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QUALITATIVE STUDY OF THE SELKOV MODEL

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Abstract. The Selkov oscillator was formulated in 1968 and now it is a classical model for studying the glycolysis. It is a differential system of two equations depending on two parameters in dimensionless variables. When the two equations are polynomials we prove that the Selkov system is not Liouvillian integrable. Additionally, we prove that the polynomial Selkov system for any integer $n \geq 1$ has nine distinct global phase portraits in the Poincaré disk.

1. Introduction and statement of the main results

The living organisms obtain energy from sugar using a process called glycolysis. Experimental observations detected that when the input rate of sugar is constant then the subproducts of the glycolysis oscillate in time. Based in these observations Higgins [13] in 1964 provided a mathematical model in order to understand better this phenomenon. Higgins' model was improved in 1968 by Selkov [18]. Thus the Selkov model is given by the differential system of two equations

$$\begin{aligned} \dot{x} &= 1 - xy^{\gamma}, \\ \dot{y} &= ay(-1 + xy^{\gamma-1}), \end{aligned}$$

depending on two parameters $a > 0$ and γ in dimensionless variables. In order to avoid technical complications with differentiability we take $\gamma = n$ a positive integer. The variables x and y are the dimensionless concentrations of ATP (adenosine triphosphate) and ADP (adenosine diphosphate), respectively, while the dot represents the derivative with respect to a dimensionless time variable.

From now on we will work with the system

$$\begin{aligned} \dot{x} &= 1 - xy^{n}, \\ \dot{y} &= ay(-1 + xy^{n-1}), \end{aligned} \tag{1}$$

where n is a positive integer and a is a positive real number.

In his seminal paper Selkov proved that his model presents a Hopf bifurcation, showing the existence of periodic motion which allowed to explain the oscillations observed experimentally. In 2010 d'Onofrio [10] studied the stability and uniqueness of these periodic orbits.

Since some of the solutions of the Selkov model are unbounded in order to understand them it is necessary to study the neighborhood of the infinity. In 2018 Brechmann and Rendall [3] used the technique of the Poincaré compactification for studying those unbounded solutions. Additionally these authors also shown that if the unique equilibrium point of the Selkov model is stable, then any bounded solution converges to it in forward time. If this equilibrium is unstable and exists a periodic orbit then this periodic orbit is unique and all bounded solutions different from the equilibrium converge to the periodic solution in forward time. If the equilibrium is unstable and does not exist a periodic orbit then all solutions distinct to the equilibrium are unbounded.

Selkov in [18] claimed that his system admits solutions which oscillate with an amplitude which grows without limit in forward time. These solutions are called solutions with unbounded oscillations. In 2020 Brechmann and Rendall [4] proved the existence of the solutions with unbounded oscillations. In [5] the authors prove the existence and uniqueness of a limit cycle trasforming system (1) into Liénard system.

We associate to the differential system (1) the vector field

$$X = (1 - xy^n)\frac{\partial}{\partial x} + ay(-1 + xy^{n-1})\frac{\partial}{\partial y}. \tag{2}$$

The two main objectives of this paper on the Selkov model (1) are essentially mathematical, but of course they have biological implications. The first one is to decide if system (1) is Liouvillian integrable for some values of its parameters, the answer is negative, see Theorem 1. The second objective is to classify the topological phase portraits of system (1) in the Poincaré disc, see Theorem 2. This second objective was solved in the particular cases $n = 2$ by Artés et al [2] and Chen and Tang [6], and for $n = 3, 4, 5, 6$ see [14].

Theorem 1. *System* (1) *for* $a > 0$ *is not Liouvillian integrable.*

Theorem 1 is proved in Section 2.

Theorem 1 says that the differential system (1) has neither a first integral nor an integrating factor given by a Darboux function, see for more details the Appendix.

