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PLANAR QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANTS OF THE FORM $ax^2 + bxy + cy^2 + dx + ey + c_1t$

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Abstract. A function $I(x, y, t)$ constant on the solutions of a differential system in $\mathbb{R}^2$ is called an invariant. We classify all planar quadratic differential systems having invariants of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ with $c_1 \neq 0$. There are 13 different families of quadratic systems having invariants of this form. As far as we know this is the first time that quadratic differential systems having an invariant different from a Darboux invariant have been classified.

1. Introduction and statement of the main results

In this paper we shall work with polynomial differential systems

$$\dot{x} = P(x, y), \;\; \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials of degree 2, called simply *quadratic systems*. As usual the dot denotes derivative with respect to the independent variable t, usually called the time.

Many natural phenomena in various branches of the sciences are modelized using quadratic systems. We can find in the literature more than one thousand published papers studying the quadratic systems, see for instance the books [3, 14, 15] and the hundreds of references which are cited therein.

Let U be an open and dense subset of $\mathbb{R}^2$, an *invariant* of system (1) in U is a non-constant C^1 function $I : U \times \mathbb{R} \to \mathbb{R}$ depending on t such that $I(x(t), y(t), t)$ is constant on all the solution curves $(x(t), y(t))$ of system (1) contained in U, i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial x}Q + \frac{\partial I}{\partial t} = 0, \tag{2}$$

for all $(x, y) \in U$. For more details on the invariants, see [7, Chapter 8] or [13].

The objective of this paper is to classify all quadratic systems

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2, \end{aligned} \tag{3}$$

having invariants of the form

$$I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t, \tag{4}$$

with $c_1 \neq 0$.

Quadratic planar differential systems play a pivotal role in mathematical modeling, offering valuable insights into the behavior of dynamic systems. Their significance extends across diverse fields, including physics, biology, and engineering, where they serve as powerful tools for understanding complex phenomena. For example the works [2, 5, 6] delve into the classification of quadratic planar differential systems and highlight their relevance in the analysis of dynamic behavior. These systems provide a mathematical framework for studying the intricate interplay of variables and offer a nuanced understanding of the underlying dynamics. The exploration and understanding of quadratic planar differential systems contribute not only to theoretical advancements but also to practical applications in various scientific disciplines.

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We note that many different classes of quadratic systems have been classified as the structurally stable, their centers, their isochronous centers, their Hopf bifurcations, their Lotka-Volterra, their Bernoulli, their chordal, their Abel, their quadratic-linear, their with a unique finite singularity, their having a polynomial first integral, their having a Hamiltonian first integral, their homogeneous, ..., see for details [3]. But there are few works on the quadratic systems having invariants, see [4, 10, 11, 12, 13], but all the invariants of these works are Darboux invariants, i.e. invariants of the form $f(x,y)e^{st}$ with $s \in \mathbb{R}\setminus\{0\}$.

As far as we know this is the first time that quadratic systems having an invariant different from a Darboux invariant are studied. Thus our first main result is the following.

Theorem 1. *The quadratic systems* (3) *admitting an invariant of the form* (4) *are one of the following* 13 *families of quadratic systems:*

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ \dot{y} &= b_0 - \frac{d}{e}(a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2), \\ I &= dx + ey - (da_0 + eb_0)t, \end{aligned} \tag{5}$$

with $e \neq 0$ *and* $da_0 + eb_0 \neq 0$.

$$\begin{aligned} \dot{x} =\ & a_0 + a_1x + \frac{1}{b^2}(b(2ca_1 + ea_4) - 4cea_3)y + a_3x^2 + a_4xy + \frac{2}{b^2}c(ba_4 - 2ca_3)y^2, \\ \dot{y} =\ & \frac{1}{2b^2c}((ba_1 - ea_3)(be - 2cd) - b^3a_0) - \frac{1}{2bc}(b^2a_1 - bea_3 + 2cda_3)x + \\ & \frac{1}{b^2}(bea_3 + 2cda_3 - b^2a_1 - bda_4)y - \frac{1}{2c}ba_3x^2 - \frac{1}{2c}ba_4xy + \frac{1}{b}(2ca_3 - ba_4)y^2, \\ I =\ & dx + ey + \frac{b^2}{4c}x^2 + bxy + cy^2 + \frac{1}{2b^2c}(b^2a_0 + e(ea_3 - ba_1))(be - 2cd)t, \end{aligned} \tag{6}$$

with $bc \neq 0$ *and* $(b^2a_0 + e(ea_3 - ba_1))(be - 2cd) \neq 0$.

$$\begin{aligned} \dot{x} =\ & a_0 + \left(\tfrac{d}{bd-2ae}(2aa_2 + bb_2) + \tfrac{e}{b}a_3 - b_2\right)x + a_2y + a_3x^2 + \left(\tfrac{b}{2ae-bd}(2aa_2 + bb_2) + \tfrac{b}{2a}a_3\right)xy \\ & - \tfrac{b^2}{2abd-4a^2e}(2aa_2 + b_2b)y^2, \\ \dot{y} =\ & -\tfrac{2}{b^2}a(a_0b + a_2d + b_2e) + \tfrac{1}{b(bd-2ae)}\left(-4a^2(a_2d + b_2e) - a_3bd^2 + 2a_3ade\right)x + b_2y - \\ & \tfrac{2}{b}aa_3x^2 + \left(\tfrac{2a}{bd-2ae}(2aa_2 + bb_2) - a_3\right)xy + \tfrac{b}{bd-2ae}(2aa_2 + bb_2)y^2, \\ I =\ & dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 - \frac{1}{b^2}((bd - 2ae)ba_0 - 2ae(da_2 + eb_2))t. \end{aligned} \tag{7}$$

with $ab(bd - 2ae) \neq 0$ *and* $(bd - 2ae)ba_0 - 2ae(da_2 + eb_2) \neq 0$.

