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Limit cycles of the discontinuous piecewise differential systems separated by a non-regular line and formed by a linear center and a quadratic one

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During the last decades the study of discontinuous piecewise differential systems has become an interesting subject of research due to the important applications of this kind of systems to model natural phenomena. In the qualitative theory of differential equations, one of the interesting problems is the detection of the number of limit cycles and their configurations which remains open to date, except for very particular families of differential equations. Here we are inspired to study the maximum number of limit cycles of the discontinuous piecewise differential systems separated by a non-regular line and formed by a linear center and one of the four classes of quadratic centers. The main tool used to prove our main results is based on the first integrals of such systems. All the computations of this paper are verified using the algebraic manipulator Mathematica.

Keywords: quadratic center, linear center, limit cycle, discontinuous piecewise differential system.

1. Introduction and statement of the main results

The first one who started in a serious way the study of piecewise differential systems was Andronov together with Vitt and Khaikin [Andronov *et al.*, 1996], when they tried to model natural phenomena. Nowadays these kinds of differential systems are used in biology, control theory, mechanics, economics, ..., see [Coombes, 2008; Di Bernardo *et al.*, 2008; Glendinning & Jeffrey, 2019; Makarenkov & Lamb, 2012]. In the qualitative theory of differential equations, the detection of the maximum number of limit cycles of planar piecewise differential systems is an essential subject of research and in general, it remains an open problem up to now, this problem is restricted to planar polynomial differential systems is called the second part of the 16th Hilbert's problem that was proposed in 1900 by David Hilbert, among the 23 problems at the International Congress of Mathematicians in Paris. For discontinuous piecewise differential systems, we can find two types of limit cycles, the crossing and the sliding ones, and it is well known that we call *limit cycle* an isolated periodic solution in the set of all the periodic solutions of the differential system, the limit cycle that has only isolated points in the separation line is called crossing limit cycle, while the limit cycle that contains arcs in the separation lines is called a sliding limit cycle.

Recently many researchers tried to solve this kind of problem, they started with the simplest family of planar discontinuous piecewise differential systems separated by a straight line and formed by two linear differential systems. Up to now is unknown if the maximum number of limit cycles of such planar discontinuous piecewise linear differential systems separated by a straight line, see [Euzébio & Llibre, 2015; Freire *et al.*, 1998, 2014, 2012, 2015; Llibre *et al.*, 2015; Llibre & Teixeira, 2018; Llibre *et al.*, 2013; Llibre & Zhang, 2018], in all these papers the authors gave examples where these systems have at least three limit cycles. But if the line of discontinuity is not a straight line in 2013 [Braga & Mello, 2014, 2013] the authors found more than three limit cycles for discontinuous piecewise linear differential systems with two zones, and they considered in [Braga & Mello, 2014] discontinuous piecewise linear differential systems with two zones, the same result was obtained in 2015 by Novaes *et al.* [Novaes & Ponce, 2015]. At the beginning of 2021 in [Zhao *et al.*, 2021] they separated two planar linear differential systems having centers by a non-regular line and they proved that such systems have at most two limit cycles. In the same year in [Llibre, 2023] Llibre solved the extension of the 16th Hilbert's problem for a family of planar continuous piecewise differential systems formed by a linear center and a quadratic center where the separation line is a parabola, he proved that the maximum number of limit cycles for this class of piecewise differential systems is at most one.

In this paper the upper bound on the maximum number of limit cycles for a class of piecewise differential systems is examined using the first integrals of the differential systems. As far as we know by first time through the intersection of graphics, instead of solving the established systems obtained by connecting the orbits of the first integrals of both systems on the line of discontinuity.

Here we will deal with the planar discontinuous piecewise vector fields

$$\begin{cases} \dot{x} = X^+(x,y), \quad \dot{y} = Y^+(x,y), \quad if \quad (x,y) \in \Sigma^+; \\ \dot{x} = Y^-(x,y), \quad \dot{y} = Y^-(x,y), \quad if \quad (x,y) \in \Sigma^-; \end{cases} \tag{1}$$

where the separation curve is the non-regular line $\Sigma = \Sigma^+ \cap \Sigma^-$ such that $\Sigma^+ = \{(x,y) : x \geq 0 \text{ and } y \geq 0\}$ and $\Sigma^- = \{(x,y) : x \leq 0\} \cup \{(x,y) : x \geq 0, \quad y \leq 0\}$. On the separation line Σ the piecewise vector field is defined following the Filippov's rules, see [Filippov, 1988]. Of course, at the origin of coordinates, where the separation line is not smooth, the vector field is not defined, so we do not consider the orbits which start or end at this point. In this paper we study the maximum number of limit cycles of systems (1) formed by a linear center and a quadratic center given in what follows.

We will use the next lemma that gives the normal form of an arbitrary linear differential center.

Lemma 1. *By doing a linear change of variables and a rescaling of the independent variable every linear center at the origin in $\mathbb{R}^2$ can be written*

$$\dot{x} = -\alpha x - (\alpha^2 + \omega^2)y, \quad \dot{y} = x + \alpha y, \quad with \ \ \alpha \neq 0, \quad \omega > 0, \tag{2}$$

with its first integral

$$H(x,y) = (x + \alpha y)^2 + \omega^2 y^2. \tag{3}$$

For a proof of Lemma 1 see [Llibre & Teixeira, 2018].

A normal form of the quadratic centers is given in the following result.

Theorem 1 [Kapteyn-Bautin Theorem]. *Any quadratic system candidate to have a center can be written after an affine transformation and a rescaling of the independent variable in the next form*

$$\dot{x} = -y - Cxy - bx^2 - dy^2, \quad \dot{y} = x + Axy + ax^2 - ay^2. \tag{4}$$

This system has a center at the origin if and only if one of the following conditions holds

(i) $C = a = 0$,
(ii) $b + d = 0$,
(iii) $A - 2b = C + 2a = 0$,
(iv) $C + 2a = A + 4b + 5d = a^2 + bd + 2d^2 = 0$.

Theorem 1 is proved in [Bautin, 1954; Dumortier *et al.*, 2006; Kapteyn, 1912, 1911].

Note that in order to obtain all quadratic systems having a center, we must take the systems (4) satisfying some of the conditions (i), (ii), (iii) and (iv) and apply to these systems an arbitrary affine transformation, see Section 2.

In this work we focus on studying the problem of Lum and Chua extended to the class of discontinuous piecewise differential systems in the plane with two pieces, and in each piece we have a linear center or a quadratic center separated by the non-regular line Σ.

We denote by $\mathcal{C}_k$ with $k = i, ii, iii, iv$ the four classes of discontinuous piecewise differential systems separated by the non-regular line Σ, and formed by two pieces, in one piece there is an arbitrary linear differential center, and in the other piece there is a quadratic center in the classification of Kapteyn-Bautin Theorem after an arbitrary affine change of variables.

In this paper we study the limit cycles of the four classes of discontinuous piecewise differential systems $\mathcal{C}_k$ with $k = i, ii, iii, iv$ that intersect with the separation line Σ at two points, where we find two possible configurations of limit cycles. The first one will be denoted by **Cnf 1** is when the limit cycles have two intersection points with $\Gamma_1 = \{(x, y) : x = 0 \text{ and } y \geq 0\}$ or $\Gamma_2 = \{(x, y) : x \geq 0 \text{ and } y = 0\}$, where $\Sigma = \Gamma_1 \cup \Gamma_2$. But the study of the limit cycles of the four classes of discontinuous piecewise differential systems $\mathcal{C}_k$ with $k = i, ii, iii, iv$ that intersect the rays either Γ_1, or Γ_2 in two points is equivalent to the study that was done in [Benabdallah *et al.*, 2023] by Benabdallah *et al.* when they considred the straight line as the separation curve.

The second configuration is denoted by **Cnf 2**, where the limit cycles have two intersection points with the separation line Σ, one point in Γ_1 and the second one in Γ_2, i.e., the first point of intersection is $(x_1, 0) \in \Gamma_1$ and the second point is $(0, y_2) \in \Gamma_2$. We notice that when we combine the two configurations **Cnf 1** and **Cnf 2** we obtain another configuration that has a combination between the two kinds of limit cycles. The case where we have two arbitrary linear differential centers (2) in each region has been solved in Theorem 3 of [Llibre *et al.*, 2015] and in Theorem 4 of [Esteban *et al.*, 2021].

In the following theorem we will give the maximum number of limit cycles of the four classes $\mathcal{C}_k$ with $k = i, ii, iii, iv$ of discontinuous piecewise differential systems separated by a straight line.

Theorem 2. *The maximum number of limit cycles satisfying* ***Cnf 1*** *of the discontinuous piecewise differential systems separated by the non-regular line Σ and formed*

(a) *the class $\mathcal{C}_i$ is at most three if $A + b = 0 \neq A$; and one limit cycle if either $A = 0 \neq b$, $b = 0 \neq A$ or $A = b = 0$;*
(b) *the class $\mathcal{C}_{ii}$ is at most three if either $A + b = 0$ and $a = 0 \neq Cb$, or $AbC(A + b)(4b(A + b) + C^2) \neq 0$, or $b = C = 0$, or $A = a = 0 \neq Cb$ or $b = a = 0 \neq AC$; two if $\Delta = a = 0$ and $C \neq 0$; and one if $A = b = 0$;*
(c) *the class $\mathcal{C}_{iii}$ is at most one;*
(d) *the class $\mathcal{C}_{iv}$ is at most four.*

Theorem 2 is proved in [Benabdallah *et al.*, 2023].

Our first main result is on the upper bound of the number of limit cycles for the four classes $\mathcal{C}_k$ with $k = i, ii, iii, iv$ of discontinuous piecewise differential systems separated by the non-regular line Σ satisfying **Cnf 2**.

Theorem A. *The maximum number of limit cycles satisfying* ***Cnf 2*** *for*

(a) *the class $\mathcal{C}_i$ is at most three if $A + b = 0 \neq A$; two if either $A = 0 \neq b$ or $b = 0 \neq A$; and one if $A = b = 0$. There are systems of this class with three limit cycles see Figure 1(a), two limit cycles in Figure 1(b); and one limit cycle in Figure 1(c), respectively;*

(b) *the class* $\mathcal{C}_{ii}$ *is at most four either if* $AbC(A+b)\Delta \neq 0 = a$ *and* $\Delta \neq 0$ *or* $C = b = 0$, *or* $A + b = 0$ *and* $a = 0 \neq Cb$; *three if either* $b = a = 0 \neq AC$ *or* $A = a = 0 \neq Cb$; *and two if* $\Delta = a = 0$, *or* $A = b = 0$. *There are systems of this class with four limit cycles in Figure 2(a), three limit cycles in Figure 2(b), and two limit cycles in Figure 2(c), respectively;*
(c) *the class* $\mathcal{C}_{iii}$ *is at most one. There are systems of this class with one limit cycle, see Figure 3(a);*
(d) *the class* $\mathcal{C}_{iv}$ *is at most four. There are systems of this class with four limit cycles, see Figure 3(b).*

Theorem A is proved in Section 3.

In our second main result we provide the maximum number of limit cycles of the four classes $\mathcal{C}_k$, $k = i, ii, iii, iv$, of discontinuous piecewise differential systems having simultaneously limit cycles satisfying simultaneously **Cnf 1** and **Cnf 2**.

Theorem B. *The maximum number of limit cycles satisfying simultaneously **Cnf 1** and **Cnf 2** for*

(a) *the class* $\mathcal{C}_i$ *is at most six if* $A + b = 0 \neq A$; *three either if* $A = 0 \neq b$ *or* $b = 0 \neq A$; *and two if* $A = b = 0$. *There are systems of this class with five limit cycles see Figure 13(a), three limit cycles in Figure 13(b), and two limit cycles in Figure 13(c), respectively;*
(b) *the class* $\mathcal{C}_{ii}$ *is at most seven either if* $AbC(A+b)\Delta \neq 0 = a$ *and* $\Delta \neq 0$ *or* $C = b = 0$, *or* $A + b = 0$ *and* $a = 0 \neq Cb$; *six either if* $b = a = 0 \neq AC$, *or* $A = a = 0 \neq Cb$; *four if* $\Delta = a = 0$; *and three if* $A = b = 0$. *There are systems of this class with five limit cycles in Figure 14(a), six limit cycles in Figure 14(b), four limit cycles in Figure 14(c), and three limit cycles in Figure 14(d), respectively;*
(c) *the class* $\mathcal{C}_{iii}$ *is at most two. There are systems of this class with two limit cycles, see Figure 15(a);*
(d) *the class* $\mathcal{C}_{iv}$ *is at most eight. There are systems of this class with five limit cycles, see Figure 15(b);*

Theorem B is proved in Section 4.

Note that while in Theorem A we have proved that the upper bounds found for the limit cycles are realizable for some differential systems, this is not the case, in general, for the upper bounds obtained in Theorem B. It is an open question to know if some of the upper bounds of Theorem B are realizable.

In this paper we only give the graphics of the functions starting with a positive sign, and we omit the graphics when they start with a negative sign that can be obtained by symmetry.

2. The quadratic centers after an affine change of variables

In this section we will make a general affine change of variables for the quadratic systems and their first integrals. We consider the change of variables

$$(x, y) = (\alpha_1 x + \gamma_1 y + \delta_1, \alpha_2 x + \gamma_2 y + \delta_2) \quad \text{with} \quad \alpha_1\gamma_2 - \alpha_2\gamma_1 \neq 0.$$

Thus the quadratic differential system (4) becomes

$$\dot{x} = \frac{1}{\alpha_2\gamma_1-\alpha_1\gamma_2}\Big(x^2\left(a\gamma_1(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)+A\alpha_1\alpha_2\gamma_1+\gamma_2\left(\alpha_1^2 b+\alpha_1\alpha_2 C+\alpha_2^2 d\right)\right)+y^2(a\gamma_1^3+\gamma_1^2 \gamma_2(A+b)\gamma_1\gamma_2^2(C-a)+\gamma_2^3 d)+\delta_1(a\gamma_1\delta_1+b\gamma_2\delta_1+\gamma_1)+\delta_2(A\gamma_1\delta_1+\gamma_2+\gamma_2 C\delta_1)+ (\gamma_2 d-a\gamma_1)y(\gamma_1\gamma_2(-2a\delta_2+A\delta_1+2b\delta_1+C\delta_2)+\gamma_1^2(2a\delta_1+A\delta_2+1)+\gamma_2^2(C\delta_1+2d \delta_2+1))+x(\alpha_1(2a\gamma_1^2 y+\gamma_1+2a\gamma_1\delta_1+\gamma_1\gamma_2 y+(A+2b)+A\gamma_1\delta_2+2b\gamma_2\delta_1+\gamma_2 C(\delta_2 +\gamma_2 y))+\alpha_2(A\delta_1\gamma_1 y+\gamma_2-2a\gamma_1(\delta_2+\gamma_2 y)+C(\delta_1+\gamma_1 y)+\gamma_2 d(\delta_2+\gamma_2 y)))\Big),$$
$$\dot{y} = \frac{1}{\alpha_1\gamma_2-\alpha_2\gamma_1}\Big(x^2\left(a\alpha_1^3+\alpha_1\alpha_2^2(C-a)+\alpha_1^2\alpha_2(A+b)+\alpha_2^3 d\right)+y^2(a\alpha_1(\gamma_1-\gamma_2)(\gamma_1+\gamma_2)+\gamma_2(A \alpha_1\gamma_1+\alpha_2\gamma_1 C+\alpha_2\gamma_2 d)+\alpha_2 b\gamma_1^2)+\delta_1(a\alpha_1\delta_1+\alpha_1+\alpha_2 b\delta_1)+\delta_2^2(\alpha_2 d-a\alpha_1)+\delta_2(A \alpha_1\delta_1+\alpha_2+\alpha_2 C\delta_1)+y(\alpha_1(2a\gamma_1\delta_1-2a\gamma_2\delta_2+A\gamma_1\delta_2+A\gamma_2\delta_1+\gamma_1)+\alpha_2(2b\gamma_1\delta_1+ \gamma_2+\gamma_1 C\delta_2+\gamma_2 C\delta_1+2\gamma_2 d\delta_2))+x(\alpha_1\alpha_2(-(2a-C)(\delta_2+\gamma_2 y)+A(\delta_1+\gamma_1 y)+2b (\delta_1+\gamma_1 y))+\alpha_1^2(2a(\delta_1+\gamma_1 y)+A(\delta_2+\gamma_2 y))+\alpha_2^2(C(\delta_1+\gamma_1 y)+2d(\delta_2+\gamma_2 y)))\Big). \tag{5}$$

I. If the quadratic system (5) **satisfies** (i) **of Theorem 1.** The corresponding first integral of this system becomes

If $A+b=0\neq A$

$$H_1^{(i)}(x,y)=\big(A(\delta_2+\alpha_2 x+\gamma_2 y)+1\big)^{2d} e^{\dfrac{Z_1(x,y)}{(1+A(\delta_2+\alpha_2 x+\gamma_2 y))^2}}, \tag{6}$$

where

$$Z_1(x,y)=A(A^2(\delta_1+\alpha_1 x+\gamma_1 y)^2-2A(\delta_2+\alpha_2 x+\gamma_2 y)+4d(\delta_2+\alpha_2 x+\gamma_2 y)-1)+3d.$$

If $A=0\neq b$, then

$$H_2^{(i)}(x,y)=\big(2b^3(\delta_1+\alpha_1 x+\gamma_1 y)^2+2b^2 d(\delta_2+\alpha_2 x+\gamma_2 y)^2+2b(b-d) (\delta_2+\alpha_2 x+\gamma_2 y)-b+d\big)e^{2b(\delta_2+\alpha_2 x+\gamma_2 y)}. \tag{7}$$

If $b=0\neq A$

$$H_3^{(i)}(x,y)=\big(1+A(\delta_2+\alpha_2 x+\gamma_2 y)\big)^{2(d-A)}e^{Z_2(x,y)}, \tag{8}$$

where

$$Z_2(x,y)=A\big(A^2(\delta_1+\alpha_1 x+\gamma_1 y)^2+Ad(\delta_2+\alpha_2 x+\gamma_2 y)^2+2(A-d)(\delta_2+\alpha_2 x+\gamma_2 y)\big).$$

If $A=b=0$

$$H_4^{(i)}(x,y)=2d\big(\delta_2+\alpha_2 x+\gamma_2 y\big)^3+3\big((\delta_1+\alpha_1 x+\gamma_1 y)^2+(\delta_2+\alpha_2 x+\gamma_2 y)^2\big). \tag{9}$$

II. If the quadratic system (5) **satisfies the second condition (ii) of Theorem 1 and** $\Delta = C^2+4b(A+b)$**.** The first integral of (5) becomes

If $A+b=0$ and $a=0\neq Cb$

$$H_1^{(ii)}(x,y)=e^{Z_3(x,y)}\big(-b(\delta_2+\alpha_2 x+\gamma_2 y)+C(\delta_1+\alpha_1 x+\gamma_1 y)+1\big)^{b^2} \big(1-b(\delta_2+\alpha_2 x+\gamma_2 y)\big)^{-b^2-C^2}, \tag{10}$$

where

$$Z_3(x,y)=\frac{bC(b(\delta_1+\alpha_1 x+\gamma_1 y)+C(\delta_2+\alpha_2 x+\gamma_2 y))}{b(\delta_2+\alpha_2 x+\gamma_2 y)-1}.$$

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta = -L^2 < 0$

$$H_2^{(ii)}(x,y) = \big(1 - \frac{1}{4b}(4b^2 + C^2 + L^2)(\delta_2 + \alpha_2 x + \gamma_2 y)\big)^r \big(b^2(\delta_2 + \alpha_2 x + \gamma_2 y)^2 \\ -b(\delta_2 + \alpha_2 x + \gamma_2 y)(C(\delta_1 + \alpha_1 x + \gamma_1 y) + 2) + \frac{1}{4}\left(C^2 + L^2\right) \\ (\delta_1 + \alpha_1 x + \gamma_1 y)^2 + C(\delta_1 + \alpha_1 x + \gamma_1 y) + 1\big)^{\frac{1}{b}} e^{Z_4(x,y)}, \tag{11}$$

where $r = \dfrac{-8b}{4b^2 + C^2 + L^2}$ and

$$Z_4(x,y) = \frac{-2C}{bL}\cot^{-1}\left(\frac{L(\delta_1 + \alpha_1 x + \gamma_1 y)}{2b(\delta_2 + \alpha_2 x + \gamma_2 y) - C(\delta_1 + \alpha_1 x + \gamma_1 y) - 2}\right).$$

If $C = b = 0$

$$H_3^{(ii)}(x,y) = e^{2ar(\delta_1+\alpha_1 x+\gamma_1 y) - 2A\coth^{-1}\left(\frac{-2a^2(\delta_2 + \alpha_2 x + \gamma_2 y) + aA(\delta_1 + \alpha_1 x + \gamma_1 y) + A}{r(a(\delta_1 + \alpha_1 x + \gamma_1 y) + 1)}\right)} \\ \left(\frac{(\delta_2 + \alpha_2 x + \gamma_2 y)\left(-\left(a^2(\delta_2 + \alpha_2 x + \gamma_2 y)\right) + aA(\delta_1 + \alpha_1 x + \gamma_1 y) + A\right)}{(a(\delta_1 + \alpha_1 x + \gamma_1 y) + 1)^2} + 1\right)^{-r} \\ (a(\delta_1 + \alpha_1 x + \gamma_1 y) + 1)^{-2r}, \tag{12}$$

where $r = \sqrt{4a^2 + A^2}$.

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$

$$H_4^{(ii)}(x,y) = \left(\frac{1}{2}\left(C - \sqrt{\Delta}\right)(\delta_1 + \alpha_1 x + \gamma_1 y) - b(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\right)^{r^-} \\ \left(\frac{1}{2}\left(C + \sqrt{\Delta}\right)(\delta_1 + \alpha_1 x + \gamma_1 y) - b(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\right)^{r^+} \\ \big(A(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\big)^{\frac{1}{A}}, \tag{13}$$

where $r^{\pm} = \dfrac{\sqrt{\Delta} \pm C}{2b\sqrt{\Delta}}$.

