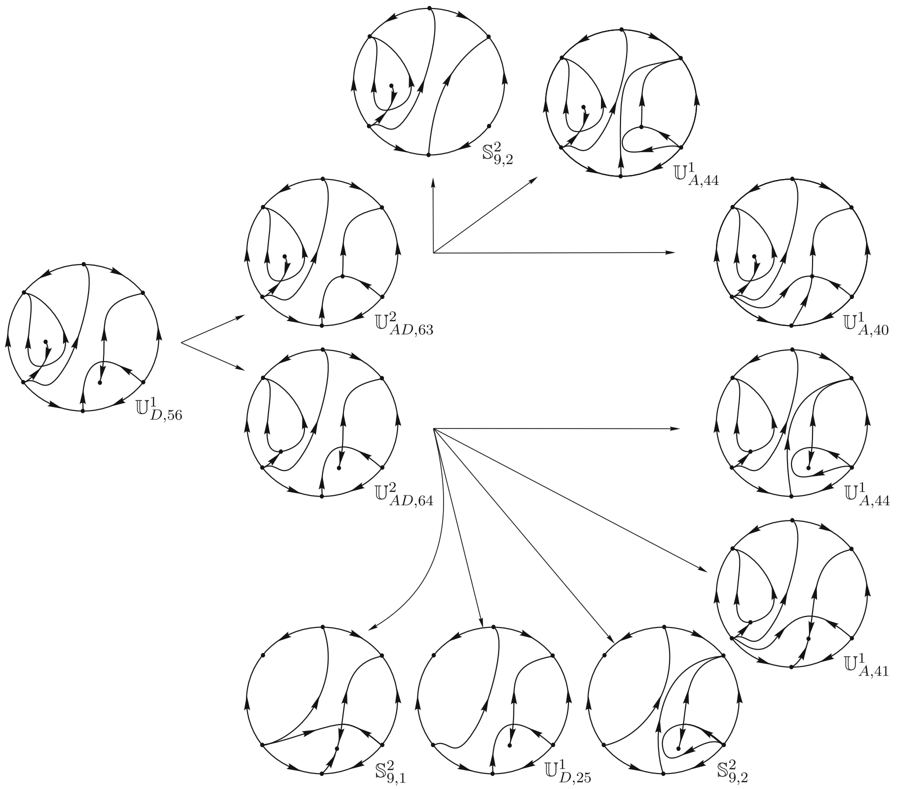


Fig. 56 Unstable phase portraits $\mathbb{U}^2_{AD,63}$ and $\mathbb{U}^2_{AD,64}$

realizable since it bifurcates into $\mathbb{U}^{1,I}_{A,49}$ which even though was shown as realizable in [6], it was finally proved impossible in [15].

After bifurcation of $\mathbb{U}^2_{AD,69}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,3}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,50}$ or $\mathbb{U}^1_{A,51}$.

After bifurcation of $\mathbb{U}^2_{AD,70}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,25}$ (or its stable bifurcations $\mathbb{S}^2_{9,1}$ and $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,47}$ or $\mathbb{U}^1_{A,53}$.

After bifurcation of $\mathbb{U}^2_{AD,71}$ by disappearance of the saddle-node, the separatrix connection may persist and we get again the phase portrait $\mathbb{U}^1_{D,25}$ (or its stable bifurcations $\mathbb{S}^2_{9,1}$ and $\mathbb{S}^2_{9,2}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,48}$ or $\mathbb{U}^1_{A,52}$.

The picture of phase portrait $\mathbb{U}^1_{D,60}$ in [6] is topologically right, but not geometrically. The separatrix connection must be part of an invariant straight line as by Corollary 3.6 of [6]. We draw it geometrically right in Fig. 60.

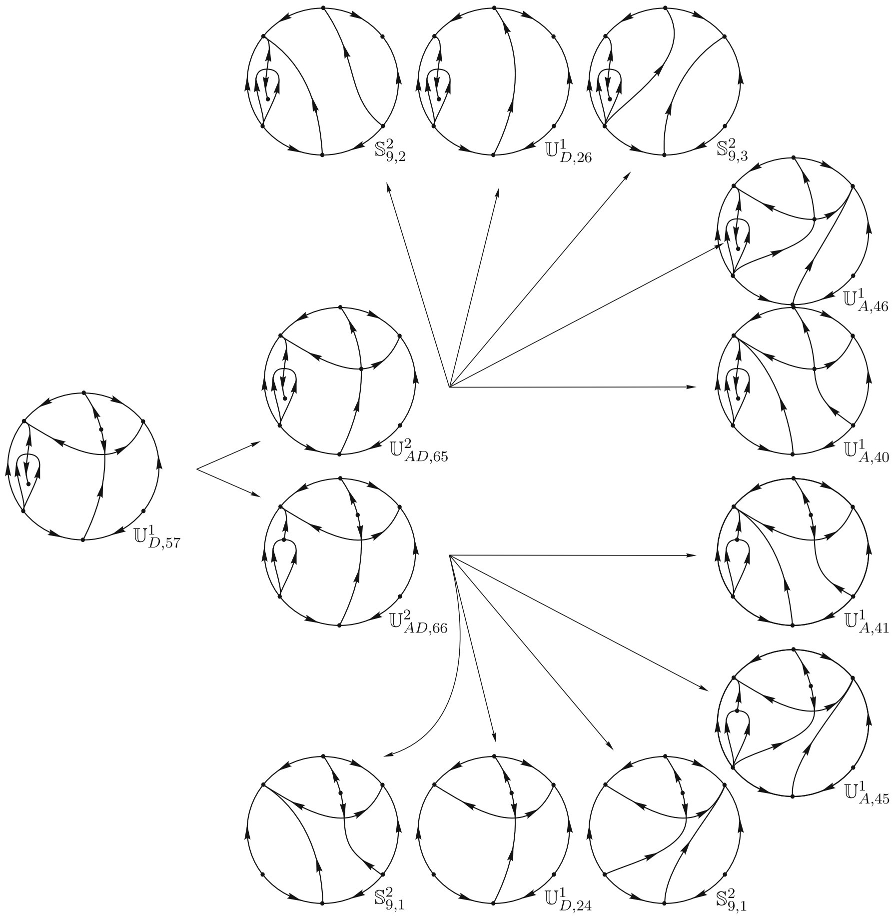


Fig. 57 Unstable phase portraits $\mathbb{U}^2_{AD,65}$ and $\mathbb{U}^2_{AD,66}$

Phase portrait $\mathbb{U}^1_{D,60}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,72}$, $\mathbb{U}^2_{AD,73}$ and $\mathbb{U}^2_{AD,74}$ (see Fig. 60). It could also produce $\mathbb{U}^{2,I}_{AD,74}$ but this phase portrait is not realizable since it bifurcates into $\mathbb{U}^1_{I,19}$ (see [6, 15]).

After bifurcation of $\mathbb{U}^2_{AD,72}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,1}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,52}$ or $\mathbb{U}^1_{A,55}$.

After bifurcation of $\mathbb{U}^2_{AD,73}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,24}$ or its stable bifurcation $\mathbb{S}^2_{9,1}$ (only one by symmetry). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,53}$ or $\mathbb{U}^1_{A,55}$.

After bifurcation of $\mathbb{U}^2_{AD,74}$ by disappearance of the saddle-node, a separatrix connection may persist in two different ways and we get again the phase portrait $\mathbb{U}^1_{D,25}$ or

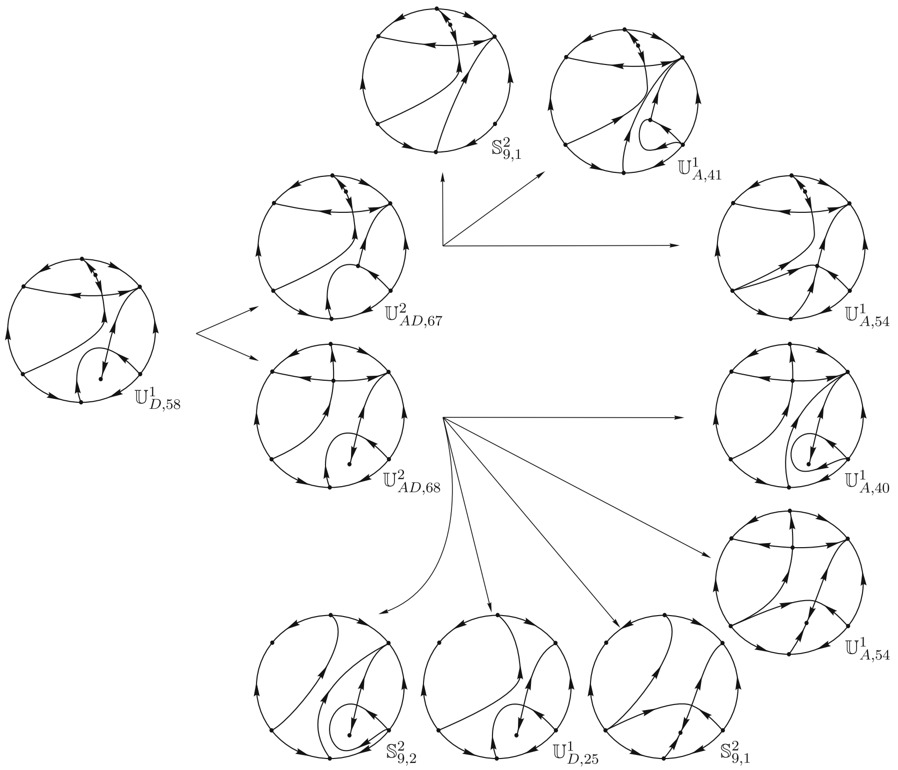


Fig. 58 Unstable phase portraits $\mathbb{U}^2_{AD,67}$ and $\mathbb{U}^2_{AD,68}$

$\mathbb{U}^1_{D,26}$ (or its stable bifurcations $\mathbb{S}^2_{9,1}$, $\mathbb{S}^2_{9,2}$ and $\mathbb{S}^2_{9,3}$). By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,51}$ or $\mathbb{U}^1_{A,55}$.