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + \frac{2}{e}ca_1xy - \frac{2}{d}cb_2y^2, \\ \dot{y} &= -\frac{1}{2c}(a_2d + b_2e) - \frac{d}{e}a_1x + b_2y, \\ I &= dx + ey + cy^2 + \frac{1}{2c}(e(da_2 + eb_2) - 2cda_0)t. \end{aligned} \tag{8}$$

with $cde \neq 0$ *and* $e(da_2 + eb_2) - 2cda_0 \neq 0$.

$$\begin{aligned} \dot{x} =\ & a_0 - \frac{e}{2a}b_3x + a_2y, \\ \dot{y} =\ & b_0 - \frac{1}{2aea_2}(e^2b_2b_3 + 4a^2a_2a_0)x + b_2y - \frac{2}{e}aa_2xy + b_3x^2, \\ I =\ & -\frac{e}{a_2}b_2x + ey + ax^2 + \left(\frac{e}{a_2}a_0b_2 - eb_0\right)t. \end{aligned} \tag{9}$$

with $aea_2 \neq 0$ *and* $\frac{e}{a_2}a_0b_2 - eb_0 \neq 0$.

$$\begin{aligned} \dot{x} &= a_0 - \frac{e}{2a}b_3x, \\ \dot{y} &= b_0 + \frac{1}{2ae}(deb_3 - 4a^2a_0)x + b_3x^2, \\ I &= dx + ey + ax^2 - (da_0 + eb_0)t. \end{aligned} \tag{10}$$

with $ae \neq 0$ *and* $da_0 + eb_0 \neq 0$.

$$\begin{aligned} \dot{x} = {} & a_0 + \frac{2}{b}aa_2x + a_2y + a_3x^2 + \frac{1}{2ad}(bda_3 - 4a^2a_2 - 2bab_2)xy - \\ & \frac{b}{2ad}(2aa_2 + bb_2)y^2, \\ \dot{y} = {} & -\frac{2}{b^2}(ba_0 + da_2)a - \frac{1}{b^2}(4a^2a_2 + bda_3)x + b_2y - \frac{2}{b}aa_3x^2 + \\ & \frac{1}{bd}(4a^2a_2 - bda_3 + 2abb_2)xy + \frac{1}{d}(2aa_2 + bb_2)y^2, \\ I = {} & dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0t. \end{aligned} \tag{11}$$

with $abd \neq 0$ *and* $a_0 \neq 0$.

$$\begin{aligned} \dot{x} = {} & a_0 + \frac{b}{2c}a_2x + a_2y + a_3x^2 + a_4xy + \frac{2}{b^2}(ba_4 - 2ca_3)cy^2, \\ \dot{y} = {} & -\frac{1}{2c}(ba_0 + da_2) - \frac{1}{4bc^2}(b^3a_2 + 4c^2da_3)x - \frac{1}{2cb^2}(2(ba_4 - 2ca_3)dc + b^3a_2)y - \\ & \frac{b}{2c}a_3x^2 - \frac{b}{2c}a_4xy + \frac{1}{b}(2ca_3 - ba_4)y^2, \\ I = {} & dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0t. \end{aligned} \tag{12}$$

with $bc \neq 0$ *and* $da_0 \neq 0$.

$$\begin{aligned} \dot{x} &= a_0 + a_2y + a_4xy + a_5y^2, \\ \dot{y} &= -\frac{d}{2c}(a_2 + a_4x + a_5y), \\ I &= dx + cy^2 - da_0t. \end{aligned} \tag{13}$$

with $c \neq 0$ *and* $da_0 \neq 0$.

$$\begin{aligned} \dot{x} = {} & a_0 + a_2y - \frac{2}{d}cb_2y^2, \\ \dot{y} = {} & b_0 + b_2y, \\ I = {} & dx - \frac{1}{b_2}(da_2 + 2cb_0)y + cy^2 + (\frac{b_0}{b_2}(da_2 + 2cb_0) - da_0)t. \end{aligned} \tag{14}$$

with $db_2 \neq 0$ *and* $\frac{b_0}{b_2}(da_2 + 2cb_0) - da_0 \neq 0$.

$$\begin{aligned} \dot{x} = {} & a_0 + a_2y + \frac{b}{e}a_2xy + \frac{b^2}{2ea}a_2y^2, \\ \dot{y} = {} & -\frac{2}{b}aa_0 - \frac{a_2}{e}(dy + 2axy + by^2), \\ I = {} & dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 - \frac{a_0}{b}(db - 2ae)t. \end{aligned} \tag{15}$$

with $abe \neq 0$ *and* $\frac{a_0}{b}(db - 2ae) \neq 0$.

$$
\begin{aligned}
\dot{x} = & \ \frac{bda_2^2}{b^2a_2+2cda_4+2bcb_2} - \frac{b}{c}a_2x + a_2y + \frac{b}{4c^2d}(b^2a_2+2cda_4+2bcb_2)x^2 + \\
& a_4xy - \frac{1}{d}(ba_2+2cb_2)y^2, \\
\dot{y} = & \ -\tfrac{a_2d(a_2b^2+a_4cd+bcb_2)}{c(a_2b^2+2c(a_4d+bb_2))} - \tfrac{(a_2b^2+a_4dc+cbb_2)}{2c^2}x + b_2y - \tfrac{b^2}{8c^3d}(a_2b^2+2c(a_4d+bb_2))x^2 - \\
& \tfrac{b}{2c}a_4xy + \tfrac{b}{2cd}(a_2b+2b_2c)y^2, \\
I = & \ dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0t.
\end{aligned} \tag{16}
$$

with $cd(b^2a_2+2cda_4+2bcb_2) \neq 0$ *and* $a_0 \neq 0$.

$$
\begin{aligned}
\dot{x} = & \ a_0 + a_4xy + \frac{2}{b}ca_4y^2, \\
\dot{y} = & \ -\frac{b}{2c}a_0 - a_4(\frac{d}{b}y + \frac{b}{2c}xy + y^2), \\
I = & \ dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0t.
\end{aligned} \tag{17}
$$

with $bc \neq 0$ *and* $da_0 \neq 0$.

