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On new infinite families of completely regular and completely transitive codes

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ABSTRACT

In two previous papers we constructed new families of completely regular codes by concatenation methods. Here we determine cases in which the new codes are completely transitive. For these cases we also find the automorphism groups of such codes. For the remaining cases, we show that the codes are not completely transitive assuming an upper bound on the order of the monomial automorphism groups, according to computational results.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of order q, where q is a prime power. For a q-ary *code* $C \subset \mathbb{F}_q^n$ of *length* n, denote by d its *minimum (Hamming) distance* between any pair of distinct codewords. The *packing radius* of C is $e = \lfloor (d-1)/2 \rfloor$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its *distance to the code* C is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$ and the *covering radius* of the code C is $\rho = \max_{\mathbf{v} \in \mathbb{F}_2^n}\{d(\mathbf{v}, C)\}$. Note that $e \leq \rho$. A *linear* $[n, k, d; \rho]_q$*-code* is a k-dimensional subspace of $\mathbb{F}_q^n$, with minimum distance d and covering radius ρ. We denote by $D = C + \mathbf{x}$ ($\mathbf{x} \in \mathbb{F}_q^n$) a *coset* of C, where $+$ means the component-wise addition in $\mathbb{F}_q$.

For a given code C of length n and covering radius ρ, define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i\}, \quad i = 0, 1, \ldots, \rho.$$

The sets $C(0) = C, C(1), \ldots, C(\rho)$ are called the *subconstituents* of C.

Say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbours* if $d(\mathbf{x}, \mathbf{y}) = 1$. Denote by $\mathbf{0}$ the all-zero vector or the all-zero matrix, depending on the context.

Definition 1.1 *([13]).* A code C of length n and covering radius ρ is *completely regular*, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbours in $C(l-1)$ and the same number b_l of neighbours in $C(l+1)$. Define $a_l = (q-1)\cdot n - b_l - c_l$ and set $c_0 = b_\rho = 0$. The parameters a_l, b_l and c_l ($0 \leq l \leq \rho$) are called *intersection numbers* and the sequence $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ is called the *intersection array* (shortly IA) of C.

Let M be a monomial matrix, i.e. a matrix with exactly one non-zero entry in each row and column. If q is prime, then the automorphism group of C, Aut(C), consists of all monomial $(n \times n)$-matrices M over $\mathbb{F}_q$ such that $\mathbf{c}M \in C$ for all

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$\mathbf{c} \in C$. If q is a power of a prime number, then $\mathrm{Aut}(C)$ also contains any field automorphism of $\mathbb{F}_q$ which preserves C. The group $\mathrm{Aut}(C)$ acts on the set of cosets of C in the following way: for all $\pi \in \mathrm{Aut}(C)$ and for every vector $\mathbf{v} \in \mathbb{F}_q^n$ we have $\pi(\mathbf{v}+C) = \pi(\mathbf{v}) + C$. Fix the following notation for groups, which we need: let C_t denote the cyclic group of order t, let S_t denote the symmetric group of order $t!$, and let $\mathrm{GL}(m,q)$ be the q-ary general linear group (formed by all nonsingular q-ary $(m \times m)$-matrices).

In this paper, we consider only monomial automorphisms. Thus, when q is not a prime, but a prime power, we omit the field automorphisms that fix the code. We denote by $\mathrm{MAut}(C)$ the monomial automorphism group of a code C.

Definition 1.2. Let C be a q-ary linear code with covering radius ρ. Then C is *completely transitive* if $\mathrm{MAut}(C)$ has $\rho+1$ orbits in its action on the cosets of C.

Note that this definition generalizes to the q-ary case the definition given in [14]. However, it is a particular case of the definition of a coset-completely transitive code given in [10], where not only monomial automorphisms are considered, but also the field automorphisms are taken into account.

Since two cosets in the same orbit have the same weight distribution, it is clear that any completely transitive code is completely regular.

The main result of the paper is Theorem 3.8, which proves that certain codes $B^{(r)}$ and $C^{(r)}$ are completely transitive. The code $B^{(r)}$ was introduced in [4] and is described in Section 2; its construction takes as input a parity check matrix for the q-ary cyclic Hamming code of length $n = (q^m-1)/(q-1)$ and produces a code of length nr with parity check matrix as in Equation (1); while $C^{(r)}$ is defined in Equation (5) just before Theorem 2.4 as a “supplementary code” to $B^{(r)}$. Theorem 3.8 applies for arbitrary q, m, r such that $\gcd(n, q-1) = 1$ and $r \le n$, and proves complete transitivity of $B^{(r)}$ and $C^{(r)}$ for certain values of q, m and r. Moreover, assuming that the monomial automorphism group $\mathrm{MAut}(B^{(r)})$ is not too large, it proves that $B^{(r)}$ and $C^{(r)}$ are not completely transitive for all other values of q, m and r. More precisely the result holds provided that $|\mathrm{MAut}(B^{(r)})| \le 12n(q-1)$; computationally it has been shown that this upper bound holds for small values of the parameters, and we conjecture that it holds in general.