Phase portrait $\mathbb{U}^1_{D,61}$ may produce by evolution phase portraits $\mathbb{U}^2_{AD,75}$ and $\mathbb{U}^2_{AD,76}$ (see Fig. 61). After bifurcation of $\mathbb{U}^2_{AD,75}$ by disappearance of the saddle-node, the separatrix connection does not persist and we get the phase portrait $\mathbb{S}^2_{9,3}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,26}$ or $\mathbb{U}^1_{A,30}$.

After bifurcation of $\mathbb{U}^2_{AD,76}$ by disappearance of the saddle-node, the separatrix connection may persist and we get the phase portrait $\mathbb{U}^1_{D,25}$ or its stable bifurcations $\mathbb{S}^2_{9,1}$ and $\mathbb{S}^2_{9,2}$. By breaking just the connection one can produce phase portraits $\mathbb{U}^1_{A,25}$ or $\mathbb{U}^1_{A,29}$.

6 Proof of Theorem 3: The Realizable Phase Portraits

Now we will give examples of all realizable structurally unstable phase portraits of codimension two* for quadratic systems of class (AD). In this case, there is no studied global family with these unstabilities. In fact, there is no global form which may

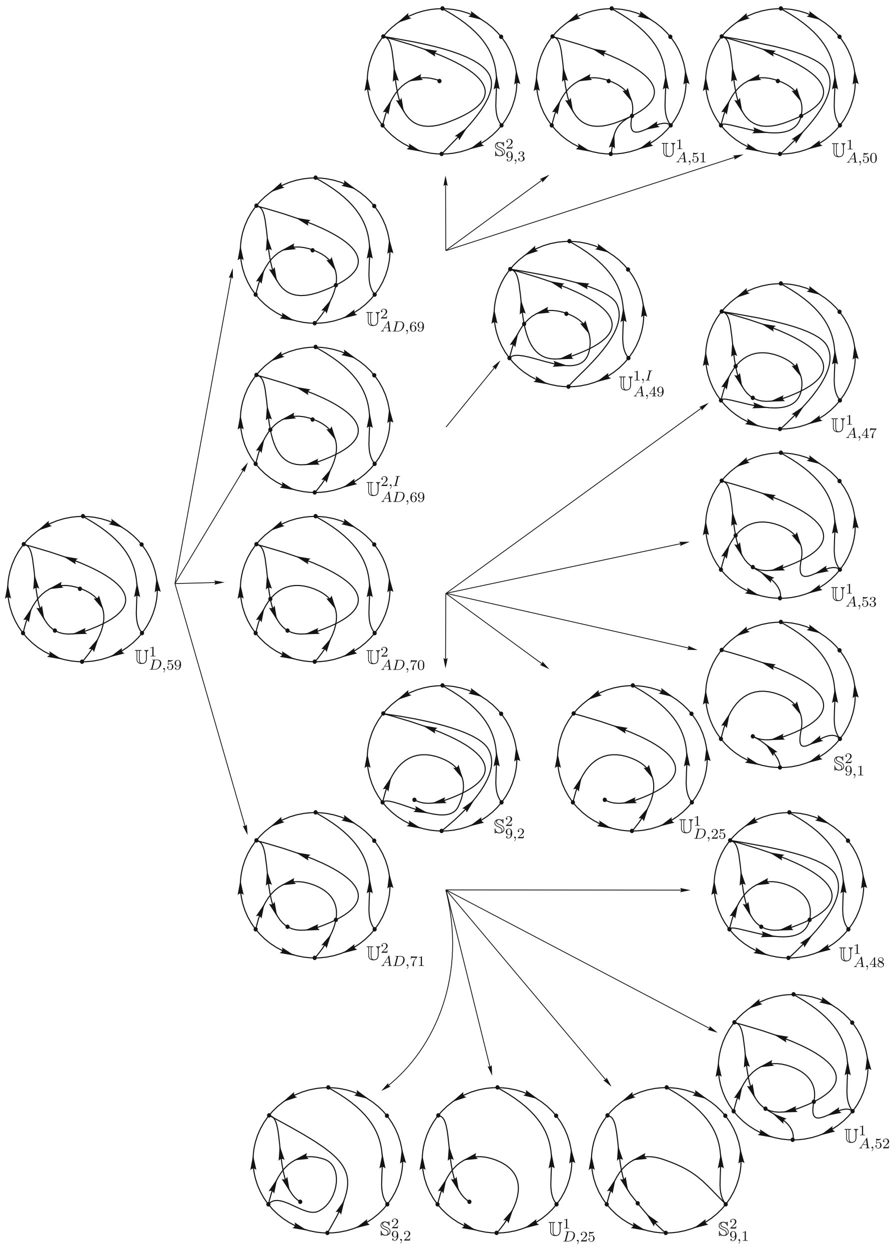


Fig. 59 Unstable phase portraits $\mathbb{U}^2_{AD,69}$, $\mathbb{U}^2_{AD,70}$ and $\mathbb{U}^2_{AD,71}$

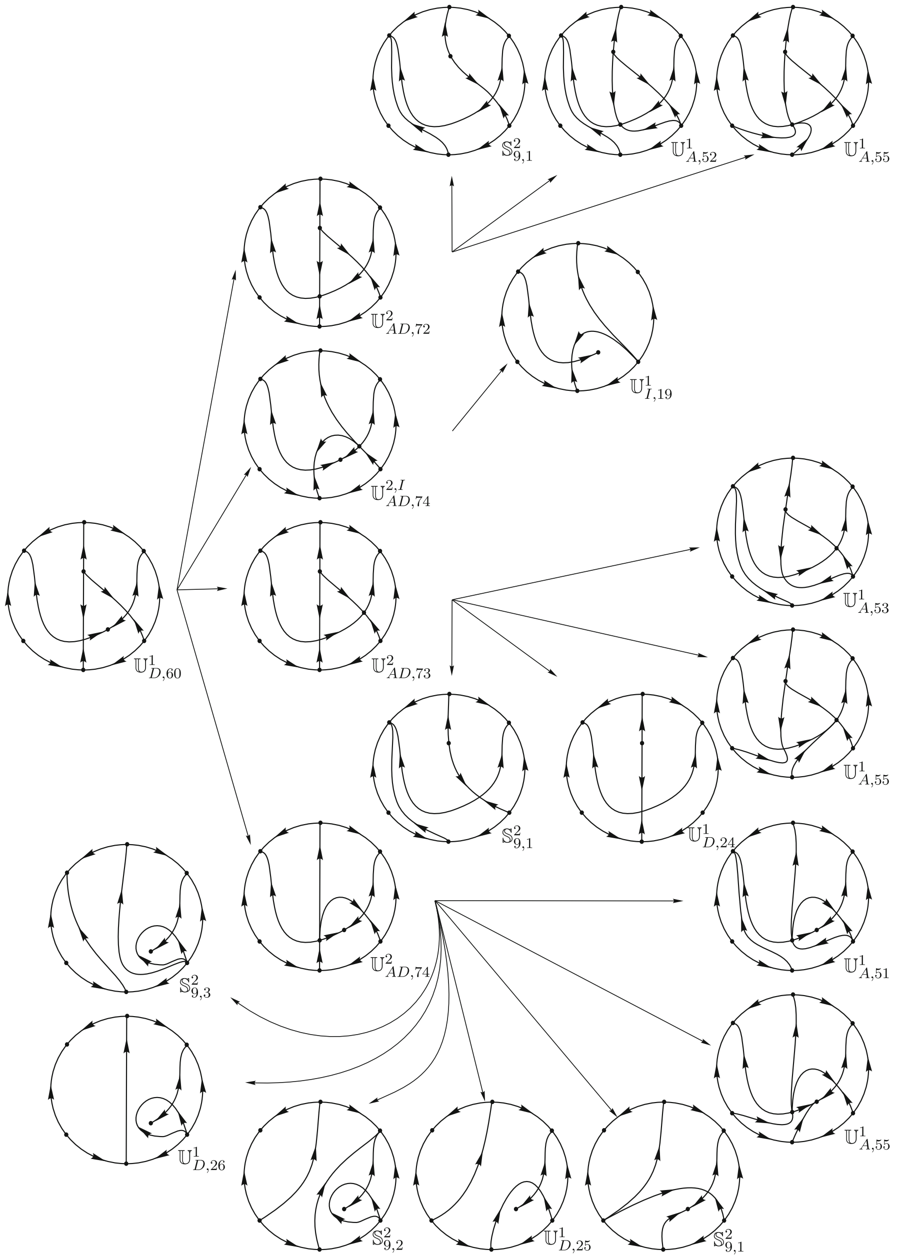


Fig. 60 Unstable phase portraits $\mathbb{U}^2_{AD,72}$, $\mathbb{U}^2_{AD,73}$ and $\mathbb{U}^2_{AD,74}$

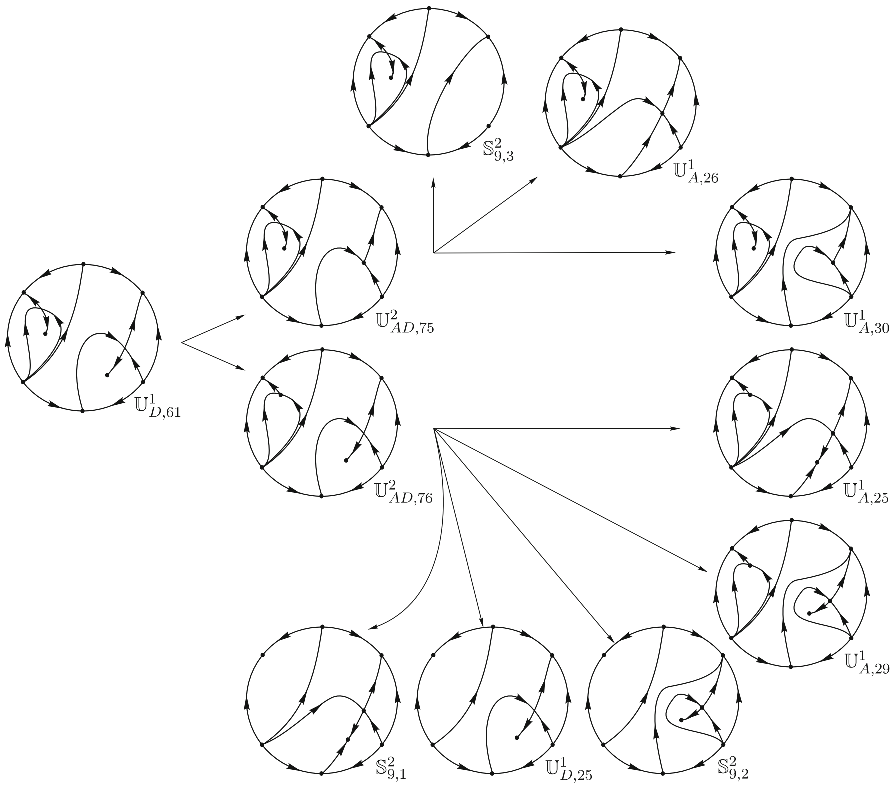


Fig. 61 Unstable phase portraits $\mathbb{U}^2_{AD,75}$ and $\mathbb{U}^2_{AD,76}$