If $b = a = 0$ and $AC \neq 0$

$$H_5^{(ii)}(x,y) = \big(C(\delta_1 + \alpha_1 x + \gamma_1 y) + 1\big)^{2A^2}\big(A(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\big)^{2C^2} e^{Z_5(x,y)}, \tag{14}$$

where

$$Z_5(x,y) = -2AC\big(A(\delta_1 + \alpha_1 x + \gamma_1 y) + C(\delta_2 + \alpha_2 x + \gamma_2 y)\big).$$

If $A = a = 0$ and $Cb \neq 0$

$$H_6^{(ii)}(x,y) = e^{\delta_2 + \alpha_2 x + \gamma_2 y}\Big(\frac{1}{2}(C - \sqrt{\Delta})(\delta_1 + \alpha_1 x + \gamma_1 y) - b(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\Big)^{r^+} \\ \Big(\frac{1}{2}(C + \sqrt{\Delta})(\delta_1 + \alpha_1 x + \gamma_1 y) - b(\delta_2 + \alpha_2 x + \gamma_2 y) + 1\Big)^{r^-}, \tag{15}$$

where $r^{\pm} = \dfrac{\sqrt{\Delta} \pm C}{2b\sqrt{\Delta}}$.

If $\Delta = a = 0$ and $C \neq 0$

$$H_7^{(ii)}(x,y) = \frac{1}{2}\left(-\frac{C^2}{4b}(\delta_2+\alpha_2 x+\gamma_2 y) - b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{-\dfrac{4b^2}{4b^2+C^2}} \Big(-2b(\delta_2+\alpha_2 x+\gamma_2 y)+C(\delta_1+\alpha_1 x+\gamma_1 y)+2\Big)e^{Z_6(x,y)}, \tag{16}$$

where

$$Z_6(x,y) = 1 + \frac{C(\delta_1+\alpha_1 x+\gamma_1 y)}{2b(\delta_2+\alpha_2 x+\gamma_2 y) - C(\delta_1+\alpha_1 x+\gamma_1 y) - 2}.$$

If $A = b = 0$

$$H_8^{(ii)}(x,y) = (1 + C(\delta_1+\alpha_1 x+\gamma_1 y))^2 e^{-C^2(\delta_2+\alpha_2 x+\gamma_2 y)^2 - 2C(\delta_1+\alpha_1 x+\gamma_1 y)}. \tag{17}$$

III. If the quadratic system (5) **satisfies the third condition (iii) of Theorem 1.** The first integral is

$$\begin{aligned} H^{(iii)}(x,y) = \frac{1}{6}(&2a(\delta_1+\alpha_1 x+\gamma_1 y)^3 + 6b(\delta_1+\alpha_1 x+\gamma_1 y)^2(\delta_2+\alpha_2 x+\gamma_2 y) + 3(\delta_1 \\ &+\alpha_1 x+\gamma_1 y)^2 - 6a(\delta_1+\alpha_1 x+\gamma_1 y)(\delta_2+\alpha_2 x+\gamma_2 y)^2 + 2d(\delta_2+\alpha_2 x \\ &+\gamma_2 y)^3 + 3(\delta_2+\alpha_2 x+\gamma_2 y)^2). \end{aligned} \tag{18}$$

IV. If the quadratic system (5) **satisfies the fourth condition (iv) of Theorem 1.** The first integral is

$$\begin{aligned} H^{(iv)}(x,y) = \big(&(a^2+d^2)(d(\delta_2+\alpha_2 x+\gamma_2 y) - a(\delta_1+\alpha_1 x+\gamma_1 y))^3 - 3ad(a^2+d^2)(\delta_1 \\ &+\alpha_1 x+\gamma_1 y)(\delta_2+\alpha_2 x+\gamma_2 y) + 3d^2(a^2+d^2)(\delta_2+\alpha_2 x+\gamma_2 y)^2 + 3d(a^2 \\ &+d^2)(\delta_2+\alpha_2 x+\gamma_2 y) + d^2\big)^2 / \big((a^2+d^2)(a(\delta_1+\alpha_1 x+\gamma_1 y) - d(\delta_2+ \\ &\alpha_2 x+\gamma_2 y))^2 + 2d(a^2+d^2)(\delta_2+\alpha_2 x+\gamma_2 y) + d^2\big)^3. \end{aligned} \tag{19}$$

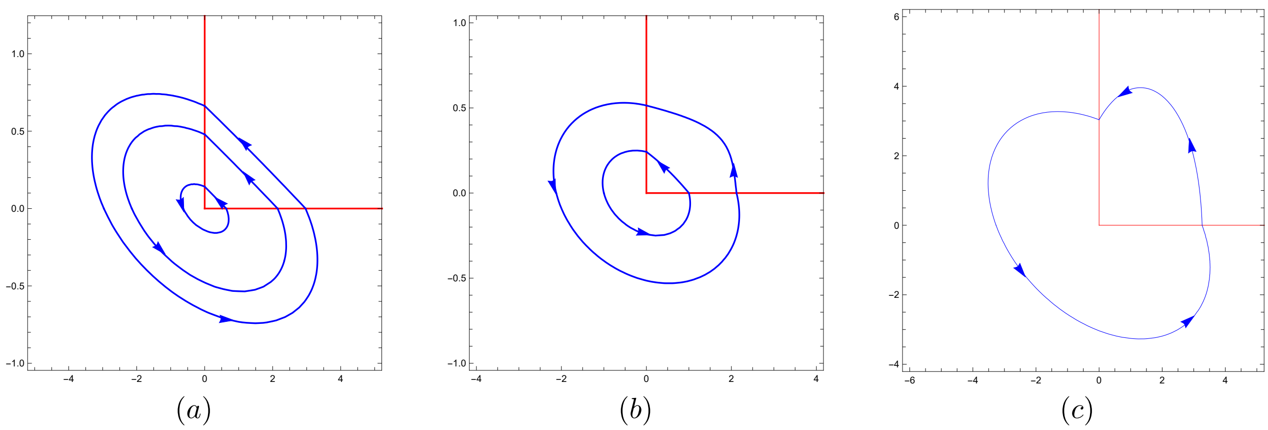


Fig. 1. (a) The three crossing limit cycles of the discontinuous piecewise differential system (21)–(22), (b) the two crossing limit cycles of the discontinuous piecewise differential system (23)–(24), and (c) the unique crossing limit cycle of the discontinuous piecewise differential system (25)–(26).

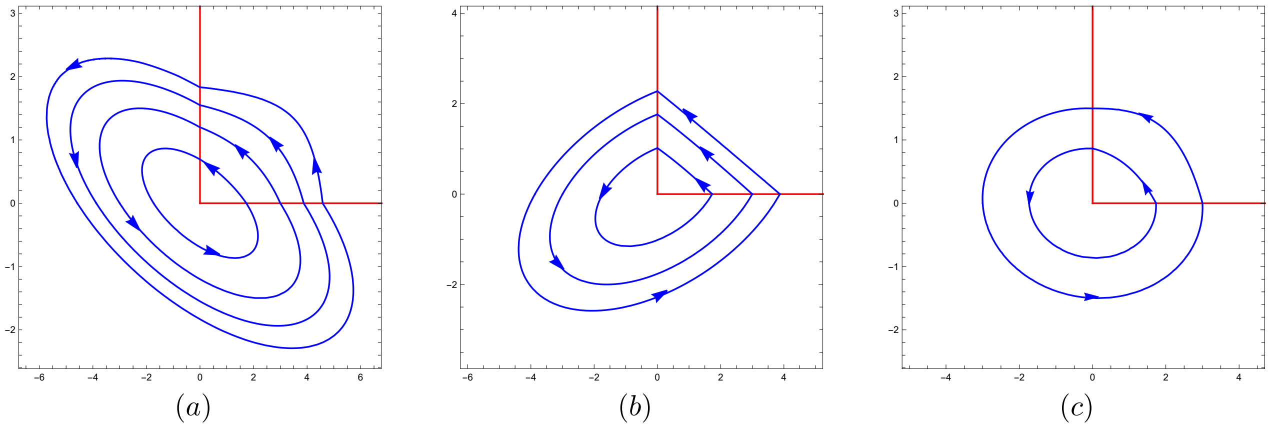


Fig. 2. (a) The four crossing limit cycles of the discontinuous piecewise differential system (27)–(28), (b) the three crossing limit cycles of the discontinuous piecewise differential system (29)–(30), and (c) the two crossing limit cycles of the discontinuous piecewise differential system (31)–(32).

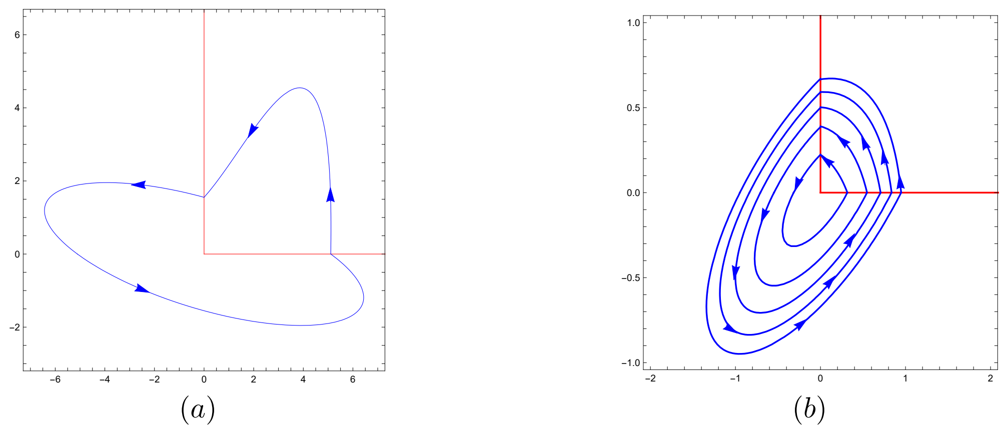


Fig. 3. (a) The unique crossing limit cycle of the discontinuous piecewise differential system (33)–(34), and (b) the four crossing limit cycles of the discontinuous piecewise differential system (35)–(36).

3. Proof of Theorem A

In one region we consider the linear differential center (2) with its first integral $H(x, y)$ given by (3). In the second region we consider the quadratic center (5) satisfying one of the four conditions of Theorem 1, with its first integral $H_k^{(j)}(x, y)$ with $k = 1, ..., 4$ for $j = i$, and $k = 1, ..., 8$ for $j = ii$ and $H^{(j)}(x, y)$ in case of $j = iii, iv$. If the discontinuous piecewise differential system (2)–(5) has a limit cycle, this limit cycle must intersect the separation line Σ in two distinct points $(x_1, 0) \in \Gamma_2$ and $(0, y_2) \in \Gamma_1$ where $x_1 > 0$ and $y_2 > 0$. These two points must satisfy the system of equations

$$\begin{aligned} &H(x_1, 0) - H(0, y_2) = h(x_1, y_2) = 0, \\ &H_k^{(j)}(x_1, 0) - H_k^{(j)}(0, y_2) = h^{(j)}(x_1, y_2) = 0, \end{aligned} \tag{20}$$

by solving $h(x_1, y_2) = 0$, we get $x_1 = \lambda y_2$ with $\lambda = \sqrt{\alpha^2 + \omega^2}$ which is a function of the variable y_2. Substituting x_1 in $h^{(j)}(x_1, y_2) = 0$ we obtain $F_k^{(j)}(y_2) = 0$ for $j = i, ii$ and $F^{(j)}(y_2) = 0$ for $j = iii, iv$ that are equations in the variable y_2.

Proof. [Proof of statement (a) of Theorem A] First we prove the statement for the first class $\mathcal{C}_i$ when $C = a = 0$. If $A + b = 0 \neq A$ corresponding to $k = 1$ and $j = i$ in system (20), the first integral of (5) is $H_1^{(i)}(x, y)$ given in (6), and the solutions of $F_1^{(i)}(y_2) = 0$ are equivalent to the solutions of the equation

$f_1^{(i)}(y_2) = g_1^{(i)}(y_2)$ where

$$f_1^{(i)}(y_2) = \left(\frac{s_1 + s_2\ y_2}{s_1 + s_3\ y_2}\right)^r \quad \text{and} \quad g_1^{(i)}(y_2) = e^{\dfrac{k_1\ y_2 + k_2\ y_2^2 + k_3\ y_2^3 + k_4\ y_2^4}{[(s_1 + s_2\ y_2)(s_1 + s_3\ y_2)]^2}},$$

and

$$\begin{aligned}
k_1 &= 2A(A\delta_2 + 1)(A^2(\lambda(\alpha_1\delta_1 + \alpha_2\delta_2) + A\delta_1(\lambda(\alpha_1\delta_2 - \alpha_2\delta_1) - \gamma_1\delta_2 + \gamma_2\delta_1) - \gamma_2\delta_2) \\
&\quad -A^2\gamma_1\delta_1 + d(2A\delta_2 + 1)(\gamma_2 - \alpha_2\lambda)), \\
k_2 &= A^2(A^3(\alpha^2(\alpha_1^2\delta_2^2 - \alpha_2^2\delta_1^2) + \delta_2(-4\alpha_2\gamma_1\delta_1\lambda + \alpha_1^2\delta_2\omega^2 - \gamma_1^2\delta_2) + 4\alpha_1\gamma_2\delta_1\delta_2\lambda - \alpha_2^2\delta_1^2\omega^2 \\
&\quad +\gamma_2^2\delta_1^2) + 2A^2(\alpha^2\delta_2(\alpha_1^2 + \alpha_2^2) + 2\alpha_1\gamma_2\delta_1\lambda - 2\alpha_2\gamma_1\delta_1\lambda + \delta_2\omega^2(\alpha_1^2 + \alpha_2^2) - \delta_2(\gamma_1^2 \\
&\quad +\gamma_2^2)) + A(-\gamma_1^2 + \alpha^2(\alpha_1^2 + \alpha_2^2(1 - 4d\delta_2)) + \omega^2(\alpha_1^2 + \alpha_2^2 - 4\alpha_2^2 d\delta_2) + \gamma_2^2(4d\delta_2 - 1)) \\
&\quad +3d(\gamma_2^2 - \alpha_2^2\lambda^2)), \\
k_3 &= 2A^3(A^2\alpha_1\gamma_2^2\delta_1\lambda - A^2\alpha_2\gamma_1^2\delta_2\lambda + A\lambda^2(\alpha_1^2(A\gamma_2\delta_2 + \gamma_2) + \alpha_2^2(\gamma_2 - A\gamma_1\delta_1)) - A\alpha_2\gamma_1^2\lambda \\
&\quad -A\alpha_2\gamma_2^2\lambda - 2\alpha_2^2\gamma_2 d\lambda^2 + 2\alpha_2\gamma_2^2 d\lambda), \\
k_4 &= -A^5\lambda^2(\alpha_2\gamma_1 - \alpha_1\gamma_2)(\alpha_1\gamma_2 + \alpha_2\gamma_1), \quad s_1 = A\delta_2 + 1, \quad s_2 = A\gamma_2, \quad s_3 = A\alpha_2\lambda, \quad r = 2d.
\end{aligned}$$

These solutions represent the intersection points between these two functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$, for that we will draw all the possible graphics of these functions.

For $r > 0$, the function $f_1^{(i)}(y_2)$ has the horizontal asymptote straight line $h_1 = \left(\frac{s_2}{s_3}\right)^r$ and the vertical asymptote straight line $y_{22} = \frac{-s_1}{s_3}$. We denote by

$$\left(f_1^{(i)}\right)'(y_2) = \frac{\eta(s_1 + s_2\ y_2)^{r-1}}{(s_1 + s_3\ y_2)^{r+1}},$$

the first derivative of the function $f_1^{(i)}(y_2)$ with $\eta = rs_1(s_2 - s_3)$. Then to draw all the possible graphics for the function $f_1^{(i)}(y_2)$ we have to study the sign of its derivative which depends on the sign of the parameter η and r. It is clear that the first derivative of the function $f_1^{(i)}(y_2)$ vanish at $y_{21} = \frac{-s_1}{s_2}$ that can have the only possible position $y_{21} < y_{22}$ with the vertical asymptote straight line y_{22}, but we have $y_{22} < y_{21}$ when $r > 1$.

Here we draw only the graphics of the function $f_1^{(i)}(y_2)$ corresponding to the case when $y_{21} < y_{22}$, and by the symmetry with the vertical asymptote straight line we get the graphics corresponding to the case $y_{22} < y_{21}$. Then,

1- if either r is even or $r = \frac{2k_1}{2k_2 + 1}$ with $k_1, k_2 \in \mathbb{N}$, the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends on the sign of the product $\eta(s_1 + s_2y_2)(s_1 + s_3y_2)$. So the only possible graphic of this function is shown in Figure 16(a);

2- if r is odd or $r = \frac{2k_1 + 1}{2k_2 + 1}$ with $k_1, k_2 \in \mathbb{N}$, the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends only on the sign of the parameter η. Here the graphics of this function are shown in Figure 16(b) if $\eta < 0$ and Figure 16(c) if $\eta > 0$;

3- if r is irrational, or $r = \frac{k_1}{2k_2}$ with k_1 an odd integer, then the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends only on the sign of η. Consequently the graphics of $f_1^{(i)}(y_2)$ are given in Figure 16(b) and (c) restricted on its definition domain.

For $r < 0$ and in a similar way we found the same graphics than the case $r > 0$.

Now to study all the possible graphics for the function $g_1^{(i)}(y_2)$, we need to study the sign of its derivative

$$\left(g_1^{(i)}\right)'(y_2) = \frac{P(y_2)}{[(s_1+s_2\ y_2)(s_1+s_3\ y_2)]^3}.e^{\frac{k_1\ y_2+k_2\ y_2^2+k_3\ y_2^3+k_4\ y_2^4}{[(s_1+s_2\ y_2)(s_1+s_3\ y_2)]^2}},$$

where

$$P(y_2) = k_1 s_1^2 + s_1(2k_2 s_1 - k_1(s_2+s_3))\ y_2 + 3(k_3 s_1^2 - k_1 s_2 s_3)\ y_2^2 + (-2k_2 s_2 s_3 \\ + k_3 s_1(s_2+s_3) + 4k_4 s_1^2)\ y_2^3 + (2k_4 s_1(s_2+s_3) - k_3 s_2 s_3)\ y_2^4.$$

Here $y_{21} = \frac{-s_1}{s_2}$ and $y_{22} = \frac{-s_1}{s_3}$ represent the two vertical asymptote straight lines for the function $g_1^{(i)}(y_2)$, and $h_2 = e^{\frac{k_4}{(s_2 s_3)^2}}$ is the horizontal asymptote straight line, and if $s_2 \neq s_3$ we know that all the possible graphics of the function $g_1^{(i)}(y_2)$ are given as follows:

1- If $P(y_2)$ has four simple real roots r_i for $i = 1,2,3,4$, and according to the possible positions of these roots with respect to the two vertical asymptotes, the graphics of $g_1^{(i)}(y_2)$ are given in Figure 20(a) and Figure 20(b) if $r_1 < r_2 < r_3 < y_{21} < r_4 < y_{22}$, or Figure 20($c$) and Figure 20($d$) if $r_1 < r_2 < y_{21} < r_3 < y_{22} < r_4$, or Figure 20($e$) and Figure 20($f$) if $r_1 < r_2 < y_{21} < r_3 < r_4 < y_{22}$, or Figure 20($g$) and Figure 20($h$) if $r_1 < y_{21} < r_2 < r_3 < y_{22} < r_4$, or Figure 20($i$) and Figure 20($j$) if $r_1 < y_{21} < r_2 < r_3 < r_4 < y_{22}$.

2- If $P(y_2)$ has one simple real root and one triple real root, or two complex roots and two simple real roots named by r_1 and r_2, then there are two possible positions for these roots with respect to the two vertical asymptotes. Then in this case the graphics of $g_1^{(i)}(y_2)$ are given in Figure 20(k) and Figure 20(l) if $r_1 < y_{21} < r_2 < y_{22}$, or Figure 20($m$) and Figure 20($n$) if $y_{21} < r_1 < r_2 < y_{22}$.

3- If $P(y_2)$ has two double real roots, the graphics are shown in Figures 20(o) and (p).

4- If $P(y_2)$ has one double real and two complex roots, the graphics are shown in Figures 20(q) and (r).

5- If $P(y_2)$ has four complex roots, the graphics are shown in Figures 20(s) and (t).

6- If $P(y_2)$ has one real root of order four, the graphics are shown in Figures 20 (u) and (v).

7- If $P(y_2)$ has one double real root r_0 and two simple real roots r_1 and r_2, the graphics of the function $g_1^{(i)}(y_2)$ with the different possible positions of these roots with the vertical asymptote are shown in Figure 19(a) and Figure 19(b) if $y_{21} < r_0 < y_{22} < r_1 < r_2$, or Figure 19($c$) and Figure 19($d$) if $r_0 < y_{21} < r_1 < r_2 < y_{22}$, or Figure 19($e$) and Figure 19($f$) if $r_0 < y_{21} < r_1 < y_{22} < r_2$, or Figure 19($g$) and Figure 19($h$) if $y_{21} < r_1 < y_{22} < r_0 < r_2$, or Figure 19($i$) and Figure 19($j$) if $r_1 < y_{21} < r_0 < r_2 < y_{22}$, or Figure 19($k$) and Figure 19($l$) if $r_1 < y_{21} < r_0 < y_{22} < r_2$.

For $s_2 = s_3$, the function $f_1^{(i)}(y_2)$ becomes a constant function $f_1^{(i)}(y_2) \equiv 1$, and $P(y_2)$ becomes a cubic polynomial which can have at most three real roots, and we omit the graphics of this case.

From the function $f_1^{(i)}(y_2)$, it is clear that the sign of the derivative $\left(f_1^{(i)}\right)'(y_2)$ can change at most three times when r is an even integer or $r = k_1/(2k_2+1)$ such that $k_1, k_2 \in \mathbb{N}$, see Figure 16(a), and this change of sign guarantees that this case gives the maximum number of the intersection points with the other function. Since the derivative of $g_1^{(i)}(y_2)$ can change its sign at most seven times, this appears in the first ten graphics of Figure 20. By considering the dependence between the horizontal and vertical asymptotes of the two functions, the maximum number of intersection points will be reduced. Therefore if $y_{21} < y_{22}$ is fixed, we have to study three distinct cases depending on the position of the horizontal asymptote straight line h_1 of the function $f_1^{(i)}(y_2)$ with the horizontal asymptote straight line h_2 of the function $g_1^{(i)}(y_2)$.

