

# THE CLASSICAL AND IMPROVED EULER-JACOBI FORMULA AND POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^n$

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**ABSTRACT.** The classical and the new Euler-Jacobi formulae for simple and double points provide an algebraic relation between the singular points of a polynomial vector field and their topological indices.

Using these formulae we obtain the geometrical configuration of the singular points together with their topological indices for several classes of polynomial differential systems in  $\mathbb{R}^n$  when these differential systems, having the maximum number of singular points, either all their singular points are simple, or at most one singular point is double (i.e. it has multiplicity two).

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Consider the polynomial differential system in  $\mathbb{R}^n$

$$(1) \quad \dot{x}_i = P_i(x_1, \dots, x_n), \quad i = 1, \dots, n \quad n \geq 2$$

where  $P_i(x_1, \dots, x_n)$  are real polynomials such that  $\deg(P_1) \in \{1, 2\}$ ,  $\deg(P_2) = m$  with  $m \in \mathbb{N}$  if  $\deg(P_1) = 1$  and  $m = 2, 3, 4, 5$  if  $\deg(P_1) = 2$ , and  $\deg(P_i) = 1$  for  $i \geq 3$ . We assume that system (1) has either  $m$  singular points, or  $m - 1$  singular points one of these singular points is double if  $\deg(P_1) = 1$ , and if  $\deg(P_1) = 2$  then it has either  $2m$  singular points, or  $2m - 1$  singular points and one of these singular points is double. In the case in which there are  $\deg(P_1)m$  finite singular points we use the classical Euler-Jacobi formula (a proof of the classical Euler-Jacobi formula can be found in [1]), and the case in which there are  $\deg(P_1)m - 1$  finite singular points, the classical Euler-Jacobi formula is not valid anymore but Gasull and Torregrosa in [6] provided a generalization of the classical Euler-Jacobi formula in the case that the system has one double point and we will use such a formula. Using these formulae we obtain all the possible distributions of the singular points of system (1) when it has either  $\deg(P_1)m$ , or  $\deg(P_1)m - 1$  singular points with  $m \in \mathbb{N}$  when either  $\deg(P_1) = 1$ , or  $m = 2, 3, 4, 5$  and  $\deg(P_1) = 2$ .

Note that assuming that there are finitely many singular points all the singular points lie in the plane  $\Pi$  not necessarily invariant given by the intersection of the  $n - 2$  hyperplanes  $P_i(x_1, \dots, x_n) = 0$  for  $i = 3, \dots, n$ . On that plane we can reduce system (1) to the planar polynomial differential system

$$(2) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

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where  $P(x, y)$  and  $Q(x, y)$  are real polynomials such that  $\deg(P) \in \{1, 2\}$  and  $\deg(Q) = m$  with  $m \in \mathbb{N}$  if  $\deg(P) = 1$ , and  $m = 2, 3, 4, 5$  if  $\deg(P) = 2$ .

It follows from geometry that given two analytic curves  $g = 0$  and  $f = 0$  and a point  $p$  such that  $f(p) = g(p) = 0$ ,  $p$  is *simple* if and only if the determinant of the Jacobian matrix of  $f$  and  $g$  at  $p$ , i.e.

$$J(f, g)(p) := J(p) = \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \Big|_{(x, y)=p}$$

is different from zero, and that it is *double* if and only if  $J(p) = 0$  and  $I(f, g)(p) := I(p) \neq 0$  where

$$\begin{aligned} I(f, g)(p) := I(p) = & \left( \frac{\partial f}{\partial y} \right)^2 \left( \frac{\partial f}{\partial x} \frac{\partial^2 g}{\partial x \partial x \partial x^2} - \frac{\partial g}{\partial x} \frac{\partial^2 f}{\partial x^2} \right) \\ & - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial x} \frac{\partial^2 g}{\partial x \partial x \partial y} - \frac{\partial g}{\partial x} \frac{\partial^2 f}{\partial x \partial y} \right) \\ & + \left( \frac{\partial f}{\partial x} \right)^2 \left( \frac{\partial f}{\partial x} \frac{\partial^2 g}{\partial x \partial y^2} - \frac{\partial g}{\partial x} \frac{\partial^2 f}{\partial y^2} \right) \Big|_{(x, y)=p}. \end{aligned}$$

For a proof see Lemma 2.2 of [6]. Moreover it is well-known that for planar polynomial differential systems (2), a simple singular point  $p$  has index 1 (if  $J(p) > 0$ ), or  $-1$  (if  $J(p) < 0$ ) (see for instance [12]), and that a double singular point of our system has index zero.

It was proved in [5] and [7] that in the case of polynomial differential systems (2) the absolute value of the sum of the topological indices of all singular points is either 0 or 2 if  $m$  is even, and it is 0 if  $m$  is odd.

Consider a differential system formed by two real polynomials of degrees 2 and  $m$  respectively in the variables  $x$  and  $y$ . If the set of singular points of that system (that we denote by  $A$ ) contains exactly  $2m$  elements, then the Jacobian determinant evaluated at each zero does not vanish (see again [12]) and for any polynomial  $R$  of degree less than or equal to  $m - 1$  we have

$$(3) \quad \sum_{a \in A} \frac{R(a)}{J(a)} = 0.$$

Using this classical Euler-Jacobi formula in [8] the authors characterized the number and distribution of the singular points of the polynomial differential systems (2) with  $m = 2, 3, 4, 5$  when these systems have  $2m$  finite singular points.

Now we consider the case in which one of the singular points is double. We will use the new Euler-Jacobi formula for double points proved in [6] which can be stated as follows. We need the following notation. We write the polynomial differential system (2) with  $\deg P = 2$  and  $\deg Q = m$  as

$$\begin{aligned} P(x, y) &= P_{10}x + P_{01}y + P_{20}x^2 + P_{11}xy + P_{02}y^2, \\ Q(x, y) &= Q_{10}x + Q_{01}y + Q_{20}x^2 + Q_{11}xy + Q_{02}y^2 + \dots, \end{aligned}$$

and given a polynomial  $R$  we also write it as

$$R(x, y) = R_{00} + R_{10}x + R_{01}y + R_{20}x^2 + R_{11}xy + R_{02}y^2 + \dots$$

The next result is proved in Theorem 3.2 of [6] for two real polynomials of degrees  $n$  and  $m$ . We state it here for the case in which  $n = 2$  when the system has  $2m - 1$  finite singular points.

**Theorem 1.** *Consider a differential system of two real polynomials of degrees 2 and  $m \geq 2$  respectively with  $m \geq 2$  in the variables  $x$  and  $y$ . If the set of all singular points of the system (that we denote by  $A$ ) contains exactly  $2m - 1$  elements ( $2m - 2$  being simple and one double that without loss of generality we can assume it is at the origin), then for any polynomial  $R$  of degree less than or equal to  $m - 1$  we have*

$$(4) \quad \sum_{a \in A_S} \frac{R(a)}{J(a)} + S(0) = 0,$$

where  $A_S$  denotes the set of simple singular points of the system and  $S(0)$  is equal to

$$S(0) = \frac{4P_{10}R_{00}N}{I(0)^2} + \frac{2P_{10}(P_{10}R_{01} - P_{01}R_{10})}{I(0)},$$

where

$$\begin{aligned} N = & P_{10}^3(Q_{10} - P_{10}Q_{03}) - P_{10}^2P_{01}(Q_{10} - P_{10}Q_{12}) \\ & + P_{10}P_{01}^2(Q_{10} - P_{10}Q_{21}) - P_{01}^3(Q_{10} - P_{10}Q_{30}) \\ & + P_{10}^3(Q_{11}P_{02} - P_{11}Q_{02}) - 2P_{10}^2P_{01}(Q_{20}P_{02} - P_{20}Q_{02}) \\ & + P_{10}P_{01}^2(Q_{20}P_{11} - P_{20}Q_{11}), \end{aligned}$$

where  $Q_{3,0} = Q_{2,1} = Q_{1,2} = Q_{0,3} = 0$  in case  $m = 2$ .

Before we state the main results of this paper we need to introduce some notations.

Let  $X = (P, Q)$  be the vector field associated to the differential system (2). We denote by  $A_X = A$  the set of points  $p \in \mathbb{R}^2$  such that  $X(p) = 0$ . Given a finite subset  $B$  of  $\mathbb{R}^2$ , we denote by  $\hat{B}$ ,  $\partial\hat{B}$  and  $\#B$  its convex hull, the boundary of the convex hull, and its cardinal, respectively.