Conjecture 1.3. *The order of* $\mathrm{MAut}(B^{(r)})$ *is not greater than* $12n(q-1)$*, for* $r > 3$ *except when* $q = 2$*,* $r \in \{n-1, n\}$*.*

The assumption of this open question has been justified via computations involving codes with small parameters. Next table shows the computational results for $q \in \{2,3,4,5\}$ and $m \in \{2,3,4,5,6,7,8\}$. The entries in the table are the range of values r such that the code $B^{(r)}$ with parameters (q,m,r) has been checked and satisfy the conjecture. The mark $*$ means that $\gcd(n, q-1) \ne 1$ and so we do not deal with this class of codes. The mark - means that MAGMA could not compute the automorphism group.

q \ m	2	3	4	5	6	7	8
2		[4..7]	[4..15]	[4..31]	[4..53−]	[4..38−]	[4..22−]
3	*	[4..13]	*	[4..13−]	*	-	*
4	[4..5]	*	[4..12−]	-	*	-	-
5	*	[4..25−]	*	-	*	-	*

Let C be a q-ary linear code of length n and dimension m and let G be a generator matrix for C. If $M \in \mathrm{MAut}(C)$, then GM is another generator matrix for C. Hence, GM can be obtained by a linear transformation from G. In other words, $GM = KG$ for some invertible matrix K, i.e. $K \in \mathrm{GL}(m,q)$. Note that $\mathrm{MAut}(C)$ fixes set-wise the set of vectors of weight one.

A Hamming code $\mathcal{H}_m$ has a parity check matrix H_m which consists of all non-zero m-dimensional column vectors such that no column is a scalar multiple of another one. Thus, $\mathcal{H}_m$ has length $n = (q^m-1)/(q-1)$. Such codes are single-error-correcting codes (with minimum distance $d = 3$) and perfect (thus $\rho = 1$) [12]. Hence, all their cosets have minimum weight at most 1.

Lemma 1.4. *The monomial automorphism group of a q-ary Hamming code,* $\mathrm{MAut}(\mathcal{H}_m)$*, is isomorphic to the general linear group* $\mathrm{GL}(m,q)$*. The action of* $\mathrm{MAut}(\mathcal{H}_m)$ *on the set of vectors of weight one is transitive and 2-transitive if and only if* $q = 2$*.*

Proof. It is well known, e.g. see [12], that the monomial automorphism group of a Hamming code $\mathcal{H}_m$ is isomorphic to the general linear group $\mathrm{GL}(m,q)$ and $\mathrm{MAut}(\mathcal{H}_m)$ is a permutation group acting on a set of cardinality (q^m-1). In the binary case, the action of $\mathrm{MAut}(\mathcal{H}_m)$ on the set of coordinate positions is doubly transitive [12] (but only in the binary case). In the general case of q-ary Hamming codes, for $q > 2$, it is easy to see that the action of $\mathrm{MAut}(\mathcal{H}_m)$ on the set of q^m-1 non-zero vectors of $\mathbb{F}_q^m$ is transitive, since we can always find a nonsingular matrix in $\mathrm{GL}(m,q)$ that takes any non-zero vector of $\mathbb{F}_q^m$ to another one. However, it is not 2-transitive nor 2-homogeneous, since any two non-zero vectors $v, w \in \mathbb{F}_q^m$ cannot be sent, for example, to $v, -v$, respectively. Indeed, in this case, due to linearity the vector $v+w$ would be sent to zero, consequently $w = -v$ and vector w would no longer be any arbitrary vector. $\square$

As a consequence, all cosets of minimum weight one are in the same orbit and thus Hamming codes are completely transitive.

Completely regular and completely transitive codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction, and enumeration of all such codes are open hard problems (see [1,5,8,11,13,15] and references there).

It is well known that new completely regular codes can be obtained by the direct sum of perfect codes or, more general, by the direct sum of completely regular codes with covering radius 1 [2,14]. In the previous papers [4,6], we extend these constructions, giving several explicit constructions of new completely regular codes, based on concatenation methods. Here, we find the monomial automorphism groups for that families in some cases. This gives mutually nonequivalent binary linear completely regular codes with the same intersection arrays and isomorphic monomial automorphism groups. We show cases in which the constructed families of codes are also completely transitive.