Now assuming that $h_1 < h_2$, we can locate four, or three, or two, or one intersection point on the left side of the vertical asymptote y_{21} between the functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$. The four intersection points between $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$ are resulting from the intersection between Figure 16(a) and Figure

$20(a)$, the three intersection points are resulting from Figure $16(a)$ and Figure $20(b)$, or Figure $20(c)$ or Figure $20(f)$, the two intersection points are resulting from Figure $16(a)$ and Figure $20(d)$, or Figure $20(e)$, or Figure $20(h)$ or Figure $20(i)$, and finally the unique intersection point coming from Figure $16(a)$ and Figure $20(g)$ or Figure $20(j)$. In a similar way and between y_{21} and y_{22} we can find four intersection points between Figure $16(a)$ and Figure $20(i)$ or Figure $20(j)$, three intersection points from Figure $16(a)$ and Figure $20(e)$-(h), two intersection points between Figure $16(a)$ and Figure $20(a)$-(d). On the right side of y_{22} we can find only one intersection point between Figure $16(a)$ and all the graphics of Figure 20. Then for $h_1 < h_2$ we know that the maximum number of intersection points between $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$ is at most seven. See for example the intersection of Figure $16(a)$ and Figure $20(a)$.

Similarly we find that the maximum number of intersection points in the graphics of Figures 16 and 20 is at most seven for $h_1 \geq h_2$. We know that if (y_1, y_2) is a solution of (20) then (y_2, y_1) is also a solution of this system. Consequently the maximum number of limit cycles is at most three.

By taking the values $\{r, s_1, s_2, s_3, k_1, k_2, k_3, k_4\} = \{2, -2, 2, 1, -0.45, 1.38, -1.32, 0.38\}$ we construct an example with exactly six intersection points between the graphics of the two functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$, these points are shown in Figure 4. Then we have three limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ with $A \neq 0$.

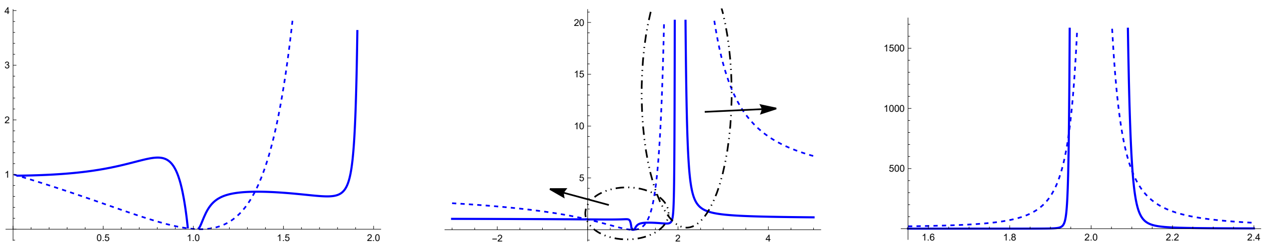


Fig. 4. Example with six intersection points between the graphics of the two functions $f_1^{(i)}(y_2)$ drawn in dashed line and $g_1^{(i)}(y_2)$ drawn in a continuous line.

In what follows we give a discontinuous piecewise differential system of the class $\mathcal{C}_i$ where $A \neq 0$ with exactly three limit cycles. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} &= -0.564609..x^2 + x(-0.669869..y - 30.2403..) + (-0.203527..y - 15.8386..)y - 430.571..,\\ \dot{y} &= 0.203527..x^2 + x(0.0459715..y + 8.26475..) + (-0.0592883..y - 3.18892..)y + 96.2791..,\end{aligned} \tag{21}$$

its first integral is

$$H_1^{(i)}(x,y) = \frac{Exp\Big(\dfrac{13.038..x^2 + x(18.979..y + 704.713..) + y(6.9071..y + 485.714..) + 9716.81..}{(x+y+30)^2}\Big)}{(0.1x + 0.1y + 3)^4}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -2x - 20y, \quad \dot{y} = x + 2y, \tag{22}$$

with the first integral

$$H(x,y) = (x+2y)^2 + 16y^2.$$

For the discontinuous piecewise differential system (21)–(22), system (20) has the three solutions $(x_1, y_1) = (2.96648.., 0.663325..)$, $(x_2, y_2) = (2.14476.., 0.479583..)$ and $(x_3, y_3) = (0.632456.., 0.141421..)$ which provide the three crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the six points $(x_j, 0)$ and $(0, y_j)$ with $j = 1, 2, 3$, see Figure $1(a)$.

If $A = 0 \neq b$ corresponding to $k = 2$ and $j = i$ in system (20), the first integral is $H_2^{(i)}(x,y)$ given in (7), the study of the solutions y_2 satisfying $F_2^{(i)}(y_2) = 0$, is equivalent to studying the solutions y_2 of the equation $f_2^{(i)}(y_2) = g_2^{(i)}(y_2)$ such that

$$f_2^{(i)}(y_2) = e^{L_0 + L_1\ y_2 + L_2\ y_2^2} \quad \text{and} \quad g_2^{(i)}(y_2) = \frac{K_0 + K_1\ y_2 + K_2\ y_2^2}{K_0 + G_1\ y_2 + G_2\ y_2^2},$$

where

$$K_0 = 2b^3\delta_1^2 + 2b^2 d\delta_2^2 + 2b^2\delta_2 - 2bd\delta_2 - b + d, \quad K_1 = (4\alpha_1 b^3\delta_1 + 2\alpha_2 b^2 + 4\alpha_2 b^2 d\delta_2 - 2\alpha_2 bd)\lambda,$$
$$K_2 = (2\alpha_1^2 b^3 + 2\alpha_2^2 b^2 d)\lambda^2, \quad G_1 = 4b^3\gamma_1\delta_1 + 2b^2\gamma_2 + 4b^2\gamma_2 d\delta_2 - 2b\gamma_2 d, \quad G_2 = 2b^3\gamma_1^2 + 2b^2\gamma_2^2 d,$$
$$L_0 = L_2 = 0, \quad L_1 = 2b\gamma_2 - 2\alpha_2 b\lambda.$$

The function $g_2^{(i)}(y_2)$ has a horizontal asymptote straight line $h = \dfrac{K_2}{G_2}$, and its first derivative is given by

$$\left(g_2^{(i)}\right)'(y_2) = \frac{P_1(y_2)}{P_2(y_2)^2},$$

with $P_1(y_2) = K_0K_1 - G_1K_0 + (2K_0K_2 - 2G_2K_0)\ y_2 + (G_1K_2 - G_2K_1)\ y_2^2$ and $P_2(y_2) = K_0 + G_1\ y_2 + G_2\ y_2^2$. We denoted by $\Delta = (2K_0K_2 - 2G_2K_0)^2 - 4(G_1K_2 - G_2K_1)(K_0K_1 - G_1K_0)$ the discriminant of the quadratic equation $P_1(y_2)$. Then according to the different kinds of solutions of the quadratic equation $P_2(y_2) = 0$ and the sign of Δ, we know that the possible graphics of $g_2^{(i)}(y_2)$ are given as follows:

1- If the quadratic equation $P_2(y_2) = 0$ has two real solutions named by y_{21}, y_{22} and $\Delta > 0$, the possible graphics of $g_2^{(i)}(y_2)$ are (a) and (b) of Figure 22. If $\Delta \leq 0$ the possible graphics of $g_2^{(i)}(y_2)$ are (c) and (d) of Figure 22.

2- If the equation $P_2(y_2) = 0$ has one double real solution y_2 and $\Delta > 0$, then the possible graphics of $g_2^{(i)}(y_2)$ are (e) and (f) of Figure 22. If $\Delta \leq 0$ the possible graphics of $g_2^{(i)}(y_2)$ are (g) and (h) of Figure 22.

3- If $P_2(y_2) = 0$ has two complex solutions and $\Delta > 0$ the possible graphics of $g_2^{(i)}(y_2)$ are (i) and (j) of Figure 22.

Since the possible graphics of $g_2^{(i)}(y_2)$ are given in Figure 22, and all the possible graphics for $f_2^{(i)}(y_2)$ are given by (c) and (d) of Figure 22 because $L_2 = 0$. Then, using the same arguments for studying the intersection points between the graphics of $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$, we obtain that the maximum number of intersection points between these two functions is at most four. So by the symmetry as in the first case when $k = 1$, the maximum number of limit cycles is at most two.

By taking the values $\{L_0, L_1, L_2, K_0, K_1, K_2, G_1, G_2\} = \{0, -3, 0, -0.1, 0.9, -0.8, 0.6, 0.4\}$. We construct an example with precisely four intersection points between the graphics of the two functions $f_2^{(i)}(y_2)$ and $g_2^{(i)}(y_2)$. These intersection points are presented in Figure 5.

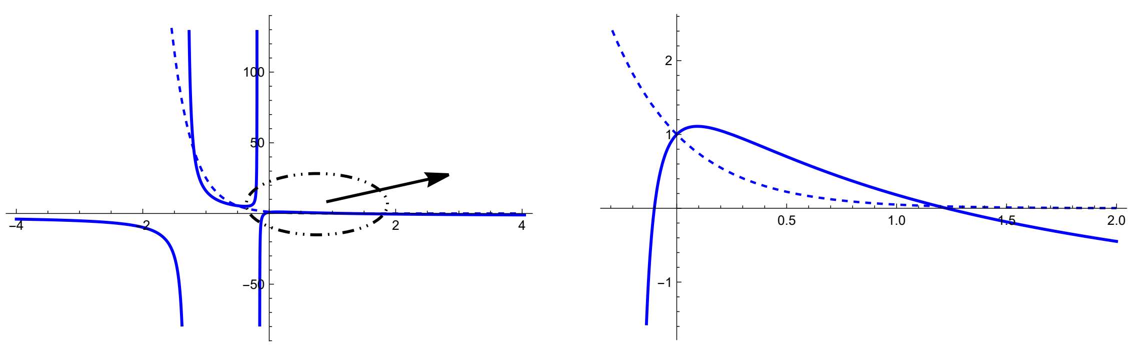


Fig. 5. The four intersection points between the graphics of the two functions $f_2^{(i)}(y_2)$ drawn in dashed line and $g_2^{(i)}(y_2)$ drawn in a continuous line.

If $b = 0 \neq A$ corresponding to $k = 3$ and $j = i$ in system (20), the first integral of the quadratic center satisfying the first condition of Theorem 1 is $H_3^{(i)}(x, y)$ given by (8), and the solutions of $F_3^{(i)}(y_2) = 0$ are equivalent to the solutions of the equation $f_3^{(i)}(y_2) = g_3^{(i)}(y_2)$ such that

$$f_3^{(i)}(y_2) = f_1^{(i)}(y_2) \quad \text{with} \quad r = 2(d - A) \quad \text{and} \quad g_3^{(i)}(y_2) = f_2^{(i)}(y_2),$$

where

$$\begin{aligned}
L_0 &= 0,\\
L_1 &= 2A\Big(A^2\delta_1(\alpha_1\lambda - \gamma_1) - A(d\delta_2 + 1)(\gamma_2 - \alpha_2\lambda) + d(\gamma_2 - \alpha_2\lambda)\Big),\\
L_2 &= A^2\Big(A\alpha_1^2\lambda^2 - A\gamma_1^2 + \alpha_2^2 d\lambda^2 - \gamma_2^2 d\Big).
\end{aligned}$$

As in the precedent case, we give all the possible graphics of $g_3^{(i)}(y_2)$ shown in Figure 17, and the possible ones for $f_3^{(i)}(y_2)$ are shown in Figure 16. Hence the maximum number of intersection points between these two functions is at most four. Then by symmetry, the maximum number of limit cycles is at most two.

By taking $\{L_0, L_1, L_2, r, s_1, s_2, s_3\} = \{0, 0.9, -0.09, 2, 2, -2, -1\}$ we produce an example with precisely four intersection points between the graphics of the function $f_3^{(i)}(y_2)$ and $g_3^{(i)}(y_2)$. These intersection points appear in Figure 6. Then we have two limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ when $b = 0 \neq A$.

In what follows we give a discontinuous piecewise differential system of this class $\mathcal{C}_i$ when $b = 0 \neq A$ with two limit cycles. In the region Σ^+ we consider the quadratic center

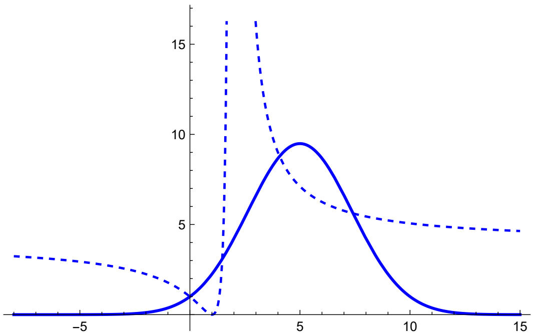


Fig. 6. The four intersection points between the graphics of the two functions $g_3^{(i)}(y_2)$ drawn in a continuous line and $f_3^{(i)}(y_2)$ drawn in dashed line.

$$\begin{aligned}
\dot{x} &= x(0.547606.. - 0.794014..y) + y(2.73803.. - 6.69883..y) + 0.10915..x^2\\
&\quad -1.82025..,\\
\dot{y} &= 0.0126244..x^2 + x(-0.0460281..y - 0.0295211..) + (-0.545752..y\\
&\quad -0.147606..)y + 0.364049..,
\end{aligned} \tag{23}$$

its corresponding first integral is

$$H_3^{(i)}(x, y) = (x + 5y)e^{0.0173389..x^2 + x(-0.299823..y - 0.081091..) + y(1.84009..y - 1.50421..)}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -x - 17y, \quad \dot{y} = x + y, \tag{24}$$

with the first integral

$$H(x, y) = (x + y)^2 + 16y^2.$$

For the discontinuous piecewise differential system (23)–(24), system (20) has the two solutions $(x_1, y_1) = (2.12132.., 0.514496..)$ and $(x_2, y_2) = (1, 0.242536..)$ which provide the two crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the four points $(x_j, 0)$ and $(0, y_j)$ with $j = 1, 2$, see Figure 1(b).

If $A = b = 0$ corresponding to $k = 4$ and $j = i$ in system (20), the first integral in this case is $H_4^{(i)}(x, y)$ given in (9), and $F_4^{(i)}(y_2)$ is a polynomial in the variable y_2 of degree three. The maximum number of real solutions of $F_4^{(i)}(y_2) = 0$ is three. Then in this case the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ when $A = b = 0$ is at most one.

In what follows we give a discontinuous piecewise differential system of the class $\mathcal{C}_i$ when $A = b = 0$ with one limit cycle. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} &= x(0.275591.. - 0.0393701..y) - 0.00393701..x^2 + (-0.0984252..y \\ &\quad -0.622047..)y - 1.25984.., \\ \dot{y} &= 0.000787402..x^2 + x(0.00787402..y + 4.94488..) + (0.019685..y \\ &\quad -0.275591..)y - 5.74803..,\end{aligned} \tag{25}$$

this system has the first integral

$$H_4^{(i)}(x, y) = 2(0.1x + 0.5y + 1)^3 + 3\left((-2.5x + 0.2y + 3)^2 + (0.1x + 0.5y + 1)^2\right).$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -0.4x - 1.1571..y, \quad \dot{y} = x + 0.4y, \tag{26}$$

with the first integral

$$H(x, y) = (x + 0.4y)^2 + 0.997096..y^2.$$

For the discontinuous piecewise differential system (25)–(26), system (20) has the unique solution $(x_1, y_1) = (3.26625.., 3.03644..)$ that provides the unique crossing limit cycle intersecting the rays Γ_1 and Γ_2 in the two points $(x_1, 0)$ and $(0, y_1)$, see Figure 1(c). This example completes the proof of statement (a). ■

Proof. [Proof of statement (b) of Theorem A] Now we will prove the statement for the second class $\mathcal{C}_{ii}$ when $b + d = 0$. If $A + b = 0$ and $a = 0 \neq Cb$ corresponding to $k = 1$ and $j = ii$ in system (20), the first integral of the quadratic center is $H_1^{(ii)}(x, y)$ given in (10). The study of the solutions y_2 satisfying $F_1^{(ii)}(y_2) = 0$ is equivalent to study the solutions y_2 of the equation $f_1^{(ii)}(y_2) = g_1^{(ii)}(y_2)$ such that

$$f_1^{(ii)}(y_2) = \left(\frac{m_1 + m_2\ y_2}{m_1 + m_3\ y_2}\right)^p \left(\frac{n_1 + n_2\ y_2}{n_1 + n_3\ y_2}\right)^q \quad \text{and} \quad g_1^{(ii)}(y_2) = e^{\dfrac{K_1\ y_2 + K_2\ y_2^2}{(m_1 + m_2\ y_2)(m_1 + m_3\ y_2)}},$$

where

$$\begin{aligned}&m_1 = 1 - b\delta_2, \quad m_2 = -b\gamma_2, \quad m_3 = -\alpha_2 b\lambda, \quad p = b^2 + C^2, \\ &n_1 = 1 - b\delta_2 + C\delta_1, \quad n_2 = (\alpha_1 C - \alpha_2 b)\lambda, \quad n_3 = \gamma_1 C - b\gamma_2, \quad q = b^2, \\ &K_1 = bC\left(b^2\left(\lambda(\alpha_2\delta_1 - \alpha_1\delta_2) + \gamma_1\delta_2 - \gamma_2\delta_1\right) + b\left(\alpha_1\lambda - \gamma_1\right) + \alpha_2 C\lambda - \gamma_2 C\right), \\ &K_2 = b^3 C\lambda(\alpha_2\gamma_1 - \alpha_1\gamma_2).\end{aligned}$$

For $p, q > 0$ and from the geometric study, we can divide the study of the function $f_1^{(ii)}(y_2)$ on two parts according to the number of the vertical asymptotes straight lines.

The function $f_1^{(ii)}(y_2)$ has one vertical asymptote straight line if either $p = 0$ and $q \neq 0$, or $q = 0$ and $p \neq 0$, or $m_2 = m_3$, or $n_2 = n_3$ in these cases the graphics of the function $f_1^{(ii)}(y_2)$ are the same as the ones of the function $f_1^{(i)}(y_2)$ shown in Figure 16.

To draw all the possible graphics of the function $f_1^{(ii)}(y_2)$ which has two vertical asymptotes straight lines $y_{21} = \dfrac{-m_1}{m_3}$ and $y_{22} = \dfrac{-n_1}{n_3}$, we need to study the sign of its first derivative that depend on the nature of the parameters p and q, the roots of the quadratic polynomial $P(y_2) = M_0 + M_1\ y_2 + M_2\ y_2^2$ and of the possible position of these two roots with the positions of y_{21} and y_{22} and on the two roots $y_{01} = \dfrac{-m_1}{m_2}$ and $y_{02} = \dfrac{-n_1}{n_2}$ of $\left(f_1^{(ii)}\right)'(y_2)$, where

$$\left(f_1^{(ii)}\right)'(y_2) = \frac{(m_1 + m_2\ y_2)^{p-1}(n_1 + n_2\ y_2)^{q-1}}{(m_1 + m_3\ y_2)^{p+1}(n_1 + n_3\ y_2)^{q+1}} P(y_2),$$

where

$$\begin{aligned} M_0 &= m_1^2 n_1 q(n_2 - n_3) + m_1 n_1^2 p(m_2 - m_3), \\ M_1 &= m_1 n_1 \big(p(m_2 - m_3)(n_2 + n_3) + q(m_2 + m_3)(n_2 - n_3)\big), \\ M_2 &= m_1 n_2 n_3 p(m_2 - m_3) + m_2 m_3 n_1 q(n_2 - n_3). \end{aligned}$$

Now we will give the possible positions of the real roots of the quadratic polynomial with respect to y_{21}, y_{22}, y_{01} and y_{02}.

I - If $P(y_2)$ has two distinct real roots r_1 and r_2, the possible positions of these two roots with respect to y_{21}, y_{22}, y_{01} and y_{02} are

(1) $y_{01} < r_1 < y_{02} < y_{21} < r_2 < y_{22}$;
(2) $y_{01} < y_{21} < y_{02} < r_1 < r_2 < y_{22}$;
(3) $y_{01} < r_1 < r_2 < y_{21} < y_{02} < y_{22}$;
(4) $r_1 < y_{01} < y_{21} < r_2 < y_{22} < y_{02}$;
(5) $y_{21} < y_{01} < r_1 < y_{02} < y_{22} < r_2$;
(6) $r_1 < r_2 < y_{21} < y_{01} < y_{22} < y_{02}$.

II - If $P(y_2)$ has one double real root r_0, the possible positions of this double root with respect to y_{21}, y_{22}, y_{01} and y_{02} are

(1) $r_0 < y_{01} < y_{02} < y_{21} < y_{22}$;
(2) $y_{01} < r_0 < y_{21} < y_{02} < y_{22}$;
(3) $y_{01} < y_{21} < r_0 < y_{02} < y_{22}$.

III - If $P(y_2)$ has two complex roots, we have $y_{01} < y_{21} < y_{02} < y_{22}$ as the only possible position.

Now all the possible graphics of the function $f_1^{(ii)}(y_2)$ are shown in Figures 21, 23 and 26 and in what follows we explain how they have been obtained.

1- If p and q are even integers, or if p is even and $q = \dfrac{k_1}{2k_2 + 1}$ with $k_1, k_2 \in \mathbb{N}$, we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 21. If $P(y_2)$ has two distinct real roots taking either the position (1), or (2), or (3), or (4), or (5) or (6), then the graphics of $f_1^{(ii)}(y_2)$ are given by (a), or (b), or (c), or (d), or (e) or (f) of Figure 21, respectively. If $P(y_2)$ has two complex roots, the graphic of $f_1^{(ii)}(y_2)$ is given by (g) in Figure 21. If $P(y_2)$ has one double real root taking either the position (1), or (2) or (3), then the graphics of $f_1^{(ii)}(y_2)$ are given by (h), or (i) or (j) of Figure 21, respectively.