Set  $A_0 = A$  and  $A_{i+1} = A_i \cap \partial\hat{A}_i$  for  $i \geq 0$ . Note that there exists a positive integer  $q$  such that  $A_q \neq \emptyset$  and  $A_{q+1} = \emptyset$ .

We say that  $A$  has the *configuration*  $(K_1; K_2; \dots; K_q)$  if  $K_i = \#A_i$  for  $i = 1, \dots, q$ . We say that  $A$  has the configuration  $(K_1; K_2; \dots; K_r; *)$  if we do not specify for the values of  $K_i$  for  $i$  between  $r + 1$  and  $q$ . We also say that the singular points of  $X$  belonging to  $A_i \cap \partial\hat{A}_i$  are on the  *$i$ -th level*.

We want to be more precise and study also the indices of the singular points of  $X = (P, Q)$  with  $P$  and  $Q$  as in (2). Then we substitute each  $K_i$  by the sign of the indices of the points of  $A_i$ , i.e. instead of  $K_i$  in the configuration we write the string  $(s_1^i, s_2^i, \dots, s_{K_i}^i)$  where  $s_j^i \in \{+, -, 0\}$ . When  $\hat{A}_i$  is a polygon, the starting  $s_1^i$  will be the point with multiplicity two (denoted by  $p_0$ ) if such point is in the  $i$ -th level and the signs  $s_2^i, \dots, s_{K_i}^i$  are the signs of the list of positive or negative indices that follow the point with multiplicity two in counterclockwise or clockwise sense according with the largest list of points with the same index between the two lists closest to the point with multiplicity two. In case that both closest lists have the same length, we choose the one with the second largest closest list, and so one. In

fact when there are  $\ell$  equal consecutive signs, for instance if they are  $+$ , then instead of  $+\dots+\ell$ -times we shall write  $\ell+$ . In the case that the  $i$ -th level does not contain the point with multiplicity two, then  $s_1^i$  is the length of the largest list of positive or negative indices of the singular point in the  $i$ -th level. The numbers  $s_2^i, \dots, s_{K_1}^i$  are chosen following the previous criteria changing the point with multiplicity two by  $s_1^i$ .

When  $\hat{A}_i$  is a segment, which does not contain the point with multiplicity two, we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. The signs of the strings are ordered starting at one of the endpoints. Then we start for the endpoint having the larger list of signs independently if this list is formed by plus or minus signs. In case that the length of the list of signs of both endpoints are equal, then we choose to start with the endpoint whose second list is larger, and so on.

If  $\hat{A}_i$  is a segment containing the point with multiplicity two again we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. Then the starting sign in the list is the sign 0 of the point of multiplicity two, and after it we choose the largest list closest to  $p_0$ . In case that the two lists of signs separated by  $p_0$  have the same length, then we choose to start with the list near  $p_0$  whose second list is larger, and so on.

With these notations we can state the main results of the paper. The first main result is when  $\deg(P) = 1$ .

**Theorem 2.** *For the polynomial differential (2) with  $\deg(P) = 1$  and  $m \in \mathbb{N}$  having  $2m$  singular points, the following statements hold.*

- (a) *If it has  $m$  singular points then only the following configurations are possible:*
  - (a.1)  $(m) = (2+, -, +, -, \dots, +, -)$  or  $(m) = (2-, +, -, +, \dots, +)$  if  $m$  is odd;
  - (a.2)  $(m) = (+, -, +, -, \dots, +, -)$  or  $(m) = (-, +, -, +, \dots, -, +)$  if  $m$  is even.
- (b) *If it has  $m-1$  singular points with one of them double then only the following configurations are possible:*
  - (b.1)  $(m-1) = (0, +, -, +, -, \dots, +, -, +)$  or  $(m) = (0, -, +, -, +, \dots, +, -)$  if  $m$  is odd;
  - (b.2)  $(m-1) = (0, +, -, +, -, \dots, +, -)$  or  $(m) = (0, -, +, -, +, \dots, -, +)$  if  $m$  is even.

Moreover there are examples of all these configurations.

The proof of Theorem 2 is given in section 3.

From now on we consider the cases in which  $\deg(P) = 2$ . The first main result under these conditions, which is the second main result of this paper, is when system (2) has  $2m$  finite singular points.

**Theorem 3.** *For the polynomial differential system (2) with  $\deg(P) = 2$ ,  $m = 2, 3, 4, 5$  and with  $2m$  singular points, the following statements hold.*

- (a) *If  $m = 2$  then only the following two configurations are possible*

- (i)  $(4) = (+, -, +, -)$ ,
- (ii)  $(3; 1) = (3+; -), (3-; +)$ ,  
and there exist examples of such configurations.
- (b) If  $m = 3$  then only the following two configurations are possible
  - (i)  $(6) = (+, -, +, -, +, -)$ ,
  - (ii)  $(4; 2) = (2+, 2-; +, -)$ ,
  - (iii)  $(3; 3) = (2+, -; 2-, +), (2-, +; 2+, -)$ ,  
and there exist examples of such configurations.
- (c) If  $m = 4$  then only the following configurations are possible
  - (i)  $(8) = (+, -, +, -, +, -, +, -)$ ,
  - (ii)  $(5; 3) = (4+, -; -, +, -), (4-, +; +, -, +), (2+, -, +, -; -, +, -),$   
 $(2-, +, -, +; +, -, +)$ ,
  - (iii)  $(4; 4) = (+, -, +, -; +, -, +, -)$ ,
  - (iv)  $(4; 3; 1) = (4+; 3-; +), (4-; 3+; -)$ ,
  - (v)  $(3; 5) = (3+; 2-, +, -, +), (3-; 2+, -, +, -)$ ,  
and there exist examples of such configurations.
- (d) If  $m = 5$  then only the following configurations are possible
  - (i)  $(10) = (+, -, +, -, +, -, +, -, +, -)$ ,
  - (ii)  $(6; 4) = (2+, 2-, +, -; +, -, +, -), (2+, 2+, -, +; +, -, +, -)$ ,
  - (iii)  $(4; 6) = (2+, 2-; +, -, +, -, +, -)$ ,
  - (iv)  $(4; 4; 2) = (2+, 2-; 2+, 2-; +, -)$ ,
  - (v)  $(4; 3; 3) = (2+, 2-; 2-, +; 2+, -), (2+, 2-; 2+, -; 2-, +)$ ,
  - (vi)  $(3; 7) = (2-, +; 2+, -, +, -, +, -), (2+, -; 2-, +, -, +, -, +)$ ,  
and there exist examples of such configurations.

The case  $m = 2$  of Theorem 3 is the well-known Berlinskii's Theorem proved in [2] and reproved in [4] using the Euler-Jacobi formula. The case  $m = 3$  was proved in [4]. The cases  $m = 4, 5$  were proved in [8]. So we do not need to prove Theorem 3. However the configurations on that papers were counted in a slightly different way.

The second main result in the paper takes into account the case in which system (2) has  $\deg(P) = 2$  and  $2m - 1$  finite singular points and  $m = 5$ .

**Theorem 4.** *For the polynomial differential (2) with  $\deg(P) = 2$ ,  $m = 2, 3, 4, 5$  and with  $2m - 1$  singular points with one double singular point, the following statements hold.*

- (a) If  $m = 2$  then only the configurations  $(3) = (0, +, -)$ , and  $(0, 2+)$  are possible, and there exist examples of such configurations.
- (b) If  $m = 3$  then only the following two configurations are possible
  - (i)  $(5) = (0, +, -, +, -)$ ,
  - (ii)  $(4; 1) = (0, 2+, -; -), (0, 2-, +; +), (2+, 2-; 0)$ ,
  - (iii)  $(3; 2) = (0, +, -; +, -), (2+, -; 0, -), (2-, +; 0, +)$ ,  
and there exist examples of such configurations.
- (c) If  $m = 4$  then only the following configurations are possible
  - (i)  $(7) = (0, +, -, +, -, +, -)$ ,
  - (ii)  $(5; 2) = (0, 2+, -, +; +, -), (0, 2-, +, -; +, -), (0, 3+, -; +, -),$   
 $(0, 3-, +; +, -), (4+, -; 0, -), (4-, +; 0, +), (0, -, +, -, +; +, -)$ ,