In Section 2 we give some preliminary results, mostly coming from the previous papers [3,4,6] (where we introduced two infinite families of CR codes $A^{(r)}$ and $B^{(r)}$), that help to us place ourselves in the problem we want to address. In Section 3 we study the complete transitivity of codes $B^{(r)}$ and calculate $\mathrm{MAut}(B^{(r)})$ for some cases. Finally, we determine, assuming an additional bound on the order of $\mathrm{MAut}(B^{(r)})$ the complete transitivity for the codes $B^{(r)}$.

2. Preliminary results

In this section we recall the results of [3,4,6]. Combining such results, in the next section, we specify cases in which the constructed infinite families of completely regular codes are also completely transitive.

For any vector $x = (x_1, \dots, x_n) \in \mathbb{F}_q^n$, denote by $\sigma(x)$ the right cyclic shift of x, i.e. $\sigma(x) = (x_n, x_1, \dots, x_{n-1})$. Define recursively $\sigma^i(x) = \sigma(\sigma^{i-1}(x))$, for $i = 2, 3, \dots$ and $\sigma^1(x) = \sigma(x)$. For $j < 0$, we define $\sigma^j(x) = \sigma^\ell(x)$, where $\ell = j \bmod n$.

The next constructions are described in [4], although the dual codes of the resulting family of q-ary completely regular codes are well known as the family SU2 in [9].

Let H be the parity check matrix of a q-ary cyclic Hamming code $\mathcal{H}_m$ of length $n = (q^m - 1)/(q - 1)$ (recall that a cyclic version of $\mathcal{H}_m$ exists exactly when n and $q - 1$ are coprime numbers). Clearly the code generated by H is also a cyclic code. For any $r \in \{1, 2, \dots, n\}$, consider the code $B^{(r)}$ of length nr with the following parity check matrix $H_b(r)$ (we call it Construction I in [4]):

$$H_b(r) = \begin{bmatrix} H & H & \cdots & H \\ H_1 & H_2 & \cdots & H_r \end{bmatrix}, \quad r = 1, \dots, n; \tag{1}$$

where H_i, for $1 \le i \le n$, is the matrix H multiplying each column by ξ^i (here, we consider the columns of H as elements $1, \xi, \xi^2, \dots, \xi^{n-1}$, where $\xi \in \mathbb{F}_{q^m}$ is a primitive nth root of unity), hence H_i is obtained from H by cyclically shifting i times its columns to the right (in these terms $H = H_n$). In [4] we also presented Construction II. In this case, the corresponding code of length $n(r + 3)$, which we denote $A^{(r)}$, has a parity check matrix $H_a(r)$ of the form:

$$H_a(r) = \begin{bmatrix} H & O & H & H & H & \cdots & H \\ O & H & H & H_1 & H_2 & \cdots & H_r \end{bmatrix}. \tag{2}$$

For the codes $A^{(r)}$ we fix the following interval for r: $r \in \{-2, -1, 0, 1, \dots, n - 1\}$ (for $r = n$ we would have pairs of linear dependent columns), where $A^{(0)}$, $A^{(-1)}$ and $A^{(-2)}$ are defined by the parity check matrices, respectively

$$H_a(0) = \begin{bmatrix} H & O & H \\ O & H & H \end{bmatrix}, \quad H_a(-1) = \begin{bmatrix} H & O \\ O & H \end{bmatrix}, \quad H_a(-2) = \begin{bmatrix} H \\ O \end{bmatrix}. \tag{3}$$

Note that the codes $B^{(1)}$ and $A^{(-2)}$ are the Hamming code of length $(q^m - 1)/(q - 1)$.

Lemma 2.1. *Let i and j be positive integers such that $i + j \le n$. The code D with parity check matrix*

$$\begin{bmatrix} H & H & \cdots & H \\ H_i & H_{i+1} & \cdots & H_{i+j-1} \end{bmatrix} \tag{4}$$

is equivalent to the code $B^{(j)}$.

Proof. The statement is evident, since the parity check matrices for the codes D and $B^{(j)}$ differ from each other by the multiplier ξ^{j-i}, if $j > i$, applied to the second bottom part of rows. $\square$

In [4] we proved the following theorem.