In order to understand the behaviour of system (1) we need to draw the global phase portraits in the Poincaré disc. Roughly speaking, the *Poincaré disc* is the closed unit disc centered at the origin of $\mathbb{R}^2$, its interior is identified with the whole plane $\mathbb{R}^2$ and the circle of its boundary is identified with the infinity of $\mathbb{R}^2$. In the plane we can go to infinity in as many directions as points has the circle. There is a unique analytic way to extend a polynomial differential system defined in $\mathbb{R}^2$ to the Poincaré disc. Working with this extended system defined in the Poincaré disc, we can study how the orbits of the polynomial differential system goes or come from infinity. For more details on the so called *Poincaré compactification* see for instance chapter 5 of [11], where there are also the expressions of the local charts for studying the extended system in the Poincaré disc.

We recall that our second objective is to present the topological classification of all global phase portraits of system (1) for all $n \geq 1$. Thus, we follow the works of Markus, Neumann and Peixoto, see [15, 16, 17] and the notion of separatrix configuration that appears there. In what follows we denote by S the number of separatrices and by R the number of the canonical regions. If there not exist a homeomorphism to bring the separatrix configuration of one phase portrait to the separatrix configuration of the other, we say that the two global phase portrait are *not topological equivalent.* Next theorem classify the topological phase portraits of system (1).

Theorem 2. *For the Selkov system* (1) *with* $a > 0$ *and* $n \geq 1$*, there are exactly nine non–topological equivalent phase portraits in the Poincaré disc:*

(a) *For* $n = 1$ *see Figure* 1(*e*) *with* $S = 15$ *and* $R = 2$*.*
(b) *For* $n \geq 2$ *and* n *odd (resp. even)*
  (i) *Figure* 1(*a*) *(resp.* 2(*a*)*) with* $a \in (0,\ 1/(n-1))$ *and* $S = 17$ *and* $R = 4$*.*
  (ii) *Figure* 1(*b*)*(resp.* 2(*b*)*) with* $a \in (1/(n-1),\ a^*)$ *and* a^* *is a unique constant in the interval* $\left(\frac{1}{n-1},\ \frac{2^n-1}{2^n-2}\right)$*. In this case,* $S = 18$ *and* $R = 5$*.*
  (iii) *Figure* 1(*c*) *(resp.* 2(*c*)*) with* $a = a^*$ *and* $S = 16$ *and* $R = 4$*.*
  (iv) *Figure* 1(*d*) *(resp.* 2(*d*)*) with* $a > a^*$ *and* $S = 17$ *and* $R = 4$*.*

Theorem 2 is proved in Section 3.