encapsulate these two unstabilities since one of them is not completely algebraic. And it is not yet done the study of the family of systems with a finite saddle-node (this implies a study in dimension 5). So, we will need to rely on different studied families of codimension two with two unstable objects related with singularities and look for the cases where a separatrix connection occurs. Concretely we will take some families having a finite saddle-node and an infinite saddle-node $\overline{\binom{0}{2}} SN$ and families which apart from the finite saddle-node have an infinite saddle-node $\overline{\binom{1}{1}} SN$. These families are studied in two papers each, [17] and [18] for the first case and [12] and [13] for the second. All four papers are done using similar techniques, and the notation used to describe the phase portraits is similar. In them, the surfaces of the bifurcation space dealing with invariant straight lines are denoted as $4S$ and the surfaces dealing with non-algebraic separatrix connections are denoted as $7S$. So the examples we will extract from them will have mainly these notations. It is worth noting the importance of these works since they show more than 115 different phase portraits having a finite saddle-node plus a separatrix connection and one infinite saddle-node. So, by breaking the infinite saddle-node into two singularities, or making it disappear, we have lots of candidates of the wished class to look for examples. We have checked every example of the four papers having a separatrix connection. So, we are sure (modulo "islands") that the phase portraits of the class (AD) which have not appeared, cannot be inside the families studied in those papers.

There are two other papers with studies of families of quadratic systems of codimension two which have been checked. These are the study of quadratic systems with a triple semi-elemental node [16] and with a triple semi-elemental saddle [19]. But the first has just one candidate to produce elements of the class (AD) and clearly it does not lead to any of the conjectured impossible phase portraits. The second paper shows three candidates from which phase portraits of class (AD) could be obtained, but only one needs to be studied carefully so to discard that it could generate any of the conjectured impossible phase portraits. This will be done in Sect. 6.6.

A very recent paper [20] has completed the bifurcation diagram of all quadratic systems having a finite saddle-node and a weak focus of first order. This is another 4-parameter family which has produced 192 topologically different phase portraits, of which 30 have a separatrix connection (and no other extra unstable object). After checking all of them, we have found a phase portrait which we were close to conjecture as impossible, but has finally become realizable. We will see this in Sect. 6.5.

With all these families (from which examples of class (AD) may be obtained) already studied, there are still some families of codimension two with a finite saddle-node to be studied so to provide new examples with a separatrix connection which could cover some of the conjectured impossible phase portraits. In order to obtain such a codimension two family, we must fix some property apart from the finite saddle-node. We can think in either a weak finite saddle, a weak infinite saddle, a finite one direction node n^d or an infinite one direction node N^d. These families are not very interesting from the geometrical point of view, but just the possibility of finding in them one of the conjectured impossible phase portraits of this class makes worth their study. Of course, the most interesting family to be studied would be the codimension one case of having just a finite saddle-node, but this is a too big family with the current tools. Anyway we have decided to proceed with the publication of this paper leaving the conjecture as it is, and excitedly waiting if someone can corroborate it or enlarge the number of 77 found phase portraits.

Of course, there is also the remote possibility that a conjectured impossible case could live in an "island" of the parameter spaces that these papers describe (but it have never been found).

Please note that several mistypes were detected in [18] and many of them were corrected in the Appendix A of [13]. In fact here we have found another mistype in the main theorem of [18] where a phase portrait drawn as a focus corresponds to a region where the anti-saddle is instead a node. This is not a big problem according to a topological classification but may be a small nuisance since the phase portraits look geometrically different. Anyway we have preferred to mention the name of the phase portrait which appears in the main theorem of those papers instead of mentioning the concrete region of the parameter space in which the focus exists so to avoid that the reader which wants to check these results, needs also to get deeper in those papers.

We will do some examples with detail, and we will add a list with the rest of phase portraits that can be obtained in a similar way. Most of the examples are given in P4 format at the link "http://mat.uab.cat/~artes/articles/P4su2AD.zip". Some examples rely on phase portraits which already need the use of very small parameters (or parameters in a very narrow interval), and so, the perturbations needed must be even smaller. This makes that the use of the numerical program P4 becomes less conclusive, and we simply apply the continuity criteria in the bifurcation. Regarding the numerical examples in P4 we must point out that we have not bifurcated every single phase portrait from the mentioned papers but there is at least one example for every phase portrait of the main theorem. In the cases where a numerical parameter is needed for a rotated vector field to recover a non-algebraic separatrix connection, the example we give in the file is an approximate value for which we have checked that such parameter modified one unit of the last digit (plus or minus) produced the bifurcation on the other sense. For example, if we see a P4 file with an "alfa" parameter being 0.0032 it means that either for $\alpha = 0.0033$ or $\alpha = 0.0031$ we obtain the other phase portrait. Assume it is for $\alpha = 0.0033$, then by continuity, there exists $\alpha^* \in (0.0032, 0.0033)$ for which the phase portrait with the separatrix connection occurs.

In the examples where we have an invariant straight line, it is relatively easy to bifurcate the infinite saddle-node while conserving the straight line and thus, we obtain directly the desired phase portrait without using a rotated vector field. In cases where the separatrix connection is not algebraic, we will not even have an exact parameter set from which we could affirm the existence of the original phase portrait. But we will have an interval (in a certain parameter) for which such phase portrait exists. Then we will be able to break the saddle-node, and this for sure will also affect the separatrix connection, but by means of a rotated vector field, we will be able to recover the connection without affecting the rest of singular points.

6.1 Examples Obtained from [13] with an Invariant Straight Line

In [13] it is proved that any quadratic system with a finite saddle-node, another finite singularity and an infinite $\overline{\binom{1}{1}}SN$ can be moved into the normal form:

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy, \\ \dot{y} &= ex + ey - ex^2 + 2mxy. \end{aligned} \tag{4}$$

The origin is a $\overline{sn}_{(2)}$ and the infinite singularity $[0 : 1 : 0]$ is a $\overline{\binom{1}{1}}SN$ (requires $h \neq 0$). Assume now that we have a system (4) with a separatrix connection on an invariant straight line. Now we make a perturbation as

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy, \\ \dot{y} &= ex + ey - ex^2 + 2mxy + \varepsilon y^2. \end{aligned} \tag{5}$$

The Jacobian matrix of the infinite singularity $[0 : 1 : 0]$ has eigenvalues $2h - \varepsilon$ and $-\varepsilon$. Whether we take ε positive or negative, we will change the sign of the determinant of the Jacobian. That is, if we take ε such that $-(2h - \varepsilon)\varepsilon < 0$ (respectively > 0) the perturbation will leave a saddle at infinity (respectively a node). If the separatrix connection is on the horizontal or vertical axis, then it will persist. If the connection is on an oblique straight line, then before the perturbation we make the change $(x, y) \rightarrow$

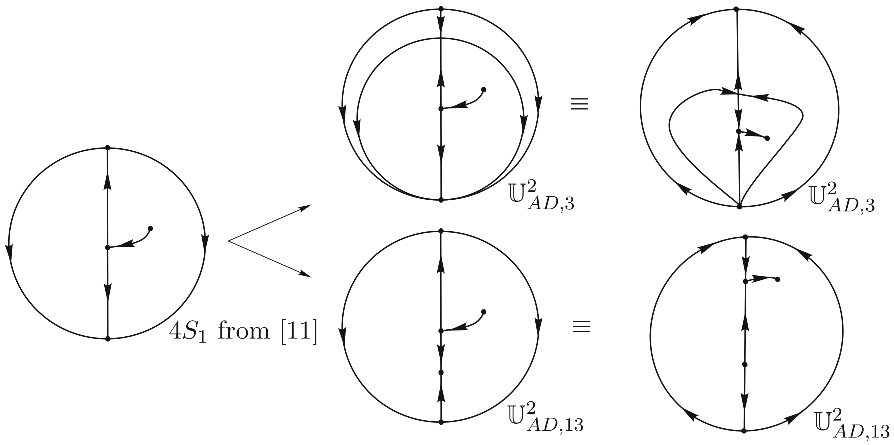


Fig. 62 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,3}$ and $\mathbb{U}^2_{AD,13}$

$(x, kx + y)$ (this is called a k-twist in [9]) so to locate the invariant straight line at the horizontal position while maintaining the infinite saddle-node unmoved. Otherwise, the perturbation would break also the separatrix connection and we would not get the desired phase portraits.

We may start then with the first detailed example. Take system (5) with $(c, e, h, m) = (0, 10, 1, 4)$ and $\varepsilon = 0$. This system has phase portrait $4S_1$ from [13] (see Fig. 62). If we perturb $\varepsilon \neq 0$ the vertical invariant straight line (which is a connection of separatrices) persists. If $\varepsilon < 0$ the point $[0 : 1 : 0]$ is a node and a finite saddle has appeared on the y-axis for y a large positive value. Then we obtain phase portrait $\mathbb{U}^2_{AD,3}$. If instead of $\varepsilon < 0$ we take $\varepsilon > 0$, the vertical invariant straight line is again a connection of separatrices which persists, the point $[0 : 1 : 0]$ is a saddle and a finite attractor node has appeared on the y-axis for y a large negative value. Trivially we obtain phase portrait $\mathbb{U}^2_{AD,13}$. In Fig. 62 we show the bifurcation of $4S_1$ from [13] into $\mathbb{U}^2_{AD,3}$ and $\mathbb{U}^2_{AD,13}$ and we also show the version of these phase portraits from Fig. 1 with some rotation and/or symmetry to check that they are the same (there may remain the need of a time change).