Theorem 1 is proved in section 2.

Roughly speaking the *Poincaré disc* $\mathbb{D}^2$ is the unit closed disc centered at the origin of coordinates whose interior is diffeomorphic with $\mathbb{R}^2$ and whose boundary, the circle $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Note that we can go to the infinity in the plane $\mathbb{R}^2$ in as many as directions as points has the circle $\mathbb{S}^1$. Any polynomial differential system can be extended analytically to the whole Poincaré disc, being the circle of the infinity invariant by this extended flow. This extension is called the *Poincaré compactification*, see for more details Chapter 5 of [7].

Now the singular points of a polynomial differential system which are in the circle of the infinity, i.e. which for its Poincaré compactification are in the circle of the infinity, are called the *infinite singular points* of the polynomial differential system, and the usual singular points of a polynomial differential system contained in $\mathbb{R}^2$, or equivalently in the open Poincaré disc are called the *finite singular points* of the polynomial differential system.

The *chordal quadratic systems* are the quadratic systems without finite singular points. The phase portraits in the Poincaré disc of all the chordal quadratic systems are classified in [8]. By the Poincaré-Bendixson Theorem (see for instance Theorem 1.25 and Corollary 1.30 of [7]) all the orbits of a chordal quadratic system starts at an infinite singular point and ends in an infinite singular point, eventually both singular points can coincide.

Corollary 2. *All quadratic systems having an invariant of the form* $I(x,y,t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ *with* $c_1 \neq 0$ *are chordal quadratic systems.*

The proof of Corollary 2 is done at the end of section 2.

We say that two phase portraits in the Poincaré disc are *topologically equivalent* if there is a diffeomorphism from $\mathbb{D}^2$ into itself such that send orbits of one phase portrait onto the orbits of the other phase portrait preserving or reversing the sense of all the orbits.

As an exercise we have classified the phase portraits of the chordal quadratic systems given by system (14) with $a_0 \neq 0$ in section 3.

2. The proofs

In this section we shall prove Theorem 1 and Corollary 2.

Proof of Theorem 1. To prove that the function $I(x, y, t) = dx + ey + ax^2 + bxy + cy^2 + c_1t$ is an invariant of system (3) we must verify the partial derivative equation (2). Thus, the following polynomial must be a zero polynomial

$$\begin{aligned}
&a_0d + b_0e + c_1 + (a_1d + 2aa_0 + b_1e + bb_0)x + (a_0b + a_2d + 2b_0c + b_2e)y+\\
&(a_3d + 2aa_1 + b_3e + bb_1)x^2 + (a_1b + a_4d + 2aa_2 + 2b_1c + b_4e + bb_2)xy+\\
&+(a_2b + a_5d + 2b_2c + b_5e)y^2 + (2aa_3 + bb_3)x^3+\\
&(a_3b + 2aa_4 + 2b_3c + bb_4)x^2y + (a_4b + 2aa_5 + 2b_4c + bb_5)xy^2 + (a_5b + 2b_5c)y^3.
\end{aligned}$$

Therefore the following algebraic system must be solved

$$\begin{aligned}
&a_0d + b_0e + c_1 = 0,\\
&a_1d + 2aa_0 + b_1e + bb_0 = 0,\\
&a_0b + a_2d + 2b_0c + b_2e = 0,\\
&a_3d + 2aa_1 + b_3e + bb_1 = 0,\\
&a_1b + a_4d + 2aa_2 + 2b_1c + b_4e + bb_2 = 0,\\
&a_2b + a_5d + 2b_2c + b_5e = 0,\\
&2aa_3 + bb_3 = 0,\\
&a_3b + 2aa_4 + 2b_3c + bb_4 = 0,\\
&a_4b + 2aa_5 + 2b_4c + bb_5 = 0,\\
&a_5b + 2b_5c = 0.
\end{aligned} \tag{18}$$

By solving this algebraic system with the help of the Mathematica software we obtain 49 solutions, but we select only the minimal number of solutions because many solutions provided by Mathematica are particular solutions of other solutions. In this work, we are interested in the case in which the independent variable t appears explicitly for that we neglect the solutions when $c_1 = 0$. Then, we get the following sets of independent solutions with $c_1 \neq 0$