- $(0, +, -, +, -, +, -, -), (0, +, 2-, +, +, -, -), (0, -, 2+, -, +, -, -),$   
 $(2-, +, -, +, +, 0, +), (2+, -, +, -, +, 0, -),$   
 (iii)  $(4; 3) = (0, 3+; 2-, +), (0, 3-; 2+, -), (4+; 0, 2-), (4-; 0, 2+),$   
 $(0, +, -, +; 2-, +), (0, -, +, -, +; 2+, -), (+, -, +, -, +; 0, +, -),$   
 (iv)  $(3; 4) = (0, 2+; +, -, +, -, -), (0, 2-; +, -, +, -, -), (3+; 0, -, +, -, -),$   
 $(3-; 0, +, -, +), (3+; 0, 2-, +), (3-; 0, 2+, -), (0, +, -, +, -, +, -),$   
*and there exist examples of such configurations.*  
 (d) *If  $m = 5$  then only the following configurations are possible*  
 (i)  $(9) = (0, +, -, +, -, +, -, +, -),$   
 (ii)  $(6; 3) = (0, 2-, +, -, +, 2+, -), (0, 2+, -, +, 2-, +),$   
 $(0, +, 2-, +, -, +, 2+, -), (0, -, 2+, -, +, 2-, +),$   
 $(2+, 2-, +, -, +, 0, +, -),$   
 (iii)  $(5; 4) = (0, 2+, 2-; +, -, +, -, -), (0, +, 2-, +; +, -, +, -),$   
 $(0, -, 2+, -; +, -, +, -), (0, +, -, +, -; +, -, +, -),$   
 (iv)  $(4; 5) = (0, 2+, -; -, +, -, +, -, -), (0, 2-, +; +, -, +, -, +),$   
 $(2+, 2-; 0, +, -, +, -, -),$   
 (v)  $(4; 4; 1) = (2+, 2-; 2+, 2-; 0), (2+, 2-; 0, 2-, +; +),$   
 $(2+, 2-; 0, 2+, -; -), (0, 2+, -; 3-, +; +),$   
 $(0, 2-, +; 3+, -; -),$   
 (vi)  $(4; 3; 2) = (2+, 2-; 2-, +; 0, +), (2+, 2-; 2+, -; 0, -),$   
 $(2+, 2-; 0, +, -, +, -, -), (0, 2-, +; 2+, -, +, -, -),$   
 $(0, 2+, -; 2-, +, +, -, -),$   
 (vii)  $(3; 6) = (0, +, -, +, -, +, -, +, -), (2-, +; 0, +, -, +, -, +),$   
 $(2+, -, +; 0, -, +, -, +, -),$   
*and there exist examples of such configurations.*

The cases  $m = 2$  and  $m = 3$  were proved in [6]. The case  $m = 4$  was proved in [11]. In these two papers with the configurations counted in a slightly different way. In the present paper we will prove the case  $m = 5$ , see also [9, 10].

## 2. PRELIMINARITIES

In the proof of Theorem 4 we will use the following auxiliary result proved in [3].

**Lemma 5.** *Let  $(P, Q)$  be a polynomial vector field with  $\max(\deg P, \deg Q) = n$ . If  $(P, Q)$  has  $n + 1$  singular points on the straight line  $L(x, y) = 0$ , then this line is full of singular points.*

First observe that if a configuration exists for a polynomial vector field  $X = (P, Q)$ , then it is possible to construct the same configuration but interchanging the index  $+1$  by the index  $-1$ . For doing that it is enough to take the vector field  $Y = (-P, Q)$  instead of the vector field  $X = (P, Q)$ . So we can restrict ourselves to the cases in which  $\sum_{a \in A} i_X(a) \geq 0$ .

Assume that the vector field  $(P, Q)$  has degrees 2 and 5 (respectively), 8 simple singular points and 1 double singular point  $p_0$ . We can consider  $p_0$  at the origin and denote by  $p_1, \dots, p_8$  the simple singular points. Clearly  $p_0$  has index 0 and the other singular points have index  $\pm 1$ . During the proof of statement (d) of Theorem 4 we will denote by  $p_j$  the singular point for which there is no information about its index, by  $p_j^+$  the singular points having positive index, and by  $p_j^-$  the singular points

having negative index. Also we will denote by  $L_{i,j}$  the straight line  $L_{p_i,p_j}(x,y) = 0$  through the points  $p_i$  and  $p_j$ , and we will denote by  $L_i$  a straight line through a singular point  $p_i$  such that for any singular point  $q$  with  $q \neq p_i$  we have  $L_i(q) \neq 0$  and  $L_i(p_i) = 0$ .

It was proved in [7], see also [5], that in the case of polynomial vector fields of degree  $(1, m)$  with  $m$  odd it holds that  $|\sum_{a \in A} i_X(a)| = 1$ , and  $\sum_{a \in A} i_X(a) = 0$  if  $m$  is even. Moreover in the case of polynomial vector fields of degree  $(2, m)$  with  $m$  odd it holds that  $\sum_{a \in A} i_X(a) = 0$  and if  $m$  is even then  $|\sum_{a \in A} i_X(a)| \in \{0, 2\}$ .

### 3. PROOF OF THEOREM 2

By statement (a) of Theorem 4 we have that if  $m$  is even then there are  $m/2$  singular points with index  $+1$  and  $m/2$  singular points with index  $-1$ . If  $m$  is odd, by the explanation in the previous section, we can assume that there are  $(m+1)/2$  points with index  $+1$  and  $(m-1)/2$  singular points with index  $-1$ .

Since  $P$  has degree 1,  $P(x, y) = 0$  is a straight line and the  $m$  finite singular points of system (2) are on this straight line. Therefore when there are  $m$  finite singular points, the unique possible configuration is  $(m)$  because any convex hull of  $m$  points on a straight line has all points in the boundary of the convex hull. For the same reason when there are  $m-1$  finite singular points the unique possible configuration is  $m-1$ .

We first study the configuration  $(m)$ . Assume that the subscripts of the points in  $A$  are in such a way that  $p_1, \dots, p_m$  are ordered in  $\partial \hat{A}$  consecutively. Take

$$C_1 = L_1 \cdots L_{m-2}, \quad C_{m-1} = L_3 L_4 \cdots L_m$$

and for  $i = 2, \dots, m-2$ ,

$$C_i = L_1 \cdots L_{m-i-1} L_{m-i+2} \cdots L_m,$$

where all the straight lines which appear in the  $C_i$ 's for  $i = 1, \dots, m-1$  are parallel.

Then the Euler Jacobi formula (4) applied with  $R = C_i$  yields

$$\frac{C_i(p_{m-i})}{J(p_{m-i})} + \frac{C_i(p_{m-i+1})}{J(p_{m-i+1})} = 0.$$

Since all the points  $p_1, \dots, p_m$  are in a straight line, the polynomial  $C_i(x, y)$  has the same sign on the two points  $p_{m-i}$  and  $p_{m-i+1}$ . So  $J(p_{m-i})J(p_{m-i+1}) < 0$  for all  $i = 1, \dots, m-1$ . Hence the indices of  $p_{m-i}$  and  $p_{m-i+1}$  are different for  $i = 1, \dots, m-1$  providing the configurations (a.1) and (a.2) of statement (a) of Theorem 2.

The configurations of statement (a) can be realized intersecting a straight line  $P(x, y) = 0$  with  $m$  parallel straight lines  $Q(x, y) = 0$ . This completes the proof of statement (a) of Theorem 2.

Now we prove statement (b). Assume that the subscripts of the points in  $A$  are in such a way that  $p_0, p_1, \dots, p_m$  are ordered in  $\partial \hat{A}$  consecutively. Take

$$C_1 = L_0^2 L_1 \cdots L_{m-3}, \quad C_{m-2} = L_0^2 L_3 L_4 \cdots L_{m-1}$$

and for  $i = 2, \dots, m-3$ ,

$$C_i = L_0^2 L_1 \cdots L_{m-i-1} L_{m-i+2} \cdots L_{m-1},$$

where all the straight lines which appear in the  $C_i$ 's for  $i = 1, \dots, m-2$  are parallel.

Then the Euler Jacobi formula (4) applied with  $R = C_i$  yields

$$\frac{C_i(p_{m-i})}{J(p_{m-i})} + \frac{C_i(p_{m-i+1})}{J(p_{m-i+1})} = 0.$$

Since all the points  $p_0, p_1, \dots, p_m$  are in a straight line, the polynomial  $C_i(x, y)$  has the same sign on the two points  $p_{m-i}$  and  $p_{m-i+1}$ . So  $J(p_{m-i})J(p_{m-i+1}) < 0$  for all  $i = 1, \dots, m-2$ . Hence the indices of  $p_{m-i}$  and  $p_{m-i+1}$  are different for  $i = 1, \dots, m-2$  providing the configurations (b.1) and (b.2) of statement (b) of Theorem 2.