Theorem 2.2 *([4]).*

(i) *For any r, $2 \le r \le n$, the code $B^{(r)}$ with parity check matrix $H_b(r)$ given in (1) is a completely regular $[nr, nr-2m, 3; 2]_q$-code with intersection array*

$$\mathrm{IA} = \{(q-1)nr, ((q-1)n-r+2)(r-1); 1, r(r-1)\}.$$

(ii) *For any r, $0 \le r \le n-1$, the code $A^{(r)}$ with parity check matrix $H_a(r)$ given in (2) is a completely regular $[(r+3)n, (r+3)n - 2m, 3; 2]_q$-code with intersection array*

$$\mathrm{IA} = \{(q-1)n(r+3), ((q-1)n-1-r)(r+2); 1, (r+2)(r+3)\}.$$

In [6] we defined the concept of *supplementary codes*. Let H_m be the parity check matrix of a q-ary Hamming code of length $n = (q^m-1)/(q-1)$. Consider a non-empty subset of $n_A < n$ columns of H_m as a parity check matrix of a code A. Call B the code that has as parity check matrix the remaining $n_B = n - n_A$ columns of H_m. The code A is the supplementary code of B, and vice versa. We summarize the results of [6] in the following theorem.

Theorem 2.3 *([6]). Let A and B be supplementary codes as defined above.*

(i) *If A has covering radius 1, then it is a Hamming code. Both codes A and B are completely regular and completely transitive.*
(ii) *If A has covering radius 2, dimension $n_A - m$, and A is completely regular, then B is completely regular with covering radius at most 2.*
(iii) *If A has covering radius 2, dimension $n_A - m$, and A is completely transitive, then B is completely transitive.*

In all cases, the parameters and intersection arrays are computed in [6].

Now, consider the parity check matrix H_{2m} of a q-ary Hamming code $\mathcal{H}_{2m}$ of length $(q^{2m}-1)/(q-1)$, such that the q-ary Hamming code $\mathcal{H}_m$ is cyclic. Therefore, $n = (q^m-1)/(q-1)$ and $q-1$ are coprime numbers. We assume that $m \ge 3$ to avoid trivial cases. In fact, for $m = 2$, $\mathcal{H}_m$ is cyclic only if q is a power of 2. Present H_{2m} as follows:

$$H_{2m} = \left[\, H_b(r) \;\middle|\; H_c(r) \,\right], \quad r \in \{1, \dots, q^m\}, \tag{5}$$

where $H_b(r)$ is the matrix (1) with $rn = r(q^m-1)/(q-1)$ columns, hence it is a parity check matrix for $B^{(r)}$. Call $C^{(r)}$ the supplementary code of $B^{(r)}$, i.e. $C^{(r)}$ has parity check matrix $H_c(r)$. By combining the results of [4] and [6], we obtain the following theorem.

Theorem 2.4. *Let $n = (q^m-1)/(q-1)$ with $\gcd(n, q-1) = 1$, and let $B^{(r)}$ and $C^{(r)}$ be the supplementary codes as defined above.*

(i) *$B^{(1)}$ is a Hamming $[n, n-m, 3; 1]_q$-code and $C^{(1)}$ is a $[q^m n, q^m n - 2m, 3, 2]_q$-code. Both codes are completely regular and completely transitive with intersection arrays*

$$\mathrm{IA}(B^{(1)}) = \{q^m - 1; 1\};$$
$$\mathrm{IA}(C^{(1)}) = \{q^m(q^m-1), q^m - 1; 1, q^m(q^m-1)\}.$$

(ii) *For $2 \le r \le n$, $B^{(r)}$ is a $[rn, rn-2m, 3; 2]_q$-code and $C^{(r)}$ is a $[(q^m+1-r)n, (q^m+1-r)n-2m, 3; 2]_q$-code. Both codes are completely regular with intersection arrays*

$$\mathrm{IA}(B^{(r)}) = \{r(q^m-1), (r-1)(q^m+1-r); 1, r(r-1)\};$$
$$\mathrm{IA}(C^{(r)}) = \{(q^m+1-r)(q^m-1), r(q^m-r); 1, (q^m+1-r)(q^m-r)\}.$$

Furthermore, $B^{(r)}$ is completely transitive if and only if $C^{(r)}$ is completely transitive.

Remark 2.5. In the binary case, $q = 2$, the matrix H_{2m} can be written as

$$H_{2m} = \left[\begin{array}{cccc|cccccc} H & H & \cdots & H & H & \cdots & H & H & \mathbf{0} & H \\ H_1 & H_2 & \cdots & H_r & H_{r+1} & \cdots & H_{n-1} & \mathbf{0} & H & H \end{array}\right], \tag{6}$$

where H is a parity check matrix of a binary cyclic Hamming code $\mathcal{H}_m$.

Indeed, H_{2m} has $n(n+2) = 2^{2m} - 1$ columns and there are no repeated columns. Therefore, in the binary case and by Lemma 2.1, the code $C^{(r)}$ is equivalent to the code $A^{(n-r-1)}$ (Construction II in [4]). $\triangle$

3. Completely transitive families

For the rest of the paper, let $B^{(r)}$ be the code with parity check matrix $H_b(r)$ given in (1). Let $T_1, \ldots, T_r$ be the n-sets, which we call *blocks*, of coordinate positions corresponding to the columns of $H_1, \ldots, H_r$. That is,

$$T_j = \{(j-1)n+1, (j-1)n+2, \ldots, jn\}, \quad \forall j = 1, \ldots, r.$$

Consider also the vectors of weight one indexed as follows:

$$e_1^1, \ldots, e_n^1, e_1^2, \ldots, e_n^2, \ldots, \ldots, e_1^r, \ldots, e_n^r;$$

where the super-indices indicate the corresponding block.