$$\begin{aligned}
s_1 = \;&\Big\{ b_1 = -\frac{d}{e}a_1,\ b_2 = -\frac{d}{e}a_2,\ b_3 = -\frac{d}{e}a_3,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\ a = 0,\\
&b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\Big\},\\
s_{11} = \;&\Big\{ a_2 = \frac{1}{b^2}(2bca_1 + bea_4 - 4cea_3),\ a_5 = \frac{2}{b^2}c(ba_4 - 2ca_3),\ b_3 = -\frac{b}{2c}a_3,\\
&b_0 = \frac{1}{2b^2c}((ba_1 - ea_3)(be - 2cd) - b^3a_0),\ b_5 = \frac{1}{b}(2ca_3 - ba_4),\\
&b_1 = -\frac{1}{2bc}(b^2a_1 - bea_3 + 2cda_3),\ b_4 = -\frac{b}{2c}a_4,\\
&a = \frac{b^2}{4c},\ b_2 = \frac{1}{b^2}(bea_3 + 2cda_3 - b^2a_1 - bda_4),\\
&c_1 = \frac{1}{2b^2c}(be - 2cd)(b^2a_0 + e(ea_3 - ba_1))\Big\},\\
s_{12} = \;&\Big\{ a_1 = \frac{1}{b^2d - 2abe}(bdea_3 + 2abda_2 + 2abeb_2 - 2ae^2a_3),\ b_3 = -\frac{2}{b}aa_3,\\
&a_4 = \frac{b}{2a(2ae - bd)}(4a^2a_2 - bda_3 + 2a(ea_3 + bb_2)),\\
&b_0 = -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a,\ b_4 = \frac{1}{2ae - bd}(bda_3 - 4a^2a_2 - 2a(ea_3 + bb_2)),\\
&b_1 = \frac{1}{b(bd - 2ae)}(-4a^2(da_2 + eb_2) + 2adea_3 - bd^2a_3),\\
&a_5 = \frac{b^2}{2a(2ae - bd)}(2aa_2 + bb_2),\ c = \frac{b^2}{4a},\ b_5 = \frac{b}{bd - 2ae}(2aa_2 + bb_2),\\
&c_1 = \frac{1}{b^2}((2ae - bd)ba_0 + 2(da_2 + eb_2)ae)\Big\},\\
s_{13} = \;&\Big\{ a_3 = 0,\ a_4 = \frac{2}{e}ca_1,\ a_5 = -\frac{2}{d}cb_2,\ b_1 = -\frac{d}{e}a_1,\ b_0 = -\frac{1}{2c}(da_2 + eb_2),\\
&b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c_1 = \frac{e}{2c}(da_2 + eb_2) - da_0\Big\},
\end{aligned}$$

$$s_{20} = \Big\{a_1 = -\frac{e}{2a}b_3,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_1 = -\frac{1}{2aea_2}(e^2b_3b_2 + 4a^2a_2a_0),$$
$$b_4 = -\frac{2}{e}aa_2,\ b_5 = 0,\ b = 0,\ c = 0,\ d = -\frac{e}{a_2}b_2,\ c_1 = \frac{e}{a_2}a_0b_2 - eb_0\Big\},$$
$$s_{21} = \Big\{a_1 = -\frac{e}{2a}b_3,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_1 = \frac{1}{2ae}(deb_3 - 4a^2a_0),$$
$$b_2 = 0,\ b_4 = 0,\ b_5 = 0,\ c_1 = -(da_0 + eb_0)\},$$
$$s_{22} = \Big\{\ a_1 = \frac{2a}{bd-2ae}(da_2 + eb_2),\ a_3 = 0,\ a_4 = -\frac{b}{bd-2ae}(2aa_2 + bb_2),$$
$$a_5 = -\frac{b^2}{2a(bd-2ae)}(2aa_2 + bb_2),\ b_0 = -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a,\ b_3 = 0,$$
$$b_1 = -\frac{4}{b(bd-2ae)}a^2(da_2 + eb_2),\ b_4 = \frac{2}{bd-2ae}a(2aa_2 + bb_2),\ c = \frac{b^2}{4a},$$
$$b_5 = \frac{b}{bd-2ae}(2aa_2 + bb_2),\ c_1 = \frac{1}{b^2}(b(2ae - bd)a_0 + 2ae(da_2 + eb_2))\Big\},$$
$$s_{24} = \Big\{a_1 = \frac{2}{b}aa_2,\ a_4 = \frac{1}{2ad}(ba_3 - 2aa_2 - bb_2),\ a_5 = -\frac{b}{2ad}(2aa_2 + bb_2),$$
$$b_0 = -\frac{2}{b^2}(ba_0 + da_2)a,\ b_1 = -\frac{1}{b^2}(4a^2a_2 + bda_3),\ b_3 = -\frac{2}{b}aa_3,$$
$$b_4 = \frac{1}{bd}(4a^2a_2 + 2abb_2 - bda_3),\ b_5 = \frac{1}{d}(2aa_2 + bb_2),\ c = \frac{b^2}{4a},\ e = 0,$$
$$c_1 = -da_0\},$$
$$s_{27} = \Big\{a_1 = \frac{b}{2c}a_2,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\ b_0 = -\frac{1}{2c}(ba_0 + da_2),$$
$$b_1 = -\frac{1}{4c^2b}(b^3a_2 + 4c^2da_3),\ b_3 = -\frac{b}{2c}a_3,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = \frac{2}{b}ca_3 - a_4,$$
$$b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3) - \frac{b}{2c}a_2,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\},$$
$$s_{29} = \Big\{a_1 = 0,\ a_3 = 0,\ b_0 = -\frac{d}{2c}a_2,\ b_1 = -\frac{d}{2c}a_4,\ b_2 = -\frac{d}{2c}a_5,\ b_3 = 0,$$
$$b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ e = 0,\ c_1 = -da_0\},$$
$$s_{31} = \Big\{a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = -\frac{2}{d}cb_2,\ b_1 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,$$
$$a = 0,\ b = 0,\ e = -\frac{1}{b_2}(da_2 + 2cb_0),\ c_1 = \frac{b_0}{b_2}(da_2 + 2cb_0) - da_0\Big\},$$
$$s_{33} = \Big\{a_1 = 0,\ a_3 = 0,\ a_4 = \frac{b}{e}a_2,\ a_5 = \frac{b^2}{2ae}a_2,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0,\ b_3 = 0,$$
$$b_2 = -\frac{d}{e}a_2,\ b_4 = -\frac{2}{e}aa_2,\ b_5 = -\frac{b}{e}a_2,\ c = \frac{b^2}{4a},\ c_1 = a_0(\frac{2}{b}ae - d)\Big\},$$
$$s_{36} = \Big\{a_1 = \frac{2}{b}aa_2,\ a_3 = 0,\ a_4 = -\frac{1}{d}(2aa_2 + bb_2),\ a_5 = -\frac{b}{2ad}(2aa_2 + bb_2),$$
$$b_0 = -\frac{2}{b^2}a(ba_0 + da_2),\ b_1 = -\frac{4}{b^2}a^2a_2,\ b_3 = 0,\ b_4 = \frac{2}{bd}a(2aa_2 + bb_2),$$
$$b_5 = \frac{1}{d}(2aa_2 + bb_2),\ e = 0,\ c = \frac{b^2}{4a},\ c_1 = -da_0\Big\},$$
$$s_{39} = \Big\{a_1 = 0,\ a_2 = 0,\ a_4 = \frac{b}{2ad}(da_3 - 2ab_2),\ a_5 = -\frac{b^2}{2ad}b_2,\ b_0 = -\frac{2}{b}aa_0,$$
$$b_1 = -\frac{d}{b}a_3,\ b_3 = -\frac{2}{b}aa_3,\ b_4 = \frac{2}{d}ab_2 - a_3,\ b_5 = \frac{b}{d}b_2,\ c = \frac{b^2}{4a}, e = 0,\ c_1 = -da_0\Big\},$$
$$s_{42} = \Big\{a_1 = 0,\ a_2 = 0,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\ b_0 = -\frac{b}{2c}a_0,\ b_1 = -\frac{d}{b}a_3,$$
$$b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3),\ b_3 = -\frac{b}{2c}a_3,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = \frac{2}{b}ca_3 - a_4,$$
$$a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\},$$