2- If p and q are odd integers we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 23. If $P(y_2)$ has two distinct real roots then the graphics of $f_1^{(ii)}(y_2)$ are given by (a) and (b) of Figure 23 when these roots taking position (1), (c) and (d) of Figure 23 when these roots taking position (2), (e) and (f) of Figure 23 when these roots taking position (3), (g) and (h) of Figure 23 when these roots taking position (4), (i) and (j) of Figure 23 when these roots taking position (5), (k) and (l) of Figure 23 when these roots taking position (6). If $P(y_2)$ has two complex roots, the graphics of $f_1^{(ii)}(y_2)$ are given in (m) and (n) of Figure 23. If $P(y_2)$ has one double real root then the graphics of $f_1^{(ii)}(y_2)$ are given by (o) and (p) of Figure 23 when this root takes position (1), (q) and (r) of Figure 23 when this root takes position (2), (s) and (t) of Figure 23 when this root takes position (3).

3- In a similar way if p odd and q even, or if p odd and $q = \dfrac{k_1}{2k_2+1}$ with $k_1, k_2 \in \mathbb{N}$, we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 26.

4- If p is odd and $q = \dfrac{k_1}{2k_2}$ and k_1 is an odd integer and $k_2 \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than in the case that both p, q are odd, but in its domain of definition.

5- If p is even and $q = \dfrac{k_1}{2k_2}$ and $k_1, k_2 \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)$. Therefore the graphics of $f_1^{(ii)}(y_2)$ are the same as in the case where q is odd and p is even, but in its domain of definition.

6- If $p = \dfrac{k_1}{2k_2}$ and $q = \dfrac{k_1'}{2k_2'}$ and k_1, k_1' are odd integers, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same as in the case p, q are odd, but in its domain of definition.

7- If $p = \dfrac{k_1}{2k_2+1}$ and $p = \dfrac{k_1'}{2k_2'+1}$ with $k_1, k_2, k_1', k_2' \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)(m_1+m_2y_2)(m_1+m_3y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are topologically equivalent to graphics of case of both p, q are even but in its domain of definition.

8- If $p = \dfrac{k_1}{2k_2}$ and $p = \dfrac{k_1'}{2k_2'+1}$ and k_1 is an odd integer with $k_2, k_1', k_2' \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than in the case p odd and q even, but in its domain of definition.

In a similar way we find the same graphics than in the case $p, q > 0$ if both p and q are negative, or if one of them is negative and the other is positive.

Now for the function $g_1^{(ii)}(y_2)$, the horizontal asymptote straight line is $h = \dfrac{K_2}{m_2m_3}$, and the first derivative of $g_1^{(ii)}(y_2)$ is

$$\left(g_1^{(ii)}\right)'(y_2) = \frac{P(y_2)}{(m_1+m_2\ y_2)(m_1+m_3\ y_2)} e^{\dfrac{K_1y_2+K_2y_2^2}{(m_1+m_2\ y_2)(m_1+m_3\ y_2)}},$$

where $P(y_2) = K_1m_1^2 + 2K_2m_1^2\ y_2 + (K_2m_1(m_2+m_3) - K_1m_2m_3)\ y_2^2$. For $m_2 \neq m_3$ and according to the different possible kinds of roots of the quadratic polynomial $P(y_2)$ and their possible positions with the two vertical asymptote straight lines y_{01} and y_{21} of the function $f_1^{(ii)}(y_2)$, all the graphics of the function $g_1^{(ii)}(y_2)$ are given in what follows.

1- If $P(y_2)$ has two distinct real roots, the graphics of the function $g_1^{(ii)}(y_2)$ are shown in (a) and (b) of Figure 18.

2- If $P(y_2)$ has two complex roots, the graphics of the function $g_1^{(ii)}(y_2)$ are shown in Figures 18(c) and 18(d).

3- If $P(y_2)$ has one double real root this root is either y_{01} or y_{21}. Here the graphics of the function $g_1^{(ii)}(y_2)$ are shown in Figures 18(e) and 18(f).

As in the precedent case and for the same reason, we only have drawn the graphics of the function $g_1^{(ii)}(y_2)$ for $m_2 \neq m_3$.

The function $g_1^{(ii)}(y_2)$ has at most three changes in the sign of its derivative which appear in (a) and (b) of Figure 18. We also know that $g_1^{(ii)}(y_2)$ is a positive function, so to get the maximum number of intersection points between the graphics (a) and (b) of Figure 18 and the graphics of $f_1^{(ii)}(y_2)$ it is sufficient to solve the problem of the intersection points between the graphics (a) and (b) of Figure 18 with the graphics of Figure 21. As we mentioned in the precedent cases the change of sign of the derivative of $f_1^{(ii)}(y_2)$ plays a main role, here we see that seven is the maximum number of the changes of $\left(f_1^{(ii)}\right)'(y_2)$ and this is shown in $(a),\cdots,(f)$ of Figure 21. In this case we remark that y_{01} and y_{21} represent a root and a vertical asymptote, respectively, of the function $f_1^{(ii)}(y_2)$ and also the vertical asymptotes for the function $g_1^{(ii)}(y_2)$. Then if we choose the horizontal asymptote of the function $g_1^{(ii)}(y_2)$ less than the one of $f_1^{(ii)}(y_2)$, we get at most four intersection points between Figure 18(a) and (a), (b), (e) and (f) of Figure 21. Similarly, we study the case when the horizontal asymptote of the function $g_1^{(ii)}(y_2)$ is greater than the one of $f_1^{(ii)}(y_2)$. These points are less than y_{01}, and at most four intersection points between y_{01} and y_{21} by intersecting Figure 18(a) with Figure 21(a) and no intersection points after y_{21}. So we remark that these graphics can intersect at most at eight points. Then the upper bound number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A+b=0$ and $a=0\neq Cb$ is four.

By taking $\{K_1,K_2,m_1,m_2,m_3,p,n_1,n_2,n_3,q\}=\{-6,0.1,-2,2.4,0.9,4,-2,5,2,4\}$ we construct an example with exactly eight intersection points between the graphics of $f_1^{(ii)}(y_2)$ and $g_1^{(ii)}(y_2)$, these points are highlighted in Figure 7.

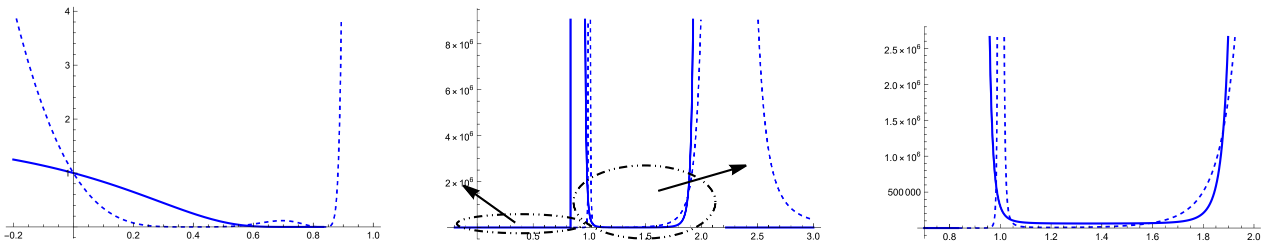


Fig. 7. The eight intersection points between the graphics of the two functions $f_1^{(ii)}(y_2)$ drawn in dashed line and $g_1^{(ii)}(y_2)$ drawn in continuous line.

If $AbC(A+b)\Delta\neq 0=a$ and $\Delta<0$ corresponding to $k=2$ and $j=ii$ in system (20), the first integral of the quadratic center (5) is $H_2^{(ii)}(x,y)$ given in (13). Then to study the solutions y_2 satisfying $F_2^{(ii)}(y_2)=0$ is equivalent to study the solutions y_2 of $f_2^{(ii)}(y_2)=g_2^{(ii)}(y_2)$ such that

$$f_2^{(ii)}(y_2)=\left(\frac{s_1+s_2\ y_2}{s_1+s_3\ y_2}\right)^r\left(\frac{K_0+K_1\ y_2+K_2\ y_2^2}{K_0+G_1\ y_2+G_2\ y_2^2}\right)^{r'}\quad\text{and}\quad g_2^{(ii)}(y_2)=e^{M\ \cot^{-1}\left(\dfrac{M_0+M_1y_2+M_2y_2^2}{N_0+N_1y_2+N_2y_2^2}\right)},$$

where

$$\begin{aligned}
M &= \frac{2C}{bL},\quad M_0=L(2b\delta_2-C\delta_1+\delta_1L-2),\\
M_1 &= L(\delta_1(2b\gamma_2-\gamma_1C+\gamma_1L)+\alpha_1\lambda(2b\delta_2-C\delta_1+\delta_1L-2)),\\
M_2 &= \alpha_1L\lambda(2b\gamma_2-\gamma_1C+\gamma_1L),\quad N_0=2\delta_1L(-2b\delta_2+C\delta_1+2),\\
N_1 &= 2L(-b\gamma_2\delta_1-b\delta_2(\gamma_1+\alpha_1\lambda)-\alpha_2b\delta_1\lambda+\gamma_1+\gamma_1C\delta_1+\alpha_1C\delta_1\lambda+\alpha_1\lambda),\\
N_2 &= -2L\lambda(\alpha_1b\gamma_2+\alpha_2b\gamma_1-\alpha_1\gamma_1C),\\
K_0 &= b^2\delta_2^2-bC\delta_1\delta_2-2b\delta_2+\frac{1}{4}\delta_1^2\left(C^2+L^2\right)+C\delta_1+1,\\
K_1 &= \frac{1}{2}\lambda\left(4\alpha_2b(b\delta_2-1)-2C(\alpha_1(b\delta_2-1)+\alpha_2b\delta_1)+\alpha_1C^2\delta_1+\alpha_1\delta_1L^2\right),\\
K_2 &= \frac{1}{4}\lambda^2\left(4\alpha_2^2b^2-4\alpha_1\alpha_2bC+\alpha_1^2\left(C^2+L^2\right)\right),
\end{aligned}$$

$$G_1 = 2b^2\gamma_2\delta_2 - 2b\gamma_2 - b\gamma_1 C\delta_2 - b\gamma_2 C\delta_1 + \frac{1}{2}\gamma_1\delta_1\left(C^2+L^2\right) + \gamma_1 C,$$
$$G_2 = b^2\gamma_2^2 - b\gamma_1\gamma_2 C + \frac{1}{4}\gamma_1^2\left(C^2+L^2\right), \quad r' = \frac{1}{b}, \quad r = \frac{8b}{4b^2+C^2+L^2},$$
$$s_1 = 1 - \frac{2\delta_2}{r}, \quad s_2 = -\frac{2\gamma_2}{r}, \quad s_3 = -\frac{2\alpha_2\lambda}{r}.$$

The solutions of $f_2^{(ii)}(y_2) = g_2^{(ii)}(y_2)$ represent the intersection points between the graphics of the two functions $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$. We have that

$$\left(f_2^{(ii)}\right)'(y_2) = \frac{(s_1+s_2\ y_2)^{r-1}\left(K_0+K_1\ y_2+K_2\ y_2^2\right)^{r'-1}}{(s_1+s_3\ y_2)^{r+1}\left(K_0+G_1\ y_2+G_2\ y_2^2\right)^{r'+1}}\ P_1(y_2),$$

and

$$\left(g_2^{(ii)}\right)'(y_2) = \frac{P_2(y_2)\ e^{M\ \cot^{-1}\left(\frac{M_0+M_1y_2+M_2y_2^2}{N_0+N_1y_2+N_2y_2^2}\right)}}{(M_0+M_1y_2+M_2y_2^2)^2+(N_0+N_1y_2+N_2y_2^2)^2},$$

with

$$\begin{aligned} P_1(y_2) = {} & K_0s_1(r's_1(K_1-G_1)+K_0r(s_2-s_3)) + (K_0s_1(-s_3(G_1(r'+r)+K_1(r-r'))-G_1r's_2+ \\ & G_1rs_2-2G_2r's_1+K_1r's_2+K_1rs_2+2K_2r's_1))\ y_2 + \Big(G_1(-K_0r's_2s_3+K_1rs_1(s_2-s_3) \\ & +K_2r's_1^2)-G_2s_1(2K_0r'(s_2+s_3)+K_0r(s_3-s_2)+K_1r's_1)+K_0K_1r's_2s_3+K_0K_2s_1(2r' \\ & (s_2+s_3)+r(s_2-s_3))\Big)\ y_2^2 + (G_1K_2s_1(s_2(r'+r)+s_3(r'-r))-G_2(2K_0r's_2s_3+K_1s_1 \\ & (s_2(r'-r)+s_3(r'+r)))+2K_0K_2r's_2s_3)\ y_2^3 + (G_1K_2r's_2s_3-G_2K_1r's_2s_3+G_2K_2rs_1 \\ & (s_2-s_3))\ y_2^4, \end{aligned}$$

and

$$P_2(y_2) = -M(M_1N_0-M_0N_1) - M(2M_2N_0-2M_0N_2)\ y_2 - M(M_2N_1-M_1N_2)\ y_2^2.$$

Let $\Delta_1 = K_1^2 - 4K_0K_2 = -L^2(\alpha^2+\omega^2)(\alpha_1-\alpha_1b\delta_2+\alpha_2b\delta_1)^2$ and $\Delta_2 = G_1^2-4K_0G_2 = -L^2(-b\gamma_1\ \delta_2+b\gamma_2\delta_1+\gamma_1)^2$ be the discriminant of the quadratic equations $K_0+K_1\ y_2+K_2\ y_2^2=0$ and $K_0+G_1\ y_2+G_2\ y_2^2=0$, respectively. It is clear that $\Delta_i \le 0$ with $i=1,2$.

1^{st} **case.** If either $\Delta_i = 0$ with $i=1,2$, or if $\Delta_1=0$ and the exponent $r'<0$, or if $\Delta_2=0$ and $r'>0$, then the graphics of the function $f_2^{(ii)}(y_2)$ are equivalent to the graphics of the function $f_1^{(ii)}(y_2)$ that are shown in Figures 21, 23 and 26.

2^{nd} **case.** If either $\Delta_i<0$ with $i=1,2$, or if $\Delta_1<0$ and $r'<0$ or if $\Delta_2<0$ and $r'>0$. Here for $r>0$ the only vertical asymptote straight line for the function $f_2^{(ii)}(y_2)$ is $y_{21} = \dfrac{-s_1}{s_3}$, and $h = \left(\dfrac{s_2}{s_3}\right)^r\left(\dfrac{K_2}{G_2}\right)^{r'}$ is the horizontal asymptote straight line, then according to the sign of $\left(f_2^{(ii)}\right)'(y_2)$ which depends on the parameter r and on the different possible kind of the roots of the quartic polynomial $P_1(y_2)$ and their possible positions with respect to the vertical asymptote y_{21}, and denoted by $r_0 = \dfrac{-s_1}{s_2} < y_{21}$ we will obtain all the graphics of the function $f_2^{(ii)}(y_2)$ as follows.

1- If either r is even or $r = \dfrac{k_1}{2k_2+1}$ with $k_1,k_2 \in \mathbb{N}$, and If

(1.a) the polynomial $P_1(y_2)$ has four real roots r_i with $i=1,2,3,4$, then the graphics of $f_2^{(ii)}(y_2)$ are shown in Figures 24(a) and 24(b) when $r_j<y_{21}$ with $j\in\{0,1,2,3,4\}$, or in Figures 24(c) and 24(d) when

$r_j < y_{21}$ with $j \in \{0,1,2,3\}$ and $r_4 > y_{21}$, or in Figures 24(e) and 24(f) when $r_j < y_{21}$ with $j \in \{0,1,2\}$ and $r_4 > r_3 > y_{21}$.

(1.b) the polynomial $P_1(y_2)$ has four complex roots, the graphics of this function are shown in (g) and (h) of Figure 24.

(1.c) the polynomial $P_1(y_2)$ has one double real root r_1 and two complex roots. The graphics of $f_2^{(ii)}(y_2)$ are shown in (i) and (j) of Figure 24 when $r_1 < r_0 < y_{21}$, or in (k) and (l) of Figure 24 when $r_0 < r_1 < y_{21}$, or in (m) and (n) of Figure 24 when $r_0 < y_{21} < r_1$.

(1.d) the polynomial $P_1(y_2)$ has one triple and one simple real root, or two simple real roots r_1 and r_2 and two complex roots. The graphics of $f_2^{(ii)}(y_2)$ are given in (o) and (p) of Figure 24 when $r_j < y_{21}$ with $j = 0,1,2$, or (q) and (r) of Figure 24 when $r_j < y_{21}$ with $j = 0,1$ and $r_2 > y_{21}$.

(1.e) the polynomial $P_1(y_2)$ has two double real roots r_1 and r_2, The graphics are given in (s) and (t) of Figure 24 when $r_1 < r_2 < r_0 < y_{21}$, or in (u) and (v) of Figure 24 when $r_1 < r_0 < r_2 < y_{21}$, or in (w) and (x) of Figure 24 when $r_1 < r_0 < y_{21} < r_2$.

2- If r is odd we have the same graphics as the case when r is even where now r_0 represents an inflection point of the function $f_2^{(ii)}(y_2)$.

3- If $r = \dfrac{k_1}{2k_2}$ with $k_1, k_2 \in \mathbb{N}$, the sign of the derivative depends only on the sign of $(K_0 + K_1\, y_2 + K_2\, y_2^2)(K_0 + G_1\, y_2 + G_2\, y_2^2)P_1(y_2)$, then the possible graphics of the function $f_2^{(ii)}(y_2)$ are the same than the ones of the case where r is an odd integer drawn on its definition domain.

For $r < 0$ and in a similar way we find the same graphics as in the case of $r > 0$.

Now for the function $g_2^{(ii)}(y_2)$ it is clear that the sign of its derivative $\left(g_2^{(ii)}\right)'(y_2)$ depends only on the sign of the quadratic polynomial $P_2(y_2)$. So to study the variation of the function $g_2^{(ii)}(y_2)$ we distinguish three different cases.

1- When the function $g_2^{(ii)}(y_2)$ has two vertical asymptotes straight lines y_{21} and y_{22}. i.e., when $N_1^2 - 4N_0N_2 > 0$. According to the different possible kinds of roots of $P_2(y_2)$ and of their possible position with respect to the vertical asymptote y_{21} and y_{22}, all the graphics of the function $g_2^{(ii)}(y_2)$ are given in Figure 25. Indeed, if $P_2(y_2)$ has two distinct real roots r_1 and r_2, the graphics are given in (a) and (b) of Figure 25 when $r_1 < r_2 < y_{21} < y_{22}$, or in (c) and (d) of Figure 25 when $r_1 < y_{21} < r_2 < y_{22}$, or in (e) and (f) of Figure 25 when $y_{21} < r_1 < r_2 < y_{22}$, or in (g) and (h) of Figure 25 when $r_1 < y_{21} < y_{22} < r_2$. If $P_2(y_2)$ has two complex roots or one double real root, then the graphics of $g_2^{(ii)}(y_2)$ are in (i) and (j) of Figure 25 are the two possible graphics.

2- When the function $g_2^{(ii)}(y_2)$ has only one vertical asymptote straight line y_{21} if $P_2(y_2)$ has two distinct real solutions r_1 and r_2, then the graphic of $g_2^{(ii)}(y_2)$ is in (k) and (l) of Figure 25 when $r_1 < r_2 < y_{21}$ or (m) and (n) of Figure 25 when $r_1 < y_{21} < r_2$. If $P_2(y_2)$ has two complex roots or one double real root, then the graphics of $g_2^{(ii)}(y_2)$ are in (o) and (p) of Figure 25 are the two possible graphics.

3- When the function $g_2^{(ii)}(y_2)$ has no vertical asymptote straight line, the only possible graphics of $g_2^{(ii)}(y_2)$ are given in (q) and (r) of Figure 25.

In order to find the maximum number of intersection points between the graphics of the two functions $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$, we begin with the function $g_2^{(ii)}(y_2)$, and since Figure 25 illustrates all the possible graphics of the function $g_2^{(ii)}(y_2)$, then the maximum number of changes in the sign of its first derivative is at most three, as shown in $(a), \ldots, (h)$ of Figure 25. We know that seven is the maximum number of changes in the sign of the first derivative of $f_2^{(ii)}(y_2)$ shown in $(a), \ldots, (f)$ of Figures 21 and 24. If we fixe (a) of Figure 21, we remark that eight is the maximum number of intersection points between this figure and Figure 25(a). Similarly we check out the maximum number of intersection points between the remaining graphics of the functions $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$. Thus these two functions can clearly intersect at a maximum of eight points. Hence $F_2^{(ii)}(y_2) = 0$ can have at most eight real solutions. Consequently the maximum number of limit cycles for the class of the discontinuous piecewise differential system $\mathcal{C}_{ii}$ under the present conditions is at most four.

By taking $\{r, s_1, s_2, s_3, r', K_0, K_1, K_2, G_1, G_2, M, M_0, M_1, M_2, N_0, N_1, N_2\} = \{2, 2, -1.5, 2.2, 1.1, -3.5, 2.8, -1.6, 4.2, 4.2, 1.5, 3.5, 15, 3.3, -0.01, 1.8, -0.8\}$ we construct an example with exactly eight intersection points between the graphics of $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$, these points are shown in Figure 8.

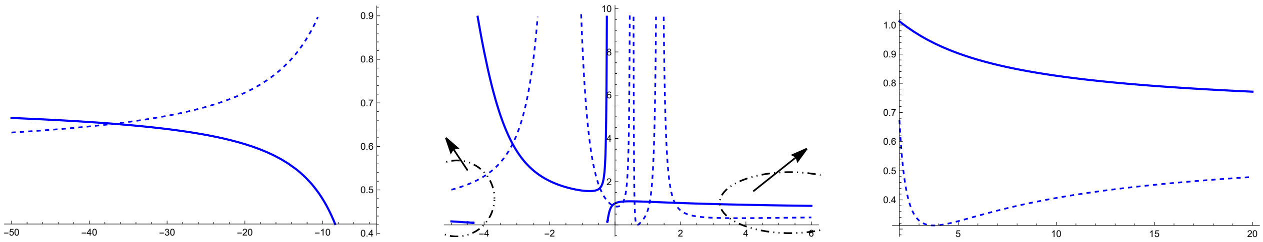


Fig. 8. The eight intersection points between the graphics of the two functions $f_2^{(ii)}(y_2)$ drawn in dashed line and $g_2^{(ii)}(y_2)$ drawn in continuous line.