In a very similar way we may obtain

$\mathbb{U}^2_{AD,30}$ and $\mathbb{U}^2_{AD,42}$ from $4S_2$, $\mathbb{U}^2_{AD,16}$ and $\mathbb{U}^2_{AD,72}$ from $4S_3$,
$\mathbb{U}^2_{AD,17}$ and $\mathbb{U}^2_{AD,42}$ from $4S_6$, $\mathbb{U}^2_{AD,30}$ and $\mathbb{U}^2_{AD,72}$ from $4S_8$,
$\mathbb{U}^2_{AD,15}$ and $\mathbb{U}^2_{AD,73}$ from $4S_{17}$, $\mathbb{U}^2_{AD,27}$ and $\mathbb{U}^2_{AD,74}$ from $4S_{22}$,
$\mathbb{U}^2_{AD,26}$ and $\mathbb{U}^2_{AD,72}$ from $4S_{33}$, $\mathbb{U}^2_{AD,16}$ and $\mathbb{U}^2_{AD,50}$ from $4S_{34}$,
$\mathbb{U}^2_{AD,17}$ and $\mathbb{U}^2_{AD,51}$ from $4S_{37}$, $\mathbb{U}^2_{AD,26}$ and $\mathbb{U}^2_{AD,50}$ from $4S_{40}$,
$\mathbb{U}^2_{AD,19}$ and $\mathbb{U}^2_{AD,66}$ from $4S_{42}$, $\mathbb{U}^2_{AD,19}$ and $\mathbb{U}^2_{AD,62}$ from $4S_{50}$,
$\mathbb{U}^2_{AD,27}$ and $\mathbb{U}^2_{AD,61}$ from $4S_{59}$, $\mathbb{U}^2_{AD,32}$ and $\mathbb{U}^2_{AD,74}$ from $4S_{65}$
and $\mathbb{U}^2_{AD,31}$ and $\mathbb{U}^2_{AD,41}$ from $4S_{70}$.

We remark that we have obtained several repeated cases and the reason is that we have checked all possible bifurcations coming from every phase portrait in [13] with a separatrix connection being an invariant straight line since we are not only interested in finding all the phase portraits from Theorem 3, but also be sure that none of the conjectured impossible phase portraits (see Conjecture 1) may be obtained from this class. The same will happen in the next classes we will study.

6.2 Examples Obtained from [13] with No Invariant Straight Line

Now we start describing some cases where the separatrix connection is not algebraic. We start again from system (5) and we choose phase portrait $7S_8$ from [13]. We can find a representative of this phase portrait in system (5) with the parameters $(c, h, m, \varepsilon, \alpha) = (1, 1, 0, 0, 0)$ and $e = e^* \in (0.62, 0.64)$. For $e = 0.62$ the phase portrait is V_{53} (without limit cycle) and for $e = 0.64$ the phase portrait is V_{52} (with limit cycle). So, by continuity, there must exist a value e^* in that interval for which we obtain the loop.

Now we perturb the system taking $\varepsilon = -0.1$. The infinite saddle-node at $[0 : -1 : 0]$ bifurcates and ejects a node into the negative y semi-plane, remaining the infinite singularity as a saddle. Visually in program P4 the loop looks exactly the same as it was without the perturbation, but since we were not even sure of the exact value of e^* for which the loop existed, we can neither be sure now that the loop exists. So we must prove that we can recover the loop connection with a sufficiently small change that does not affects the singular points.

So we make a rotated vector field like

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy, \\ \dot{y} &= ex + ey - ex^2 + 2mxy + \varepsilon y^2 + \alpha(cx + cy - cx^2 + 2hxy). \end{aligned} \tag{6}$$

Sometimes we may prefer to use the rotated vector field

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy + \alpha(ex + ey - ex^2 + 2mxy + \varepsilon y^2), \\ \dot{y} &= ex + ey - ex^2 + 2mxy + \varepsilon y^2. \end{aligned} \tag{7}$$

By using of the parameter α it is simple to confirm that there is no limit cycle for $\alpha = 0.01$ and there is limit cycle if $\alpha = -0.01$. The finite singularities remain unaffected because this is a rotated vector field, and the rotation has been small enough so not to affect the infinite singularities. So, by continuity, there must exist a value $\alpha^* \in (-0.01, 0.01)$ for which the loop exists and thus we get phase portrait $\mathbb{U}^2_{AD,12}$. In the same way, if we make first the perturbation $\varepsilon = 0.1$, the infinite saddle-node at $[0 : 1 : 0]$ bifurcates and ejects a saddle into the positive y semi-plane, and again the separatrix connection may be recovered with the use of α obtaining phase portrait $\mathbb{U}^2_{AD,9}$, see Fig. 63.

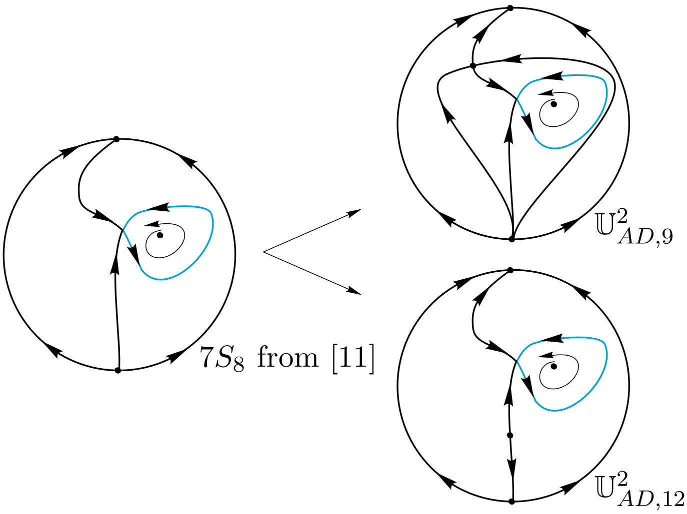


Fig. 63 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,9}$ and $\mathbb{U}^2_{AD,12}$

In a very similar way we may obtain

$\mathbb{U}^2_{AD,18}$ and $\mathbb{U}^2_{AD,45}$ from $7S_1$, $\mathbb{U}^2_{AD,23}$ and $\mathbb{U}^2_{AD,56}$ from $7S_3$,
$\mathbb{U}^2_{AD,29}$ and $\mathbb{U}^2_{AD,46}$ from $7S_4$, $\mathbb{U}^2_{AD,33}$ and $\mathbb{U}^2_{AD,69}$ from $7S_5$,
$\mathbb{U}^2_{AD,25}$ and $\mathbb{U}^2_{AD,54}$ from $7S_6$, $\mathbb{U}^2_{AD,34}$, $\mathbb{U}^2_{AD,49}$ and $\mathbb{U}^2_{AD,71}$ from $7S_7$,
$\mathbb{U}^2_{AD,28}$ and $\mathbb{U}^2_{AD,67}$ from $7S_9$, $\mathbb{U}^2_{AD,35}$ and $\mathbb{U}^2_{AD,76}$ from $7S_{11}$,
$\mathbb{U}^2_{AD,18}$ and $\mathbb{U}^2_{AD,67}$ from $7S_{12}$, $\mathbb{U}^2_{AD,37}$ and $\mathbb{U}^2_{AD,64}$ from $7S_{13}$,
$\mathbb{U}^2_{AD,40}$ and $\mathbb{U}^2_{AD,68}$ from $7S_{14}$, $\mathbb{U}^2_{AD,33}$ and $\mathbb{U}^2_{AD,75}$ from $7S_{17}$,
$\mathbb{U}^2_{AD,28}$ and $\mathbb{U}^2_{AD,45}$ from $7S_{18}$, $\mathbb{U}^2_{AD,18}$ and $\mathbb{U}^2_{AD,44}$ from $7S_{21}$
and $\mathbb{U}^2_{AD,24}$ and $\mathbb{U}^2_{AD,56}$ from $7S_{22}$.

The needed perturbation values of ε and α may be different from case to case.

Notice that $7S_7$ may bifurcate in 3 different ways. When a finite saddle bifurcates from $[0:1:0]$ we can maintain the separatrix connection of the finite saddle-node with the infinite saddle as we had in $7S_7$ and obtain $\mathbb{U}^2_{AD,71}$ or we can maintain the connection with the separatrix of the new finite saddle and produce a new graphic. In this case we obtain $\mathbb{U}^2_{AD,49}$, see Fig. 64. When a finite node bifurcates from $[0:1:0]$ we obtain phase portrait $\mathbb{U}^2_{AD,34}$.

6.3 Examples Obtained from [12]

The study of the quadratic systems with a finite saddle-node and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ was divided into two families. The family (B) that we have used

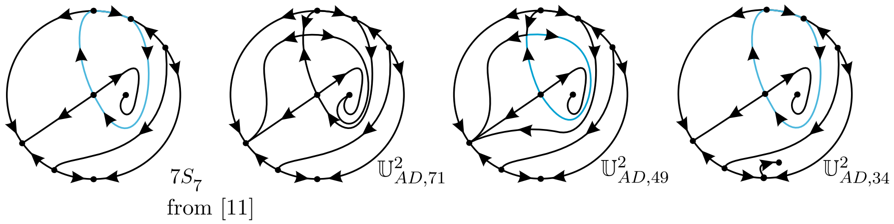


Fig. 64 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,34}$, $\mathbb{U}^2_{AD,49}$ and $\mathbb{U}^2_{AD,71}$

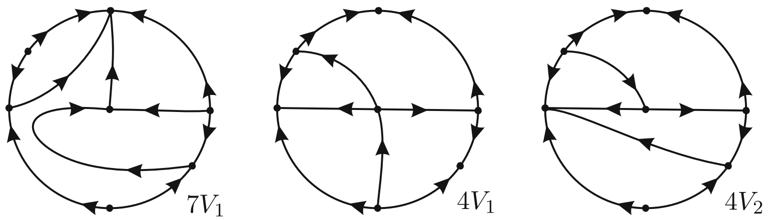


Fig. 65 Unstable phase portraits $7V_1$, $4V_1$ and $4V_2$ from [12]

in the previous two subsections and family (A) which has the property of having two singularities of the type $\overline{\binom{1}{1}}SN$. Then, a system in that family having a separatrix connection (and no other unstability from those already mentioned) will offer us at least 4 possibilities for quadratic systems in class (AD). There are 3 cases in that family with a separatrix connection, to know: $7V_1$, $4V_1$ and $4V_2$ (see Fig. 65).