$$
\begin{aligned}
s_{43} = \ & \Big\{ a_0 = \frac{bda_2^2}{b^2a_2 + 2c(da_4 + bb_2)},\ a_3 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2)),\\
& a_1 = -\frac{b}{2c}a_2,\ a_5 = -\frac{1}{d}(ba_2 + 2cb_2),\ b_0 = d(\frac{a_2}{2c} - \frac{2aa_2^2}{b^2a_2 + 2c(da_4 + bb_2)}),\\
& b_1 = \frac{1}{2c}(2aa_2 - \frac{1}{2c}(b^2a_2 + 2c(da_4 + bb_2))),\ a = \frac{b^2}{4c},\ e = 0,\\
& b_4 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))(\frac{4}{b^2}ac - 1) - \frac{2}{b}aa_4,\ c_1 = -da_0,\\
& b_3 = -\frac{2}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))a,\ b_5 = \frac{1}{2cd}(b^2a_2 + 2c(da_4 + bb_2)) - a_4 \Big\},\\
s_{44} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b}ca_4,\ b_0 = -\frac{b}{2c}a_0,\ b_1 = 0,\ b_2 = -\frac{d}{b}a_4,\\
& b_3 = 0,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = -a_4,\ e = 0,\ a = \frac{b^2}{4c},\ c_1 = -da_0 \Big\},\\
s_{45} = \ & \Big\{ a_0 = \frac{bda_2^2}{b^2a_2 + 2dca_4 + 2bcb_2},\ a_1 = -\sqrt{\tfrac{a_2^2bd}{a_2b^2+2a_4cd+2bb_2c}}\sqrt{\tfrac{b(a_2b^2+2a_4cd+2bb_2c)}{c^2d}},\\
& a_3 = -\frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2)),\ a_5 = -\tfrac{1}{d}(a_2b + 2b_2c),\ b_4 = -\tfrac{ba_4}{2c},\\
& b_3 = -\frac{b^2}{8c^3d}(b^2a_2 + 2c(da_4 + bb_2)),\\
& b_1 = -\tfrac{1}{4c^2}(-bc\sqrt{\tfrac{a_2^2bd}{a_2b^2+2a_4cd+2bb_2c}}\sqrt{\tfrac{b(a_2b^2+2a_4cd+2bb_2c)}{c^2d}} + a_2b^2 + 2a_4cd + 2bb_2c),\\
& b_0 = \tfrac{d}{2b}(\sqrt{\tfrac{a_2^2bd}{a_2b^2+2a_4cd+2bb_2c}}\sqrt{\tfrac{b(a_2b^2+2a_4cd+2bb_2c)}{c^2d}} - \tfrac{a_2^2b^3}{c(a_2b^2+2c(a_4d+bb_2))}),\\
& b_5 = \tfrac{b}{2cd}(ba_2 + 2cb_2),\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0 \Big\},
\end{aligned}
$$

In what follows we explain how we have eliminated the other solutions provided by Mathematica. The solutions s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 and s_{10} are particular solutions of s_1 when one or more parameters of s_1 are zero, then all these solutions provides system (5).

The solutions s_{11}, s_{12} and s_{13} produce systems (6), (7) and (8), respectively.

The solution s_{14} is a particular solution of s_1 when $d = 0$ and $a_3 = 0$, so it contained in system (5).

The solutions s_{15}, s_{16}, s_{17}, s_{18} and s_{19} are particular solutions of the solution s_{13} when one or more parameters vanish.

The solutions s_{20}, s_{21} and s_{22} yield systems (9), (10) and (11), respectively.

The solution s_{23} is a particular solution of s_{22} when we take $2aa_2 = bb_2$.

The solution s_{24} produces system (11).

The solution s_{25} is a particular solution of the solution s_{24} when $2aa_2 + bb_2 = 0$, so we do not consider it.

The solution s_{26} is a particular solution of s_{24} when $2aa_2 + bb_2 = 0$ and $a_3 = 0$, again we do not need to consider it.

The solution s_{27} gives system (12).

The solution s_{28} is a particular solution of the solution s_{27} when $a_2 = 0$.

The solution s_{29} yields system (13).

The solutions s_{30}, s_{32} and s_{34} produce linear differential systems instead of quadratic ones.

The solutions s_{31} and s_{33} correspond to system (14) and (15), respectively.

The solutions s_{36} and s_{39} represent particular cases of the solution s_{24} when $a_3 = 0$ and $a_2 = 0$, respectively. Therefore, we will not consider them.