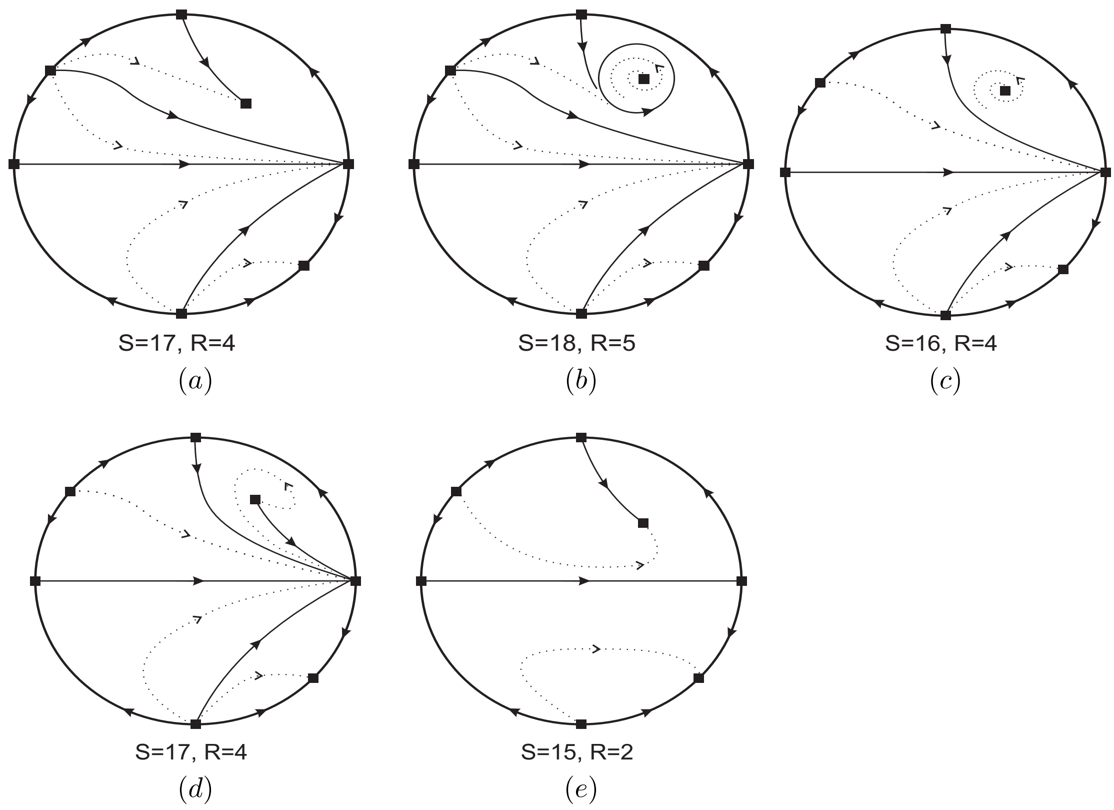

FIGURE 1. The global phase portraits of systems (1) for n odd, $n \geq 1$.

2. Proof of Theorem 1

In the proof of Theorem 1 we shall use Lemmas 3 and 4.

Lemma 3. *The unique irreducible invariant algebraic curve of system* (1) *is* $y = 0$.

Proof. From (1) we have that $\dot{y}|_{y=0} = 0$, therefore the straight line $y = 0$ is invariant under the flow of system (1). Now we must prove that there is no other irreducible invariant algebraic curve.

Consider an irreducible polynomial $F(x, y)$ distinct of the polynomial y. We write $F(x, y) = F_0(x) + F_1(x)y + F_2(x)y^2 + \cdots + F_k(x)y^k$ where F_i is a polynomial in the variable x for $i = 0, \cdots, k$ with $F_k(x) \neq 0$. Note that $F_0(x) \neq 0$, otherwise F is not irreducible because y would be a factor of F. We assume that $F = 0$ is an invariant algebraic curve of system (1) with cofactor $K = K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n$ with K_i polynomials in the variable x for $i = 0, \cdots, n-1$. Since system (1) is of degree $n+1$ we have that the degree of the cofactor K is at most n and consequently k_n must be a constant.

Since $F = 0$ is an invariant algebraic curve of system (1) we have that

$$(1 - xy^n)F_0'(x) = (K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n)F_0(x), \tag{3}$$

if $k = 0$; and

(4)
$$(1 - xy^n)(F_0'(x) + F_1'(x)y + \cdots + F_k'(x)y^k) - a(y - xy^n)(F_1(x) + 2F_2(x)y + \cdots + kF_k(x)y^{k-1})$$
$$= (K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n)(F_0(x) + F_1(x)y + \cdots + F_k(x)y^k).$$