The configurations of statement (b) can be realized intersecting a straight line  $P(x, y) = 0$  with  $m$  straight lines,  $Q(x, y) = 0$ , being  $m-1$  parallel straight lines and the other straight line intersecting  $P$  and one of the other  $m-1$  straight lines in the same point. This completes the proof of statement (b) of Theorem 2.

#### 4. PROOF OF STATEMENT (d) OF THEOREM 4

In principle we could have the configurations (9), (8;1), (7;2), (6;3), (5;4), (5;3;1), (4;5), (4;4;1), (4;3;2), (3;6), (3;5;1), (3;4;2), (3;3;3) and (2;\*). Note that since the polynomial  $P$  has degree two,  $P(x, y) = 0$  is a conic and the nine singular points of system (2) are on this conic. Clearly configurations of the form  $(2+;*)$  cannot occur because the seven singular points would be on a straight line, and by Lemma 5 this straight line will be full of singular points, a contradiction. Moreover the configurations (8;1) and (7;2) are only possible if the conic  $P = 0$  is formed by two straight lines intersecting at a real point but then either seven or six singular points would be on the same straight line, and by Lemma 5 this straight line will be full of singular points, a contradiction. The configuration (6;3) is only possible with two straight lines intersecting at a real point. The configuration (5;4) is only possible with two straight lines intersecting at a real point. The configuration (5;\*;\*) is not possible because any convex hull of 5 points on a conic with five points at least in the first level can only be supported by a conic formed by two straight lines intersecting to a real point and this configuration do not support having points in the 2nd-level. The configurations (4;5), (4;4;1) and (4;3;2) are only possible with either a hyperbola or two straight lines intersecting at a real point, and the configuration (3;6) is only possible with a hyperbola. The configurations (3;5;1), (3;4;2), (3;3;3) are not possible since no real conic (ellipse, parabola, hyperbola, two parallel straight lines, two straight lines intersecting in a real point, one double straight line, two parallel straight lines, or one point) do not support such configurations and the configurations.

In short the unique possible configurations are: (9) (realized with either an ellipse, or two straight lines intersecting at a point), (6;3) and (5;4) (both realized with two straight lines intersecting at a real point), (4;5) (realized with either a hyperbola or two straight lines intersecting at a real point) and (4;4;1), (4;3;2), (3;6) (all realized with a hyperbola). We study them separately.



**Configuration (9).** We first show that two consecutive points  $p_{k_1}, p_{k_2}$  none of them being  $p_0$  must have opposite index, otherwise applying formula (4) with  $R(x, y) = L_{p_0, p_{k_3}} L_{p_0, p_{k_4}} L_{p_{k_5}, p_{k_6}} L_{p_{k_7}, p_{k_8}}$  with  $p_{k_i}$  for  $i \in \{3, \dots, 8\}$  being all different and different from  $p_{k_1}, p_{k_2}$  we get

$$\frac{R(p_{k_1})}{J(p_{k_1})} + \frac{R(p_{k_2})}{J(p_{k_2})} = 0,$$

which is a contradiction because  $R(p_{k_1})R(p_{k_2}) > 0$  and  $J(p_{k_1}) = J(p_{k_2})$ .

In short, the only possible configurations are  $(0, +, -, +, -, +, -, +, -)$  (and of course  $(0, -, +, -, +, -, +, -)$ ). System (2) with

$$\begin{aligned} P(x, y) &= xy, \\ Q(x, y) &= -96 + 224x + 219.2y - 190x^2 - 180y^2 + 75x^3 + 224xy^2 + 68y^3 \\ &\quad - 14x^4 - 224x^3y + 224x^2y^2 + 224xy^3 - 12y^4 + x^5 + 224x^4y \\ &\quad + 224x^3y^2 + 224x^2y^3 + 0.8y^5, \end{aligned}$$

has the singular points

$$(4, 0), (0, 5), (0, 4), (0, 3), (0, 2), (0, 1), (1, 0), (2, 0), (3, 0),$$

in the configuration  $(0, +, -, +, -, +, -, +, -)$  (we recall that the other configuration can be obtained reversing the sing in  $P$ ).

**Configuration (6; 3).** We note that the configuration  $(6; 3)$  only can be realized with a conic formed by two straight lines ( $R_1$  and  $R_2$ ) intersecting at a point  $q$ . Without loss of generality we can assume that five singular points are in  $R_1$  and five or four singular points are in  $R_2$  depending if the intersection point  $q$  is a singular point or not. Note that all points of  $R_1$  are in the 1-st level.

If  $q$  is not a singular point then  $p_0$  is in  $R_2$ , otherwise applying formula (4) with  $R(x, y) = R_1 L_{p_0, q_1} L_{p_0, q_2} L_{p_0, q_3}$  where  $q_1, q_2$  and  $q_3$  are three singular points on  $R_2$ . Then we reach a contradiction. Moreover if  $q$  is a singular point then  $q = p_0$ . Indeed, if  $p_0$  is in  $R_1$  and  $p_0 \neq q$  the previous argument also provides in this case a contradiction. Also if  $p_0 \neq q$  is in  $R_2$  we can repeat the same argument. So  $q = p_0$ .

We separate the proof in two cases.

*Csse 1:*  $q$  is not a singular point. Therefore  $p_0 \in R_2$ .

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  not separated by  $q$ . Applying formula (4) to  $R(x, y) = R_2 L_{p_0, p_{k_3}} L_{p_0, p_{k_4}} L_{p_0, p_{k_5}}$  where  $p_{k_j}$  for  $j = 3, 4, 5$  are singular points in  $R_1$  and different from  $p_{k_1}, p_{k_2}$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Let  $p_{k_1}$  and  $p_{k_2}$  be two singular points in  $R_1$  such that the segment having them as endpoints contains the point  $q$  and not other singular points. Applying formula (4) to  $R(x, y) = R_2 L_{p_0, p_{k_3}} L_{p_0, p_{k_4}} L_{p_0, p_{k_5}}$  where  $p_{k_j}$  for  $j = 3, 4, 5$  are singular points in  $R_1$  and different from  $p_{k_1}, p_{k_2}$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have the same index.

Let  $p_{\ell_1}$  and  $p_{\ell_2}$  be two consecutive singular points in  $R_2$  different from  $p_0$ . Applying formula (4) to  $R(x, y) = R_1 L_{p_0, p_{k_1}} L_{p_0, p_{k_2}} L_{p_{\ell_3}, p_{k_5}}$  where  $p_{k_j}$  for  $j = 1, 2, 3$  are singular points in  $R_1$  and  $p_{\ell_3}$  is a singular point in  $R_2$  and different from  $p_{\ell_1}$  and  $p_{\ell_2}$ , we get that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

Let  $p_{\ell_1}$  and  $p_{\ell_2}$  be two singular points in  $R_2$  such that the segment having them as endpoints contains only a singular point, which is  $p_0$ . Applying formula (4) to  $R(x, y) = R_1 L_{p_0, p_{k_1}} L_{p_0, p_{k_2}} L_{p_{\ell_3}, p_{k_5}}$  where  $p_{k_j}$  for  $j = 1, 2, 3$  are singular points in  $R_1$  and  $p_{\ell_3}$  is a singular point in  $R_2$  and different from  $p_{\ell_1}$  and  $p_{\ell_2}$ , we get that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

*Csse 2:*  $q$  is a singular point. Then  $q = p_0$ .

