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ORIGINAL PAPER

Contributions of the anthropological theory of the didactic to the epistemological programme of research in mathematics education

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Abstract

In the early 1980s, the *theory of didactic situations* put the analysis of mathematical activity at the core of the didactic analysis, thus initiating a new research programme in mathematics education: the *epistemological programme*. This paper describes and interprets some of the contributions of the *anthropological theory of the didactic* (ATD) to the development of this programme. These contributions come from the new institutional and ecological perspective provided by the ATD to approach the didactic problem and from the significance it assigns to macrodidactic phenomena characterised by didactic transposition. They are illustrated with research works involving different fields of school mathematics in which the ATD team led by Spanish researchers played a central role: research in school algebra and its relations with functional modelling, differential calculus, and real numbers. The paper concludes by briefly describing the contributions of the ATD to the dialogue between didactic theories, another domain that has notable Spanish contributions.

Keywords Theory of didactic situations · Epistemological programme · Anthropological theory of the didactic · Elementary algebra · Real numbers · Limits of functions · Elementar differential calculus

Our history of ideas derives from our ideas about history, that is, from our own intellectual point of view.
Adventures of ideas.
Whitehead (1933).

1 Preamble

In this paper, I will use the *rational reconstruction* (Lakatos, 1971) of one of the lines of evolution of the didactics of mathematics proposed in (Gascón, 2003). This reconstruction essentially contemplates two successive extensions and transformations of the object of study of didactics that give rise, respectively, to two *research programmes* (Lakatos, 1978a) in mathematics education: the *cognitive programme* and the *epistemological programme*. A *scientific research programme* is made up of a series of developing theories related to each other historically and logically in such a way that the latter arise from the preceding ones. It is made up of three main components: a *hard core*, formed by a few postulates shared by the theories that are part of the programme and that is considered provisionally irrefutable by methodological decision; a *protective belt*, formed by auxiliary hypotheses that can be modified to protect the core from possible falsifiability; and a *heuristic*, or set of methodological rules with two aspects, positive and negative. The *positive heuristic* provides techniques for solving problems and largely determines the selection of the type of problems that the programme will privilege. The *negative heuristic* tells which research paths should be avoided to develop the programme in a *progressive* manner. This description of research programmes will be used in the paper as a metadiscourse to explain the epistemological programme of research in mathematics education and the contributions of the ATD in it.

The cognitive programme can be approached to what Brousseau called "the classical approach in didactics", while the epistemological programme includes what he designated as "fundamental didactics" in the first developments of the *theory of didactic situations* (TDS). Brousseau characterises the "classical approach in didactics" as the approach which, in the explanation of didactic facts, considers the *cognitive activity of the subject* to be central (Brousseau, 1986). For

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lack of space, and because it is not part of the objectives of this paper, I will not analyse the emergence or the development of the cognitive programme either. I will begin by describing, in a very schematic way, the birth of the TDS and the consequent emergence of the epistemological programme. The contributions of the *anthropological theory of the didactic* (ATD) to this programme will then be located at that moment, without forgetting that both theories continue to develop with mutual influences.

A first objective of this paper is to show that ATD shares with TDS some of the postulates that characterize the epistemological programme of research in mathematics education. But the main objective is to show how ATD takes up and develops these postulates, providing new theoretical and methodological instruments that allow the formulation and approach of new types of problems, and the explanation of new didactic phenomena. In this sense, one of the most relevant achievements of the ATD is the new institutional approach to didactic problems and the corresponding enlargement of the unit of analysis to include all the stages of didactic transposition. This new approach makes it possible to prioritize the construction and study of macrodidactic phenomena by means of a methodology that emphasizes the ecological dimension of didactic problems. The four investigations briefly described in Sect. 4 exemplify the pertinence of these instruments and show the relevance of the contributions of ATD to the development of the epistemological programme.

2 The *Brousseaunian* revolution as the origin of the epistemological programme

The TDS (Brousseau, & Warfield, 2020) was initiated and developed by Guy Brousseau in the 1980s.[1] At that moment, it radically broadened the object of study of didactics of mathematics by including the need to model the school mathematical activity, given that every discipline models what it questions and what it seeks to explain. Moreover, the TDS modified the nature of didactic research problems because it inverted the problem posed by the "classical approach in didactics":

From the perspective of the theory of didactic situations, learners become the revealers of the characteristics of the situations to which they react (it is important to note this reversal of position concerning approaches in psychology, where situations are usually studied as a tool for revealing the learner's knowledge) (Brousseau, 2007, p. 24, our translation).

Consequently, the TDS considers "didactic situations" as its main object of study and, at the same time, the model to propose alternative descriptions of mathematical knowledge. A didactic situation includes the *types of interactions* of a *generic subject* having a certain *environment* (the *milieu*) that determine certain *mathematical knowledge*. These interactions are considered as the optimal resource available to the subject to reach or maintain a favourable state in this environment. This inversion of the object of study requires questioning the usual ways of describing and interpreting *mathematical knowledge.* Indeed, the TDS questions the conceptions of the different fields of mathematics that prevail in different educational and mathematical institutions. From the domain of didactics, it asks how to conceive knowledge objects or fields like proportionality, decimal numbers, counting, geometry, the relationship between statistics and probability, among others.