If $C = b = 0$ corresponding to $k = 3$ and $j = ii$ in system (20), the first integral of the quadratic center is $H_3^{(ii)}(x, y)$ given in (12), the study of the solutions y_2 satisfying $F_3^{(ii)}(y_2) = 0$ is equivalent to study the solutions y_2 of the equation $f_3^{(ii)}(y_2) = g_3^{(ii)}(y_2)$ such that

$$f_3^{(ii)}(y_2) = \left(\frac{k_0 + k_1\ y_2 + k_2\ y_2^2}{k_0 + g_1\ y_2 + g_2\ y_2^2}\right)^A e^{M\ y_2} \quad \text{and} \quad g_3^{(ii)}(y_2) = (g_2^{(i)}(y_2))^r,$$

where

$$\begin{aligned}
&K_0 = -a^2\delta_2^2 + A\delta_2(a\delta_1 + 1) + (a\delta_1 + 1)^2, \quad K_1 = \lambda((a\delta_1 + 1)(2a\alpha_1 + A\alpha_2) + a\delta_2(A\alpha_1 - 2a\alpha_2)),\\
&K_2 = a\lambda^2(a(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) + A\alpha_1\alpha_2), \quad G_1 = (a\delta_1 + 1)(2a\gamma_1 + A\gamma_2) + a\delta_2(A\gamma_1 - 2a\gamma_2),\\
&G_2 = a(a(\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) + A\gamma_1\gamma_2), \quad r = \sqrt{4a^2 + A^2}, \quad M = 2ar\left(\gamma_1 - \alpha_1\lambda\right),\\
&k_0 = \ -4a^4\delta_2^2 + 4a^2A\delta_2(a\delta_1 + 1) - (aA\delta_1 + A)^2 + (a\delta_1 r + r)^2,\\
&k_1 = \ -a\lambda(-2a\alpha_2 + A\alpha_1 - \alpha_1 r)\left((a\delta_1 + 1)(A + r) - 2a^2\delta_2\right) - a(\gamma_1(A + r) - 2a\gamma_2)((a\delta_1 + 1)\\
&\qquad (A - r) - 2a^2\delta_2),\\
&k_2 = \ -a^2\lambda(-2a\alpha_2 + A\alpha_1 - \alpha_1 r)(\gamma_1(A + r) - 2a\gamma_2),\\
&g_1 = \ -a\lambda(\alpha_1(A + r) - 2a\alpha_2)\left((a\delta_1 + 1)(A - r) - 2a^2\delta_2\right) - a(-2a\gamma_2 + A\gamma_1 - \gamma_1 r)((a\delta_1 + 1)\\
&\qquad (A + r) - 2a^2\delta_2),\\
&g_2 = \ -a^2\lambda(\alpha_1(A + r) - 2a\alpha_2)(-2a\gamma_2 + A\gamma_1 - \gamma_1 r).
\end{aligned}$$

Since $\Delta_1 = (r\lambda)^2(-a\alpha_1\delta_2 + a\alpha_2\delta_1 + \alpha_2)^2$ and $\Delta_2 = r^2(-a\gamma_1\delta_2 + a\gamma_2\delta_1 + \gamma_2)^2$ are the discriminants of the numerator and the denominator of the function $g_3^{(ii)}(y_2)$, respectively, and they are positive, the possible graphics of the function $g_3^{(ii)}(y_2)$ are the ones drawn in Figure 16 if $\Delta_i = 0$ with $i = 1, 2$, Figures 21 and 23 if either $\Delta_i > 0$ with $i = 1, 2$, or $\Delta_1 > 0$ and $\Delta_2 = 0$, or $\Delta_1 = 0$ and $\Delta_2 > 0$.

For the function $f_3^{(ii)}(y_2)$ we have that

$$\begin{aligned}
\Delta_1 = a^2(&\lambda(\alpha_1(A - m) - 2a\alpha_2)\left((a\delta_1 + 1)(A + m) - 2a^2\delta_2\right) - (\gamma_1(A + m) - 2a\gamma_2)((a\delta_1 + 1)\\
&(A - m) - 2a^2\delta_2))^2,\\
\Delta_2 = a^2(&\lambda(\alpha_1(A + m) - 2a\alpha_2)\left((a\delta_1 + 1)(A - m) - 2a^2\delta_2\right) - (\gamma_1(A - m) - 2a\gamma_2)((a\delta_1 + 1)\\
&(A + m) - 2a^2\delta_2))^2,
\end{aligned}$$

are the discriminants of the numerator and the denominator of the function $f_3^{(ii)}(y_2)$, respectively. It is clear that since $\Delta_i \geq 0$, and

$$\left(f_3^{(ii)}\right)'(y_2) = \frac{\left(k_0 + k_1\ y_2 + k_2\ y_2^2\right)^{A-1}}{\left(k_0 + g_1\ y_2 + g_2\ y_2^2\right)^{A+1}} P(y_2) e^{M\ y_2},$$

where $P(y_2) = k_0(A(k_1 - g_1) + k_0 M) + k_0(2A(k_2 - g_2) + M(g_1 + k_1))\ y_2 + (A(g_1 k_2 - g_2 k_1) + g_1 k_1 M + k_0 M(g_2 + k_2))\ y_2^2 + M(g_1 k_2 + g_2 k_1)\ y_2^3 + g_2 k_2 M\ y_2^4$, the function $f_3^{(ii)}(y_2)$ has the same variation than the function $f_2^{(ii)}(y_2)$, i. e, they have the same graphics, the only difference is at infinity. If $M > 0$, $f_3^{(ii)}(y_2)$ has a parabolic branch at $+\infty$ but if $M < 0$, the parabolic branch is at $-\infty$.

1- If $\Delta_i = 0$ with $i = 1, 2$ the graphics of the function $f_3^{(ii)}(y_2)$ are equivalent to the graphics of the function $f_2^{(ii)}(y_2)$ when both discriminants of the function $f_3^{(ii)}(y_2)$ are strictly negative, see Figure 24. But there is a parabolic branch at infinity instead of a horizontal asymptote.
2- If either $\Delta_i > 0$ with $i = 1, 2$, or if $\Delta_1 = 0$ and $\Delta_2 > 0$, or if $\Delta_1 > 0$ and $\Delta_2 = 0$, then the graphics of the function $f_3^{(ii)}(y_2)$ are equivalent to the graphics of the function $f_2^{(ii)}(y_2)$ that are shown in Figures 21, 23 and 26.

Since $f_2^{(ii)}(y_2)$ and $f_3^{(ii)}(y_2)$ have the same behavior we conclude that $(a), \ldots, (f)$ of Figures 21 and 24 with the graphics of the function $g_3^{(ii)}(y_2)$ are the ones that are going to give the maximum number of intersection points between the graphics of these functions. For the function $g_3^{(ii)}(y_2)$, Figure 21 is one that will give the maximum number of the intersection points between $g_3^{(ii)}(y_2)$ and $f_3^{(ii)}(y_2)$ because of the positive sign of $f_3^{(ii)}(y_2)$. Then it is clear that these two functions can intersect at most in eight points. Hence the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $C = b = 0$ is at most four.

By taking $\{r, K_0, K_1, K_2, G_1, G_2, A, M, k_1, k_2, k_3, k_4, k_5, k_6, k_7\} = \{2, -2, 2.4, 0.9, -2, 5, 2, 0.5, 1.5, 4, 1, 6, 0.8\}$ we build an example with exactly eight intersection points between the graphics of the two functions $f_3^{(ii)}(y_2)$ and $g_3^{(ii)}(y_2)$, these points are shown in Figure 9.

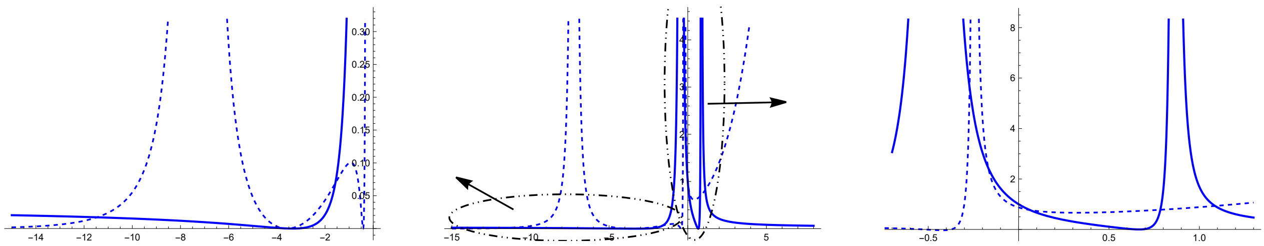


Fig. 9. The eight intersection points between the graphics of the two functions $f_3^{(ii)}(y_2)$ drawn in dashed line and $g_3^{(ii)}(y_2)$ drawn in a continuous line.

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ corresponding to $k = 4$ and $j = ii$ in system (20), the first integral of the quadratic center (5) is $H_4^{(ii)}(x, y)$ given in (13). Then to study the solutions y_2 satisfying $F_4^{(ii)}(y_2) = 0$ is equivalent to study the solutions y_2 of $f_4^{(ii)}(y_2) = g_4^{(ii)}(y_2)$ such that

$$f_4^{(ii)}(y_2) = f_1^{(i)}(y_2) \ \text{ with } \ r = \frac{1}{A} \ \text{ and } \ g_4^{(ii)}(y_2) = \left(\frac{m_1^+ + m_2^+\ y_2}{m_1^+ + m_3^+\ y_2}\right)^{p^+} \left(\frac{m_1^- + m_2^-\ y_2}{m_1^- + m_3^-\ y_2}\right)^{p^-},$$

where

$$m_1^\pm = \frac{1}{2}\delta_1(C \pm \sqrt{\Delta}) - b\delta_2 + 1, \quad m_2^\pm = \frac{1}{2}\gamma_1(C \pm \sqrt{\Delta}) - b\gamma_2,$$
$$m_3^\pm = (\frac{1}{2}\alpha_1(C \pm \sqrt{\Delta}) - b\alpha_2)\lambda, \quad p^\pm = \frac{1}{2b}\Big(1 \pm \frac{C}{\sqrt{\Delta}}\Big).$$

Figures 21, 23 and 26 represent all the possible graphics of the function $g_4^{(ii)}(y_2)$. The function $f_4^{(ii)}(y_2)$ is drawn in the previous cases and all its graphics are shown in Figure 16. As in the proof of statement (a) of Theorem 4 for the function $f_4^{(ii)}(y_2)$ we chose Figure 16(a) to get the maximum number of the intersections points between this figure and the graphics of $g_4^{(ii)}(y_2)$. Since Figure 16(a) of $f_4^{(ii)}(y_2)$ is up to the y_2-axis, the number of the changes of the sign of $\left(g_4^{(ii)}\right)'(y_2)$ and the position of the graphics $g_4^{(ii)}(y_2)$ play a main role in finding the maximum number of the intersection points. Hence the graphic of the function $g_4^{(ii)}(y_2)$ is up to the y_2-axis when both p and q are even and $\left(g_4^{(ii)}\right)'(y_2)$ has seven changes of the sing in $(a), \cdots, (f)$ of Figure 21. Then the maximum number of the intersection points between these two functions is at most eight. By symmetry, we obtain that the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ is at most four.

By taking $\{r, s_1, s_2, s_3, p^+, m_1^+, m_2^+, m_3^+, p^-, m_1^-, m_2^-, m_3^-\} = \{8, -2, 2, 0.8, 2, -2, 2.4, 0.9, 6, -2, 5, 2\}$ we build an example with exactly eight intersection points between the graphics of the two functions $f_4^{(ii)}(y_2)$ and $g_4^{(ii)}(y_2)$, these points are shown in Figure 11. Then we have four limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$.

In what follows we give a discontinuous piecewise differential system of the class $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ with four limit cycles. In the region Σ^+ we consider the quadratic center

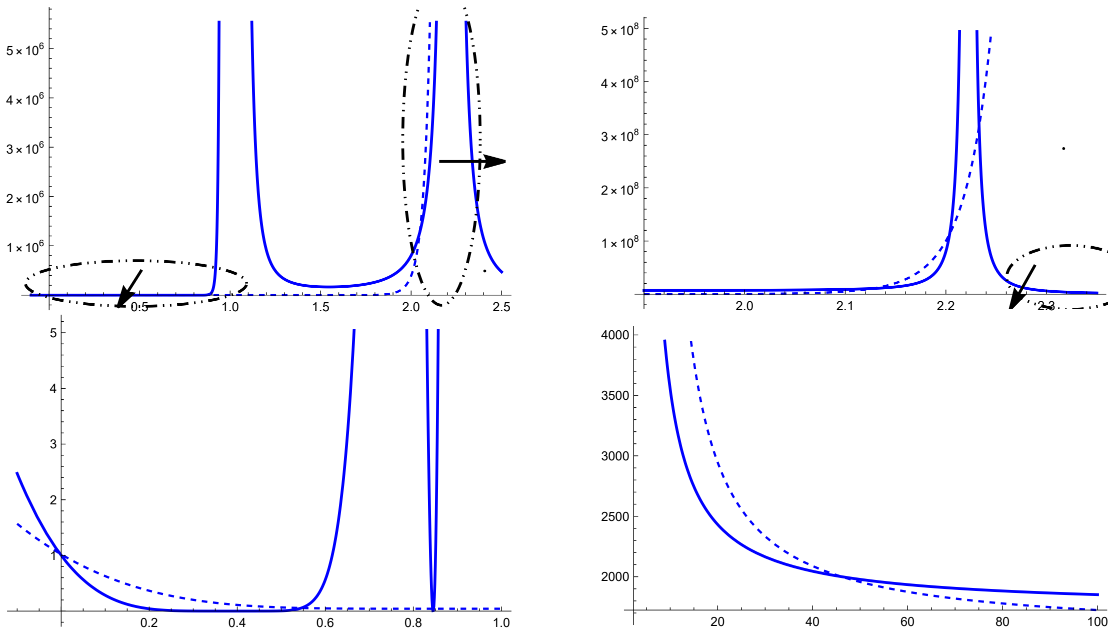


Fig. 10. The eight intersection points between the graphics of the two functions $f_4^{(ii)}(y_2)$ drawn in dashed line and $g_4^{(ii)}(y_2)$ drawn in a continuous line.

$$
\begin{aligned}
\dot{x} &= -73350.3x^2 + x(0.224264..y + 758819) + (3.622145914919896..\text{`*^-7}y \\
&\quad -1.16003..)y - 1.96252..\,10^6, \\
\dot{y} &= -0.0239117..x^2 + x(73350.3y - 151764) + (-0.0747547..y - 379409)y \\
&\quad +785009,
\end{aligned} \tag{27}
$$

this system has the first integral

$$
\begin{aligned}
H_4^{(ii)}(x,y) = {} & (103733.x - 0.3y - 536564.)(-103733.x - 0.0885618..y + 536568.) \\
& (-3.167224349454045..\text{`*^-8}x + 0.194281..y - 0.401973..)^2.
\end{aligned}
$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{3x}{2} - \frac{25y}{4}, \quad \dot{y} = x + \frac{3y}{2}, \tag{28}$$

with the first integral

$$H(x,y) = \left(x + \frac{3y}{2}\right)^2 + 4y^2.$$

For the discontinuous piecewise differential system (27)–(28), system (20) has the four solutions $(x_1, y_1) = (4.58258.., 1.83303..)$, $(x_2, y_2) = (3.87298.., 1.54919..)$, $(x_3, y_3) = (3, 1.2)$ and $(x_4, y_4) = (1.73205.., 0.69282..)$ which provides the four crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the eight points $(x_j, 0)$ and $(0, y_j)$ with $j = 1, 2, 3, 4$, see Figure 2(a).

If $b = a = 0 \neq AC$ corresponding to $k = 5$ and $j = ii$ in system (20), the first integral of the quadratic center is $H_5^{(ii)}(x,y)$ given in (14). The equation $F_5^{(ii)}(y_2) = 0$ is equivalent to the equation $f_5^{(ii)}(y_2) = g_5^{(ii)}(y_2)$ such that

$$f_5^{(ii)}(y_2) = f_1^{(ii)}(y_2) \quad \text{and} \quad g_5^{(ii)}(y_2) = f_2^{(i)}(y_2),$$

where

$$\begin{aligned} &L_0 = L_2 = 0, \quad L_1 = 2AC\left((A\alpha_1 + \alpha_2 C)\lambda - A\gamma_1 - \gamma_2 C\right), \\ &m_1 = A\delta_2 + 1, \quad m_2 = A\alpha_2\lambda, \quad m_3 = A\gamma_2, \quad p = 2C^2, \\ &n_1 = C\delta_1 + 1, \quad n_2 = \alpha_1 C\lambda, \quad n_3 = \gamma_1 C, \quad q = 2A^2. \end{aligned}$$

Due to $L_2 = 0$ we show the graphics of the function $g_5^{(ii)}(y_2)$ in Figures 17(c) and 17(d). For the function $f_5^{(ii)}(y_2)$ the possible graphics are shown in Figures 21, 23 and 26.

Since the function $g_5^{(ii)}(y_2) = e^{L_1 y_2}$ is a positive function and it has the x-axis as a horizontal asymptote straight line and as we mentioned in the precedent cases the number and the location of the extrema according to the x-axis is important for the maximum number of the intersection points between the graphics of $g_5^{(ii)}(y_2)$ and $f_5^{(ii)}(y_2)$. For that reason we guarantee that the maximum number of intersection points between the graphics of these two functions can be precisely between the graphics of (c), (d) of Figure 17 and the graphics of Figure 21. Then we remark that the graphics of these functions can intersect at most in seven points. By symmetry the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $b = a = 0 \neq AC$ is at most three.

By taking $\{p, q, m_1, m_2, m_3, n_1, n_2, n_3, L_0, L_1, L_2\} = \{2, 4, -2.4, 2.3, 1, -2, 6, 1.8, 0, 5.5, 0\}$ we build an example with exactly seven intersection points between the graphics of the two functions $f_5^{(ii)}(y_2)$ and $g_5^{(ii)}(y_2)$, these points are shown in Figure 11.

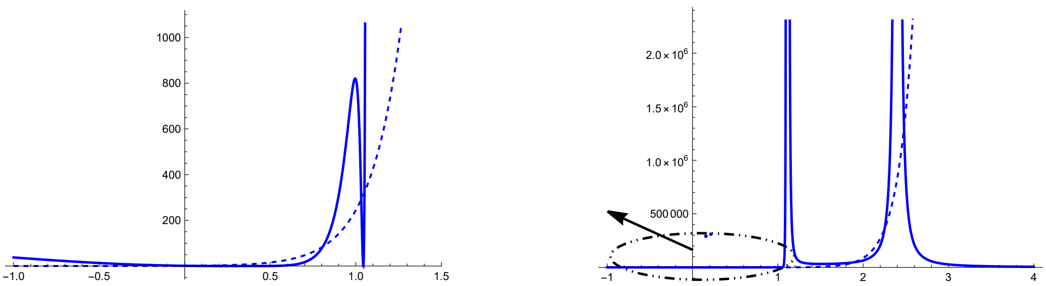


Fig. 11. The seven intersection points between the graphics of the two functions $f_5^{(ii)}(y_2)$ drawn in a continuous line and $g_5^{(ii)}(y_2)$ drawn in dashed line.

If $A=a=0\neq Cb$ corresponding to $k=6$ and $j=ii$ in system (20), the first integral in this case is $H_6^{(ii)}(x,y)$ given in (15), the solutions of $F_6^{(ii)}(y_2)=0$ are equivalent to the solutions of the equation $f_6^{(ii)}(y_2)=g_6^{(ii)}(y_2)$ such that

$$f_6^{(ii)}(y_2)=g_4^{(ii)}(y_2)\quad\text{and}\quad g_6^{(ii)}(y_2)=f_2^{(i)}(y_2),$$

where

$$\begin{aligned}
&m_1^{\pm}=\frac{1}{2}\delta_1(C\pm\sqrt{\Delta})-b\delta_2+1,\quad m_2^{\pm}=\frac{1}{2}\gamma_1(C\pm\sqrt{\Delta})-b\gamma_2,\\
&m_3^{\pm}=\frac{1}{2}(\alpha_1(C\pm\sqrt{\Delta})-2\alpha_2 b)\lambda,\quad p^{\pm}=\frac{1}{2b}\Big(1\pm\frac{C}{\sqrt{\Delta}}\Big),\\
&L_0=2\delta_2,\quad L_1=\alpha_2\lambda-\gamma_2,\quad L_2=0.
\end{aligned}$$

Here we remark that this case is similar to the precedent one where $b=0\neq A$. In the precedent case we have proved that three is the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $b=a=0\neq AC$. Hence the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A=0\neq b$ is at most three.

In what follows we give a discontinuous piecewise differential system of the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A=0\neq b$ with three limit cycles. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}
\dot{x}&=-0.198885..x^2+x(-0.109869..y-0.912492..)+(0.387068..y-1.53815..)y\\
&\quad-1.22818..,\\
\dot{y}&=y(0.599866..-0.096767..y)+0.0497213..x^2+x(0.0274673..y+0.834053..)\\
&\quad+0.672247..,
\end{aligned}\tag{29}$$

this system has the first integral

$$H_6^{(ii)}(x,y)=(-1.x-1.69835..y+4.22541..)^2(1.x-1.14593..y+5.99997..)e^{0.2x+0.8y}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x}=\frac{4x}{5}-\frac{289y}{100},\quad \dot{y}=x-\frac{4y}{5},\tag{30}$$

with the first integral

$$H(x,y)=\left(x-\frac{4y}{5}\right)^2+\frac{9y^2}{4}.$$

For the discontinuous piecewise differential system (29)–(30), system (20) has the three solutions $(x_1,y_1)=(3.87298..,2.27823..)$, $(x_2,y_2)=(3,1.76471..)$ and $(x_3,y_3)=(1.73205..,1.01885..)$ that provide the three crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the six points $(x_j,0)$ and $(0,y_j)$ with $j=1,2,3$, see Figure 2(b).