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  different from  $p_0$ . Applying formula (4) to  $R(x, y) = R_2^2 L_{p_{\ell_1}, p_{k_3}} L_{p_{\ell_2}, p_{k_4}}$  where  $p_{k_j}$  for  $j = 3, 4$  are singular points in  $R_1$  and different from  $p_{k_1}, p_{k_2}$  and  $p_{\ell_1}, p_{\ell_2}$  singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Let  $p_{k_1}$  and  $p_{k_2}$  be two singular points in  $R_1$  such that the segment having them as endpoints contains only a singular point, which is  $p_0$ . Applying formula (4) to  $R(x, y) = R_2^2 L_{p_{\ell_1}, p_{k_3}} L_{p_{\ell_2}, p_{k_4}}$  where  $p_{k_j}$  for  $j = 3, 4$  are singular points in  $R_1$  and different from  $p_{k_1}, p_{k_2}$  and  $p_{\ell_1}, p_{\ell_2}$  singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Let  $p_{\ell_1}$  and  $p_{\ell_2}$  be two consecutive singular points in  $R_2$  different from  $p_0$ . Applying formula (4) to  $R(x, y) = R_1^2 L_{p_{\ell_3}, p_{k_1}} L_{p_{\ell_4}, p_{k_2}}$  where  $p_{k_j}$  for  $j = 1, 2$  are singular points in  $R_1$  and  $p_{\ell_3}, p_{\ell_4}$  singular points in  $R_2$  and different from  $p_{\ell_1}$  and  $p_{\ell_2}$ , we get that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

The possible configurations are  $(0, 2-, +, -, +; 2+, -)$ ,  $(0, +, 2-, +, -; 2+, -)$ ,  $(2+, 2-, +, -; 0, +, -)$  (and of course,  $(0, 2+, -, +, -; 2-, +)$ ,  $(0, -, 2+, -, +; 2-, +)$ ).

System (2) with

$$\begin{aligned} P(x, y) &= x(y + 1), \\ Q(x, y) &= x + x^2 + xy + \frac{1}{36}y^2 - \frac{3}{8}x^3 + \frac{7}{3}x^2y - \frac{5}{8}xy^2 + \frac{17}{72}y^3 + x^4 + x^3y + x^2y^2 \\ &\quad + xy^3 + \frac{5}{8}y^4 + x^5 - \frac{5}{12}x^4y + x^3y^2 + x^2y^3 + xy^4 + \frac{1}{2}y^5, \end{aligned}$$

has the singular points

$(0, 0)$ ,  $(-1, -1)$ ,  $(-2/3, -1)$ ,  $(-1/2, -1)$ ,  $(-1/4, -1)$ ,  $(1, -1)$  in the 1st level, and  $(0, -2/3)$ ,  $(0, -1/3)$ ,  $(0, -1/4)$  in the 2nd level,

in the configuration  $(0, 2-, +, -, +; 2+, -)$  (configuration  $(0, 2+, -, +, -; 2-, +)$  can be obtained reversing the sing in  $P$ ).

System (2) with

$$\begin{aligned} P(x, y) &= -x(y + 1), \\ Q(x, y) &= x + x^2 + xy - \frac{1}{36}y^2 - \frac{11}{8}x^3 + \frac{7}{6}x^2y + \frac{3}{8}xy^2 - \frac{17}{72}y^3 + x^4 + x^3y + x^2y^2 \\ &\quad + xy^3 - \frac{5}{8}y^4 + x^5 + \frac{11}{12}x^4y + x^3y^2 + x^2y^3 + xy^4 - \frac{1}{2}y^5, \end{aligned}$$

has the singular points

$(0, 0)$ ,  $(1, -1)$ ,  $(-1, -1)$ ,  $(-1/2, -1)$ ,  $(-1/4, -1)$ ,  $(2/3, -1)$  in the 1st level, and  $(0, -2/3)$ ,  $(0, -1/3)$ ,  $(0, -1/4)$  in the 2nd level,

in the configuration  $(0, +, 2-, +, -; 2+, -)$  (configuration  $(0, -, 2+, -, +; 2-, +)$  can be obtained reversing the sing in  $P$ ).

System (2) with

$$\begin{aligned} P(x, y) &= x(1 + y), \\ Q(x, y) &= x - y/9 + x^2 - \frac{19}{18}y^2 - \frac{16}{9}x^3 - \frac{65}{18}x^2y - \frac{14}{9}xy^2 - \frac{11}{3}y^3 + x^3y + xy^3 \\ &\quad - \frac{11}{2}y^4 + x^5 + \frac{1}{2}x^4y + x^2y^3 - 3y^5, \end{aligned}$$

has the singular points

$(1, -1)$ ,  $(0, 0)$ ,  $(-2, -1)$ ,  $(1/3, -1)$ ,  $(1/2, -1)$ ,  $(2/3, -1)$  in the 1st level, and  
 $(0, -2/3)$ ,  $(0, -1/2)$ ,  $(0, -1/3)$  in the 2nd level,

in the configuration  $(2+, 2-, +, -; 0, +, -)$ .

**Configuration (5; 4).** We note that the configuration (5; 4) only can be realized with a conic formed by two straight lines ( $R_1$  and  $R_2$ ) intersecting at a point  $q$ . Without loss of generality we can assume that four singular points are in  $R_1$  (all in the 1st-level) and five singular points are in  $R_2$ .

Note that the intersection point  $q$  is not a singular point otherwise one of the straight lines would have six singular points, and by Lemma 5 this straight line will be full of singular points, a contradiction.

We note that  $p_0$  is in  $R_1$ . Otherwise if  $p_0$  is in  $R_2$  then applying (4) with  $R(x, y) = R_2L_{p_0, q_1}L_{p_0, q_2}L_{p_0, q_3}$  where  $q_1, q_2$  and  $q_3$  are three singular points on  $R_1$  we reach to a contradiction. So  $p_0$  must be in  $R_1$ , and so in the 1st-level.

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  different from  $p_0$  and not separated by  $q$ . Applying formula (4) to  $R(x, y) = R_2L_{p_0, p_{\ell_1}}L_{p_0, p_{\ell_2}}L_{p_{\ell_3}, p_{k_3}}$  where  $p_{k_3}$  is a singular point in  $R_1$  different from  $p_{k_1}$  and  $p_{k_2}$ , and  $p_{\ell_j}$  for  $j = 1, 2, 3$  are singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  such that the segment having them as endpoints does not contain  $q$  and contains only a singular point, which is  $p_0$ . Applying formula (4) to  $R(x, y) = R_2L_{p_0, p_{\ell_1}}L_{p_0, p_{\ell_2}}L_{p_{\ell_3}, p_{k_3}}$  where  $p_{k_3}$  is a singular point in  $R_1$  different from  $p_{k_1}$  and  $p_{k_2}$ , and  $p_{\ell_j}$  for  $j = 1, 2, 3$  are singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  different from  $p_0$  and separated by  $q$ . Applying formula (4) to  $R(x, y) = R_2L_{p_0, p_{\ell_1}}L_{p_0, p_{\ell_2}}L_{p_{\ell_3}, p_{k_3}}$  where  $p_{k_3}$  is a singular point in  $R_1$  different from  $p_{k_1}$  and  $p_{k_2}$ , and  $p_{\ell_j}$  for  $j = 1, 2, 3$  are singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have the same index.

Let  $p_{k_1}$  and  $p_{k_2}$  be two consecutive singular points in  $R_1$  such that the segment having them as endpoints contains only  $q$  and one singular point which is  $p_0$ . Applying formula (4) to  $R(x, y) = R_2L_{p_0, p_{\ell_1}}L_{p_0, p_{\ell_2}}L_{p_{\ell_3}, p_{k_3}}$  where  $p_{k_3}$  is a singular point in  $R_1$  different from  $p_{k_1}$  and  $p_{k_2}$ , and  $p_{\ell_j}$  for  $j = 1, 2, 3$  are singular points in  $R_2$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have the same index.

Let  $p_{\ell_1}$  and  $p_{\ell_2}$  be two consecutive singular points in  $R_2$ . Applying formula (4) to  $R(x, y) = R_1 L_{p_0, p_{\ell_3}} L_{p_0, p_{\ell_4}} L_{p_0, p_{\ell_5}}$  where  $p_{\ell_j}$  for  $j = 3, 4, 5$  are singular points in  $R_2$  and different from  $p_{\ell_1}, p_{\ell_2}$ , we get that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

The possible configurations are  $(0, 2+, 2-; +, -, +, -)$ ,  $(0, +, 2-, +; +, -, +, -)$ ,  $(0, +, -, +, -; +, -, +, -)$  (and of course  $(0, -, 2+, -; +, -, +, -)$ ),

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(1 + y), \\ Q(x, y) &= x - \frac{1}{12}y + x^2 + xy - \frac{7}{8}y^2 - \frac{5}{4}x^3 + \frac{9}{4}x^2y + \frac{1}{4}xy^2 - \frac{79}{24}y^3 + x^4 + x^3y \\ &\quad + x^2y^2 + xy^3 - \frac{21}{4}y^4 + x^5 + x^3y^2 + x^2y^3 + xy^4 - 3y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-1, -1), (-1/2, -1), (1/2, -1) \text{ in the 1st level, and} \\ &(0, -2/3), (0, -1/2), (0, -1/3), (0, -1/4) \text{ in the 2nd level,} \end{aligned}$$

in the configuration  $(0, 2+, 2-; +, -, +, -)$ .