To answer these questions, and thus to be able to tackle the new didactic problems, the TDS constructs epistemological models of these fields of knowledge. This is the origin of the name "experimental epistemology" (of mathematics) that Brousseau initially gave to didactics of mathematics, moved by the ambition of explaining not only the *development* and *dissemination* of mathematics, but also its *genesis*.

Didactic theory can revolutionise epistemology and transform the description of the construction of [mathematical] knowledge [...]. It is a challenge, but it is understood that this challenge is linked to the ambition of constructing didactics (Brousseau, 1996, p. 40, our translation).

In what sense do we speak of a *Brousseaunian revolution in mathematics education* (Gascón, 2013)? How does the TDS transform the nature of the *hard core* of didactics? How does it change and expand its object of study, proposing a new *heuristic*? To begin with, let us say that the postulates that constituting the hard core of TDS are based on an epistemological model of mathematics that expands what was considered "mathematical" in the usual epistemological theories[2]. In the TDS, a *mathematical piece of knowledge* is defined by the *situations* that determine it, that is, by a set of situations for which such knowledge is suitable because it provides the optimal solution in the context of a certain institution (Brousseau, 1994). Situations contain the "raison d'être" of the piece of knowledge they define, i.e.,

[1] A collection of his works published between 1970 and 1990 can be found in (Brousseau, 1997). The website http://guy-brousseau.com/ contains a large part of his scientific output.

[2] The *Euclidean Programme*, consisting of *logicism* (Russell, 1903), *formalism* (Hilbert, 1923) and *intuitionism* (Heyting, 1956), was initially characterised by Lakatos (1978b) and did not require any empirical basis because it identified the problem of what mathematics is with a purely logical problem. With the evolution of this problem, epistemology became an experimental discipline and required successive extensions of its empirical basis, considering historical data (like Lakatos' work), then psychogenesis data (Piaget & García, 1982). The emergence of the epistemological program required to also integrate didactic facts (Gascón, 2001).

the problems that give it meaning, as well as the conditions under which it manifests itself, the constraints that limit its use in a given institution, and its potential applications. Thus, given knowledge such as, for example, *proportionality*, the idea is to produce an *adidactic situation* (in Brousseau's language) for which this knowledge provides the optimal solution. In this case, the students, who lack mathematical knowledge about proportionality, must enlarge the pieces of a puzzle so that a segment of one of the pieces, which measures 4 cm, must measure 7 cm in the reproduction (Brousseau, 1997).

As for *positive heuristics*, the new research programme initiated by the TDS changes the old questions that constituted the primary objects of research in mathematics education: how students learn mathematics, what difficulties they encounter, through what mechanisms or cognitive processes they acquire mathematical concepts (or how they construct them), what methods are the most appropriate for teaching these concepts, etc. (Dubinsky, 1991; Harel & Kaput, 1991; Novak, 1977; Tall & Vinner, 1981). These objects are replaced by questions of a different nature: What conditions should a situation satisfy to enact the specific knowledge it models? What are the foreseeable effects of this operation on the subjects involved and their productions? What game should be played for the knowledge to be needed? What information or relevant feedback should the subjects receive from the environment to orient their choices and enact one kind of knowledge instead of another? These questions lead to consider the milieu (the environment) as an autonomous system, antagonistic to the subject, of which it is appropriate to make a model, as a kind of automaton (Brousseau, 2007). In the above example, the puzzle situation constitutes a device that is part of the milieu and that "responds to the subject" by following certain rules. If the students end up producing a winning strategy, it is not thanks to their ability to decipher the teacher's didactic intentions, but because the feedback from the milieu (the fact that the pieces of the puzzle do not match) will have rejected the inoperative strategies and validated the correct ones.

The TDS thus transformed a problem centred on the cognitive mechanisms or processes that describe and explain *how students learn* into one aimed at explaining the conditions that situations should fulfil to *facilitate (or hinder*) *the learning of certain pieces of knowledge* and the *effects of such learning* as manifested in the subject's productions. The *negative heuristic* of the TDS considers the enquiry into the cognitive processes of learners and teachers as research paths to be avoided. For the TDS, what has to be explained (and thus modelled) are not the *psycho-cognitive mechanisms* underlying the learning process, but the *situations*, considered as models of bodies of mathematical knowledge. In short, the TDS changes the hard core and the methodological rules that constitute the heuristics of classical didactics. By doing this, it initiated and began to develop a new research programme in mathematics education: the *epistemological programme* (Gascón, 2003). In the language of (Lakatos, 1978a), the transition from the cognitive programme to the epistemological programme constitutes a *progressive change of the problems addressed*, with the consequent increase in the *heuristics power* of the new research programme. This increase is corroborated by the appearance of new types of problems, new auxiliary theories and the anticipation of new facts and phenomena.