If $\Delta=a=0$ corresponding to $k=7$ and $j=ii$ in system (20), the first integral of the quadratic center (5) is $H_7^{(ii)}(x,y)$ given in (16). Then to study $F_7^{(ii)}(y_2)=0$ it is enough to study the solutions y_2 of the equation $f_7^{(ii)}(y_2)=g_7^{(ii)}(y_2)$ such that

$$f_7^{(ii)}(y_2)=f_1^{(ii)}(y_2)\quad\text{and}\quad g_7^{(ii)}(y_2)=g_1^{(ii)}(y_2),$$

where

$$\begin{aligned}
&m_1=-2b\delta_2+C\delta_1+2,\quad m_2=\gamma_1C-2b\gamma_2,\quad m_3=(\alpha_1C-2\alpha_2b)\lambda,\quad p=1,\\
&n_1=(4b^2+C^2)\delta_2-4b,\quad n_2=\alpha_2\left(4b^2+C^2\right)\lambda,\quad n_3=\gamma_2\left(4b^2+C^2\right),\quad q=\frac{4b^2}{4b^2+C^2},\\
&K_1=-bC\Big(b^2\left(\lambda(\alpha_2\delta_2-\alpha_1\delta_2)+\gamma_1\delta_2-\gamma_2\delta_2\right)+b(\alpha_1\lambda-\gamma_1)+\alpha_2C\lambda-\gamma_2C\Big),\\
&K_2=-b^3C(\alpha_2\gamma_1-\alpha_1\gamma_2)\lambda.
\end{aligned}$$

It is clear that since $q = 1$ all the possible graphics of the function $f_7^{(ii)}(y_2)$ are shown in Figure 23 if q is an odd number and in Figure 26 if q is an even number. For the function $g_7^{(ii)}(y_2)$ we show all its possible graphics in Figure 18.

Since $g_7^{(ii)}(y_2)$ is a positive function and the maximum number of changes of the sign of the derivative of this function is when $\left(g_7^{(ii)}\right)'(y_2)$ has two extrema. Figures 22(a) and 22(b) are the ones that give the maximum number of the intersection points with the graphics of $f_7^{(ii)}(y_2)$. Because the first vertical asymptote straight line of $f_7^{(ii)}(y_2)$ is also a vertical asymptote straight line for $g_7^{(ii)}(y_2)$ and the second vertical asymptote of $f_7^{(ii)}(y_2)$ is an extremum of $g_7^{(ii)}(y_2)$, the maximum number of intersection points between these two functions is at most four. Then by symmetry the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $\Delta = a = 0$ is two.

By taking $\{K_1, K_2, m_1, m_2, m_3, p, n_1, n_2, n_3, q\} = \{-0.5, 2, -2, 2, 1, 1, 1, 2, 0.5, 2\}$ we build an example with exactly four intersection points between the graphics of the two functions $f_7^{(ii)}(y_2)$ and $g_7^{(ii)}(y_2)$, these points are shown in Figure 12.

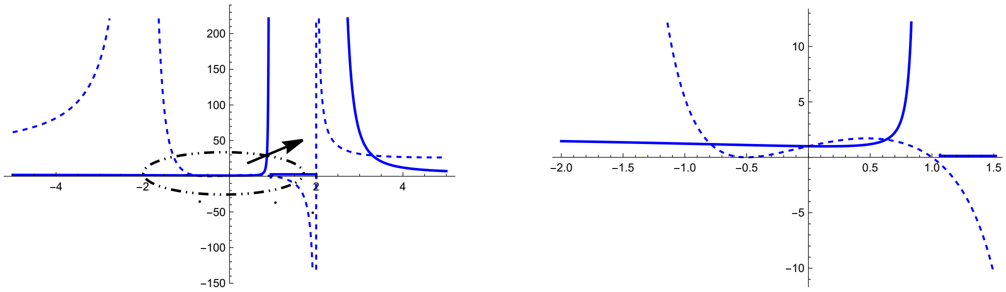


Fig. 12. The four intersection points between the graphics of the two functions $f_7^{(ii)}(y_2)$ drawn in a continuous line and $g_7^{(ii)}(y_2)$ drawn in dashed line.

If $A = b = 0$ corresponding to $k = 8$ and $j = ii$ in system (20), the first integral in this case is $H_8^{(ii)}(x, y)$ given in (17), the study of the solutions y_2 satisfying $F_8^{(ii)}(y_2) = 0$ is equivalent to study the solutions y_2 of the equation $f_8^{(ii)}(y_2) = g_8^{(ii)}(y_2)$ such that

$$f_8^{(ii)}(y_2) = f_1^{(i)}(y_2) \quad \text{with} \quad r = 2 \quad \text{and} \quad g_8^{(ii)}(y_2) = f_2^{(i)}(y_2),$$

where

$$L_0 = 0, \quad L_1 = -2C\left(\gamma_1 + \gamma_2 C\delta_2 - (\alpha_1 + \alpha_2 C\delta_2)\lambda\right), \quad L_2 = C^2(\alpha^2\alpha_2^2 + \alpha_2^2\omega^2 - \gamma_2^2),$$
$$s_1 = C\delta_1 + 1, \quad s_2 = \alpha_1 C\lambda, \quad s_3 = \gamma_1 C.$$

Due to the fact that the graphics of $f_8^{(ii)}(y_2)$ are equivalent to the graphics of $f_3^{(i)}(y_2)$ with $r = 2$, and to the fact that the graphics of $g_8^{(ii)}(y_2)$ are equivalent to the graphics of $g_3^{(i)}(y_2)$, it follows from the proof of the case $b = 0$ of statement (a) of Theorem 4, that the maximum number of intersection points between the graphics of the two functions $f_8^{(ii)}(y_2)$ and $g_8^{(ii)}(y_2)$ is at most four. Then the maximum number of limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A = b = 0$ is at most two.

Now we give an example having two limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A = b = 0$. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} &= x(0.741461.. - 0.529638..y) + 0.0173273..x^2 + (0.294564..y + 2.21484..)y \\ &\quad -5.42593.., \\ \dot{y} &= x(1.50454.. - 0.934655..y) + 0.0305776..x^2 + (0.519819..y - 1.97381..)y \\ &\quad +1.78917..,\end{aligned} \tag{31}$$

this system has the first integral

$$H_8^{(ii)}(x,y) = (1.x - 30.y + 47.9582..)^2 e^{-0.81(1.x-0.566667..y+1.36927..)^2+0.2x-6y}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{x}{10} - \frac{401y}{100}, \quad \dot{y} = x + \frac{y}{10}, \tag{32}$$

with the first integral

$$H(x,y) = \left(x + \frac{y}{10}\right)^2 + 4y^2.$$

For the discontinuous piecewise differential system (31)–(32), system (20) has the two solutions $(x_1, y_1) = (1, 1.49813..)$ and $(x_2, y_2) = (1.73205.., 0.864945..)$ that provide the two crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the four points $(x_j, 0)$ and $(0, y_j)$ with $j = 1, 2$, see Figure 2(c). This example completes the proof of statement (b). ■

Proof. [Proof of statement (c) of Theorem A] Now we will prove this statement for the third class $\mathcal{C}_{iii}$ when $A - 2b = C + 2a = 0$. In this case $j = iii$ in system (20), then the equation $F^{(iii)}(y_2) = 0$ where

$$\begin{aligned} F^{(iii)}(y_2) = \Big(&\alpha_1\delta_1\lambda + \alpha_2\delta_2\lambda + a\alpha_1\delta_1^2\lambda - \gamma_1(a\delta_1^2 - a\delta_2^2 + 2b\delta_1\delta_2 + \delta_1) - \gamma_2\delta_2(-2a\delta_1 + d\delta_2 + 1) \\ &-a\alpha_1\delta_2^2\lambda + 2a\alpha_2\delta_1\delta_2\lambda + b\delta_1(\lambda(2\alpha_1\delta_2 + \alpha_2\delta_1) - \gamma_2\delta_1) + \alpha_2 d\delta_2^2\lambda\Big)\ y_2 + \frac{1}{2}\Big(4\gamma_1\gamma_2(a\delta_2 \\ &-b\delta_1) - \gamma_1^2(2a\delta_1 + 2b\delta_2 + 1) + \alpha^2(\alpha_1^2(2a\delta_1 + 2b\delta_2 + 1) + 4\alpha_1\alpha_2(b\delta_1 - a\delta_2) + \alpha_2^2(-2 \\ &a\delta_1 + 2d\delta_2 + 1)) + \gamma_2^2(2a\delta_1 - 2d\delta_2 - 1) + \omega^2(\alpha_1^2(2a\delta_1 + 2b\delta_2 + 1) + 4\alpha_1\alpha_2(b\delta_1 - a \\ &\delta_2) + \alpha_2^2(-2a\delta_1 + 2d\delta_2 + 1))\Big)\ y_2^2 + \frac{1}{3}\Big(a(\alpha_1\lambda^3(\alpha_1^2 - 3\alpha_2^2) - \gamma_1^3 + 3\gamma_1\gamma_2^2) + 3\alpha_1^2\alpha_2 b\lambda^3 \\ &-3b\gamma_1^2\gamma_2 + \alpha_2^3 d\lambda^3 - \gamma_2^3 d\Big)\ y_2^3, \end{aligned}$$

is a cubic equation in the variable y_2. This equation has at most three real solutions. Then by the symmetry the class of discontinuous piecewise differential system $\mathcal{C}_{iii}$ when $A - 2b = C + 2a = 0$ can have at most one limit cycle. So we have proved that there is at most one limit cycle for the class of discontinuous piecewise differential system $\mathcal{C}_{iii}$ when $A - 2b = C + 2a = 0$.

To complete the proof of this case we build an example of the class of discontinuous piecewise differential system $\mathcal{C}_{iii}$ when $A - 2b = C + 2a = 0$ with one limit cycle. In the region Σ^+ we consider the quadratic center

$$\begin{aligned} \dot{x} &= \frac{1}{1300}\left(-169x^2 + x(988y + 6720) - 2y(247y + 3760) - 27900\right), \\ \dot{y} &= \frac{1}{650}\left(2678x^2 + x(169y - 3290) - y(247y + 3360) - 9700\right), \end{aligned} \tag{33}$$

this system has the first integral

$$\begin{aligned} H^{(iii)}(x,y) = {}& 5356x^3 + x^2(507y - 9870) - 6x(y(247y + 3360) + 9700) + 2y(y(247y + 5640) \\ &+41850) + 201000. \end{aligned}$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -2x - 10.8228..y, \quad \dot{y} = x + 2y, \tag{34}$$

with the first integral

$$H(x,y) = (x + 2y)^2 + 6.82276..y^2.$$

For the discontinuous piecewise differential system (33)–(34), system (20) has only the solution $(x_1, y_1) = (0.567633.., 2.66406..)$ which provides the unique crossing limit cycle intersecting the rays Γ_1 and Γ_2 in the two points $(x_1, 0)$ and $(0, y_1)$, see Figure 3(a). This example completes the proof of statement (c). ■

Proof. [Proof of statement (d) of Theorem A] Here we prove this statement for the fourth class $\mathcal{C}_{iv}$ when $C+2a=A+4b+5d=a^2+bd+2d^2=0$ and $j=iv$ in system (20), then $F^{(iv)}(y_2)=0$ is a polynomial equation of degree nine and because of the large expression of this equation we omit it. This equation has at most nine real solutions. In fact these nine solutions represent four real solutions of (20) because of the symmetry. Then all these nine solutions provide the four limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{iv}$ when $C+2a=A+4b+5d=a^2+bd+2d^2=0$. So we have proved that there are at most four limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{iv}$ when $C+2a=A+4b+5d=a^2+bd+2d^2=0$.

To complete the proof of this case we introduce an example with four limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{iv}$ when $C+2a=A+4b+5d=a^2+bd+2d^2=0$. In the region Σ^- we consider the quadratic center

$$\begin{aligned}\dot{x} &= x(0.898898..-0.154472..y)+0.0752303..x^2+(0.00129847..y-1.58169..)y\\ &\quad -1.51959..,\\ \dot{y} &= x(1.93316..-0.196016..y)+0.150808..x^2+(-0.047621..y-1.81001..)y\\ &\quad +1.01708..,\end{aligned} \tag{35}$$

this system has the first integral

$$\begin{aligned}H^{(IV)}(x,y) = \Big(&x^2(36.2311-1.67779y)+y(y(12.5561-0.174923y)-269.803)+x^3+x\\ &((0.938325y-42.7139)y+434.695)+1710.3\Big)^2\Big/\Big(x(24.1541-1.11853y)\\ &+x^2+(0.312775y-14.9674)y+142.483\Big)^3.\end{aligned}$$

In the region Σ^+ we consider the linear differential center

$$\dot{x}=x-2y, \quad \dot{y}=x-y, \tag{36}$$

with the first integral

$$H(x,y)=(x-y)^2+y^2.$$

For the discontinuous piecewise differential system (35)–(36), system (20) has the four solutions $(x_1,y_1)=(0.83666..,0.591608..)$, $(x_2,y_2)=(0.707107..,0.5..)$, $(x_3,y_3)=(0.547723..,0.387298..)$ and $(x_4,y_4)=(0.316228..,0.223607..)$ that provide the four crossing limit cycles intersecting the rays Γ_1 and Γ_2 in the eight points $(x_i,0)$ and $(0,y_i)$ with $i=1,2,3,4$, see Figure 3(b). This example completes the proof of Theorem A. ∎

4. Proof of Theorem B

We simultaneously study the existence of limit cycles of configuration **Cnf 1** and **Cnf 2**. The limit cycles of configuration **Cnf 1** intersect the separation line Σ_2 in two distinct points $(0,y_1)$ and $(0,y_2)$ with $0\le y_1<y_2$. These two points must satisfy the system of equations

$$\begin{aligned}H(0,y_1)-H(0,y_2)&=(y_1-y_2)h(y_1,y_2)=0,\\ H_k^{(j)}(0,y_1)-H_k^{(j)}(0,y_2)&=h_k^{(j)}(y_1,y_2)=0,\end{aligned} \tag{37}$$

with $k=1,...,4$ for $j=i$, and $k=1,...,8$ for $j=ii$. For $j=iii,iv$ we have $H_k^{(j)}(x,y)=H^{(j)}(x,y)$ and $h_k^{(j)}(y_1,y_2)=h^{(j)}(y_1,y_2)$. On the other hand, the two intersection points of limit cycles of configuration **Cnf 2** with the separation line Σ must satisfy (20). In Theorems 2 and A the maximum number of limit cycles is already provided for each one of the configurations **Cnf 1** and **Cnf 2**, respectively. Then we have the following results.

Proof. [Proof of statement (a) of Theorem B] We give an example with five limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ when $A+b=0\neq A$. In what follows we give a discontinuous

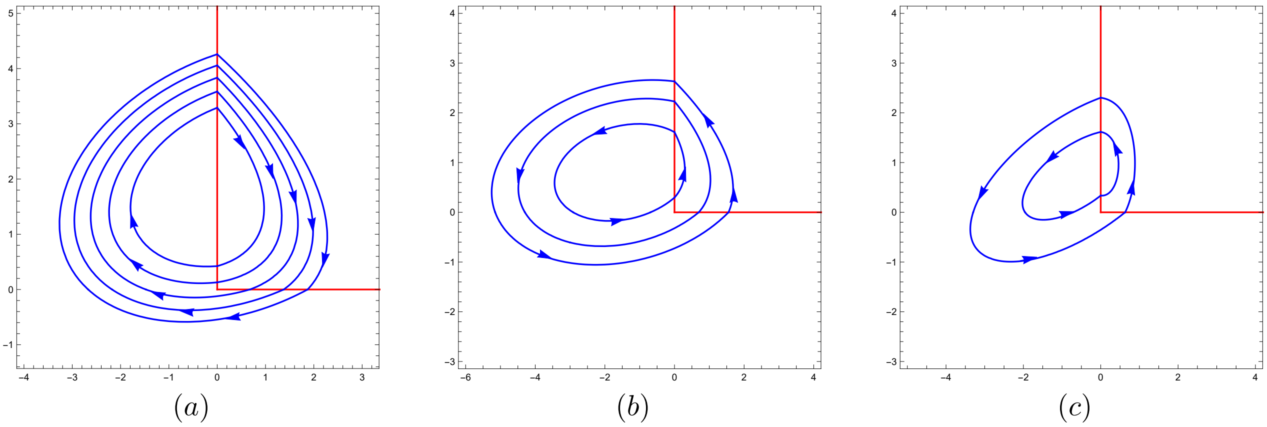


Fig. 13. (a) The five crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (38)–(39), (b) the three crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (40)–(41), and (c) the two crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (42)–(43).

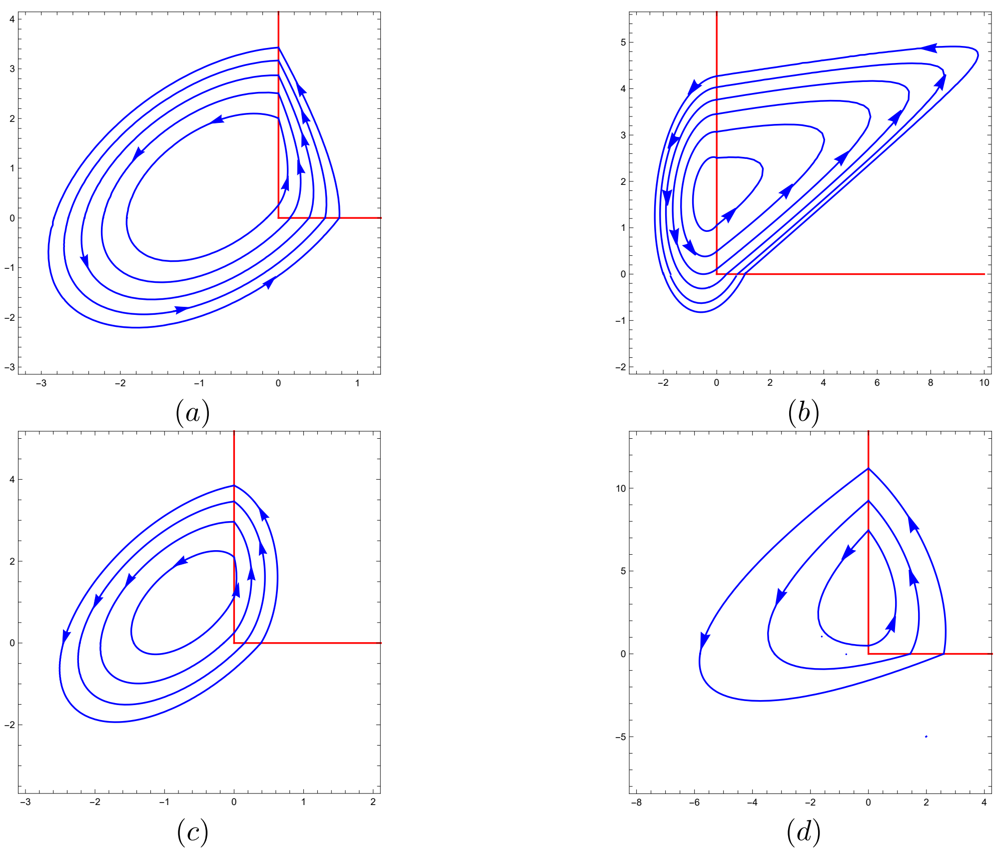
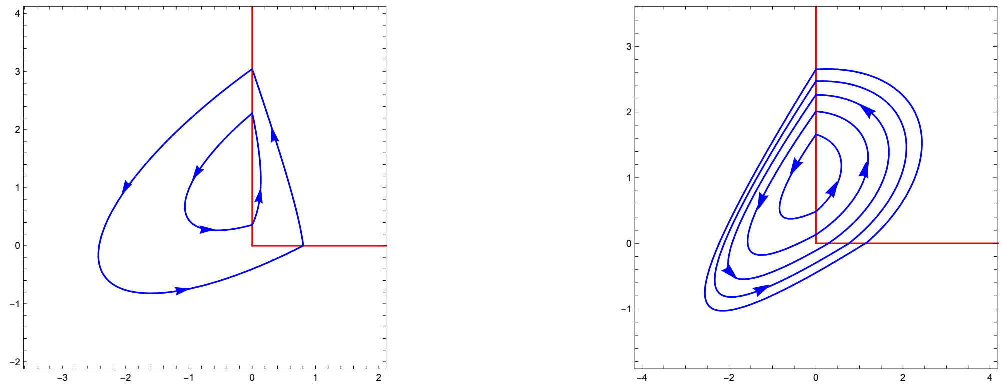


Fig. 14. (a) The five crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (44)–(45), (b) the six crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (46)–(47), (c) the four crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (48)–(49), and (d) the three crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (50)–(51).

piecewise differential system of the class (2)–(5) when $A + b = 0 \neq A$ with five limit cycles two satisfying



(a) (b)

Fig. 15. (a) The two crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (52)–(53), (b) the five crossing limit cycles satisfying **Cnf 1** and **Cnf 2** of the discontinuous piecewise differential system (54)–(55).

Cnf 1 and three satisfying **Cnf 2**. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} &= y(2.05262 - 0.0538376y) + 0.00875921x^2 + x(0.00952768y + 0.72735) - 3.62673,\\ \dot{y} &= -4.884707923516701..\text{`*\^{}-}7x^2 + x(0.00882947..y - 0.907179..) + (0.00615901..y\\ &\quad -0.719153..)y - 0.362589..,\end{aligned} \tag{38}$$

with the first integral

$$H_1^{(i)}(x,y) = Exp\left(\frac{184.31..x^2 + x(305.64..y + 8913.59..) + y(126.70..y - 4217) + 3.474.. \times 10^7}{(1x - 47.945..y + 5034.93..)^2}\right)$$
$$(0.000204571..x - 0.00980828..y + 1.03..).$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{11x}{5} + \frac{1346y}{125} - 20, \quad \dot{y} = -5x + \frac{11y}{5} - 2, \tag{39}$$

with the first integral

$$H(x,y) = -8(11x+100)y + 20x(5x+4) + \frac{5384y^2}{25}.$$

For the discontinuous piecewise differential system (38)–(39), system (37) has the two solutions $(y_1, y_2) = ((2500 - 25\sqrt{5962})/1346, (25\sqrt{5962} + 2500)/1346)$ and $(y_3, y_4) = ((2500 - 25\sqrt{8654})/1346, (25\sqrt{8654} + 2500)/1346)$, which provide two crossing limit cycles intersecting Γ_1 in the four points $(0, y_i)$ with $i = 1,2,3,4$, and system (20) has the three solutions $(x_5, y_5) = ((1/5)\left(\sqrt{29} - 2\right), (25\sqrt{11346} + 2500)/1346))$, $(x_6, y_6) = ((1/5)\left(\sqrt{79} - 2\right), (25\sqrt{14038} + 2500)/1346)$ and $(x_7, y_7) = ((1/5)\left(\sqrt{129} - 2\right), (25\sqrt{16730} + 2500)/1346)$, which provide the six intersecting points $(x_j, 0)$ and $(0, y_j)$ with $j = 5,6,7$ of the three crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (38)–(39) has exactly five crossing limit cycles, see Figure 13(a).