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(1 + y), \\ Q(x, y) &= x - \frac{1}{24}y + x^2 - \frac{7}{16}y^2 - \frac{5}{4}x^3 + \frac{21}{8}x^2y - \frac{3}{4}xy^2 - \frac{79}{48}y^3 + x^3y + x^2y^2 \\ &\quad + xy^3 - \frac{21}{8}y^4 + x^5 - \frac{1}{2}x^4y + x^3y^2 + xy^4 - \frac{3}{2}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-1, -1), (-1/2, -1), (1/2, -1) \text{ in the 1st level, and} \\ &(0, -2/3), (0, -1/2), (0, -1/3), (0, -1/4) \text{ in the 2nd level,} \end{aligned}$$

in the configuration  $(0, +, 2-, +; +, -, +, -)$  (note that  $(0, -, 2+, -; +, -, +, -)$  can be obtained reversing the sing in  $P$ ).

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y + 1), \\ Q(x, y) &= x + \frac{1}{18}y + x^2 + xy + \frac{7}{12}y^2 - \frac{5}{3}x^3 + x^2y + \frac{2}{3}xy^2 + \frac{79}{36}y^3 + x^4 + x^3y \\ &\quad + x^2y^2 + xy^3 + \frac{7}{2}y^4 + x^5 + \frac{5}{6}x^4y + x^3y^2 + x^2y^3 + xy^4 + 2y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-1, -1), (1/3, -1), (1/2, -1) \text{ in the 1st level, and} \\ &(0, -2/3), (0, -1/2), (0, -1/3), (0, -1/4) \text{ in the 2nd level,} \end{aligned}$$

in the configuration  $(0, +, -, +, -; +, -, +, -)$ .

**Configuration (4; 5).** We note that the configuration (4; 5) can only be realized in a hyperbola or the conic formed by two straight lines intersecting to a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola,  $B_1$  has the two singular points  $p_8, p_9$  ordered in counterclockwise sense which are both of them in the 1st level and the other branch of the hyperbola  $B_2$  has the seven singular points  $p_1, p_2, p_3, p_4, p_5, p_6, p_7$  ordered in counterclockwise sense. Note that  $p_1, p_7$  are in the 1st level and  $p_2, p_3, p_4, p_5, p_6$  are in the 2nd level. Note that one of the  $p_i$ 's must be  $p_0$  but we will make it explicit during the proof.

Assume that  $p_0 \in B_2$ . We show that two singular points  $p_{\ell_1}, p_{\ell_2}$  in  $B_2$  none of them being  $p_0$  either are consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is  $p_0$ , must have different index. Otherwise, applying formula (4) to  $R(x, y) = L_{p_8, p_9} L_{p_0, p_{\ell_3}} L_{p_0, p_{\ell_4}} L_{p_{\ell_5}, p_{\ell_6}}$  being  $p_{\ell_i}$  for  $i = 3, \dots, 6$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$  we reach a contradiction.

Assume that  $p_0 \in B_1$ . Without loss of generality  $p_0 = p_8$ . Then two consecutive singular points  $p_{\ell_1}, p_{\ell_2}$  of the branch  $B_2$  have different index. Indeed, applying formula (4) to  $R(x, y) = L_{p_0, p_9} L_{p_0, p_{\ell_3}} L_{p_{\ell_4}, p_{\ell_5}} L_{p_{\ell_6}, p_{\ell_7}}$  being  $p_{\ell_i}$  for  $i = 3, \dots, 7$  singular points in  $B_2$  different from  $p_{\ell_1}$  and  $p_{\ell_2}$  it follows that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

This characterized completely the indices of the singular points of the branch  $B_2$ .

Assume first that  $p_0$  is in the 1st level. Then by the arguments above the unique possible configurations in the 2nd-level must be  $(+, -, +, -, +)$  or  $(-, +, -, +, -)$ . If  $p_0$  is in the branch  $B_1$  then, again by the arguments above, the unique possible configuration is  $(0, 2-, +; +, -, +, -, +)$  in the first case, and  $(0, 2+, -; -, +, -, +, -)$  in the second case. If  $p_0$  is in the branch  $B_2$  then without loss of generality we can assume that it is  $p_7$  and applying formula (4) to  $R(x, y) = L_{p_0, p_8} L_{p_0, p_2} L_{p_3, p_4} L_{p_5, p_6}$  we get that  $p_1$  and  $p_9$  must have the same index and so the unique possible configurations are again  $(0, 2-, +; +, -, +, -, +)$  and  $(0, 2+, -; -, +, -, +, -)$ .

If  $p_0$  is in the 2nd-level it must be in  $B_2$ . Then the unique possible configurations in the 2nd level is  $(0, +, -, +, -)$ . Applying formula (4) to  $R(x, y) = L_{p_7, p_8} L_{p_0, p_{\ell_1}} L_{p_0, p_{\ell_2}} L_{p_{\ell_3}, p_{\ell_4}}$  with  $p_{\ell_i}$  for  $i = 1, \dots, 4$  the singular points in  $B_2$  different from  $p_0$  and  $p_1$ , we get that  $p_1$  and  $p_9$  have the same index. In short the unique possible configuration is  $(2+, 2-; 0, +, -, +, -)$ .

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(1 + y), \\ Q(x, y) &= -\frac{32}{117}y^2 + \frac{5}{18}x^3 + \frac{25}{18}x^2y + \frac{13}{18}xy^2 - \frac{118}{117}y^3 - y^4 - x^5 - \frac{3}{2}x^4y - \frac{2}{13}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-1, -1), (0, -16/3) \text{ in the 1st level, and} \\ &(0, -1/2), (0, -2/3), (1/3, -1), (1/2, -1), (2/3, -1) \text{ in the 2nd level,} \end{aligned}$$

in the configuration  $(0, 2+, -; -, +, -, +, -)$  (the configuration  $(0, 2-, +; +, -, +, -, +)$  can be obtained reversing the sing in  $P$ ).

The differential system (2) with

$$P(x, y) = y(y - x), \quad Q(x, y) = x(x^2 - 1)(x^2 - 4),$$

has the singular points

$$(-2, -2), (-2, 0), (2, 2), (2, 0) \text{ in the 1st level, and}$$

$$(1, 0), (1, 1), (-1, -1), (-1, 0), (0, 0) \text{ in the 2nd level,}$$

in the configuration  $(2+, 2-; 0, +, -, +, -)$ .

**Configuration**  $(4; 4; 1)$ . We note that the configuration  $(4; 4; 1)$  can only be realized in a hyperbola or the conic formed by two straight lines intersecting to a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola,  $B_1$  has the four singular points  $p_6, p_7, p_8, p_9$  ordered in counterclockwise sense being  $p_6, p_9$  in the 1st-level and  $p_7, p_8$  in the 2nd level, and in the other branch  $B_2$  of the hyperbola there are the five singular points  $p_1, p_2, p_3, p_4, p_5$  ordered in counterclockwise sense. Note that  $p_1, p_5$  are in the 1st level,  $p_2, p_4$  are in the 2nd-level and  $p_3$  is in the 3rd-level. Note that one of the  $p_i$ 's must be  $p_0$  but we will make it explicit during the proof.