In order to establish in which cases the codes of Theorem 2.4 are completely transitive, we need several previous results.

Lemma 3.1. *Let $n = (q^m-1)/(q-1)$ with $\gcd(n, q-1) = 1$. For $1 < r \leq n$, let $B^{(r)}$ be the $[nr, nr-2m, 3; 2]_q$-code with parity check matrix given in (1). Then, the number of cosets of $B^{(r)}$ with minimum weight 2 is $q^{2m} - 1 - r(q^m-1)$.*

Proof. Since $B^{(r)}$ has minimum distance 3, we have that the number of cosets of weight 1 is $rn(q-1) = r(q^m-1)$. The total number of cosets is $q^{rn}/(q^{rn-2m}) = q^{2m}$. Hence, the number of cosets with minimum weight greater than 1 is $q^{2m} - 1 - r(q^m-1)$. All such cosets have minimum weight 2, because the covering radius of $B^{(r)}$ is $\rho = 2$. $\square$

Lemma 3.2. *Let $B^{(r)}$ be the $[nr, nr-2m, 3; 2]_q$-code with parity check matrix given in (1), where $n = (q^m-1)/(q-1)$ and $\gcd(n, q-1) = 1$. Then,*

(i) $\mathrm{MAut}(B^{(1)}) \cong \mathrm{GL}(m, q)$.
(ii) $\mathrm{MAut}(B^{(2)}) \cong \mathrm{GL}(m, q) \times \mathrm{GL}(m, q) \rtimes C_2$.
(iii) $\mathrm{MAut}(B^{(3)}) \cong \mathrm{GL}(m, q) \times S_3$.

Proof. The statement (i) is trivial, since $B^{(1)}$ is a Hamming code of length $(q^m-1)/(q-1)$. It is well known that its automorphism group is isomorphic to $\mathrm{GL}(m, q)$ [12].

For (ii), note that, after linear operations on the rows, the parity check matrix $H_b(2)$ of the code $B^{(2)}$ can be presented in the form

$$H_b(2) = \left[\begin{array}{c|c} H & O \\ O & H \end{array}\right].$$

Since the Hamming code of length $(q^m-1)/(q-1)$ (defined by the parity check matrix H) is stabilized by the group $\mathrm{GL}(m, q)$ by left multiplication, we have that the code $B^{(2)}$ is invariant under the action of any matrix G of type

$$G = \left[\begin{array}{c|c} G_1 & O \\ O & G_2 \end{array}\right], \tag{7}$$

where both matrices G_1 and G_2 are arbitrary matrices from the group $\mathrm{GL}(m, q)$. We conclude that the monomial automorphism group $\mathrm{MAut}(B^{(2)})$ contains as a subgroup the group $\mathrm{GL}(m, q) \times \mathrm{GL}(m, q)$. From the other side, we can clearly interchange the blocks corresponding to the columns of G_1 and G_2, respectively, without changing the code $B^{(2)}$. Hence the group $\mathrm{MAut}(B^{(2)})$ contains C_2 as a subgroup of order 2. It is easy to see that any monomial automorphism of $B^{(2)}$ contains only automorphisms from these groups above. Indeed, assume that the matrix G belongs to $\mathrm{GL}(2m, q)$, but not to $\mathrm{GL}(m, q) \times \mathrm{GL}(m, q)$. Then we can see (by solving the corresponding system of linear equations) that any column of $H_b(2)$ can be moved to any other column. But this leads to a contradiction to the shape of $H_b(2)$. Hence, we conclude that $\mathrm{MAut}(B^{(2)})$ contains as subgroups only $\mathrm{GL}(m, q) \times \mathrm{GL}(m, q)$ and C_2. However, C_2 is not normal in $\mathrm{MAut}(B^{(2)})$. Taking $g \in \mathrm{GL}(m, q)$ and σ the non-trivial element in C_2 we have:

$$\left[\begin{array}{c|c} g & 0 \\ 0 & I \end{array}\right] \sigma \left[\begin{array}{c|c} g^{-1} & 0 \\ 0 & I \end{array}\right] = \left[\begin{array}{c|c} g & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{c|c} 0 & I \\ I & 0 \end{array}\right] \left[\begin{array}{c|c} g^{-1} & 0 \\ 0 & I \end{array}\right] =$$
$$\left[\begin{array}{c|c} 0 & g \\ I & 0 \end{array}\right] \left[\begin{array}{c|c} g^{-1} & 0 \\ 0 & I \end{array}\right] = \left[\begin{array}{c|c} 0 & g \\ g^{-1} & 0 \end{array}\right],$$

and the result does not belong to C_2. Here, I stands for the $m \times m$ identity matrix.