Notice that the family studied in [12] (B) is not 4-dimensional but 5-dimensional, so the generic regions in it are hyper-volumes, and the bifurcations are volumes. Thus $4V_1$ and $4V_2$ correspond to a bifurcation with a separatrix connection in an invariant straight line and $7V_1$ corresponds to a case with a separatrix connection but no invariant straight line. However, an easy test of their possible bifurcations gives us already obtained phase portraits. Concretely $7V_1$ bifurcates in $\mathbb{U}^2_{AD,18}$, $\mathbb{U}^2_{AD,28}$, $\mathbb{U}^2_{AD,45}$ and $\mathbb{U}^2_{AD,67}$. On its own, $4V_1$ bifurcates in $\mathbb{U}^2_{AD,16}$, $\mathbb{U}^2_{AD,26}$, $\mathbb{U}^2_{AD,50}$ and $\mathbb{U}^2_{AD,72}$. And finally $4V_2$ bifurcates in $\mathbb{U}^2_{AD,15}$, $\mathbb{U}^2_{AD,16}$, $\mathbb{U}^2_{AD,17}$, $\mathbb{U}^2_{AD,19}$, $\mathbb{U}^2_{AD,30}$, $\mathbb{U}^2_{AD,42}$ and $\mathbb{U}^2_{AD,72}$. The first two cases bifurcate in just 4 possibilities since the saddle parts of the infinite saddle-nodes are not adjacent. However in $4V_2$ those parts are adjacent, and thus the infinite arc which joins them plays the role of a separatrix connection and this connection may move to the affine plane and there break in two different ways, and moreover, we may maintain that connection and broke the original invariant line in two different ways (see Fig. 66). Note that we have used a still unnumbered phase portrait of codimension three as an intermediate step to describe the bifurcation.

In summary, the bifurcations from phase portraits in [12] with a separatrix connection into class (AD) do not bring any new phase portrait from those obtained from [13].

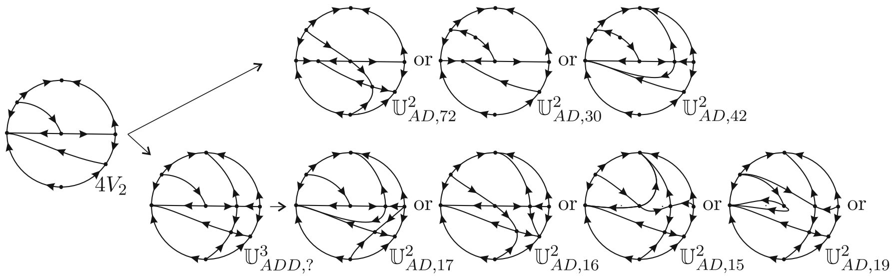


Fig. 66 Bifurcation of phase portraits $4V_2$ from [12] into class (AD)

6.4 Examples Obtained from [17, 18] with an Invariant Straight Line

In a very similar way, we are going to use papers [17] and [18] to obtain most of the rest of phase portraits of class (AD).

In [17] it is described that any quadratic system with a finite saddle-node and an infinite $\overline{\binom{0}{2}}SN$ can be moved into three different normal forms depending on the position of the infinite saddle-node. Since the finite saddle-node has both eigenvectors already fixed in this normal form, the position of the infinite saddle-node is relevant. This paper studies the two more degenerate forms and the most generic of them is studied in [18]. The normal form which collects all three characteristics is:

$$
\begin{aligned}
\dot{x} &= gx^2 + 2hxy + ky^2,\\
\dot{y} &= y + \ell x^2 + 2mxy + ny^2, \quad \text{with}\\
\eta &= -27\ell^2k^2 - 36\ell ghk + 18\ell gkn + 32\ell h^3 - 48\ell h^2 n + 72\ell hkm + 24\ell hn^2\\
&\qquad -36\ell kmn - 4\ell n^3 - 4g^3k + 4g^2h^2 - 4g^2hn + 24g^2km + g^2n^2 - 16gh^2m\\
&\qquad +16ghmn - 48gkm^2 - 4gmn^2 + 16h^2m^2 - 16hm^2n + 32km^3 + 4m^2n^2 = 0.
\end{aligned}
\tag{8}
$$

The polynomial η corresponds to the conditions needed to produce the coalesence of two infinite singularities, applied to this normal form. This, as many more invariants, are presented in [9].

6.4.1 Examples Obtained from [17] (A)

Assume now that we have a system (8) and the infinite singularity $[1:0:0]$ is a $\overline{\binom{0}{2}}SN$. Then $\ell = 0 = g - 2m$ and $m \neq 0$, which can be set as $m = 1/2$ and the system has an horizontal invariant straight line. In order to obtain a phase portrait of class (AD) we just need to split the infinite double point into two real infinite singularities. This can be done with a perturbation like

$$
\begin{aligned}
\dot{x} &= x^2 + 2hxy + ky^2 + \varepsilon x^2,\\
\dot{y} &= y + xy + ny^2.
\end{aligned}
\tag{9}
$$

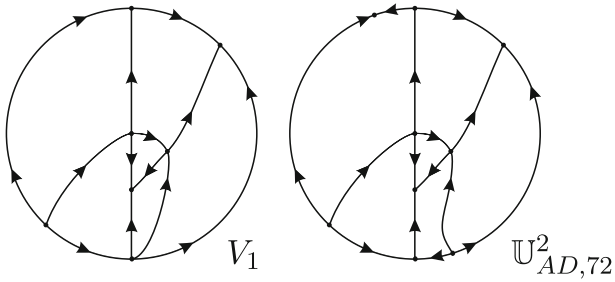


Fig. 67 Bifurcation of phase portraits V_1 from [17] (B) into class (AD)

We make a detailed example of this case. Take system (9) with $(h, k, n, \varepsilon) = (1, 2, -1, 0)$. This system has phase portrait V_1 from [17] (A). If $\varepsilon \neq 0$ the point $[1 : 0 : 0]$ splits into a node and a saddle. But we want to maintain the invariant straight line as a connection of separatrices, so we must take $\varepsilon < 0$ in order to fix the saddle at $[1 : 0 : 0]$. Then we obtain phase portrait $\mathbb{U}^2_{AD,20}$.

In a very similar way we may obtain

$$\begin{aligned}
&\mathbb{U}^2_{AD,65} \text{ from } V_3, && \mathbb{U}^2_{AD,74} \text{ from } V_6,\\
&\mathbb{U}^2_{AD,61} \text{ from } V_9, && \mathbb{U}^2_{AD,27} \text{ from } V_{12},\\
&\mathbb{U}^2_{AD,32} \text{ from } V_{15} \quad \text{and} && \mathbb{U}^2_{AD,74} \text{ from } V_{16}.
\end{aligned}$$

6.4.2 Examples Obtained from [17] (B)

Assume now that we have a system (8) and the infinite singularity $[0 : 1 : 0]$ is a $\overline{\binom{0}{2}}SN$. Then $k = 0 = n - 2h$ and $g \neq 0$ which can be set as $g = 1$ and the system has a vertical invariant straight line. In order to obtain a phase portrait of class (AD) we just need to split the infinite double point into two real infinite singularities This can be done with a perturbation as

$$\begin{aligned}
\dot{x} &= x^2 + 2hxy,\\
\dot{y} &= y + \ell x^2 + 2mxy + 2hy^2 + \varepsilon y^2.
\end{aligned} \tag{10}$$

We make a detailed example of this case. Take system (10) with $(h, \ell, m, \varepsilon) = (1, 1, 0, 0)$. This system has phase portrait V_1 from [17] (B).[1] If $\varepsilon \neq 0$ the point $[0 : 1 : 0]$ splits into a node and a saddle. But we want to maintain the invariant straight line as a connection of separatrices, so we must take $\varepsilon < 0$ in order to fix the saddle at $[0 : 1 : 0]$. Then we obtain phase portrait $\mathbb{U}^2_{AD,72}$. We add here the phase portrait V_1 from [17] (B) as well as its bifurcation into $\mathbb{U}^2_{AD,72}$, see Fig. 67.

[1] In [17] the phase portraits where denoted with an extra superscript that was just a counting value of phase portraits. Since such superscript has not been used in other papers of similar type, and does not apport any relevant information, we have prefered to omit it here.

In a very similar way we may obtain $\mathbb{U}^2_{AD,73}$ from V_2, $\mathbb{U}^2_{AD,26}$ from V_3, $\mathbb{U}^2_{AD,31}$ from V_6 and $\mathbb{U}^2_{AD,30}$ from V_7.

6.4.3 Examples Obtained from [18] with an Invariant Straight Line

The starting normal form in [18] is

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + (n - g - 2h)y^2, \\ \dot{y} &= y + \ell x^2 + (2g + 2h - 2\ell - n)xy + (\ell + 2n - 2g - 2h)y^2. \end{aligned} \tag{11}$$

Please note that in paper [18] there appear several typos in phase portraits and some of their labels which were lately corrected in an appendix of [13]. Thus the phase portraits we will use here in order to produce the required phase portraits of class (AD) are the right ones from the corrected version. This will affect also the next subsection.

In [18] we find phase portraits with an oblique invariant straight line (which passes through the infinite saddle-node $[1 : 1 : 0]$) or a different invariant line which passes through an elemental infinite singularity. The normal form implies then that this line is horizontal or vertical. We must study these cases separately.