3 Contributions of the anthropological theory of the didactic to the epistemological programme

The TDS, like any scientific theory, provides a new way of interpreting a part of the world. The ATD (Chevallard, 1992, 2019a, 2019b; Chevallard & Bosch, 2020a, b), which emerged within the TDS, assumes the vision of the didactic universe proposed by the TDS, locating itself openly in the epistemological programme of research in mathematics education and sharing some of its main assumptions:

- The ATD shares with the TDS the inclusion of school mathematical activity as a primary object of research and the corresponding questioning of the epistemological models of mathematics prevailing in school institutions. Consequently, it assumes the need to *construct alternative models* as a starting point for studying didactic phenomena.
- The ATD agrees with the TDS in that didactic phenomena refer not only to the *dissemination*, but also to the *genesis* and *development* of mathematics, which establishes that "the mathematical" and "the didactic" are, in a certain sense, inseparable. This broadens the domain of *didactic phenomena* and establishes a new relationship between didactics and the epistemology of mathematics as experimental disciplines (see note 2).
- The ATD, like the TDS, is concerned with the *conditions that facilitate* (*or hinder*) *learning* and is interested in the *effects* of learning on subjects' productions. The psycho-cognitive processes involved in learning are not part of its object of study. *Neither the TDS nor the ATD claim to be theories of individual learning*.

In what follows, we will see that many of the contributions of the ATD to the epistemological programme go in the direction of developing, specifying, and analysing these characteristics that initially defined the epistemological programme. Other contributions of the ATD refer to the importance it assigns to the *institutional dimension* of didactic processes, to the *macrodidactic phenomena* characterised by *didactic transposition*, and the *economic-ecological problems*, among others. This general description of the contributions will be later illustrated and further explained with four cases of study.

3.1 Relationship between persons, institutions and praxeologies

In the ATD, the notions of *person* and *institution* are primitive. A very general notion of institution is used, which is compatible with the one used in (Douglas, 1986) and with the one proposed in (Mosterín, 2008).

Knowledge is modelled by the notion of *praxeology*, i.e., a dialectical and indissoluble union of *know-how* or *praxis* [T, τ] −procedural *knowledge*− formed by *types of tasks* T and *techniques* τ; and *knowledge* or *logos* [θ, Θ] −*declarative* or *propositional* knowledge−, which consists of two successive levels of description and justification of the praxis: a *technology* θ and a *theory* Θ, and which is expressed through reasoned discourse (Chevallard, 1992). The *praxeological equipment* of a person or an institution is the complex of praxeologies that can be mobilised by this person or institution (Chevallard, 2011). Finally, the notion of *learning* can be described as a change in the (personal or institutional) *praxeological equipment*.

The personal relation to a praxeology is the result of the history of the person's institutional subjections, past and present. Reciprocally, an institution - and the praxeologies it hosts, contains, and develops - could not exist without the subjects' actions within the institution. Despite the person-institution dialectic, in the current state of development of didactics and for methodological reasons, the ATD prioritises the institutional perspective to approach didactic problems in front of the individual one. Institutional praxeologies are assumed to condition and limit what individuals do, especially what and how they are taught, what they are taught for and, to a large extent, what they can learn. Moreover, the analysis of institutional praxeologies provides the terms to describe and explain the individuals' behaviour.

3.2 Didactic transposition, units of analysis and macrodidactic phenomena

Consequently, for the ATD, the explanation of didactic facts is first of all based on the analysis of the praxeologies available in the teaching institution, which are the result of a *process of transposition*. The *theory of didactic transposition* (Chevallard, 1985/1991), which constitutes the germ of the ATD (Bosch & Gascón, 2006; Chevallard 1992, 1999; Chevallard et al., 1997), emphasises that, for the knowledge coming from an institution I_1 to be adapted to the epistemological ecology of another institution I_2, it must undergo a transposition process that usually consists of the complex reconstruction and reorganisation of the praxeologies that constitute it. The analysis of the adaptations and transposition transformations that institutional praxeologies undergo is therefore central to the ATD research methodology because these changes condition the possible *modalities of study* and, consequently, the possible learning processes that can take place in a teaching institution.

The ATD makes explicit and extends the *fundamental methodological principle* of the TDS by stressing that, to adequately interpret didactic phenomena, it is not only necessary to take school mathematics as an object of study. It is also unavoidable to question and model the way mathematics is interpreted and manipulated in all institutions that produce, develop, use and disseminate mathematics. This shatters the illusion of a single, transparent, and unquestionable vision of knowledge assumed by school institutions. The notion of a *didactic phenomenon* is thus further extended and remains irreducible to the associated cognitive, sociological, linguistic, or semiotic phenomena.