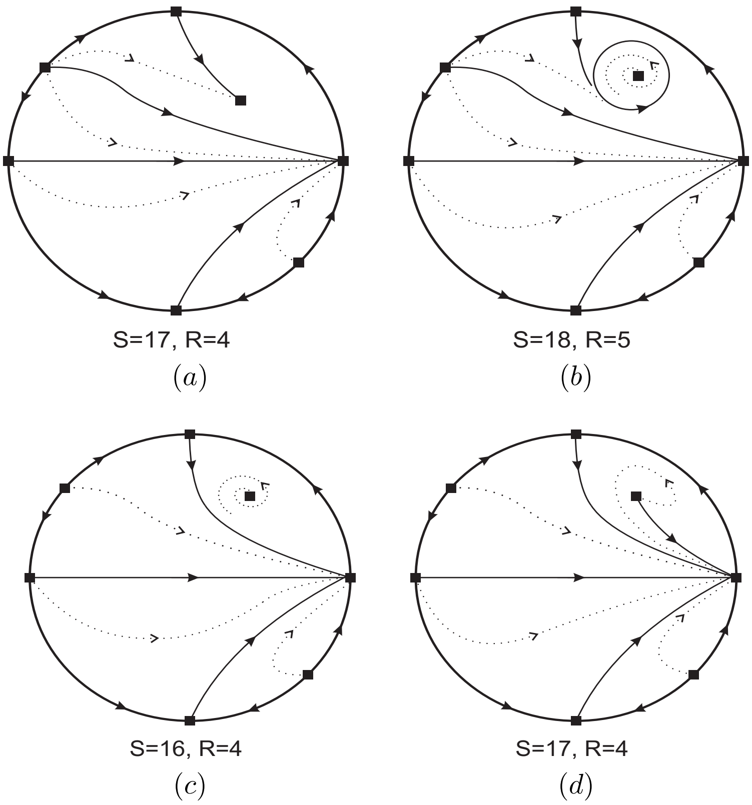

FIGURE 2. The global phase portraits of systems (1) for n even, $n \geq 2$.

if $k \geq 1$.

We consider the coefficients of y^0 in (3) and (4) and we obtain $F_0'(x) = K_0(x)F_0(x)$, and so $K_0(x) = 0$ and $F_0(x) = f_0 \in \mathbb{R}$ with $f_0 \neq 0$. Therefore, $F(x,y) = f_0 + yH(x,y)$ and since $K_0(x) = 0$ the cofactor K is divisible by y

In summary if $k = 0$ then $F(x,y) = F_0(x) = f_0 \neq 0$, in contradiction that $F(x,y) = 0$ is an invariant algebraic curve. Hence $k \geq 1$.

We separate the rest of the proof in two cases.

Case 1: $n = 1$. Then from (4) we have

$$\begin{aligned}(1 - xy)(F_1'(x)y + \cdots + F_k'(x)y^k) - a(y - xy)(F_1(x) + 2F_2(x)y + \cdots + kF_k(x)y^{k-1}) \\ = k_1 y(f_0 + F_1(x)y + \cdots + F_k(x)y^k).\end{aligned} \tag{5}$$

The coefficient of y in (5) is

$$F_1'(x) - a(1-x)F_1(x) = k_1 f_0.$$

The solution of this linear differential equation is

$$F_1(x) = C_1 e^{ax - \frac{ax^2}{2}} + \sqrt{\frac{\pi}{2a}} k_1 f_0 e^{-\frac{ax^2}{2} + ax - \frac{a}{2}} \operatorname{erfi}\left(\frac{\sqrt{a}(x-1)}{\sqrt{2}}\right).$$

where C_1 is a constant and erfi is the imaginary error function, see [20]. Since $F_1(x)$ must be a polynomial, we have that the $C_1 = k_1 = 0$ and $F_1(x) = 0$. Then (5) becomes

$$(1 - xy)(F_2'(x)y^2 + \cdots + F_k'(x)y^k) - a(y - xy)(2F_2(x)y + \cdots + kF_k(x)y^{k-1}) = 0. \tag{6}$$

Now the coefficient of y^2 in (6) is

$$F_2'(x) - 2a(1 - x)F_2(x) = 0.$$

Therefore $F_2(x) = C_2 e^{ax(2-x)}$ where C_2 is a constant. Since $F_2(x)$ must be a polynomial, we have that the $C_2 = 0$ and $F_2(x) = 0$. Repeating this process we obtain that $F_j(x) = 0$ for $j = 3, \ldots, k$, a contradiction with the assumption that $F_k(x) \neq 0$. So the lemma is proved if $n = 1$.