Assume first that we have a system (8) with an oblique invariant straight line. This means that the infinite saddle-node is located at the end of such a line and this also implies $\ell = g$. The invariant straight line is $y = x - 1/n$. Now we first move the invariant straight line so to pass through the origin with $y \to y - 1/n$. After we make a -1-twist and having $\ell = g$ we obtain system

$$\begin{aligned} \dot{x} &= -x + nxy, \\ \dot{y} &= \frac{n - 2h - \ell}{n^2} + \left(1 - \frac{2h}{n}\right) x + 2\frac{h + \ell - n}{n} y \\ &\quad + \ell x^2 + 2\left(h + \ell - \frac{n}{2}\right) xy + ny^2, \end{aligned} \tag{12}$$

which clearly has the vertical axis as an invariant line. Now the finite saddle-node is at $(-1/n, 1/n)$ and we translate it back to the origin $(x, y) \to (x + 1/n, y - 1/n)$ and obtain:

$$\begin{aligned} \dot{x} &= -y + nxy, \\ \dot{y} &= y + \ell x^2 + (2h + 2\ell - n)xy + ny^2, \end{aligned} \tag{13}$$

which has the vertical invariant straight line $x = 1/n$. Finally we apply a perturbation which does not affect the invariant line, and breaks the infinite saddle-node:

$$\begin{aligned} \dot{x} &= -y + nxy, \\ \dot{y} &= y + \ell x^2 + (2h + 2\ell - n)xy + ny^2 + \varepsilon y^2. \end{aligned} \tag{14}$$

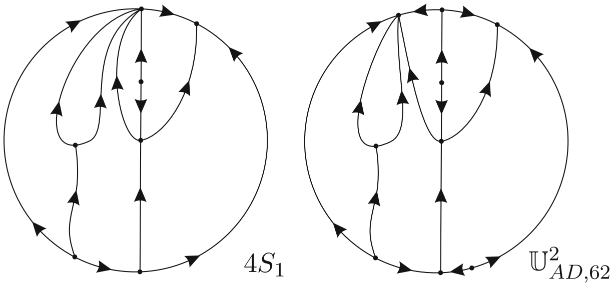


Fig. 68 Obtaining unstable phase portrait $\mathbb{U}^2_{AD,62}$ from $4S_1$ in [18]

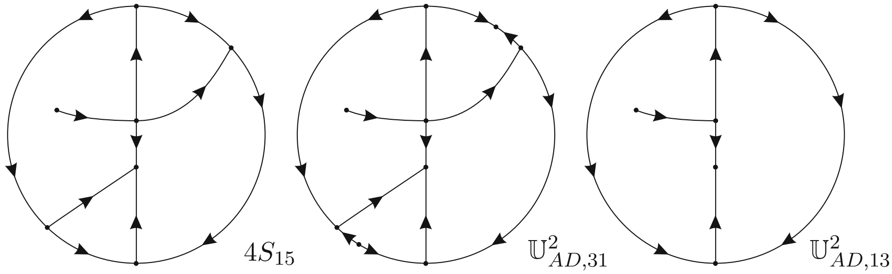


Fig. 69 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,31}$ and $\mathbb{U}^2_{AD,13}$ from $4S_{15}$ in [18]

Notice that the invariant straight line will persist whichever sign of ε we take. This means that for every ε small enough we will always split the infinite saddle-node into two real singularities.

We make a detailed example of this case. Take system (14) with $(g, h, \ell, n, \varepsilon) = (1, -3, 1, 3, 0)$. This system has phase portrait $4S_1$ from [18]. If $\varepsilon \neq 0$ the point $[0 : 1 : 0]$ splits into a node and a saddle. Then we obtain phase portrait $\mathbb{U}^2_{AD,62}$ (see Fig. 68).

In a very similar way we may obtain $\mathbb{U}^2_{AD,22}$ from $4S_3$, $\mathbb{U}^2_{AD,73}$ from $4S_6$, $\mathbb{U}^2_{AD,73}$ from $4S_8$, $\mathbb{U}^2_{AD,22}$ from $4S_{20}$ and $\mathbb{U}^2_{AD,66}$ from $4S_{31}$.

Assume now that we have a system (11) and a vertical invariant straight line. This implies $h = (n - g)/2$. Then a simple perturbation as

$$\begin{aligned}\dot{x} &= gx^2 + (n - g)xy,\\ \dot{y} &= y + \ell x^2 + (g - 2\ell)xy + (\ell + n - g)y^2 + \varepsilon y^2,\end{aligned} \tag{15}$$

will keep the invariant straight line $x = 0$ and will break the saddle-node at $[1 : 1 : 0]$ in real or complex singularities.

We make a detailed example of this case. Take system (15) with $(g, h, \ell, \varepsilon) = (1, 2, -2, 0)$. This system has phase portrait $4S_{15}$ from [18]. If $\varepsilon > 0$ the point $[1 : 1 : 0]$ splits into a node and a saddle and we obtain phase portrait $\mathbb{U}^2_{AD,31}$. If $\varepsilon < 0$ the point $[1 : 1 : 0]$ splits into complex singularities and we obtain phase portrait $\mathbb{U}^2_{AD,13}$ (see Fig. 69).

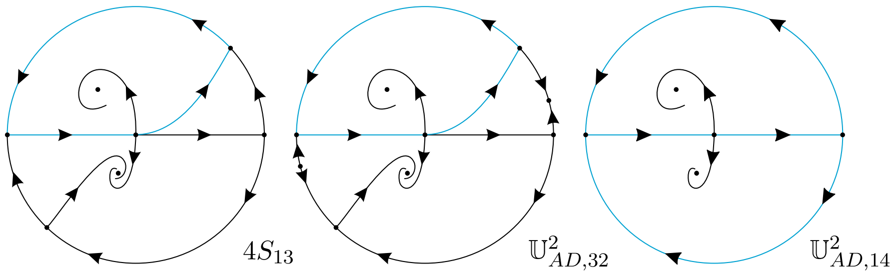


Fig. 70 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,32}$ and $\mathbb{U}^2_{AD,14}$ from $4S_{13}$ in [18]

In a very similar way we may obtain

$$\begin{aligned} &\mathbb{U}^2_{AD,3} \text{ and } \mathbb{U}^2_{AD,41} \text{ from } 4S_{16}, && \mathbb{U}^2_{AD,4} \text{ and } \mathbb{U}^2_{AD,52} \text{ from } 4S_{29},\\ &\mathbb{U}^2_{AD,4} \text{ and } \mathbb{U}^2_{AD,51} \text{ from } 4S_{32}, && \mathbb{U}^2_{AD,4} \text{ and } \mathbb{U}^2_{AD,50} \text{ from } 4S_{33},\\ &\mathbb{U}^2_{AD,13} \text{ and } \mathbb{U}^2_{AD,26} \text{ from } 4S_{36}, && \mathbb{U}^2_{AD,3} \text{ and } \mathbb{U}^2_{AD,42} \text{ from } 4S_{42}\\ \text{and } &\mathbb{U}^2_{AD,13} \text{ and } \mathbb{U}^2_{AD,30} \text{ from } 4S_{44}. \end{aligned}$$

By the way, we have skipped $4S_9$ from [18] since it should not be there because it has no separatrix connection and it is topologically equivalent to its neighbors V_{44} and V_{45}.

Finally assume that we have a system (11) and an horizontal invariant straight line. This implies $\ell = 0$. Then a simple perturbation as

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + (n - g - 2h)y^2,\\ \dot{y} &= y + (2g + 2h - n)xy + (2n - 2g - 2h)y^2 + \varepsilon y^2, \end{aligned} \tag{16}$$

will keep the invariant straight line $y = 0$ and will break the saddle-node at $[1 : 1 : 0]$ in real or complex singularities.

We make a detailed example of this case. Take system (16) with $(g, h, n, \varepsilon) = (1, 3, 5, 0)$. This system has phase portrait $4S_{13}$ from [18]. If $\varepsilon < 0$ the point $[1 : 1 : 0]$ splits into a node and a saddle and we obtain phase portrait $\mathbb{U}^2_{AD,32}$. If $\varepsilon > 0$ the point $[1 : 1 : 0]$ splits into complex singularities and we obtain phase portrait $\mathbb{U}^2_{AD,14}$ (see Fig. 70).

In a very similar way we may obtain $\mathbb{U}^2_{AD,14}$ (with limit cycle) and $\mathbb{U}^2_{AD,27}$ from $4S_{51}$.

6.4.4 Examples Obtained from [17, 18] with No Invariant Straight Line

As well as with the papers [12] and [13] now we must focus our attention in the cases from [17] and [18] where we have a separatrix connection which is not an invariant straight line. Namely, we have just one case in [17] (A) which is $7S_1$, none in [17] (B) and 48 in [18] (all those named $7S_x$). We can tear apart those having a limit cycle for reasons already explained.

In all these systems we will need the starting normal form, plus a perturbation to break the infinite saddle-node, plus a rotated vector field to recover the separatrix connection. These cases will be trickier than those from papers [12] and [13], because now we have to split a $\overline{\binom{0}{2}}SN$ instead of a $\overline{\binom{1}{1}}SN$. Now after the perturbation we will have two close infinite singularities, and the rotated vector field needed to recover the separatrix connection will affect those infinite singularities and it could make them coalesce again and disappear.

We start with normal form (8) as we did in a previous case. The infinite singularity $[1:0:0]$ is a $\overline{\binom{0}{2}}SN$. Then $\ell = 0 = g - 2m$ and $m \neq 0$, which can be set as $m = 1/2$ and the system has an horizontal invariant straight line. Since we want to start from phase portrait $7S_1$ we can take $(h, k, n) = (1, 4/5, n^*)$ with $n^* \in (0.49, 0.50)$. This phase portrait has two separatrix connections, the invariant straight line $y = 0$ as all the family has, plus a loop. So we can obtain a system of class (AD) in four different forms. We may keep the horizontal straight line, break the loop in two different ways, and split the infinite saddle-node into two real singularities, but all this is equivalent to make the bifurcations from cases V_3 and V_9 from [17] (A) that we have already described. So, what we need to do is breaking the straight line and keeping the loop. This can be done with a perturbation as

$$\begin{aligned} \dot{x} &= x^2 + 2hxy + ky^2, \\ \dot{y} &= y + xy + ny^2 + \varepsilon xy + \alpha(x^2 + 2hxy + ky^2). \end{aligned} \tag{17}$$

Take system (17) with $(h, k, n, \varepsilon, \alpha) = (1, 4/5, n^*, 0, 0)$ and n^* being some value in the interval $(0.49, 0.50)$. We have the invariant straight line, the infinite saddle-node and the loop, leading to phase portrait $7S_1$ (see Fig. 71).

Take system (17) with $(h, k, n, \varepsilon, \alpha) = (1, 4/5, n^*, -0.05, 0)$. We still have the invariant straight line, but the loop is clearly broken and the infinite saddle-node splits into two singularities. So, we have phase portrait $\mathbb{U}^2_{AD,61}$ with limit cycle.