Every experimental discipline takes, more or less explicitly, a unit of analysis which is both the basic theoretical construct and the domain in which all empirical data will be analysed. Because of considering transposition phenomena, the *unit of analysis* of the ATD encompasses all the stages involved in the process of didactic transposition, including, in addition to the *community of study* and the *school institution*, the *noosphere* and the *scholarly* (or *knowledge-producing*) *institutions*, together with the corresponding forms of knowledge (Fig. 1) (Bosch & Gascón, 2006).

The *unit of analysis* of didactic processes is thus extended by giving priority (for methodological reasons) to *macrodidactic phenomena*, i.e., phenomena that encompass all the institutions involved in didactic transposition processes. Some examples of macrodidactic phenomena will be illustrated in Sect. 4.

The ATD thus draws attention to the fact that didactic phenomena (even those apparently confined to the community of study) can only be explained properly when they are interpreted as part of or concern *macrodidactic phenomena*.

3.3 The central role of ecological issues

The ATD broadens and specifies the universe considered by didactics, and studies the *ecological problem* in more detail, that is, the set of conditions that affect the didactic universe. The main tool to approach this problem is the *scale of levels of didactic co-determinacy* (Chevallard, 2002). This scale ranges from disciplinary and sub-disciplinary conditions to pedagogical, school, social, civilisational and

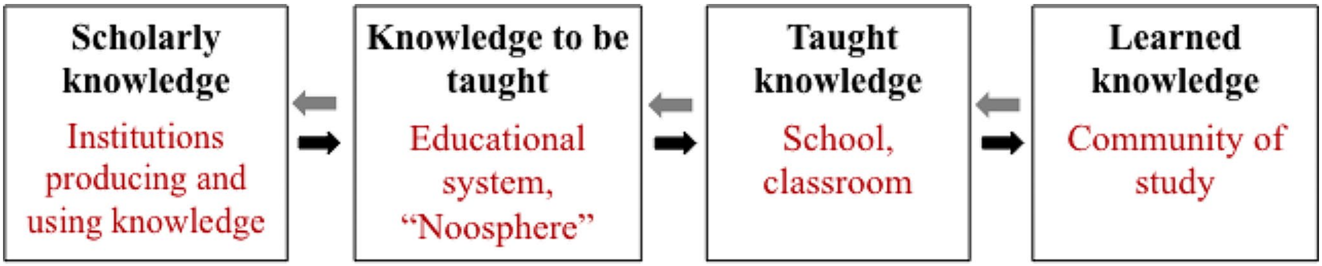

Fig. 1 Stages in the process of didactic transposition

Fig. 2 Scale of levels of didactic co-determinacy

anthropological conditions (Fig. 2). At each level, specific conditions emerge which, in principle, can influence any of the other levels (the arrows in the diagram suggest such interactions).

3.4 Praxeological analysis as a gateway to didactic analysis

We call *praxeological analysis* the analysis of (personal and institutional) knowledge which, as we have said, is modelled in praxeological terms. To carry out this analysis, the ATD constructs *reference epistemological models* (REMs), which are *limit cases*, i.e., *ideal types* in the sense of (Weber, 1904/2009). An REM can be described in terms of an arborescence of *mathematical praxeologies* which, in turn, can be interpreted as tentative answers to questions that might arise in the course of a hypothetical process of enquiry. The theory of didactic transposition has taught us that there is no privileged, absolute, and universal reference system for the knowledge at stake (as is the case with reference systems in mechanics). However, this fact does not make it less essential to use an REM as a provisional, relative, and local reference, appropriate to each situation.

To construct an REM, empirical data are taken from all the institutions involved in didactic transposition processes. An REM is first constructed, to support the praxeological analysis of a certain curricular *field*, designated as an object of study and constituted by a *diffused praxeological conglomerate*, such as "proportionality", "analytical geometry", "differential calculus", or "conditional probability". The praxeological analysis of a field of mathematics is a *means* to advance didactic research. It starts when a *didactic phenomenon* becomes apparent, that is, a set of *didactic facts* that are remarkable and surprising (from a certain perspective), that admit a generic description, that are regularly repeated in certain circumstances, and therefore require an explanation. Here appears the need for an REM to support the praxeological analysis of the knowledge at stake, as we will see in the cases of Sect. 4, which will also partially illustrate the REMs' *phenomenotechnical function*[3] (Lucas et al., 2019).

[3] REMs are constructed in didactics of mathematics as heuristic tools to *make visible certain didactic phenomena*. More precisely, REMs can be helpful as design principles for didactic engineering, which aim

Through praxeological analysis, we obtain a *representation* (of some features) of the knowledge at play, which we call the *current epistemological model* (CEM) in the considered institution, and which depends on the REM taken as a reference. We usually speak of an REM, and of the corresponding CEM, around a field F of mathematics, designated as an object of study in a school institution *I*. We will denote them, respectively, by $REM_I(F)$ and $CEM_I(F)$. For the same field F, different $REM_I(F)$ can be constructed in association with different didactic phenomena, as we will also see in the examples.