Case 2: $n > 1$. Now we write the irreducicle invariant algebraic curve $F(x, y) = 0$ of system (1) as $F(x, y) = G_0(y) + G_1(y)x + G_2(y)x^2 + \cdots + G_\ell(y)x^\ell$, where G_i is a polynomial in the variable y for $i = 0, \ldots, \ell$ with $G_\ell(y) \neq 0$. Since F is irreducible $G_0(y) \neq 0$. And we write its cofactor as $K(x, y) = k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1} + k_n x^n$ with k_i a polynomial in the variable y for $i = 0, \cdots, n$. Since the degree of the cofactor K is at most n we have that k_n is a constant. But since the cofactor K is divisible by y it follows that $k_n = 0$ and y divides $k_i(y)$ for $i = 0, 1, \ldots, k - 1$. Moreover, since $F(x, y) = f_0 + yH(x, y)$ we have that y divides $G_j(y)$ for $j = 1, \ldots, \ell$ and $G_0(y) = f_0 + g(y)$ with $f_0 \neq 0$.

Since $F = 0$ is an invariant algebraic curve of system (1) we have that

$$-a(y - xy^n)G_0'(y) = (k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1})G_0(y), \tag{7}$$

if $\ell = 0$; and

$$\begin{aligned}(1 - xy^n)(G_1(y) + 2G_2(y)x + \cdots + \ell G_\ell(y)x^{\ell-1}) - a(y - xy^n)(G_0'(y) + G_1'(y)x + \cdots + G_\ell'(y)x^\ell) \\ = (k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1})(G_0(y) + G_1(y)x + \cdots + G_\ell(y)x^\ell),\end{aligned} \tag{8}$$

if $\ell \geq 1$.

Assume $\ell = 0$, then from (7) we get

$$-a(y - xy^n)G_0'(y) = (k_0(y) + k_1(y)x)G_0(y), \tag{9}$$

or equivalently

$$-ayG_0'(y) = k_0(y)G_0(y), \qquad ay^n G_0'(y) = k_1(y)G_0(y). \tag{10}$$

Dividing the second of these equations by the first one we have that $k_1(y) = -k_0(y)y^{n-1}$. Solving (10) we obtain

$$G_0(y) = C_0 e^{-\int \frac{k_0(y)}{ay}\, dy},$$

where C_0 is a constant. Since $G_0(y) \neq 0$ and it must be a polynomial we have that $k_0(y) = k_0$ must be a constant. Then $G_0(y) = C_0 y^{-k_0/a}$, and since $G_0(y) = f_0 + g(y)$ with $f_0 \neq 0$ we have that $k_0 = 0$ and so $G_0 = C_0 = f_0$ and $g(y) = 0$. But then $F(x, y) = G_0 = f_0$, a contradiction, the invariant curve cannot be a constant number.

From now on we assume that $\ell \geq 1$.

The coefficient of $x^{n+\ell-1}$ in the polynomial (8) must satisfy

$$k_{n-1}(y)G_\ell(y) = 0 \quad \text{if } n + \ell - 1 > 1 + \ell, \tag{11}$$

or

$$k_{n-1}(y)G_\ell(y) = ay^n G_\ell'(y) \quad \text{if } n + \ell - 1 = 1 + \ell. \tag{12}$$

If (12) holds, then $n=2$ and

$$G_\ell(y) = C_\ell e^{\int \frac{k_1(y)}{ay^2}}\, dy,$$

where C_ℓ is a constant. Since $G_\ell(y) \neq 0$ must be a polynomial, we get that $k_1(y) = ky$ and so $G_\ell(y) = C_\ell y^{k/a}$ with $k/a \geq 1$ a non-negative integer and $C_\ell \neq 0$, because y divides $G_\ell(y)$. Therefore from (8) we obtain

$$\begin{aligned}
&(1-xy^2)(G_1(y) + 2G_2(y)x + \cdots + \ell C_\ell y^{k/a} x^{\ell-1}) \\
&\quad -a(y-xy^2)(G_0'(y) + G_1'(y)x + \cdots + G_{\ell-1}'(y)x^{\ell-1} + G_\ell'(y)x^\ell) \\
&= (k_0(y) + kyx)(G_0(y) + G_1(y)x + \cdots + C_\ell y^{k/a} x^\ell).
\end{aligned} \tag{13}$$

Here $k_0(y) = k_{01}y + k_{02}y^2$, because $n=2$ and the cofactor must be divisible by y.