Finally, we have that

$$\mathrm{MAut}(B^{(2)}) = \mathrm{GL}(m, q) \times \mathrm{GL}(m, q) \rtimes C_2\,.$$

For (iii), the parity check matrix $H_b(3)$ (see (1)) of $B^{(3)}$ can be transformed, by linear operations on the rows and column permutations, to the following form:

$$H_b(3) \simeq \left[\begin{array}{c|c|c} H & O & H \\ O & H & H \end{array}\right].$$

The exclusive property of this matrix is that any of all three blocks is stabilized by all matrices G of the type

$$G = \left[\begin{array}{c|c} G_1 & O \\ O & G_1 \end{array}\right],$$

where G_1 is an arbitrary matrix from the group $\mathrm{GL}(m,q)$. We conclude that the group $\mathrm{MAut}(B^{(3)})$ contains as a subgroup the group $\mathrm{GL}(m,q)$. Now we can see that any codeword $z=(x,y,u)$ of $B^{(3)}$ can be presented in the form $z=(u+x_1, u+x_2, -u+x_3)$, where x_i is a codeword of the Hamming code (of length $n=(q^m-1)/(q-1)$, i.e. it has the support only in the block T_i, $i=1,2,3$, and u is an arbitrary vector of length n). We deduce that all three blocks can be arbitrary permuted without changing of the code $B^{(3)}$ and, therefore, $\mathrm{MAut}(B^{(3)})$ contains as a subgroup the group S_3. Similarly to the previous case, we can see that these are the only monomial automorphisms of the code $B^{(3)}$. Indeed, if for example $G_1 \neq G_2$, where G is of the form (7) (see case (ii)), we change the third block of $H_b(3)$ and obtain a column which does not belong to the third block of $H_b(3)$. Since any element of $\mathrm{GL}(m,q)\times \mathrm{GL}(m,q)$ cannot interchange the blocks and since they intersect only in the identity element, we conclude that the group $\mathrm{MAut}(B^{(3)})$ is the direct product of groups $\mathrm{GL}(m,q)$ and S_3, i.e.

$$\mathrm{MAut}(B^{(3)}) = \mathrm{GL}(m,q) \times S_3\,.$$

Evidently, the actions of both $\mathrm{GL}(m,q)$ and S_3 do not depend on each other. $\square$

The next statement shows some cases in which the code $B^{(r)}$ is completely transitive (hence, so is $C^{(r)}$).

Proposition 3.3. *The codes $B^{(1)}$ and $B^{(2)}$ are completely transitive. Furthermore, for $q=2$, the codes $B^{(3)}$, $B^{(n-1)}$ and $B^{(n)}$ are also completely transitive (here $n=(q^m-1)/(q-1)$ and $\gcd(n,q-1)=1$).*

Proof. The code $B^{(1)}$ is a Hamming code. Its monomial automorphism group is isomorphic to $\mathrm{GL}(m,q)$ [12] and $\mathrm{MAut}(B^{(1)})$ acts transitively over the set of vectors of weight one. Hence, all cosets with minimum weight one are in the same orbit and thus $B^{(1)}$ is completely transitive. Recall that a Hamming code has covering radius 1.

For $B^{(2)}$, we have that its monomial automorphism group is isomorphic to $\mathrm{GL}(m,q)\times \mathrm{GL}(m,q) \rtimes C_2$ (Lemma 3.2). Hence, $\mathrm{MAut}(B^{(2)})$ acts transitively over the set of vectors of weight one in each block and also both blocks can be interchanged. Therefore all cosets with minimum weight one are in the same orbit. By Theorem 2.2, the covering radius of $B^{(2)}$ is 2. Hence, we only need to prove that the cosets with minimum weight two are also in the same orbit. Note that any vector at distance 2 from the code has its non-zero coordinates in different blocks of coordinate positions. Since we can act to these blocks independently, we have that the cosets with minimum weight two of $B^{(2)}$ are in the same orbit.