Now take system (17) with $(h, k, n, \varepsilon, \alpha) = (1, 4/5, n^*, -0.05, \alpha^*)$ and α^* being some value in the interval $(0, 0.01)$. We have broken the horizontal invariant straight line, the infinite singularities remain isolated, and we have recovered the loop so to obtain $\mathbb{U}^2_{AD,55}$. If we take $(h, k, n, \varepsilon, \alpha) = (1, 4/5, n^*, 0.4, 0)$ we have split the infinite singularity into two simple singularities, but we still maintain the horizontal invariant straight line. And the loop is clearly broken. We have phase portrait $\mathbb{U}^2_{AD,65}$.

We now need to make a rotation with $\alpha < 0$ in order to recover the separatrix connection and we check that there must be a value $\alpha^* \in (-0.02, -0.01)$ which produces the connection. But $\alpha < 0$ also helps us in breaking the invariant straight line exactly in the way we need and it is small enough so that the infinite singular points are still split. For $\alpha < -0.03$ we see that they have coalesced. So, we have obtained phase portrait $\mathbb{U}^2_{AD,60}$ (see Fig. 71).

Now we study the examples that can be obtained from [18].

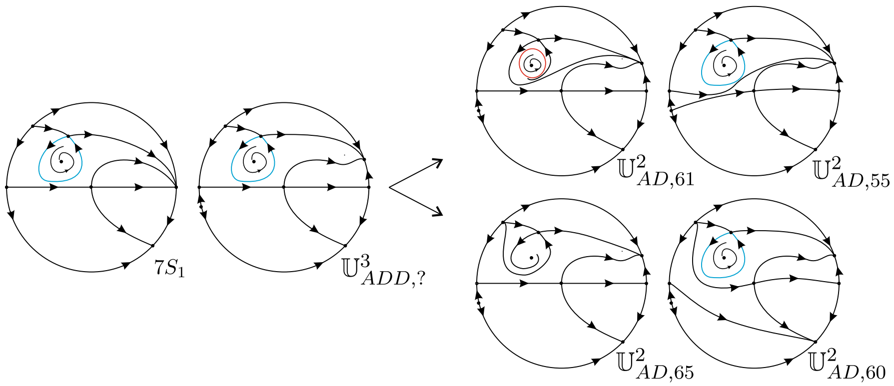


Fig. 71 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,61}$, $\mathbb{U}^2_{AD,65}$, $\mathbb{U}^2_{AD,55}$ and $\mathbb{U}^2_{AD,60}$ from $7S_1$ in [17] (A)

We start with normal form (8) as we did in a previous case. The infinite singularity [1 : 1 : 0] is a $\overline{\binom{0}{2}}SN$. In [18] it is proved that such system can be written as

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + (n - g - 2h)y^2, \\ \dot{y} &= y + \ell x^2 + (2g + 2h - 2\ell - n)xy + (\ell + 2n - 2g - 2h)y^2. \end{aligned} \tag{18}$$

For this group of phase portraits, we are going to use a different idea so to obtain the desired phase portrait. For these examples, using a perturbation to break the infinite saddle-node plus a rotated vector field to recover the separatrix connection has been proved to be very hard, at least in several cases. But we can convert a difficulty into a tool to obtain what we need even in an easier way. As we know, we do not have the exact parameter for which a phase portrait has a non-algebraic separatrix connection. At most we know that, for a parameter being in some interval, we are crossing the bifurcation where the separatrix connection exists. Then we know that the perturbation to modify the infinite saddle-node will affect other orbits and it would break the separatrix connection. Even though this perturbation is not a rotated vector field, for a very small perturbation enough to break the saddle-node, the separatrix connection will move the separatrices to one or another side. So, we just need to use a starting system where we are very close to the separatrix connection and we are in the side so that the perturbation needed to break the infinite saddle-node will move the separatrices in the required direction. For a very small perturbation, it will not be even enough to move the separatrices to the other side, but then making a little bigger perturbation we will get the separatrix connection. In fact, we will just see that we are at the other side of the separatrix connection, and in this way we will have proved the existence of the required phase portrait. To tell the truth, this "perturbation" is not really a perturbation since it needs to be large enough to produce the separatrix connection. Thus, this perturbation could cross another bifurcation of the system. But since the starting system may be as close as needed to the point of the separatrix connection, then the perturbation we do is a good one.

Let us see all this with an example. We start with the system:

$$\begin{aligned}\dot{x} &= gx^2 + 2hxy + (n - g - 2h)y^2,\\ \dot{y} &= y + \ell x^2 + (2g + 2h - 2\ell - n)xy + (\ell + 2n - 2g - 2h)y^2 + \varepsilon_x x^2.\end{aligned} \tag{19}$$

Take system (19) with $(g, n, h, \ell, \varepsilon_x) = (1, 2, -2.46, \ell^*, 0)$ and ℓ^* being some value in the interval (1.70, 1.71). For $\ell = 1.70$, we have phase portrait V_7 from [18], and for $\ell = 1.71$, we have phase portrait V_{17} (it has a limit cycle). So, for a value between them we must have phase portrait $7S_2$. Since we do not know the value of ℓ^*, the computations in P4 must be done with a value of ℓ which will put us either in V_7 or V_{17}. Let us try to produce first $\mathbb{U}^2_{AD,57}$. For that we will need a positive ε_x. For the way that a positive ε_x will affect the separatrices forming the connection, we decide to start on V_7, that is, we take $\ell = 1.70$. Then we see in P4 that any small positive ε_x will split the infinite saddle-node into an infinite saddle and an infinite node. For example $\varepsilon_x = 0.001$ does it. But this perturbation is not enough to obtain the separatrix connection, and we even see that for $\varepsilon_x = 0.008$ we still have it not. But for $\varepsilon_x = 0.009$ we have already crossed it and there is a limit cycle. So, for a value $\varepsilon_x^* \in (0.008, 0.009)$ the system $(g, n, h, \ell, \varepsilon_x) = (1, 2, -2.46, 1.7, \varepsilon_x^*)$ we have phase portrait $\mathbb{U}^2_{AD,57}$. Notice that if for a value $\varepsilon_x \in (0.008, 0.009)$ we could see any other bifurcation affected, we could have always started with a smaller interval for ℓ, that is, we could have started much closer to the phase portrait $7S_2$, and then, consequently, the interval for ε_x would also be smaller.

Now we look for phase portrait $\mathbb{U}^2_{AD,10}$. Now we start at the other side of the bifurcation, that is on $\ell = 1.71$ (so we have in fact phase portrait V_{17} having a limit cycle). Now any perturbation with negative ε_x will convert the infinite saddle-node into complex singularities. We detect that for $\varepsilon_x = -0.001$ we got this, but the relative position of the separatrices intended to form the loop have already changed positions, thus, we have moved beyond we wanted to arrive. And this also happens for $\varepsilon_x = -0.0001$. But it does not for $\varepsilon_x = -0.00001$. Now we see that the two interesting separatrices are still in the same relative position as they were in V_{17}, that is, the limit cycle still persists. But then, playing a bit with ε_x we see that there is a change in the relative position of separatrices while moving from $\varepsilon_x = -0.00007$ to $\varepsilon_x = -0.00008$. In conclusion, for some $\varepsilon_x^* \in (-0.00008, -0.00007)$ we obtain phase portrait $\mathbb{U}^2_{AD,10}$ (see Fig. 72). In a very similar way we may obtain

$$\begin{aligned}
&\mathbb{U}^2_{AD,35} \text{ from } 7S_1, && \mathbb{U}^2_{AD,76} \text{ from } 7S_3,\\
&\mathbb{U}^2_{AD,2} \text{ and } \mathbb{U}^2_{AD,44} \text{ from } 7S_4, && \mathbb{U}^2_{AD,21} \text{ from } 7S_6,\\
&\mathbb{U}^2_{AD,8} \text{ and } \mathbb{U}^2_{AD,53} \text{ from } 7S_7, && \mathbb{U}^2_{AD,6} \text{ and } \mathbb{U}^2_{AD,48} \text{ from } 7S_8,\\
&\mathbb{U}^2_{AD,70} \text{ from } 7S_9, && \mathbb{U}^2_{AD,8} \text{ and } \mathbb{U}^2_{AD,53} \text{ from } 7S_{10},\\
&\mathbb{U}^2_{AD,6} \text{ and } \mathbb{U}^2_{AD,48} \text{ from } 7S_{15}, && \mathbb{U}^2_{AD,70} \text{ from } 7S_{16},\\
&\mathbb{U}^2_{AD,12} \text{ and } \mathbb{U}^2_{AD,24} \text{ from } 7S_{17}, && \mathbb{U}^2_{AD,10} \text{ and } \mathbb{U}^2_{AD,59} \text{ from } 7S_{22},\\
&\mathbb{U}^2_{AD,1} \text{ and } \mathbb{U}^2_{AD,43} \text{ from } 7S_{23}, && \mathbb{U}^2_{AD,21} \text{ from } 7S_{26},
\end{aligned}$$