Assume  $p_0 \in B_2$ . We show that two singular points  $p_{\ell_1}, p_{\ell_2}$  in  $B_2$  none of them being  $p_0$  that are either consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is  $p_0$ , must have different index. Otherwise, applying formula (4) to  $R(x, y) = L_{p_6, p_9} L_{p_7, p_8} L_{p_0, p_{\ell_3}} L_{p_0, p_{\ell_4}}$  being  $p_{\ell_3}, p_{\ell_4}$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$ , we reach a contradiction. Moreover, we show that two consecutive singular points  $p_{k_1}, p_{k_2}$  in  $B_1$  have different index. Indeed, applying formula (4) to  $R(x, y) = L_{p_0, p_{k_3}} L_{p_0, p_{k_4}} L_{p_{\ell_1}, p_{\ell_2}} L_{p_{\ell_3}, p_{\ell_4}}$  being  $p_{k_3}, p_{k_4}$  singular points in  $B_1$  different from  $p_{k_1}, p_{k_2}$  and  $p_{\ell_i}$  for  $i = 1, \dots, 4$  singular points in  $B_2$  different from  $p_0$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Assume  $p_0 \in B_1$ . We show that two consecutive singular points  $p_{\ell_1}, p_{\ell_2} \in B_2$  have different index. Indeed, applying the Euler-Jacobi formula (4) to  $R(x, y) = L_{p_0, p_{\ell_3}} L_{p_0, p_{k_1}} L_{p_{k_2}, p_{k_3}} L_{p_{\ell_4}, p_{\ell_5}}$  being  $p_{k_i}$  for  $i = 1, \dots, 4$  the singular points in  $B_1$  different from  $p_0$ , and  $p_{\ell_i}$  for  $i = 3, 4, 5$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$  we reach a contradiction. Finally, we show that two singular points  $p_{k_1}, p_{k_2}$  in  $B_1$  none of them being  $p_0$  that are either consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is  $p_0$ , must have different index. Otherwise, applying formula (4) to  $R(x, y) = L_{p_0, p_{k_3}} L_{p_0, p_{\ell_1}} L_{p_{\ell_2}, p_{\ell_3}} L_{p_{\ell_4}, p_{\ell_5}}$  being  $p_{k_3}$  a singular point in  $B_1$  different from  $p_{k_1}, p_{k_2}, p_0$  and  $p_{\ell_i}$  for  $i = 1, \dots, 5$  the five singular points in  $B_2$ , we reach a contradiction.

If  $p_0$  is in the 3rd-level then  $p_0 = p_3$  and we show that  $p_5$  and  $p_6$  have the same index. Indeed, applying formula (4) to  $R(x, y) = L_{p_7, p_8} L_{p_1, p_9} L_{p_0, p_4} L_{p_0, p_2}$  we get that  $p_5$  and  $p_6$  have the same index. So the unique possible configurations are  $(2+, 2-; 2+, 2-; 0)$ .

If  $p_0$  is in the 2nd-level then without loss of generality we can assume that it is  $p_4$  (otherwise we can make the same argument if it is  $p_2$ ). Applying formula (4) to  $R(x, y) = L_{p_7, p_8} L_{p_1, p_9} L_{p_0, p_3} L_{p_0, p_2}$  we get that  $p_5$  and  $p_6$  have the same index. So the unique possible configurations are  $(2+, 2-; 0, 2-, +; +)$  and of course  $(2+, 2-; 0, 2+, -; -)$ .

If  $p_0$  is in the 1st-level then without loss of generality we can assume that it is  $p_5$  (otherwise we can make the same argument if it is  $p_1$ ). Applying formula (4) to  $R(x, y) = L_{p_0, p_2} L_{p_0, p_6} L_{p_3, p_4} L_{p_7, p_8}$  we get that  $p_1$  and  $p_9$  have the same index. So the unique possible configurations are  $(0, 2+, -; 3-, +; +)$  and of course  $(0, 2-, +; 3+, -; -)$ .

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y+1), \\ Q(x, y) &= x + \frac{1}{8}y + x^2 + xy + \frac{15}{16}y^2 - \frac{17}{16}x^3 + \frac{25}{8}x^2y + \frac{1}{16}xy^2 + \frac{39}{16}y^3 + x^4 \\ &\quad + x^3y + x^2y^2 + xy^3 + \frac{5}{2}y^4 + x^5 - x^4y + x^3y^2 + x^2y^3 + xy^4 + \frac{3}{4}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-2, -1), (0, -2) \text{ in the 1st level,} \\ &(1/4, -1), (0, -1/3), (-1, -1), (-1/4, -1) \text{ in the 2nd level, and} \\ &(0, -1/2) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(2+, 2-; 2+, 2-; 0)$ .

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y+1), \\ Q(x, y) &= x + \frac{1}{2}y + x^2 + xy + \frac{35}{12}y^2 - \frac{11}{8}x^3 + \frac{7}{6}x^2y + \frac{3}{8}xy^2 + \frac{38}{9}y^3 + x^4 + x^3y \\ &\quad + x^2y^2 + \frac{7}{3}y^4 + x^5 + \frac{11}{12}x^4y + x^3y^2 + xy^3 + x^2y^3 + xy^4 + \frac{4}{9}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(0, 0), (-1, -1), (0, -2), (1, -1) \text{ in the 1st level,} \\ &(0, -3/2), (-1/2, -1), (0, -1/4), (2/3, -1) \text{ in the 2nd level, and} \\ &(-1/4, -1) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(2+, 2-; 0, 2-, +; +)$  (the configuration  $(2+, 2-; 0, 2+, -; -)$  can be obtained reversing the sing in  $P$ ).

The differential system (2) with

$$\begin{aligned} P(x, y) &= -x(y+1), \\ Q(x, y) &= x - \frac{1}{4}y + x^2 + xy - \frac{3}{2}y^2 - \frac{17}{16}x^3 + \frac{25}{8}x^2y + \frac{1}{16}xy^2 - \frac{45}{16}y^3 + x^4 + x^3y \\ &\quad + x^2y^2 + xy^3 - \frac{29}{16}y^4 + x^5 - x^4y + x^3y^2 + x^2y^3 + xy^4 - \frac{3}{8}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-2, -1), (0, -2) \text{ in the 1st level,} \\ &(1/4, -1), (0, -1/3), (-1, -1), (-1/4, -1) \text{ in the 2nd level, and} \\ &(0, -1/2) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(0, 2+, -, 3-, +, +)$  (the configuration  $(0, 2-, +, 3+, -, -)$  can be obtained reversing the sing in  $P$ ).

**Configuration  $(4; 3; 2)$ .** We note that the configuration  $(4; 3; 2)$  can only be realized in a hyperbola or in a conic formed by two straight lines intersecting into a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola,  $B_1$  has the three singular points  $p_7, p_8, p_9$  ordered in counterclockwise sense being  $p_7, p_9$  in the 1st-level and  $p_8$  in the 2nd level, and in the other branch  $B_2$  of the hyperbola there are the six singular points  $p_1, p_2, p_3, p_4, p_5, p_6$  ordered in counterclockwise sense. Note that  $p_1, p_6$  are in the 1st level,  $p_2, p_5$  are in the 2nd-level and  $p_3, p_4$  are in the 3rd-level. Note that one of the  $p_i$ 's must be  $p_0$  but we will make it explicit during the proof.

Assume  $p_0 \in B_2$ . We show that two singular points  $p_{\ell_1}, p_{\ell_2}$  in  $B_2$  none of them being  $p_0$  that are either consecutive, or are consecutive and such that in the arc of the hyperbola having them as endpoints contains only a singular point, which is  $p_0$ , must have different index. Otherwise, applying formula (4) to  $R(x, y) = L_{p_0, p_6} L_{p_7, p_9} L_{p_0, p_{\ell_3}} L_{p_{\ell_4}, p_{\ell_5}}$  being  $p_{\ell_3}, p_{\ell_4}, p_{\ell_5}$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$ , we reach a contradiction. Moreover, we show that two consecutive singular points  $p_{k_1}, p_{k_2}$  in  $B_1$  have different index. Indeed, applying formula (4) to  $R(x, y) = L_{p_0, p_{k_3}} L_{p_0, p_{\ell_1}} L_{p_{\ell_2}, p_{\ell_3}} L_{p_{\ell_4}, p_{\ell_5}}$  being  $p_{k_3}$  a singular point in  $B_1$  different from  $p_{k_1}, p_{k_2}$  and  $p_{\ell_i}$  for  $i = 1, \dots, 5$  singular points in  $B_2$  different from  $p_0$ , we get that  $p_{k_1}$  and  $p_{k_2}$  have different index.

Assume  $p_0 \in B_1$ . We show that two consecutive singular points  $p_{\ell_1}, p_{\ell_2} \in B_2$  have different index. Indeed, applying the Euler-Jacobi formula (4) to  $R(x, y) = L_{p_0, p_{k_1}} L_{p_0, p_{k_2}} L_{p_{\ell_3}, p_{\ell_4}} L_{p_{\ell_5}, p_{\ell_6}}$  being  $p_{k_i}$  for  $i = 1, 2$  the singular points in  $B_1$  different from  $p_0$ , and  $p_{\ell_i}$  for  $i = 3, \dots, 6$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$  we reach a contradiction. Finally, we show that two singular points  $p_{k_1}, p_{k_2}$  in  $B_1$  none of them being  $p_0$  that are either consecutive, or are consecutive and such that in the arc of the hyperbola having them as endpoints contains only a singular point, which is  $p_0$ , must have different index. Otherwise, applying formula (4) to  $R(x, y) = L_{p_0, p_{\ell_1}} L_{p_0, p_{\ell_2}} L_{p_{\ell_3}, p_{\ell_4}} L_{p_{\ell_5}, p_{\ell_6}}$  being  $p_{\ell_i}$  for  $i = 1, \dots, 6$  the six singular points in  $B_2$ , we reach a contradiction.