The solutions s_{35}, s_{38} and s_{41} are particular solutions of the solution s_{36}, the first when $a_2 = 0$, the second when $2aa_2 + bb_2 = 0$ and third when $a_2 = (2a_1c)/b$, $b_2 = -(a_1b + a_4d)/b$ and $a = b^2/(4c)$.

The solutions s_{35} and s_{37} are same solution.

The solution s_{40} is a specific case of s_{39} when $b_2 = 0$, and s_{42} is a specific case of s_{27} when $a_2 = 0$. Hence, we shall omit them from consideration.

The solutions s_{43} and s_{44} and give rise to systems (16) and (17), respectively.

In the solution s_{45} we simplify the value $(a_2^2bd/(a_2b^2 + 2a_4cd + 2bb_2c))^{1/2}(b(a_2b^2 + 2a_4cd + 2bb_2c)/c^2d)^{1/2}$ and it becomes $-a_2b/c$ that to get the simple form appears in system (16). The solution s_{46} is a particular solution of s_{45} when $a_3 = 0$.

The solution s_{47} is a particular solution of s_{44} when $a_4 = -bb_2/d$ and $c = b^2/(4a)$.

The solutions s_{48} and s_{49} are particular solutions of s_{47}, when $b_2 = 0$ and $b_2 = -da_4/b$, respectively. So we do not consider the solutions s_{47}, s_{48} and s_{49}.

Hence all the solutions together produce the eighteen different quadratic systems with invariants of the form (4). This completes the proof of the theorem. □

Proof of Corollary 2. All quadratic systems having an invariant of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ with $c_1 \neq 0$ have no finite singular points, because if (x_0, y_0) is a finite singular point then $I(x_0, y_0, t)$ would not be constant when t varies. □

3. Phase portrait of system (14)

System (14) depends on five parameters a_0, a_2, c, d, b_0 and b_2. Clearly $cdb_2 \neq 0$, otherwise the differential system (14) would not be a quadratic system. Here we study the phase portraits of system (14) with $a_0 \neq 0$ in the Poincaré disc, from Corollary 2 we know that they are chordal quadratic systems.

Doing the rescaling $(x, y) = (a_0X, b_2Y)$ system (14) becomes

$$\dot{X} = 1 + aY + bY^2, \qquad \dot{Y} = c + Y, \tag{19}$$

where

$$a = \frac{a_2}{a_0}, \quad b = -\frac{2cb_2}{da_0}, \quad c = \frac{b_0}{b_2}.$$

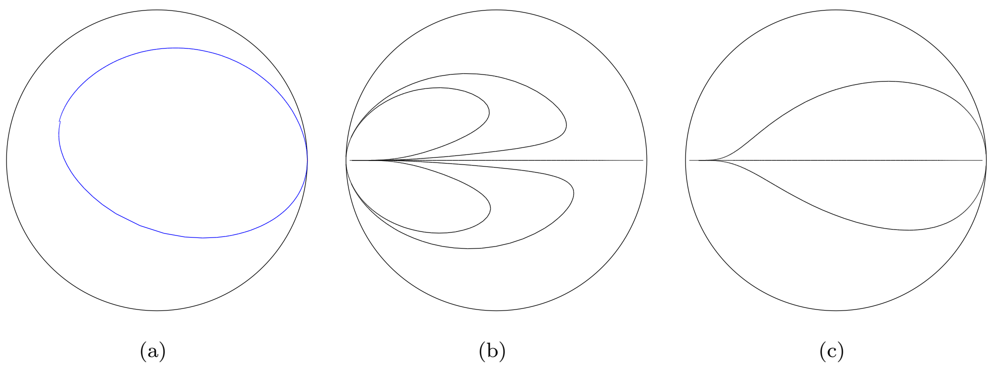

Figure 1. The three different topological phase portrait of system (19) in the Poincaré disc.

The unique singular points of the differential system (19) are the infinite singular points localized at the endpoints of the x-axis, the local phase portraits at these points can be studied doing the changes of variables called blow-ups (see [1]) and using the coordinates of the Poincaré disc (see Chapter 5 of [7]). Then we obtain three topologically non-equivalent phase portraits in the Poincaré disc for the differential system (19), the one of Figure 1(a) when $c = -1$ and $1+a+b = 0$, the one of Figure 1(b) when either $c > 1$ and $b < 0$, or $c = -1$, $1+a+b = 0$ and $a > 0$, or $c = -1$, $1 + a + b < 0$ and $b > 0$, and finally the one of Figure 1(c) otherwise.

4. Conclusion

In conclusion, this article successfully presents a classification of planar quadratic polynomial differential systems exhibiting invariants of the form $I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1t$ with $c_1 \neq 0$. The study identifies a total of 14 distinct families of quadratic systems exhibiting such kind of invariants. Notably, this research marks a pioneering effort, as it is the first instance of classifying quadratic differential systems with an invariant that deviates from a Darboux invariant. This achievement opens avenues for further exploration and understanding in the field of planar quadratic systems with diverse invariants.