We give an example with three limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ when $b = 0 \neq A$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $b = 0 \neq A$ with three limit cycles one satisfying **Cnf 1** and two satisfying **Cnf 2**. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} &= y(15.0335.. - 0.254667..y) + 0.739273..x^2 + x(2.47274..y + 1.86447..)\\ &\quad -14.1078..,\\ \dot{y} &= -0.607274..x^2 + x(-0.678545..y - 4.62004..) + (0.0739273..y - 4.18426..)y\\ &\quad -6.3485..,\end{aligned} \tag{40}$$

this system has the first integral

$$H_3^{(i)}(x,y) = e^{(x-0.1y+4.80783..)^2+(-0.8x-2.62059..y+2.31979..)^2}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -2.2x + 20.968..y - 20, \quad \dot{y} = -5x + 2.2y - 9, \tag{41}$$

with the first integral

$$H(x,y) = x(360\ - 88y) + 100.x^2 + y(419.36y - 800).$$

For the discontinuous piecewise differential system (40)–(41), system (37) has the unique solution $(y_1, y_2) = (0.295896.., 1.61177..)$, which provides a unique crossing limit cycle intersecting Γ_1 in the two points $(0, y_i)$ with $i = 1, 2$, and system (20) has the two solutions $(x_3, y_3) = (0.697999.., 2.22866..)$ and $(x_4, y_4) = (1.55261.., 2.63237..)$, which provide the four intersecting points $(x_j, 0)$, $(0, y_j)$ with $j = 3, 4$ of the two crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (40)–(41) has exactly three crossing limit cycles, see Figure 13(b).

We give an example with two limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_i$ when $A = b = 0$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $A = b = 0$ with two limit cycles one satisfying **Cnf 1** and the other one satisfying **Cnf 2**. In the region Σ^+ we consider the quadratic center

$$\begin{aligned} \dot{x} &= y(0.810541..\ - 0.00095672..y) - 0.095672..x^2 + x(0.0191344..y + 0.056492..) \\ &\quad -0.789731.., \\ \dot{y} &= -0.95672..x^2 + x(0.191344..y - 1.43508..) + (-0.0095672..y - 0.056492..)y \\ &\quad +0.0611942.., \end{aligned} \tag{42}$$

this system has the first integral

$$\begin{aligned} H_4^{(i)}(x,y) = 3.12x^2 + 1.6(x - 0.1y + 0.1)^3 + x(0.379428..y - 0.35502..) + 2.0285..y^2 \\ -3.95739..y + 1.93013... \end{aligned}$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -6x + \frac{123y}{8} - 15, \quad \dot{y} = -6x + 6y - 8, \tag{43}$$

with the first integral

$$H(x,y) = -144(2x+5)y + 48x(3x+8) + 369y^2.$$

For the discontinuous piecewise differential system (42)–(43), system (37) has the unique solution $(y_1, y_2) = (0.335447.., 1.61577..)$ which provides one crossing limit cycle intersecting Γ_1 in the two points $(0, y_1)$ and $(0, y_2)$, and system (20) has the unique solution $(x_3, y_3) = (0.63163.., 2.3040..)$, which provides the two intersecting points $(x_3, 0)$, $(0, y_3)$ of the crossing limit cycle with the separation line Σ. Then the discontinuous piecewise differential system (42)–(43) has exactly two crossing limit cycles, see Figure 14(c). This example completes the proof of statement (a). ■

Proof. [Proof of statement (b) of Theorem B] We give an example with five limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ with five limit cycles one limit cycle from the first configuration and four limit cycles from the second configuration. In the region Σ^+ we consider the quadratic center

$$\begin{aligned} \dot{x} &= 0.514763..x^2 + x(-0.224056..y - 5.81653..) + (-0.439253..y - 3.53537..)y \\ &\quad +4.46104.., \\ \dot{y} &= x(8.61407..\ - 0.796302..y) - 1.4867..x^2 + (0.365704..y + 8.05432..)y \\ &\quad +5.53532.., \end{aligned} \tag{44}$$

this system has the first integral

$$H_4^{(ii)}(x,y) = (-0.829829..x - 0.417505..y + 2.68699..)^2(0.5x - 0.3y - 3.85171..)(1.15966..x \\ +1.13501..y + 2.47773..).$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{461y}{580} + \frac{9}{10}, \quad \dot{y} = \frac{1}{20}(29x - 10y + 30), \tag{45}$$

with the first integral

$$H(x,y) = -116(5x+9)y + 29x(29x+60) + 461y^2.$$

For the discontinuous piecewise differential system (44)–(45), system (37) has the unique solution $(y_1, y_2) = (0.259656.., 2.00499..)$ which provides the unique crossing limit cycle intersecting Γ_1 in the two points $(0, y_1)$ and $(0, y_2)$, and system (20) has the four solutions $(x_3, y_3) = (0.770518.., 3.42873..)$, $(x_4, y_4) = (0.590237.., 3.16837..)$, $(x_5, y_5) = (0.387278.., 2.86942..)$ and $(x_6, y_6) = (0.150039.., 2.50692..)$, which provide the eight intersecting points $(x_k, 0)$, $(0, y_k)$ with $k = 3, 4, 5, 6$ of the four crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (44)–(45) has exactly five crossing limit cycles, see Figure 14(b).

We give an example with six limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A = a = 0 \neq Cb$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $A = a = 0 \neq Cb$ with three limit cycles from each configuration. In the region Σ^- we consider the quadratic center

$$\begin{aligned} \dot{x} &= 0.0208423..x^2 + x(-0.294257..y - 0.687356..) + (0.498832..y + 2.99943..)y \\ &\quad -7.00792.., \\ \dot{y} &= y(1.7807.. - 0.166277..y) - 0.00694744..x^2 + x(0.0980857..y - 0.587584..) \\ &\quad -3.24858.., \end{aligned} \tag{46}$$

this system has the first integral

$$H_6^{(ii)}(x,y) = (1.x - 12.1481..y + 55.7642..)^2(1.x - 1.97015..y - 1.18853..)e^{0.1x+0.3y}.$$

In the region Σ^+ we consider the linear differential center

$$\dot{x} = -\frac{3x}{2} + \frac{53y}{8} - \frac{59}{5}, \quad \dot{y} = -10x + \frac{3y}{2} - 5, \tag{47}$$

with the first integral

$$H(x,y) = -8(15x+118)y + 400x(x+1) + 265y^2.$$

For the discontinuous piecewise differential system (46)–(47), system (37) has the three solutions $(y_1, y_2) = (1.05249.., 2.50978..)$, $(y_3, y_4) = (0.491559.., 3.0707..)$ and $(y_5, y_6) = (0.109285.., 3.45298..)$ which provide three crossing limit cycles intersecting Γ_1 in the six points $(0, y_i)$ with $i = 1, 2, 3, 4, 5, 6$, and system (20) has the three solutions $(x_7, y_7) = (0.366025.., 3.76284..)$, $(x_8, y_8) = (0.724745.., 4.0304..)$ and $(x_9, y_9) = (1, 4.26936..)$, which provide the six intersecting points $(x_j, 0)$, $(0, y_j)$ with $j = 7, 8, 9$ of the three crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (46)–(47) has exactly six crossing limit cycles, see Figure 15(c).

We give an example with four limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $\Delta = a = 0$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $\Delta = a = 0$ with three limit cycles from each configuration. In the region Σ^- we consider the quadratic center

$$\begin{aligned} \dot{x} &= x(0.225773.. - 0.133734..y) - 0.00957539..x^2 + (0.00407421..y - 0.593254..)y \\ &\quad +0.94234.., \\ \dot{y} &= y(0.532632.. - 0.173414..y) + 0.749123..x^2 + x(0.0191508..y + 3.16459..) \\ &\quad +1.55252.., \end{aligned} \tag{48}$$

this system has the first integral

$$H_7^{(ii)}(x,y)=\frac{x+0.146272..y+3.84598..}{\sqrt{x-0.254197..y+3.64465..}}e^{\dfrac{0.333333..(-x-0.947209..y+0.751366..)}{x+0.146272..y+3.84598..}}.$$

In the region Σ^+ we consider the linear differential center

$$\dot{x}=\frac{x}{2}-\frac{65y}{116}+\frac{9}{10},\quad \dot{y}=\frac{1}{20}(29x-10y+30), \tag{49}$$

with the first integral

$$H(x,y)=-116(5x+9)y+29x(29x+60)+325y^2.$$

For the discontinuous piecewise differential system (46)–(47), system (37) has the two solutions $(y_1,y_2)=((-2/25)\left(\sqrt{6371}-261\right),(2/325)\left(\sqrt{6371}+261\right))$ and $(y_3,y_4)=((-2/325)(\sqrt{48621}-261),(2/325)(\sqrt{48621}+261))$ which provide two crossing limit cycles intersecting Γ_1 in the four points $(0,y_i)$ with $i=1,2,3,4$, and system (20) has the two solutions $(x_5,y_5)=((2/29)\left(\sqrt{295}-15\right),(2/325)(11\sqrt{751}+261))$ and $(x_6,y_6)=((10/29)\left(\sqrt{17}-3\right),(2/325)\left(\sqrt{133121}+261\right))$, which provide the six intersecting points $(x_j,0)$, $(0,y_j)$ with $j=5,6$ of the two limit crossing cycles with the separation line Σ. Then the discontinuous piecewise differential system (48)–(49) has exactly four crossing limit cycles, see Figure 15(c).

We give an example with three limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{ii}$ when $A=b=0$. In what follows we give a discontinuous piecewise differential system of the class (2)–(5) when $A=b=0$ with three limit cycles one satisfying **Cnf 1** and two satisfying **Cnf 2**. In the region Σ^- we consider the quadratic center

$$\begin{aligned}\dot{x}&=-0.0464516..x^2+x(0.18871..y+1.82258..)+(-0.148065..y-2.36774..)y\\&\quad+10.6129..,\\ \dot{y}&=-0.0154839..x^2+x(0.0629032..y+3.27419..)+(-0.0493548..y-3.62258..)y\\&\quad+1.87097..,\end{aligned} \tag{50}$$

this system has the first integral

$$H_8^{(ii)}(x,y)=(-x+1.0625..y+4.79167..)^2e^{-0.0009..(x-3.y+60.)^2+0.48x-0.51y}.$$

In the region Σ^+ we consider the linear differential center

$$\dot{x}=-0.2x-0.202784..y+0.806423..,\quad \dot{y}=x+0.2y+0.117971.., \tag{51}$$

with the first integral

$$H(x,y)=4(x+0.2y)^2+8(0.117971..x-0.806423..y)+0.651134..y^2.$$

For the discontinuous piecewise differential system (50)–(51), system (37) has the unique solution $(y_1,y_2)=(0.493476..,7.46006..)$ which provides one crossing limit cycle intersecting Γ_1 in the two points $(0,y_1)$ and $(0,y_2)$, and system (20) has the two solutions $(x_3,y_3)=(1.4422..,9.2446..)$ and $(x_4,y_4)=(2.59897..,11.1981..)$, which provide the four intersecting points $(x_j,0)$, $(0,y_j)$ with $j=3,4$ of the two crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (50)–(51) has exactly three crossing limit cycles, see Figure 15(d). The proof of statement (b) is completed with this example. ■

Proof. [Proof of statement (c) of Theorem B] Here we give an example with two limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{iii}$ when $A-2b=C+2a=0$, one satisfying **Cnf 1** and the other satisfying **Cnf 2**. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x}&=-0.108378..x^2+x(6.81021..y+15.0732..)+y(3.33019..y+2.60493..)\\&\quad-10.2841..,\\ \dot{y}&=0.268378..x^2+x(0.216755..y-1.01461..)+y(-3.40511..y-15.0732..)\\&\quad-14.6995..,\end{aligned} \tag{52}$$

this system has the first integral

$$\begin{aligned}H^{(iii)}(x,y) = {} & x^2(1.89027.. - 0.403825..y) - 0.333333..x^3 + x(y(12.6877..y + 56.164..)\\ & +54.7718..) + y(y(4.1362..y + 4.85311..) - 38.3196..) - 57.3826...\end{aligned}$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -6x + 9.83871..y - 13, \quad \dot{y} = -6.2x + 6y - 5, \tag{53}$$

with the first integral

$$H(x,y) = 4(6y - 6.2x)^2 - 49.6(13y - 5x) + 100y^2.$$

For the discontinuous piecewise differential system (52)–(53), system (37) has the unique solution $(y_1, y_2) = (0.35892.., 2.283..)$ which provides one crossing limit cycle intersecting Γ_1 in the two points $(0, y_1)$ and $(0, y_2)$, and system (20) has the unique solution $(x_3, y_3) = (0.80645.., 3.0462..)$, which provides the two intersecting points $(x_3, 0)$, $(0, y_3)$ of the crossing limit cycle with the separation line Σ. Then the discontinuous piecewise differential system (52)–(53) has exactly two crossing limit cycles, see Figure 15(a). With this example, statement (c) proof is complete. ■

Proof. [Proof of statement (d) of Theorem B] Finally we give an example with five limit cycles for the class of discontinuous piecewise differential system $\mathcal{C}_{iv}$ when $C + 2a = A + 4b + 5d = a^2 + bd + 2d^2 = 0$, two satisfying **Cnf 1** and three satisfying **Cnf 2**. In the region Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} = {} & x(4.38066.. - 1.97521..y) + 0.766728..x^2 + (0.992492..y - 5.78735..)y\\ & +5.07476..,\\ \dot{y} = {} & x(2.05083.. - 0.597499..y) + 0.27823..x^2 + (0.135742..y - 2.38066..)y\\ & +1.84875..,\end{aligned} \tag{54}$$

this system has the first integral

$$\begin{aligned}H^{(IV)}(x,y) = {} & \Big(x^2(15.8595.. - 6.24882..y) + y(y(40.0457.. - 9.03711..y) + 1.28941..)\\ & +x^3 + x(y(13.0159..y - 52.26..) - 81.759..) - 343.77..\Big)^2 \Big/ \Big(x(10.573..\\ & -4.16588..y) + x^2 + y(4.33863..y - 12.8171..) + 37.4499..\Big)^3.\end{aligned}$$

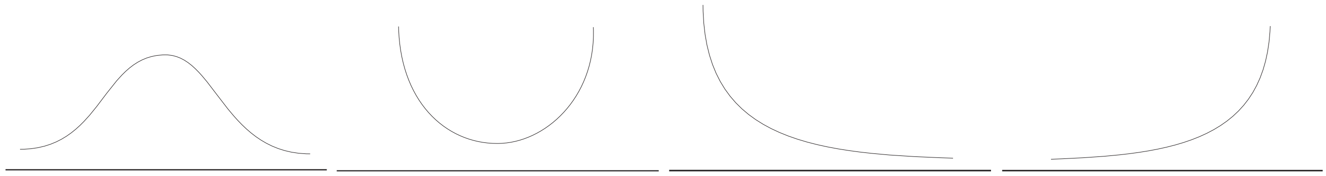
In the region Σ^- we consider the linear differential center

$$\dot{x} = \frac{7x}{2} - \frac{373y}{20} + 20, \quad \dot{y} = 5x - \frac{7y}{2} + 8, \tag{55}$$

with the first integral

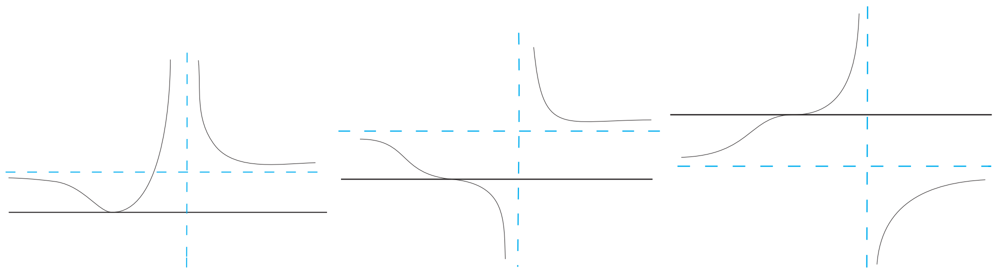
$$H(x,y) = -20(7x + 40)y + 20x(5x + 16) + 373y^2.$$

For the discontinuous piecewise differential system (54)–(55), system (37) has the two solutions $(y_1, y_2) = (0.484404.., 1.66037..)$ and $(y_3, y_4) = (0.133283.., 2.01149..)$ that provide two crossing limit cycle intersecting Γ_1 in the four points $(0, y_i)$ with $i = 1, 2, 3, 4$, and system (20) has the three solutions $(x_5, y_5) = (0.286796.., 2.26323..)$, $(x_6, y_6) = (0.757965.., 2.47035..)$ and $(x_7, y_7) = (1.14955.., 2.65052..)$, which provide the six intersecting points $(x_j, 0)$, $(0, y_j)$ with $j = 5, 6, 7$ of the three crossing limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (54)–(55) has exactly five crossing limit cycles, see Figure 15(b). With this example the proof of Theorem B is complete. ■



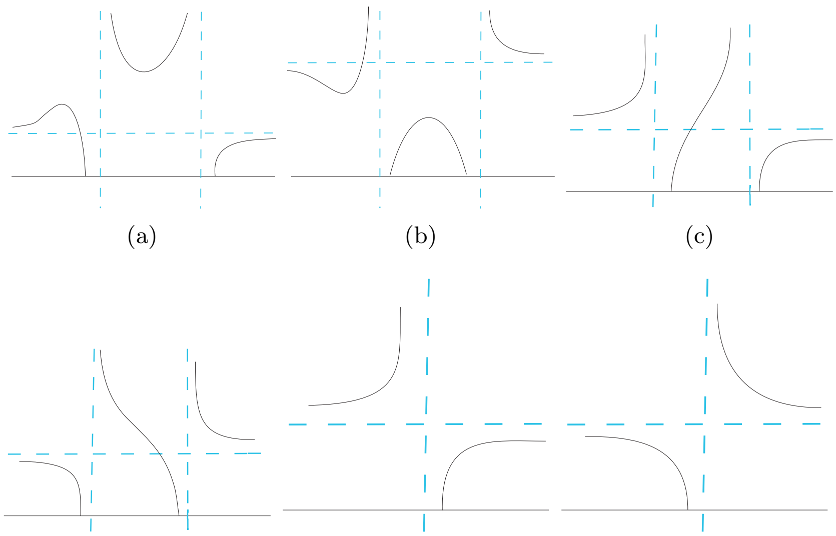
(a) (b) (c) (d)

Fig. 17. The graphics of the function $f_2^{(i)}(y_2)$, the horizontal straight line is the y_2-axis.



(a) (b) (c)

Fig. 16. The graphics of the function $f_1^{(i)}(y_2)$. The dashed lines represent the asymptote straight lines, and the horizontal straight line is the y_2-axis.



(a) (b) (c)

(d) (e) (f)

Fig. 18. The graphics of the function $g_1^{(ii)}(y_2)$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2-axis.

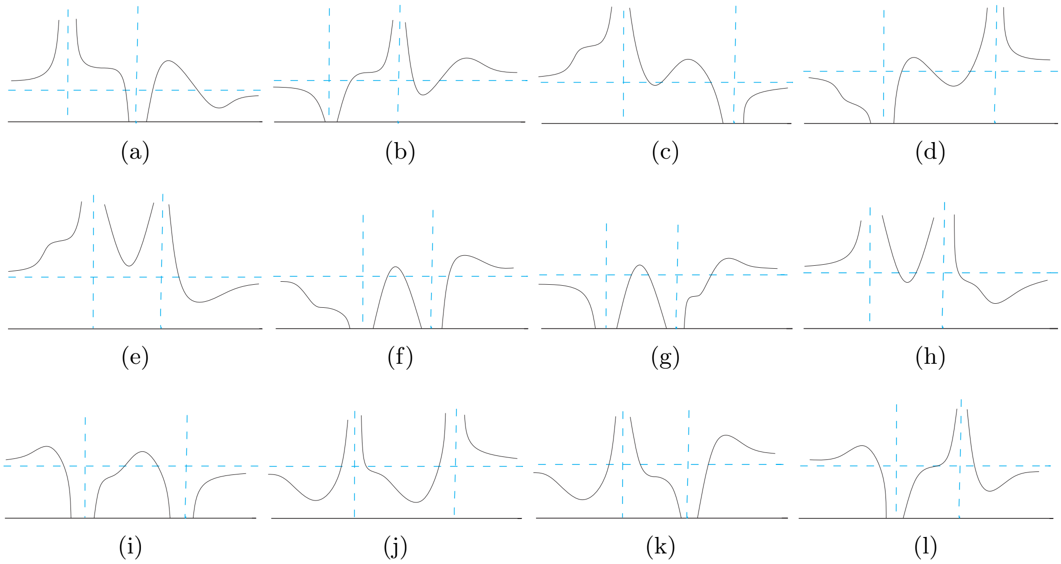


Fig. 19. Continuous of the graphics of the function $g_1^{(i)}(y_2)$ with $s_2 \neq s_3$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2-axis.