3.5 Didactics as the science of study and modalities of study

Through the notion of *study*, the ATD proposes a unitary framework to jointly describe and analyse all that can be done with knowledge in social institutions: its genesis, teaching-learning, use, and dissemination. This leads to considering didactics as the *science of study* (Chevallard et al., 1997; Gascón, 1997) and, by extension, of *modalities of study* and *their impact on learning.* Among the most basic modalities of study (in the case of mathematics globally considered) are *theoricism*, *technicism* and *modernism*. These are *ideal* modalities of study (which have never existed in any institution) constructed from research. *Theoricism* identifies mathematical education with showing students perfectly finished and crystallized mathematical theories. *Technicism* constitutes a first reaction to the technical vacuum caused by theoricism. In order to "go back to basics", it emphasizes the most rudimentary aspects of the work of the techniques, focusing essentially on algorithmic procedures. And finally, *modernism* reacts against both modalities, and identifies the learning of mathematics with the free exploration of non-trivial problems (Gascón, 2001).

Each possible modality of study in an institution *I* around a certain field F can be characterised (or modelled) by the notion of *didactic paradigm*. The *theory of didactic paradigms* (Gascón & Nicolás, 2019, 2021; Gascón, 2024) constitutes one of the latest contributions of the ATD to the epistemological programme, but, for lack of space, we will not deal with it here.

Given a praxeological field F in an institution *I*, each possible modality of study characterises and governs a particular type of *study processes* (of F in *I*) that will be conducted by a *study community* [*X*, *Y*], where *X* is a group of students and *Y* the teachers or directors of the study, which is constituted as a *didactic system*, *S*(*X*, *Y*, F), in *I*. The analysis of such study processes (called *didactic analysis*) enables describing, interpreting, and evaluating the modality of study that governs them. Ultimately, therefore, *didactic analysis* deals with *modalities of study*, not with *isolated study processes*.

The current modality of study in an educational institution is influenced by multiple conditions. On the one hand, there are the conditions (and transformations) caused by transposition processes. On the other hand, there are all the pedagogical, school, social, religious, economic, etc. conditions from all levels of the scale of co-determinacy (Chevallard, 2002) that have historically delimited the modality of study in question.

These conditions largely determine the *didactic gestures* that *X* and *Y* can carry out, act on *what students can learn* and the possible effects of this learning. As a result, an important part of the object of study of didactics will be to investigate *what* the current (or possible) modalities of study in school institutions *are* and *how they condition learning*, i.e., what the rules and principles that regulate their structure and functioning (their *economy*) are. In addition, didactics will investigate the set of conditions of all kinds that have influenced and influence these modalities of study, i.e., *why they* have come to be the way they are, *how they could be modified* in a certain direction, and *what the consequences* of such a modification *would be* (their *ecology*). In short, an important part of the subject of didactics focuses on the analysis of the *economy and ecology of current* (or *possible*) *modalities of study* in an institution. These analyses, globally considered, and supported by the praxeological analysis, constitute the *didactic analysis*.

The ATD assumes, in coherence with its basic assumptions, modalities of study based on enquiry. The *didactic means* proposes to achieve the *didactic ends* it advocates are based on *study and research paths* (SRP) (Bosch, 2018; Chevallard, 2015; García et al., 2019), which are *didactic devices* particularly suitable for overcoming the restrictions that hinder the institutional life of the activity of enquiry and the activity of mathematical modelling (Barquero et al., 2018). SRPs have been designed and tested mainly in secondary education (García, 2005; Llanos & Otero, 2015; Parra et al., 2013; Ruiz-Munzón, 2010) and in university education (Barquero, 2009; Barquero et al., 2022a; Florensa, 2018; Lucas, 2015; Sureda & Rossi, 2024).

The notion of SRP in the domain of teacher education (SRP-TE) was introduced in (Ruiz-Olarría, 2015) and is characterised by a problematic question for the teaching profession, such as: *what to teach and how to teach secondary school students about integers*, and by a modular structure. Two types of SRPs-TE can be distinguished: those based on an SRP previously experienced by students at a certain educational level (Barquero et al., 2018, 2022a, b;

to *create didactic phenomena* with the help of didactic devices. In this way, didactics can emancipate itself from the current epistemological model in the concerned institutions and autonomously construct its own objects of study.

Ruiz-Olarría, 2015) and those designed directly for teacher education (Bosch et al., 2023; Licera, 2017; Sierra, 2006).

4 Research illustrating some of the contributions of the anthropological theory of the didactic

To illustrate the above-mentioned contributions of the ATD, I will describe a sample of research by Spanish, Portuguese, and Latin American didacticians. I will focus exclusively on a few fields of school mathematics: elementary algebra, real numbers, limits of functions, and elementary differential calculus. I will end by highlighting some of the contributions of the ATD to the dialogue between didactic theories.