$$\begin{array}{ll}
\mathbb{U}^2_{AD,11} \text{ and } \mathbb{U}^2_{AD,60} \text{ from } 7S_{29}, & \mathbb{U}^2_{AD,12} \text{ and } \mathbb{U}^2_{AD,24} \text{ from } 7S_{31}, \\
\mathbb{U}^2_{AD,9} \text{ and } \mathbb{U}^2_{AD,56} \text{ from } 7S_{32}, & \mathbb{U}^2_{AD,7} \text{ and } \mathbb{U}^2_{AD,46} \text{ from } 7S_{33}, \\
\mathbb{U}^2_{AD,67} \text{ from } 7S_{37}, & \mathbb{U}^2_{AD,68} \text{ from } 7S_{38}, \\
\mathbb{U}^2_{AD,11} \text{ and } \mathbb{U}^2_{AD,55} \text{ from } 7S_{41}, & \mathbb{U}^2_{AD,40} \text{ from } 7S_{42}, \\
\mathbb{U}^2_{AD,64} \text{ from } 7S_{44}, & \mathbb{U}^2_{AD,63} \text{ from } 7S_{45}, \\
\mathbb{U}^2_{AD,11} \text{ and } \mathbb{U}^2_{AD,58} \text{ from } 7S_{52}, & \mathbb{U}^2_{AD,75} \text{ from } 7S_{55}, \\
\mathbb{U}^2_{AD,37} \text{ from } 7S_{57}, & \mathbb{U}^2_{AD,28} \text{ from } 7S_{58}, \\
\mathbb{U}^2_{AD,33} \text{ from } 7S_{60}, & \mathbb{U}^2_{AD,69} \text{ from } 7S_{61}, \\
\mathbb{U}^2_{AD,5} \text{ and } \mathbb{U}^2_{AD,47} \text{ from } 7S_{62}, & \mathbb{U}^2_{AD,69} \text{ from } 7S_{63}, \\
\mathbb{U}^2_{AD,5} \text{ and } \mathbb{U}^2_{AD,47} \text{ from } 7S_{64}, & \mathbb{U}^2_{AD,5} \text{ and } \mathbb{U}^2_{AD,45} \text{ from } 7S_{65}, \\
\mathbb{U}^2_{AD,39} \text{ from } 7S_{67}, & \mathbb{U}^2_{AD,71} \text{ from } 7S_{68}, \\
\mathbb{U}^2_{AD,71} \text{ from } 7S_{69}, & \mathbb{U}^2_{AD,7} \text{ and } \mathbb{U}^2_{AD,49} \text{ from } 7S_{70}, \\
\mathbb{U}^2_{AD,7} \text{ and } \mathbb{U}^2_{AD,49} \text{ from } 7S_{71}, & \mathbb{U}^2_{AD,34} \text{ from } 7S_{72}, \\
\mathbb{U}^2_{AD,29} \text{ from } 7S_{75}, & \mathbb{U}^2_{AD,38} \text{ from } 7S_{76}, \\
\mathbb{U}^2_{AD,9} \text{ and } \mathbb{U}^2_{AD,54} \text{ from } 7S_{77}, & \mathbb{U}^2_{AD,9} \text{ and } \mathbb{U}^2_{AD,54} \text{ from } 7S_{78}, \\
\mathbb{U}^2_{AD,12} \text{ and } \mathbb{U}^2_{AD,25} \text{ from } 7S_{79}, & \mathbb{U}^2_{AD,12} \text{ and } \mathbb{U}^2_{AD,23} \text{ from } S_{82}
\end{array}$$

and $\mathbb{U}^2_{AD,36}$ from $7S_{85}$.

Several of the previous cases bifurcate in only one phase portrait of class (AD) since their separatrix connection needs the infinite saddle-node. When this singularity disappears, so does the connection.

There is another mistype in [18]: the focus inside the graphic of phase portrait $7S_{57}$ must be attractor since it is the bifurcation between V_{137} and V_{138}. This is an interesting mistake to point out, since its bifurcation (in the wrong mode) would have produced conjectured impossible phase portrait ($\mathbb{U}^{1,l}_{D,38}$ here) while in fact it produces $\mathbb{U}^1_{D,38}$ which was already presented in the statement and proved its existence.

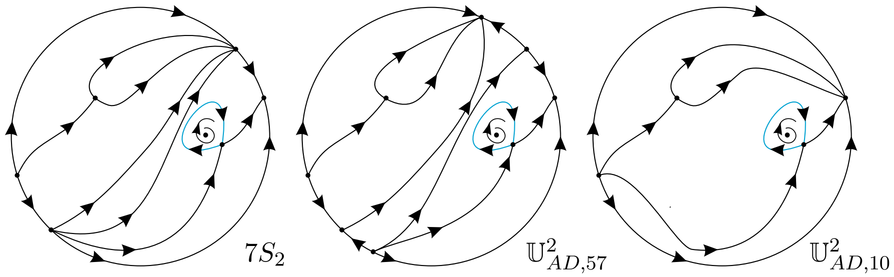


Fig. 72 Obtaining unstable phase portraits $\mathbb{U}^2_{AD,57}$ and $\mathbb{U}^2_{AD,10}$ from $7S_2$ in [18]

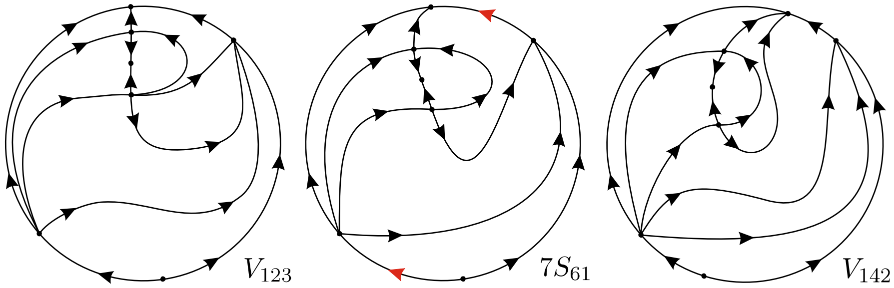


Fig. 73 Correcting a mistype of $7S_{61}$ in [18]

And another mistype in [18]: The phase portrait $7S_{61}$ has two arrows at infinity wrong. The right version is shown in Fig. 73 accompanied by two generic phase portraits which are its neighbors. The corrected arrows are indicated in red color. It is from this corrected version that $\mathbb{U}^2_{AD,69}$ may bifurcate. The merit of finding this type is not mine but of a very good referee I have had.

In summary, we have looked for examples in four big families of already studied systems from which phase portraits of class (AD) may bifurcate. We have obtained examples which confirm the existence of the phase portraits mentioned in Theorem 3, most of them from different sources, and we have not found a single example of those conjectured impossible (see Figs. 4 and 5). Only phase portrait $\mathbb{U}^2_{AD,77}$ is still missing. As already mentioned, another big family which is the quadratic systems with a weak focus and a finite saddle-node [20] has recently been studied, and $\mathbb{U}^2_{AD,77}$ has appeared in it.

6.5 Example $\mathbb{U}^2_{AD,77}$ from [20]

In paper [20] where quadratic systems with a finite semi-elemental saddle-node and a weak focus are studied, we found phase portrait $7S_{13}$ which is directly our $\mathbb{U}^2_{AD,77}$. For quite a long time we had thought that this system would be in the set of conjectured impossible phase portraits with the name of $\mathbb{U}^{2,I}_{AD,8}$ but we decided to delay a bit the ending of this paper for if some new example could appear there, as it has happened. While looking for examples of class (AD) inside codimension two* families which have one finite saddle-node plus another geometrical property which helps to reduce the number of parameters, we are just studying hyper-surfaces of the codimension one family of quadratic systems with a finite saddle-node. So there is always the possibility that some of the conjectured impossible phase portraits may exist without intersection far from the hyper-surfaces studied.

6.6 Example $7S_1$ from [19]

Even though we are not going to find any new phase portrait here, this is a case which is worth studying. In paper [19] where quadratic systems with a finite triple semi-elemental saddle are studied, we found phase portrait $7S_1$. All the generic phase

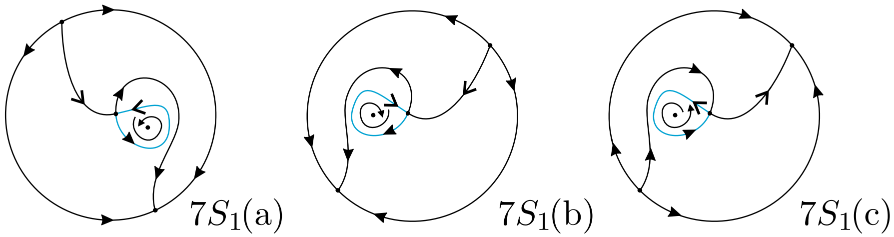


Fig. 74 Phase portrait $7S_1$ from [19] and a second view of it

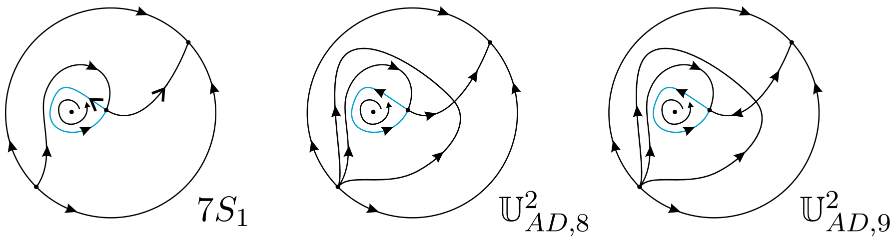


Fig. 75 Perturbations of $7S_1$ into class (AD)

portraits in this family are topologically equivalent to structurally stable quadratic systems due to the fact that the semi-elemental saddle behaves like an elemental one. And thus, the phase portrait $7S_1$ is topologically equivalent to $\mathbb{U}^1_{D,1}$ from [6]. However, there is an important fact regarding $7S_1$ which was not relevant in [19] but which is critical now. The triple finite saddle has one eigenvalue zero and another different from zero. The focus inside the graphic is stable. So, when dealing with perturbations it is very important to know if the non-null eigenvalue is positive or negative. By a simple check of the bifurcation diagram in [19], we conclude that it is positive (see Fig. 74a in which we have drawn with a thinner arrow the separatrices with zero eigenvalue). We make a vertical symmetry (b) and a time change (c) so to compare it easily with the skeletons of separatrices of the phase portraits $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,9}$ from Fig. 1 in order to see which of the phase portraits $\mathbb{U}^2_{AD,8}$, $\mathbb{U}^2_{AD,9}$, $\mathbb{U}^2_{AD,77}$ or $\mathbb{U}^{2,I}_{AD,9}$ may be obtained.

Now we can perturb $7S_1$ in two different ways while maintaining the loop and splitting a saddle-node, that is, we can split a saddle-node and let the remaining saddle form the loop, or vice versa. But what we obtain is just already known phase portraits $\mathbb{U}^2_{AD,8}$ and $\mathbb{U}^2_{AD,9}$ (see Fig. 75). Notice if the case of $7S_1$ had been the opposite (with negative eigenvalue, or the focus being unstable) we would have obtained instead the phase portraits $\mathbb{U}^2_{AD,77}$ and the conjectured impossible $\mathbb{U}^{2,I}_{AD,9}$. Far from being a proof of the impossibility of $\mathbb{U}^{2,I}_{AD,9}$, this reinforces our feeling that the conjecture is true.

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Data availability The data that support the findings of this study are openly available at https://mat.uab.cat/~artes/articles/SU2AD/SU2AD.html. (The symbol ∼ which is part of the address may be a problem if the reader tries to click the address from an electronic form of the paper. In this case, it is suggested to type the whole address manually.)

Declarations

Conflict of interest This work does not have any conflict of interest.

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