5. Appendix

Deleted solutions. Here we give the sets of solutions that we have eliminated in the proof of Theorem 1.

$$
\begin{aligned}
s_2 = \; & \Big\{a_2 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = -\frac{d}{e}a_3,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\\
& a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\Big\},\\
s_3 = \; & \{b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ d = 0,\\
& c_1 = -eb_0\},\\
s_4 = \; & \Big\{a_2 = 0,\ a_4 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = -\frac{d}{e}a_3,\ b_4 = 0,\ b_5 = -\frac{d}{e}a_5,\\
& a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\Big\},\\
s_5 = \; & \Big\{a_3 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = -\frac{d}{e}a_2,\ b_3 = 0,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\\
& a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\Big\},\\
s_6 = \; & \Big\{a_2 = 0,\ a_3 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = 0,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\\
& a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0),\Big\}\\
s_7 = \; & \{a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\\
& d = 0,\ c_1 = -eb_0\},\\
s_8 = \; & \Big\{\ a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = -\frac{d}{e}a_5,\\
& a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\Big\},\\
s_9 = \; & \{b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ d = 0,\\
& c_1 = -eb_0\},\\
s_{10} = \; & \{\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ e = 0, c_1 = -da_0\},\\
s_{14} = \; & \{a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\\
& c = 0,\ d = 0,\ c_1 = -eb_0\},
\end{aligned}
$$

$$
\begin{aligned}
s_{15} = \; & \Big\{a_1=0,\ a_3=0,\ a_4=0,\ a_5=-\frac{2}{d}b_2c,\ b_0=-\frac{1}{2c}(da_2+eb_2),\\
& b_1=-\frac{d}{e}a_1,\ b_3=0,\ b_4=0,\ b_5=0,\ a=0,\ b=0,\ c_1=-(da_0+eb_0)\Big\},\\
s_{16} = \; & \Big\{a_3=0,\ a_4=\frac{2}{e}ca_1,\ a_5=0,\ b_0=-\frac{d}{2c}a_2,\ b_2=0,\ b_1=-\frac{d}{e}a_1,\ b_3=0,\\
& b_4=0,\ b_5=0,\ a=0,\ b=0,\ c_1=\frac{d}{2c}ea_2-da_0\Big\},\\
s_{17} = \; & \Big\{a_2=-\frac{e}{d}b_2,\ a_3=0,\ a_4=0,\ a_5=0,\ b_1=-\frac{d}{e}a_1,\ b_3=0,\ b_4=0,\\
& b_5=0,\ a=0,\ b=0,\ c=0,\ c_1=-(da_0+eb_0)\},\\
s_{18} = \; & \{a_3=0,\ a_4=0,\ b_1=0,\ b_2=0,\ b_3=0,\ b_4=0,\ b_5=0,\ a=0,\ b=0,\\
& c=0,\ d=0,\ c_1=-eb_0\},\\
s_{19} = \; & \{b_1=0,\ b_2=0,\ b_3=0,\ b_4=0,\ b_5=0,\ a=0,\ b=0,\ c=0,\ d=0,\\
& c_1=-eb_0\},\\
s_{23} = \; & \Big\{a_1=\frac{2}{b}aa_2,\ a_3=0,\ a_4=0,\ a_5=0,\ b_0=\frac{1}{b^3}(4a^2ea_2-2ab(ba_0+da_2)),\\
& b_1=-\frac{4}{b^2}a^2a_2,\ b_2=-\frac{2}{b}aa_2,\ b_3=0,\ b_4=0,\ c=\frac{b^2}{4a},\ b_5=0,\\
& c_1=\frac{1}{b^3}(2ae-bd)(b^2a_0-2aea_2)\Big\},\\
s_{25} = \; & \Big\{a_1=\frac{2}{b}aa_2,\ a_4=\frac{b}{2a}a_3,\ a_5=0,\ b_0=-\frac{2}{b^2}(ba_0+da_2)a,\\
& b_1=-\frac{1}{b^2}(4a^2a_2+bda_3),\ b_2=-\frac{2}{b}aa_2,\ b_3=-\frac{2}{b}aa_3,\ b_4=-a_3,\ b_5=0,\\
& c=\frac{b^2}{4a},\ e=0,\ c_1=-da_0,\Big\},\\
s_{26} = \; & \Big\{a_1=\frac{2}{b}aa_2,\ a_3=0,\ a_4=0,\ a_5=0,\ b_0=-\frac{2}{b^2}(ba_0+da_2)a,\\
& b_1=-\frac{4}{b^2}a^2a_2,\ b_2=-\frac{2}{b}aa_2,\ b_3=0,\ b_4=0,\ b_5=0,\ c=\frac{b^2}{4a},\ e=0, c_1=-da_0\Big\},\\
s_{28} = \; & \Big\{a_1=0,\ a_2=0,\ a_5=\frac{2}{b^2}(ba_4-2ca_3)c,\ b_0=-\frac{b}{2c}a_0,\ b_1=-\frac{d}{b}a_3,\\
& b_2=-\frac{d}{b^2}(ba_4-2ca_3),\ b_3=-\frac{b}{2c}a_3,\ b_4=-\frac{b}{2c}a_4,\ b_5=\frac{2}{b}ca_3-a_4,\\
& a=\frac{b^2}{4c},\ e=0,\ c_1=-da_0\Big\},\\
s_{30} = \; & \{a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,\ a_5=0,\ b_3=0,\ b_4=0,\ b_5=0,\\
& a=0,\ b=0,\ c=0,\ e=0,\ c_1=-da_0\},\\
s_{32} = \; & \Big\{a_1=0,\ a_3=0,\ a_4=0,\ a_5=0,\ b_0=-\frac{d}{2c}a_2,\ b_1=0,\ b_2=0,\ b_3=0,\\
& b_4=0,\ b_5=0,\ a=0,\ b=0,\ c_1=da_0-\frac{d}{2c}ea_2,\ \Big\},\\
s_{34} = \; & \Big\{a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,\ a_5=0,\ b_0=-\frac{2}{b}aa_0,\ b_1=0,\ b_2=0,\\
& b_3=0,\ b_4=0,\ b_5=0,\ c=\frac{b^2}{4a},\ c_1=a_0(\frac{2}{b}ae-d)\Big\},\\
s_{35} = \; & \Big\{a_1=0,\ a_2=0,\ a_3=0,\ a_4=-\frac{b}{d}b_2,\ a_5=-\frac{b^2}{2ad}b_2,\ b_0=-\frac{2}{b}aa_0,\\
& b_1=0,\ b_3=0,\ b_4=\frac{2}{d}ab_2,\ b_5=\frac{b}{d}b_2,\ c=\frac{b^2}{4a},\ e=0,\ c_1=-da_0\Big\},\\
s_{37} = \; & \Big\{\ a_1=0,\ a_2=0,\ a_3=0,\ a_4=-\frac{b}{d}b_2,\ a_5=-\frac{b^2}{2ad}b_2,\ b_0=-\frac{2}{b}aa_0,\\
& b_1=0,\ b_3=0,\ b_4=\frac{2}{d}ab_2,\ b_5=\frac{b}{d}b_2,\ c=\frac{b^2}{4a},\ e=0,\ c_1=-da_0\Big\},
\end{aligned}
$$