4.1 Elementary algebra

The first works on the didactic transposition of algebra (Chevallard, 1989) showed that algebra is first an instrument – the algebraic instrument– which culminates in the process of *algebraic modelling* and completely transforms the conditions of mathematical work. From the perspective of this epistemological position, which can be taken as an REM (later made explicit in Ruiz-Munzón, 2010), a praxeological analysis of elementary algebra in secondary education was carried out. This analysis found a set of "undesirable" didactic facts, such as the purely *formal* manipulation of algebraic expressions, the absence of *technological questioning* of techniques, and the separation between the uses of *formulae* (which do not play the role of models) and the *functional language* (Bolea, 2003). In short, the *pre-algebraic nature of school mathematics* was confirmed. Consistent with the above, the current epistemological model of algebra in secondary school, CEM_S(Alg), was characterised as a *generalised arithmetic* that manifests itself in the identification of elementary algebra with "algebraic symbolism" (or algebraic language), as opposed to, but also as a development of, a supposed "arithmetical language". The study processes governed by the current modality of study, based on the CEM_S(Alg), have important "limitations", among which the disconnections between generalised arithmetic and *functional modelling* and the *didactic obstacle* caused by the consideration of whole numbers as *arithmetical objects* stand out (Gascón et al., 2017).

The precise formulation of a reference epistemological model of elementary algebra, REM_S(Alg), which interprets algebra as an *algebraisation tool*, is described in (Bosch, 2015; Gascón et al., 2017; Ruiz-Munzón, 2010). This model allows us to begin to account for the above phenomena and provides reasons for why the model that identifies elementary algebra with a kind of generalised arithmetic is still valid in educational institutions. The REM_S(Alg) describes the *process of algebraisation* in three stages. Each of them takes the form of a mathematical praxeology that contains and extends the previous one and is generated by a type of problem whose general structure is characterised beforehand. This REM_S(Alg) supports a new modality of study, with new *didactic ends* and new *didactic means*.

Regarding the *ecological* restrictions limiting the school development of elementary algebra as a modelling tool, we can cite, in the first place, those coming from the higher levels of the scale of didactic codetermination (*Civilisations* and *Societies*). In our civilisation, the role that scientific formalisms can play as instruments of scientific thought is not sufficiently valued and, consequently, working with algebraic expressions is not a work that *can easily be culturized* (in the sense of being accepted by everyday culture), since it is a formalism that *was born as a written language* and does not always have a clear referent in verbal discourse (Bosch, 2015; Chevallard, 1989).

There are also restrictions coming from the pedagogical level due to the growing influence of the *psychopedagogical* paradigm (Gascón, 2024), which tends to eliminate some of the most characteristic aspects of the *mathematical discipline* with the "good intention" of avoiding the *bewilderment of students* and having them drop out. Thus, the mathematics teaching process is fragmented, turning school mathematics into an *atomized* set of isolated activities, eliminating long-term objectives and, thus, hindering the development of algebra as a modelling tool (García et al., 2006). This atomization, in turn, is reinforced by the current *monumentalism* in school institutions (Chevallard, 2015). Finally, there are restrictions stemming from specifically mathematical levels, such as the *exclusive and unidirectional linking of school algebra with numerical work*, which leads to its isolation from the rest of the fields of school mathematics and makes it extremely difficult to develop elementary algebra as a modelling tool for any type of system (Gascón et al., 2017).

4.2 Real numbers

Faced with the ambiguities and contradictions in the school's treatment of decimal approximations, a *teaching problem* arises: what to teach and how to teach "real numbers" in the last stage of secondary education? Logically, the research begins by analysing the institutional response to this question, i.e., the response of the current modality of study of real numbers. To this end, in (Licera, 2017) a REM_S(R) about real numbers is sketched as a scientific hypothesis. This REM is constructed based on the following general criteria: to cover the entire institutional sphere concerning the teaching of mathematics in secondary schools, including teacher education; to recover the *raisons d'être* for the existence of real numbers, that is, to delimit problematic issues whose approach highlights the relevance and fruitfulness of extending the number system beyond rational numbers; to take into consideration, in a fundamental way, the problem of the measurement of continuous magnitudes and their relationship with real numbers; and to solve the technical problems related to the representation, comparison, and calculation with real numbers and the unavoidable "decimal approximations".

From the perspective provided by this REM, questions are raised that are part of the *economic* problem of the current modality of study of real numbers: how are real numbers currently taught in secondary education, what are the *didactic ends* officially pursued with their study, what are the CEM_S(R) in different secondary school institutions, what relationship is established in school mathematics between measurement activities and calculation with approximate values and, finally, what *didactic phenomena* can be detected concerning this teaching?

The empirical analysis of curricular documents, guided by these questions, enables identifying an important transposition phenomenon: *the disconnections between real numbers and the measurement of quantities.* Related to this, there is another phenomenon that we synthetically call the *avoidance of irrationals* in secondary education or, more precisely, the avoidance of the problems caused using irrational numbers.

The answer to some questions that are part of the *ecological* problem of the current modality of study of real numbers involves moving from the initial *teaching problem* to the *problem of the curriculum* (Licera et al., 2019). It is postulated that explicit teaching of real numbers that would effectively respond to the problem of the measurement of continuous magnitudes and solve the technical problem related to working with approximate decimal numbers would require a *deep and global transformation of secondary school curricula*. This change is beyond the scope of the *school institution*, it falls under the responsibility of the *education system* as a whole and should start by being reflected in the *institution of teacher education*.