For $q=2$, by Lemma 3.2, the automorphism group of $B^{(3)}$ is isomorphic to $\mathrm{GL}(m,2)\times S_3$. Therefore, $\mathrm{MAut}(B^{(3)})$ clearly acts transitively over the coordinate positions. Hence, the cosets of $B^{(3)}$ with minimum weight one are in the same orbit. Consider two cosets of $B^{(3)}$ with minimum weight two, $B^{(3)}+\mathbf{x}$ and $B^{(3)}+\mathbf{y}$, where $\mathbf{x}=e_i^a+e_j^b$ and $\mathbf{y}=e_k^c+e_\ell^d$. Note that

$$i\neq j, k\neq \ell, a\neq b, c\neq d;$$

otherwise both vectors $\mathbf{x}$ and $\mathbf{y}$ would not be at distance 2 from the code. Since $\mathrm{GL}(m,2)$ is doubly transitive [12], there exists $\phi \in \mathrm{MAut}(B^{(3)})$ such that $\phi(e_i^a+e_j^b)=e_k^a+e_\ell^b$. Now, consider $\theta\in S_3$ such that $\theta(T_a)=T_c$ and $\theta(T_b)=T_d$. We obtain that $\theta(\phi(\mathbf{x}))=\mathbf{y}$. Therefore all cosets with minimum weight 2 are in the same orbit. Thus, $B^{(3)}$ is completely transitive since its covering radius is 2.

For $q=2$, the supplementary codes of $B^{(n-1)}$ and $B^{(n)}$ have, respectively, parity check matrices $H_a(0)$ and $H_a(-1)$ defined in (3). We have seen that these codes are equivalent to $B^{(3)}$ and $B^{(2)}$, respectively. From Theorem 2.4, we deduce that $B^{(n-1)}$ and $B^{(n)}$ are completely transitive codes. $\square$

Lemma 3.4. *The code $B^{(3)}$ is completely transitive if and only if $q=2$.*

Proof. By Proposition 3.3 we know that $B^{(3)}$ is completely transitive for $q=2$. We have to prove that it is not for $q>2$.

The code $B^{(3)}$ is equivalent to the code C that has parity check matrix:

$$H_b(3) = \left[\begin{array}{c|c|c} H & O & H \\ O & H & H \end{array}\right].$$

Consider the set of vectors

$$S = \{\alpha e_i^1 + e_j^2 \mid (i, \alpha) \neq (j, 1),\ \ \alpha \in \mathbb{F}_q \setminus \{0\},\ \ i, j \in \{1, \ldots, n\}\}.$$

Clearly, any vector in S is at distance two from C. Indeed, if $\mathbf{x} = \alpha e_i^1 + e_j^2 \in S$, then $H_b(3)\mathbf{x}^T$ is a non-zero vector which is not a column of $H_b(3)$. Moreover, given two such vectors $\mathbf{x} = \alpha e_i^1 + e_j^2$ and $\mathbf{y} = \alpha' e_{i'}^1 + e_{j'}^2$, where $i \neq i'$ or $j \neq j'$ or $\alpha \neq \alpha'$, we have that $H_b(3)(\mathbf{x} - \mathbf{y})^T$ is a non-zero vector, thus $\mathbf{x} - \mathbf{y} \notin C$. Therefore, all vectors in S are in different cosets of C.

Compute the cardinality of S:

(i) Vectors of the form $\alpha e_i^1 + e_i^2$, $(\alpha \neq 1)$. There are $n(q-2)$ such vectors.
(ii) Vectors of the form $\alpha e_i^1 + e_j^2$, where $i \neq j$. There are $n(q-1)(n-1)$ such vectors.

Hence,

$$|S| = n(q-2) + n(q-1)(n-1) = n(q^m - 2) = \frac{(q^m-1)(q^m-2)}{q-1}.$$

As a consequence, we can find $(q^m-1)(q^m-2)/(q-1)$ vectors which are all in different cosets of minimum weight 2. Thus, if all such cosets are in the same orbit, we have that the monomial automorphisms in $\mathrm{GL}(q, m)$ of the form

$$g = \begin{bmatrix} g_1 & O \\ O & g_1 \end{bmatrix}$$

act 2-homogeneously on the first two blocks (indeed, for any vector $\alpha e_i^1 + \beta e_j^2$, we can consider a multiple $\alpha' e_i^1 + e_j^2$ in the same coset). But this implies that the action of $\mathrm{GL}(m, q)$ in only one block is 2-homogeneous, since any automorphism like g acts identically in both blocks. Thus, we have $q = 2$, by Lemma 1.4. $\square$

Remark 3.5. According to Lemma 3.1, the total number of cosets of minimum weight 2 is $(q^m-1)(q^m-2)$. Note that for $q = 2$, the number of cosets induced by the set S coincides and since the action of $\mathrm{GL}(m, 2)$ is doubly transitive, all the cosets of minimum weight 2 are in the same orbit. This is an alternative argument to see that $B^{(3)}$ is completely transitive in the binary case (Proposition 3.3). $\triangle$