If  $p_0$  is in the 3rd-level then the possible configurations are  $(2+, 2-, 2-, +; 0, +)$  and  $(2+, 2-, 2+, -, 0, -)$ .

If  $p_0$  is in the 2nd-level then it can be either in  $B_1$  or  $B_2$ . If  $p_0$  is in  $B_2$  the unique possible configuration is  $(2+, 2-, 0, +, -, +, -)$ . If  $p_0$  is in  $B_1$  then it is  $p_8$ . Applying formula (4) to  $R(x, y) = L_{p_3, p_0} L_{p_0, p_4} L_{p_5, p_2} L_{p_6, p_9}$  we get that  $p_1$  and  $p_7$  have the same index. Again the unique possible configuration is  $(2+, 2-, 0, +, -, +, -)$ .



If  $p_0$  is in the 1st-level then it can be either in  $B_1$  or  $B_2$ . If  $p_0$  is in  $B_2$  then the unique possible configurations are  $(0, 2-, +; 2+, -, +, -)$  and  $(0, 2+, -, 2-, +; +, -)$ . If  $p_0$  is in  $B_1$  then without loss of generality we can assume that it is  $p_9$  (if it is  $p_7$  the arguments are analogous). Applying formula (4) to  $R(x, y) = L_{p_0, p_7} L_{p_0, p_6} L_{p_5, p_1} L_{p_2, p_4}$  we get that  $p_3$  and  $p_8$  have different index. Again the unique possible configurations are  $(0, 2-, +; 2+, -, +, -)$  and  $(0, 2+, -, 2-, +; +, -)$ .

The differential system (2) with

$$P(x, y) = y^2 - (x - 1)^2 + 1, \quad Q(x, y) = y(2x + y)(x - 3)(x + 4)(x + 5),$$

has the singular points

$$\begin{aligned} &(3, -\sqrt{3}), (3, \sqrt{3}), (-5, \sqrt{35}), (-5, -\sqrt{35}) \text{ in the 1st level,} \\ &(2, 0), (-4, 2\sqrt{6}), (-4, -2\sqrt{6}) \text{ in the 2nd level, and} \\ &(-2/3, 4/3), (0, 0) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(2+, 2-; 2-, +; 0, +)$  (the configuration  $(2+, 2-; 2+, -, 0, -)$  can be obtained reversing the sing of  $P$ ).

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y + 1), \\ Q(x, y) &= x + \frac{3}{8}y + \frac{43}{16}y^2 - \frac{1}{4}x^3 + \frac{5}{8}x^2y - \frac{7}{4}xy^2 + \frac{107}{16}y^3 + x^3y + x^2y^2 + \frac{27}{4}y^4 \\ &\quad + x^5 - \frac{1}{2}x^4y + x^2y^3 + xy^4 + \frac{9}{4}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1), (0, 0), (-1, -1), (0, -3/2) \text{ in the 1st level,} \\ &(-1/2, -1), (1/2, -1), (0, -1/3) \text{ in the 2nd level, and} \\ &(0, -2/3), (0, -1/2) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(2+, 2-; 0, +, -, +, -)$ .

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y + 1), \\ Q(x, y) &= x + \frac{1}{24}y + x^2 + xy + \frac{19}{48}y^2 - \frac{11}{8}x^3 + \frac{15}{8}x^2y + \frac{3}{8}xy^2 + \frac{61}{48}y^3 + x^4 \\ &\quad + x^3y + x^2y^2 + xy^3 + \frac{37}{24}y^4 + x^5 + \frac{1}{4}x^4y + x^3y^2 + xy^4 + \frac{1}{2}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-1, -1), (0, -2), (1, -1), (0, 0), \text{ in the 1st level,} \\ &(-1/4, -1), (1/2, -1), (0, -1/4) \text{ in the 2nd level, and} \\ &(0, -1/3), (0, -1/2) \text{ in the 3rd level,} \end{aligned}$$

in the configuration  $(0, 2-, +; 2+, -, +, -)$  (the configuration  $(0, 2+, -, 2-, +; +, -)$  can be obtained reversing the sing in  $P$ ).

**Configuration (3;6).** We note that the configuration (3;6) only can be realized with a conic formed by a hyperbola. Moreover one branch  $B_1$  of the hyperbola has one point  $p_9$  (which is in the 1st-level) and the other branch  $B_2$  of the hyperbola has eight points. We denote them by  $p_1, \dots, p_8$  ordered in counterclockwise sense. The points  $p_1, p_8$  are in the 1st level and the remaining points are in the 2nd-level.

If  $p_0 \in B_1$ , i.e.,  $p_0 = p_9$ , then we show that two consecutive singular points in  $p_{\ell_1}, p_{\ell_2} \in B_2$  have different index. Indeed, applying formula (4) to  $R(x, y) = L_{p_0, p_{\ell_3}} L_{p_0, p_{\ell_4}} L_{p_{\ell_5}, p_{\ell_6}} L_{p_{\ell_7}, p_{\ell_8}}$  being  $p_{\ell_i}$  for  $i = 1, \dots, 8$  singular points in  $B_2$  different from  $p_{\ell_1}$  and  $p_{\ell_2}$ , we get that  $p_{\ell_1}$  and  $p_{\ell_2}$  have different index.

If  $p_0 \in B_2$  we show that two singular points  $p_{\ell_1}, p_{\ell_2} \in B_2$  none of them being  $p_0$  that are either consecutive or are consecutive and such that in the arc of hyperbola having them as endpoints contains only a singular point, which is  $p_0$  have different index. Otherwise applying formula (4) to  $R(x, y) = L_{p_0, p_9} L_{p_0, p_{\ell_3}} L_{p_{\ell_4}, p_{\ell_5}} L_{p_{\ell_6}, p_{\ell_7}}$  being  $p_{\ell_i}$  for  $i = 1, \dots, 7$  singular points in  $B_2$  different from  $p_{\ell_1}, p_{\ell_2}$  and  $p_0$ , we get a contradiction.

If  $p_0$  is in the 1st-level then the 2nd-level must be  $(+, -, +, -, +, -)$  and in this case the 1st-level can only be  $(0, +, -)$ . Hence we have the configurations  $(0, +, -; +, -, +, -, +, -)$ .

If  $p_0$  is in the 2nd-level the 2nd-level must be  $(0, +, -, +, -, +)$  (or  $(0, +, -, +, -, +)$ ) and in this case the 1st-level can only be  $(2-, +)$  (or  $(2+, -)$ ). Hence we have the configurations  $(2-, +; 0, +, -, +, -, +)$  and  $(2+, -; 0, -, +, -, +, -)$ .

The differential system (2) with

$$P(x, y) = y^2 - (x - 1)^2 + 1, \quad Q(x, y) = x(x - 5)(x - 3)(x - 4)(x - 6),$$

has the singular points

$(0, 0)$ ,  $(6, -2\sqrt{6})$ ,  $(6, 2\sqrt{6})$  in the 1st level, and

$(5, \sqrt{15})$ ,  $(4, 2\sqrt{2})$ ,  $(3, \sqrt{3})$ ,  $(3, -\sqrt{3})$ ,  $(4, -2\sqrt{2})$ ,  $(5, -\sqrt{15})$  in the 2nd level,

in the configuration  $(0, +, -; +, -, +, -, +, -)$ .

The differential system (2) with

$$P(x, y) = y^2 - (x - 1)^2 + 1, \quad Q(x, y) = -y(2x + y)(x + 3)(x + 4)(x + 5),$$

has the singular points

$(-5, -\sqrt{35})$ ,  $(-5, \sqrt{35})$ ,  $(2, 0)$  in the 1st level, and

$(-4, -2\sqrt{6})$ ,  $(-4, 2\sqrt{6})$ ,  $(-3, -\sqrt{15})$ ,  $(-3, \sqrt{15})$ ,  $(-2/3, 4/3)$ ,  $(0, 0)$  in the 2nd level,

in the configuration  $(2-, +; 0, +, -, +, -, +)$  (the configuration  $(2+, -; 0, -, +, -, +, -)$  can be obtained reversing the sing in  $P$ ).

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