4.3 Limits of functions

How to explain the didactic facts related to the "algebraic" treatment of the calculus of *limits of functions* (LF) in secondary education? To answer this question, Espinoza (1998) and Barbé et al. (2005) began by representing the CEM_S(LF) from the perspective of a certain REM_S(LF) constructed as a scientific hypothesis. This REM integrates two mathematical praxeologies around the limits of functions. The first is a praxeology around the *algebra of limits* that starts from the assumption of the existence of the limit of functions and poses the problem of how to determine its value (calculate it) for certain families of functions. The second praxeology revolves around the *topology of limits* and aims to address the problem of the nature of the object "function limit" and, consequently, answers the essential question of the *existence of the limit* of certain types of functions.

From the perspective provided by this REM, the representation obtained from the CEM_S(LF), extracted from the empirical data (provided by curricular documents, textbooks, and classroom observations), is formed by the union of two blocks. First, the *practical block* around algebra of limits −the technology and theory corresponding to this practice being absent in the curriculum−. Second, the block formed by a few *technological-theoretical elements* (definitions and presumably justifying comments) referring to the topology of limits, whose curricular function is merely decorative, since neither in the curriculum, nor in the curricular documents, nor in the classroom does any mathematical practice appear that needs to be described or justified using these technological elements.

Subsequently, a technological discourse appropriate to the practice of calculating the limits of functions that can effectively be developed in secondary schools is not constructed. Nor is a practice proposed that is appropriate to the mathematical theory of the limits of functions suggested in the curriculum and which, it is assumed, ultimately legitimises school mathematical activity. This surprising *praxeological feature*, which we have called the *bicephaly* of CEM_S(LF), has important didactic consequences, and strongly conditions the modality of study (and the possibilities of learning) of this field in secondary education. Barbé et al. (2005) describe the teacher's difficulties in deciphering, from the curricular data, what the *praxeology to be taught* concerning the limits of functions is. These difficulties lead to the impossibility of "making sense" of the calculation of limits without detaching from the subject proposed by the syllabus. The absence in the curriculum of a technology capable of interpreting and justifying the mathematical techniques used by the students prevents the *didactic ends* of this study from being clarified and severely limits the *didactic means* that the teacher can use.

4.4 Elementary differential calculus

How can we explain the enormous difficulties of making sense of *elementary differential calculus* (EDC) in the transition from secondary school to university (SU)? The starting point is Ruiz-Munzón's conjecture, which essentially states that the *raison d'être* of differential calculus, that is, the problematic questions that give meaning to the study of differential calculus in the last stage of secondary school, should be in the field of functional modelling (Ruiz-Munzón, 2010). To analyse the current modality of study of EDC, we began by redefining *functional modelling* (FM) using a map of possible modelling processes, both discrete and continuous, in which the role of EDC is specified. This field is schematically materialised in an *activity diagram* that represents the proposal of an REM_{SU}(FM) that assigns an *alternative raison d'être* to the study of EDC in SU (Lucas, 2015; Lucas et al., 2019).

From the perspective of this REM_{SU}(FM), the empirical analysis of curricular documents in various countries has brought to light the didactic phenomenon of the *lack of school visibility of the activity of functional modelling* and the consequent school absence of the activities of *construction*, *comparison*, and *interpretation* of functional models. This phenomenon can be considered a particular case of *applicationism* (Barquero, 2009) in differential calculus.

School mathematics does not explore the role that EDC could play in the construction of models from discrete data, in the comparison of the fit of models to empirical data or the interpretation of model parameters in terms of the variation of one variable of the system concerning another. The representation obtained from the CEM_{SU}(EDC) shows that the *official raison d'être* of EDC at SU, i.e., the type of tasks assigned to it in the curricular documents, is focused on the analysis of the properties of certain types of functions and on solving optimisation problems.

To empirically contrast the proposed REM_{SU}(FM), several *study and research paths* (SRPs) were designed and tested, based on this REM. These SRPs, initially conceived as didactic devices for teaching differential calculus in the first year of Nuclear Medicine (Lucas, 2015), had the ambition of teaching EDC as a tool for the construction and study of functional models useful for increasing knowledge about certain types of systems. Although there were difficulties related to the limitations of the current modality of study, the SRPs experienced enabled giving academic visibility to FM at SU, an essential condition for justifying the study of EDC in that institution. It also helped to connect different mathematical praxeologies that usually arise in an atomized form (for example, the resolution of differential equations, the calculation of primitives, and the graphic representation of functions) by integrating them into functional modelling processes.