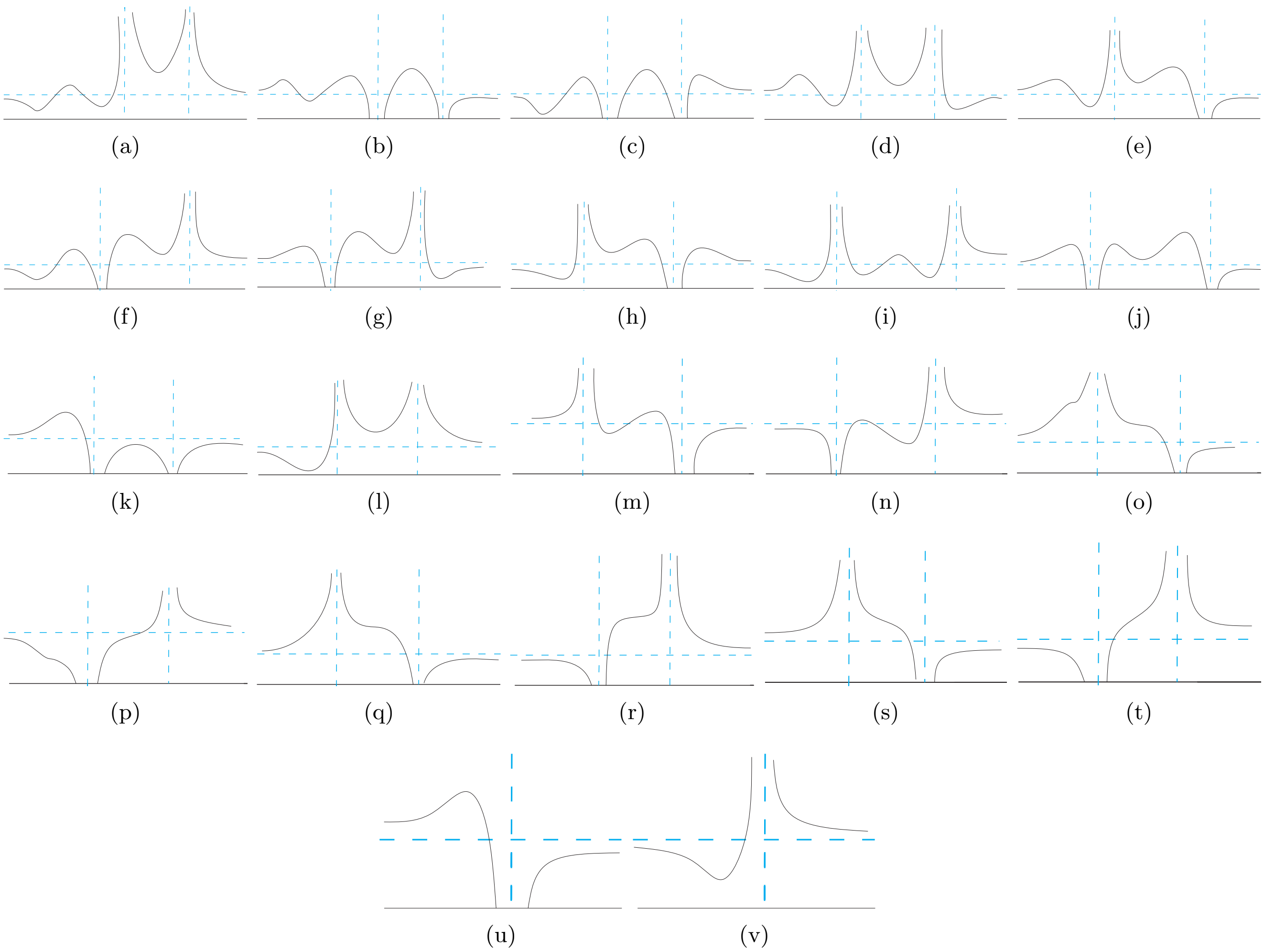


Fig. 20. The graphics of the function $g_1^{(i)}(y_2)$ with $s_2 \neq s_3$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2-axis.

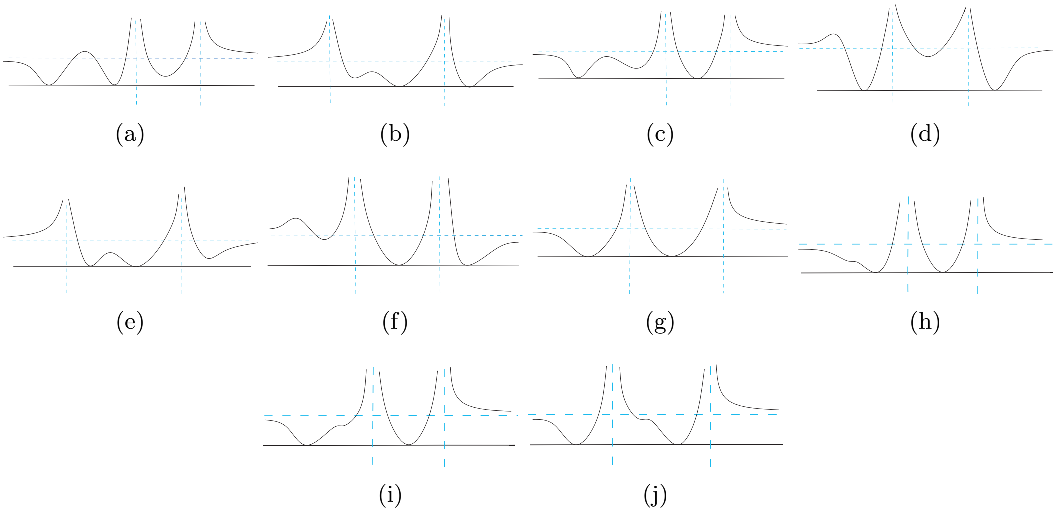


Fig. 21. The graphics of the function $f_1^{(ii)}(y_2)$ if p and q are even or if p is even and $q = \dfrac{k_1}{2k_2+1}$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the vertical asymptote straight lines, and the horizontal straight line is the y_2-axis.

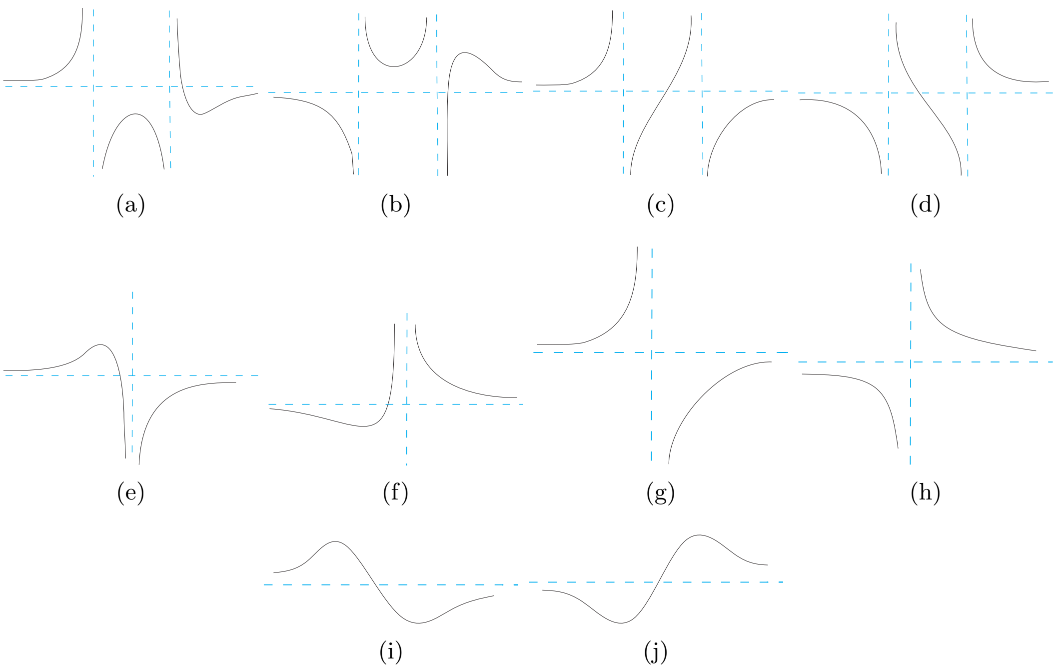


Fig. 22. The graphics of the function $g_2^{(i)}(y_2)$. The dashed lines represent the asymptotes straight lines.

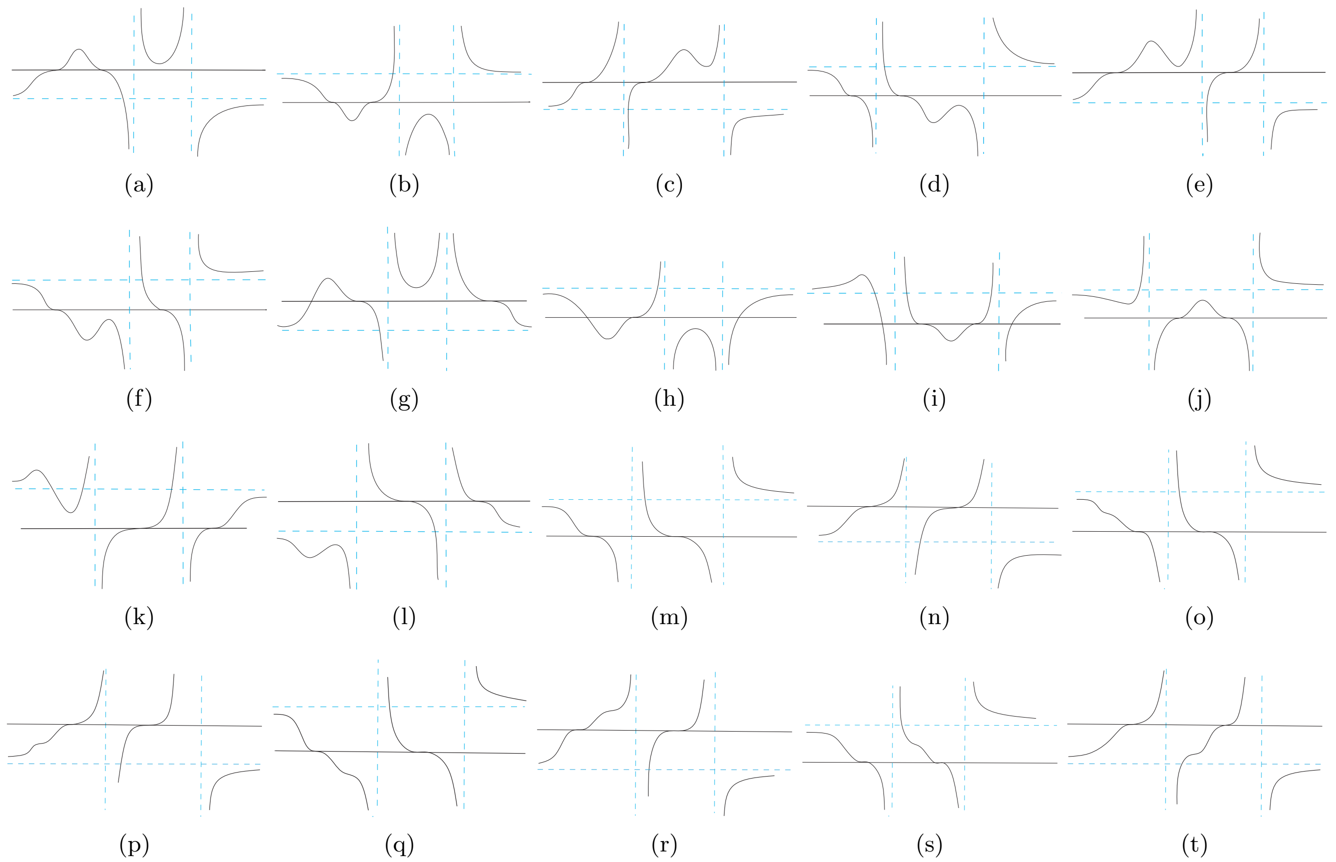


Fig. 23. The graphics of the function $f_1^{(ii)}(y_2)$ if p and q are odd. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2-axis.

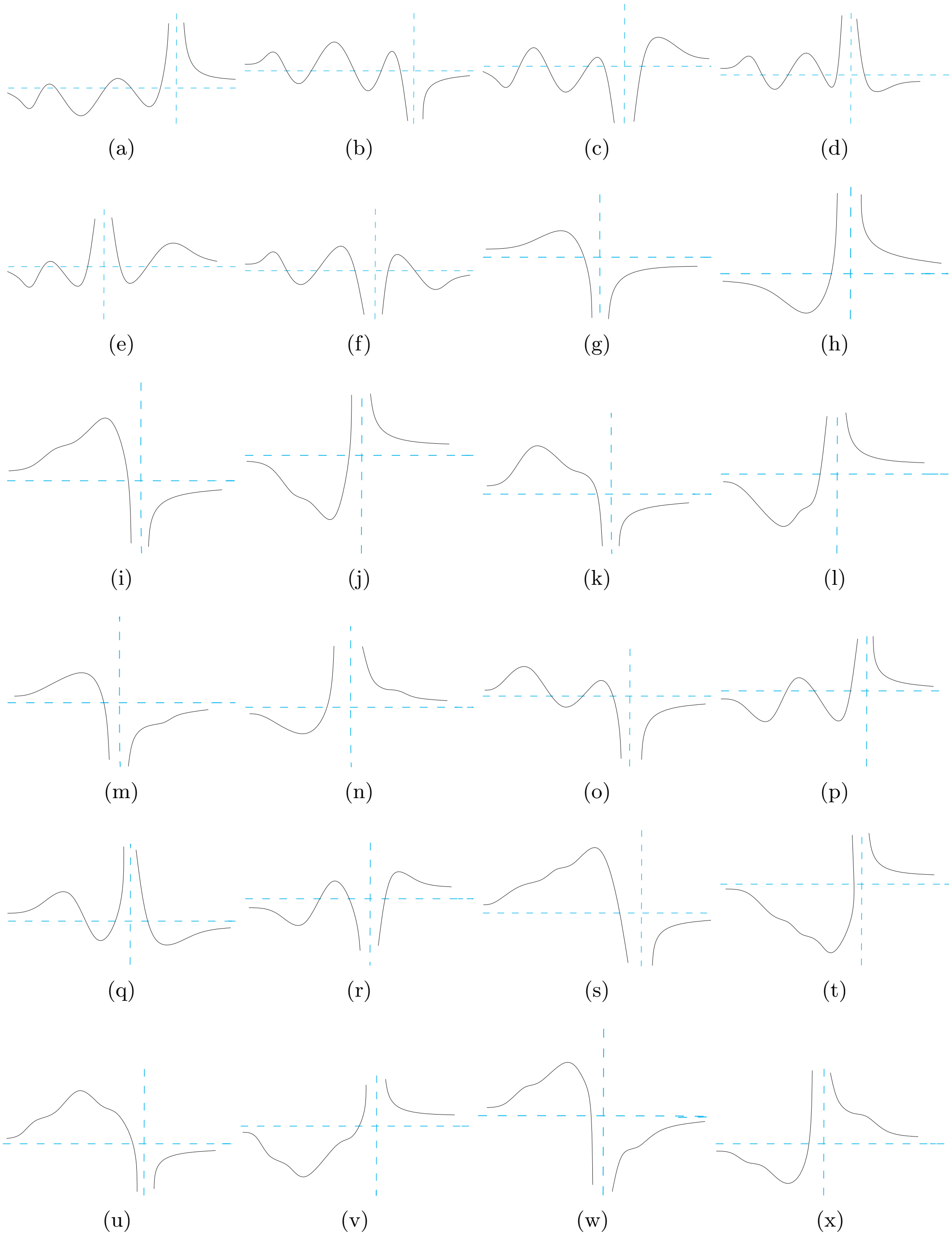


Fig. 24. The graphics of the function $f_2^{(ii)}(y_2)$ with r is an even number, or $r = \dfrac{k_1}{2k_2+1}$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the asymptotes straight lines.

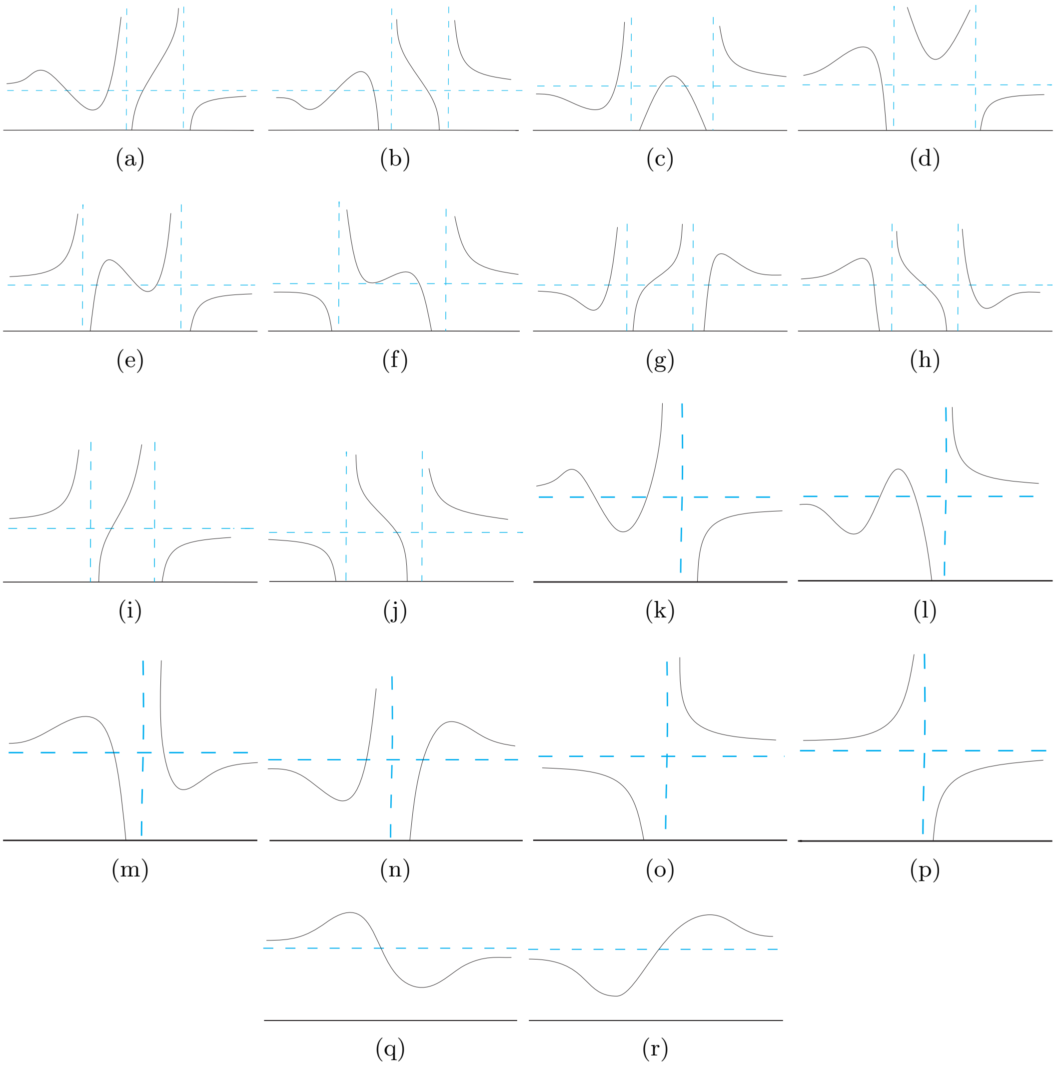


Fig. 25. The graphics of the function $g_2^{(ii)}(y_2)$. The dashed lines represent the asymptotes straight lines, and the horizontal straight line is the y_2-axis.

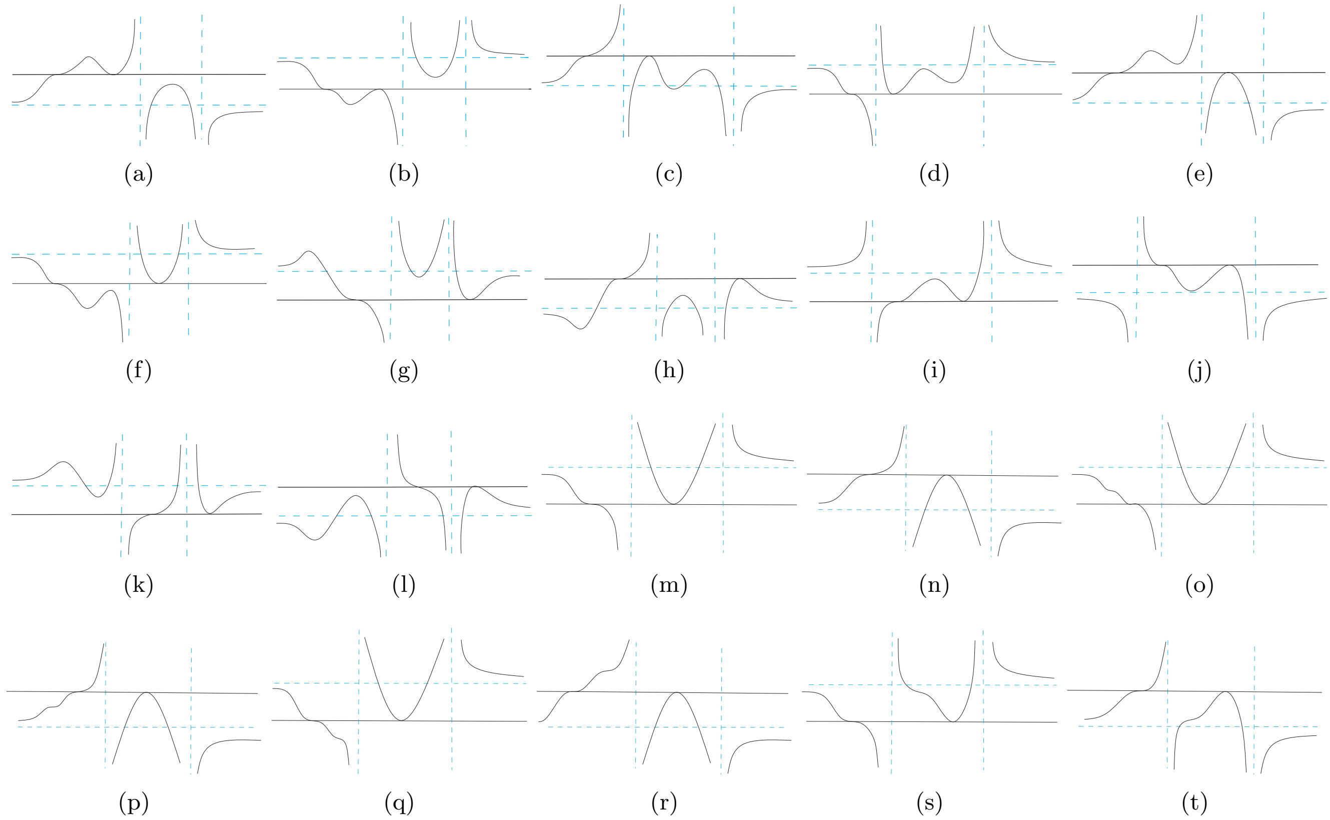


Fig. 26. The graphics of the function $f_1^{(ii)}(y_2)$ if p odd and q even, or if p odd and $q = \dfrac{k_1}{2k_2+1}$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2-axis.

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