$$s_{38} = \Big\{a_1 = \frac{2}{b}aa_2,\ a_4 = 0,\ a_3 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b^2}a(ba_0 + da_2),$$
$$b_1 = -\frac{4}{b^2}a^2a_2,\ b_2 = -\frac{2}{b}aa_2,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a}, e = 0,\ c_1 = -da_0\Big\},$$
$$s_{40} = \Big\{a_1 = 0,\ a_2 = 0,\ a_4 = \frac{b}{2a}a_3,\ a_5 = 0,\ b_0 = -\frac{2}{b}aa_0,\ b_2 = 0,$$
$$b_1 = -\frac{d}{b}a_3,\ b_3 = -\frac{2}{b}aa_3,\ b_4 = -a_3,\ b_5 = 0,\ c = \frac{b^2}{4a},\ e = 0, c_1 = -da_0\Big\},$$
$$s_{41} = \Big\{\ a_2 = \frac{2}{b}ca_1,\ a_3 = 0,\ a_5 = \frac{2}{b}ca_4,\ b_0 = -\frac{1}{2cb}(b^2a_0 + 2cda_1),\ b_1 = -\frac{b}{2c}a_1,$$
$$b_2 = -\frac{1}{b}(ba_1 + da_4),\ b_3 = 0,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = -a_4,\ a = \frac{b^2}{4c},\ e = 0,$$
$$c_1 = -da_0\},$$
$$s_{46} = \Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b^2}bca_4,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0,\ b_2 = -\frac{d}{b}a_4,$$
$$b_3 = 0,\ b_4 = -\frac{2}{b^2}aba_4,\ b_5 = -a_4,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\},$$
$$s_{47} = \Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{b}{d}b_2,\ a_5 = -\frac{b^2}{2ad}b_2,\ b_0 = -\frac{2}{b}aa_0,$$
$$b_1 = 0,\ b_3 = 0,\ b_4 = \frac{2}{d}ab_2,\ b_5 = \frac{b}{d}b_2,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0\Big\},$$
$$s_{48} = \Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0,$$
$$b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0\Big\},$$
$$s_{49} = \Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b}ca_4,\ b_0 = -\frac{b}{2c}a_0,\ b_1 = 0,\ b_2 = -\frac{d}{b}a_4,$$
$$b_3 = 0,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = -a_4,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\}.$$

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References

[1] M.J. Álvarez, A. Ferragut and J. Jarque, *A survey on the blow up technique*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **21** (2011), 3103–3118.

[2] M. Alberich-Carramiñana, A. Ferragut, and J. Llibre, *Quadratic planar differential systems with algebraic limit cycles via quadratic plane cremona maps*, Advances in Mathematics, **389** (2021), 107924.

[3] J.C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems. A global classification on the quadratic case*, Birkhäuser, 2021.

[4] Y. Bolaños, J. Llibre and C. Valls, *Phase portraits of quadratic Lotka–Volterra systems with a Darboux invariant in the Poincaré disc*, Comm. in Contemporary Math. **16** (2014), 1350041, 23 pp.

[5] D. Boularas, *A new classification of planar homogeneous quadratic systems*, Qualitative Theory of Dynamical Systems, **2** (2001), 93–110.

[6] J. Chavarriga, B. García, J. Llibre, J. S. P. del Rio, and J. A. Rodríguez, *Polynomial first integrals of quadratic vector fields*, Journal of Differential Equations, **230(2)** (2006) 393–421.

[7] F. Dumortier, J. Llibre and J.C. Artés,, *Qualitative Theory of Planar Differential Systems*, UniversiText, Springer–Verlag, New York, 2006.

[8] A. Gasull, L.R. Sheng and J. Llibre, *Chordal quadratic systems*, Rocky Mountain J. Math. **16** (1986), no. 4, 751–782.

[9] Y.A. Kuznetzov, *Elements of applied bifurcation theory*, second editions, Appl. Math. Sci. **112**, Springer, New York, 1998.

[10] J. Llibre, M. Messias and A.C. Reinol, *Darboux invariants for planar polynomial differential systems having an invariant conic*, Zeitschrift fuer Angewandte Mathematik und Physik, **65** (2014), 1127–1136.

[11] J. Llibre, M. Messias and A.C. Reinol, *Normal forms for polynomial differential systems in* $\mathbb{R}^3$ *having an invariant quadric and a Darboux invariant*, Int. J. Bifurcation and Chaos **25** (2015), 1550015, 16 pp.

[12] J. Llibre and R.D.S. Oliveira, *Quadratic systems with invariant straight lines of total multiplicity two having Darboux invariants*, Comm. in Contemporary Math. **17** (2015), 1450018, 17 pp.

[13] J. Llibre and R. Oliveira, *Quadratic systems with an invariant conic having Darboux invariants*, Commun. Contemp. Math. Vol. 20, No. 4 (2018) 1750033 (15 pages).

[14] J. Reyn, *Phase portraits of planar quadratic systems*, Mathematics and Its Applications **583**, Springer, New York, 2007.

[15] Y. Yanqian et al., *Theory of Limit Cycles, Translations of Mathematics Monographs* , Vol. 66 (American Mathematics Society, Providence, RI, 1986).

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