4.5 Contributions to the dialogue between didactic theories

Let us move on to another line of research, the relationships between theoretical approaches in mathematics education (Artigue & Mariotti, 2014; Bikner-Ahsbahs & Prediger, 2014; Radford, 2008). The ATD proposes the notion of *research praxeology* (RP) to replace and extend the notion of *scientific theory* and, in the case of didactic theories, it proposes the notion of *didactic research praxeology* (DRP). An RP, and particularly a DRP, functionally integrates *scientific know-how* (*procedural* knowledge) and *scientific knowledge* (*declarative* or *propositional* knowledge) (Artigue et al., 2011). Based on the notion of DRP, a fruitful dialogue has developed between the ATD and APOS theory (Bosch et al., 2017). Starting from the theoretical components of both DRPs, a parallel was drawn between the notion of *genetic decomposition* of APOS and the notion of *reference epistemological model* of the ATD. Starting from the technical and technological components of both DRPs, it became clear that each DRP can provide instruments to develop notions of the other, consistent with the internal logic of the latter and without either of them having to give up on their basic postulates or assumptions.

Another line of dialogue between the ATD and other didactic theories stems from the discussion on the *role that value judgements can legitimately play* in science and, in particular, in didactic science (Artigue, 2022; Godino et al., 2022; Margolinas, 2022; Trigueros, 2022). In this dialogue, we have stressed the *transformative*, *non-normative character* of didactic science in different works, pointing out the limits of science and, therefore, of didactics. Our position is closely related to the structure, in terms of *means-ends*, that we propose for didactic research problems and to the priority that the ATD assigns to ecological issues. From this perspective, we conclude that empirical science, such as didactics, cannot tell anyone *what they should do* (what and how to teach), but only *what they can do* and *how they can do it* to achieve previously determined ends, as well as what the intended and unintended consequences of their action might be (Bartolini Bussi, 2018; Davis, 2018; Gascón & Nicolás, 2019, 2021; Godino et al., 2019; Lerman, 2018; Oktaç et al., 2019).

5 Conclusions

Leaving aside the contributions of the ATD to the dialogue between didactic theories, which is situated in the domain of the *epistemology of didactics*, the remaining research clearly illustrates the contributions of the ATD to the epistemological programme in mathematics education. It is worth noting that in all cases the starting point is a *set of remarkable and surprising* (from the ATD perspective) *didactic facts* that require an explanation. For instance, the school prevalence of the formal manipulation of algebraic expressions and the separation between formulas and functions; the ambiguities and contradictions in the school use of decimal approximations linked to the avoidance of irrationals; the absence of meaning and justification of the algebraic techniques used in the calculation of the limits of functions; and the difficulties (of the school system) in making sense of elementary differential calculus whose school use seems to be reduced to the analysis of the properties of certain types of functions.

To try to explain these facts, a *praxeological analysis* of the considered *praxeological field* F of the knowledge involved is carried out in each case, from the perspective of a REM_I(F) constructed as a scientific hypothesis, which *redefines and broadens the praxeological field* initially considered. In the first case considered, this redefinition articulates elementary algebra with functional modelling and integrates the integers as algebraic objects. In the case of real numbers, the scope of the initial praxeological field is extended to include the measurement of continuous quantities and some technical work with decimal approximations. The bicephaly and incompleteness of the current modality of study of the calculation of limits of functions is revealed; and, in the case of elementary differential calculus, an REM is constructed around functional modelling as a field in which a possible raison d'être of differential calculus can be found.

In all cases, the *institutional dimension* of the didactic problems addressed is emphasised, and *macrodidactic phenomena* are studied. Specifically, in the cases considered we study the phenomena caused by the prealgebraic nature of school mathematics; those due to the disconnections between real numbers and the measurement of quantities; those related to the bicephaly of $CEM_S(LF)$; and those caused by the absence from school of mathematical activities related to the use of the tools provided by differential calculus to construct and interpret functional models of systems of all kinds.

Finally, some aspects of the *economy* and *ecology* of the *modalities of study historically constructed in I* around a field F, are investigated from the perspective of a certain $REM_I(F)$ that is elaborated for this purpose from the research. In the cases considered, for example, we study the ecological constraints of all kinds that limit the school development of elementary algebra as a modelling tool. The study of the ecological problem, in the case of real numbers, involves moving from an initial teaching problem (how to teach real numbers?) to the problem of the curriculum (what to teach about real numbers?). The economy of the current modality of study of the limits of functions is studied to bring to light and explain some of the didactic phenomena that emerge in the didactic processes governed by this modality. In the case of differential calculus, to answer the ecological question of how the current modality of study could be modified in a given direction, various SRPs have been designed and tested that have enabled making functional modelling visible and give meaning to the school study of differential calculus.

To conclude, all the contributions of the ATD to the development of the epistemological programme of research in mathematics education point in the direction of clarifying and specifying the idea that didactics is interested in the conditions that promote (or hinder) studying and the effects of this study on the individuals' productions. Didactics is the *science of study* and, by extension, of the *current or possible study modalities* in institutions. The question "How do we learn?" in the sense of "What are and how do the *psychoneurological mechanisms* involved in learning work?" is not the question that didactics asks. Didactics does not aim (or should not aim) to become a theory of learning.

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