To determine the monomial automorphism group of $B^{(r)}$ in the general case seems to be a challenging problem. However, many computational results using MAGMA [7], suggest that the order of the monomial automorphism group is not greater than $12n(q-1)$, for $r > 3$ except when $q = 2$ and $r \in \{n-1, n\}$ (see Conjecture 1.3). In these last two cases we have

$$\mathrm{MAut}(B^{(n-1)}) = \mathrm{MAut}(B^{(3)}) \cong \mathrm{GL}(m, 2) \times S_3, \text{ and}$$
$$\mathrm{MAut}(B^{(n)}) = \mathrm{MAut}(B^{(2)}) \cong \mathrm{GL}(m, 2) \times \mathrm{GL}(m, 2) \rtimes C_2,$$

respectively. It is clear that in these two cases the order of $\mathrm{MAut}(B^{(r)})$ is greater or equal than $12n(q-1)$. Assuming such upper bound we have that the codes not mentioned in Proposition 3.3 are not completely transitive.

Proposition 3.6. *If $B^{(r)}$ is completely transitive, then $|\mathrm{MAut}(B^{(r)})| = cn(q-1)$, where*

$$c \geq \max\{r, n(q-1) + 2 - r\}.$$

Proof. If $B^{(r)}$ is completely transitive, then $\mathrm{MAut}(B^{(r)})$ must be transitive over the set of vectors of weight one since all cosets of minimum weight one must be in the same orbit. Hence $|\mathrm{MAut}(B^{(r)})|$ is a multiple of $rn(q-1)$, say $cn(q-1)$. Moreover, $cn(q-1) \geq rn(q-1)$ since $B^{(r)}$ has $rn(q-1)$ cosets of minimum weight one. By Lemma 3.1, the number of cosets of minimum weight two is $n(q-1)(n(q-1)+2-r)$. All such cosets must be in the same orbit and thus $c \geq n(q-1)+2-r$. $\square$

Corollary 3.7. *Assume that Conjecture 1.3 is true. Then,*

(i) *If $q = 2$ and $4 \leq r \leq n-2$, then $B^{(r)}$ is not completely transitive.*
(ii) *If $q > 2$ and $4 \leq r \leq n$, then $B^{(r)}$ is not completely transitive.*

Proof. From Proposition 3.6, we have $r \leq 12$ and $n(q-1)+2-r \leq 12$. Combining both inequalities, we obtain $n(q-1) \leq 22$. Hence, $\frac{q^m-1}{q-1}(q-1) \leq 22$, so $q^m \leq 23$. Thus,

(i) If $q = 2$, then $m \leq 4$. Hence, the only possible cases are $m = 3$, $r \in \{4, 5\}$ and $m = 4$, $r \in \{4, \cdots, 12\}$.

(ii) If $q > 2$, then the only possible cases are $q = 3$, $m = 2$, $r = 4$, which is out of our scope since $\gcd(n, q-1) \neq 1$; and $q = 4$, $m = 2$, $r \in \{4, 5\}$.

For these remaining cases, $(q, m, r) \in \{(2, 3, 4), (2, 3, 5), (2, 4, 4), \ldots, (2, 4, 12), (4, 2, 4), (4, 2, 5)\}$, we have verified with MAGMA that the corresponding codes are not completely transitive. $\square$

From Proposition 3.3, Lemma 3.4, and Corollary 3.7 we can state the main result of this paper.

Theorem 3.8. *Let $B^{(r)}$ be the code over $\mathbb{F}_q$ of length $rn = r(q^m - 1)/(q - 1)$ ($1 \leq r \leq n$), with $\gcd(n, q-1) = 1$ and parity check matrix given in (1). Let $C^{(r)}$ be the supplementary code of $B^{(r)}$, such that the concatenation of the parity check matrices of $B^{(r)}$ and $C^{(r)}$ is H_{2m}, the parity check matrix of a Hamming code of length $(q^{2m} - 1)/(q - 1)$. Then,*

(i) *For $q = 2$, the codes $B^{(r)}$ and $C^{(r)}$ are completely transitive if $r \in \{1, 2, 3, n-1, n\}$.*
(ii) *For $q > 2$, the codes $B^{(r)}$ and $C^{(r)}$ are completely transitive if $r \in \{1, 2\}$.*
(iii) *For $q > 2$, the codes $B^{(3)}$ and $C^{(3)}$ are not completely transitive.*
(iv) *For the remaining cases (neither in (i), (ii) nor (iii)) the codes $B^{(r)}$ and $C^{(r)}$ are not completely transitive, assuming that in these cases the order of $\mathrm{MAut}(B^{(r)})$ is not greater that $12n(q-1)$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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