The coefficient of x^0 of the polynomial (13) is

$$G_1(y) - ayG_o'(y) = (k_{01}y + k_{02}y^2)G_0(y).$$

So

$$G_0(y) = e^{-\frac{1}{a}(k_{01}y + \frac{1}{2}k_{02}y^2)} \left(C_0 + \frac{1}{a}\int y^{-1} G_1(y) e^{\frac{1}{a}(k_{01}y+\frac{1}{2}k_{02}y^2)} dy \right).$$

where C_0 is a constant. Since $G_0(y)$ and $G_1(y)$ are polynomials we have that $k_{01} = k_{02} = 0$. Therefore

$$G_0(y) = C_0 + \frac{1}{a}\int y^{-1} G_1(y) dy.$$

Consequently $C_0 = f_0 \neq 0$ and we know that $G_1(y)$ is divisible by y.

The coefficient of x^ℓ of the polynomial (13) is $-\ell C_\ell y^{k/a+2} - kC_\ell y^{k/a} + ay^2 G_{\ell-1}'(y) = kyG_{\ell-1}(y)$. Therefore we obtain

$$G_{\ell-1}(y) = \frac{C_\ell}{a}\left(\ell y - \frac{k}{y}\right) y^{\frac{k}{a}} + C_{\ell-1} y^{\frac{k}{a}}.$$

Hence if $\ell > 1$ the integer $k/a > 1$ because y divides $G_{\ell-1}(y)$, and if $\ell = 1$ then $k/a = 1$ because $G_0(y) = f_0 + yg(y)$ and consequently $f_0 = -kC_\ell/a = -C_\ell = -C_1$.

Assume $\ell > 1$. Denoting the integer $k/a = m > 1$. Then (13) becomes

$$\begin{aligned}
&(1-xy^2)\left(G_1(y) + 2G_2(y)x + \cdots + (\ell-1)\left(C_\ell\left(\frac{\ell}{a}y^{m+1} - my^{m-1}\right) + C_{\ell-1}y^m\right)x^{\ell-2} + \ell C_\ell y^m x^{\ell-1}\right) \\
&-a(y-xy^2)\left(G_0'(y) + G_1'(y)x + \cdots + \left(C_\ell\left(\frac{\ell(m+1)}{a}y^m - m(m-1)y^{m-2}\right) + mC_{\ell-1}y^{m-1}\right)x^{\ell-1} + mC_\ell y^{m-1}x^\ell\right) \\
&= mayx\left(G_0(y) + G_1(y)x + \cdots + \left(C_\ell\left(\frac{\ell}{a}y^{m+1} - my^{m-1}\right) + C_{\ell-1}y^m\right)x^{\ell-1} + C_\ell y^m x^\ell\right).
\end{aligned} \tag{14}$$

The coefficient of $x^{\ell-1}$ in (14) is

$$\begin{aligned}
&\ell C_\ell y^m - (\ell-1)\left(C_\ell\left(\frac{\ell}{a}y^{m+3} - my^{m+1}\right) + C_{\ell-1}y^{m+2}\right) \\
&\quad -a\left(C_\ell\left(\frac{\ell(m+1)}{a}y^{m+1} - m(m-1)y^{m-1}\right) + mC_{\ell-1}y^m\right) + ay^2 G_{\ell-2}'(y) = mayG_{\ell-2}(y).
\end{aligned}$$

So
$$G_{\ell-2}(y) = \quad C_{\ell-2}y^m + \frac{y^{m-2}}{2a^2}\Big((\ell-1)\ell C_\ell y^4 + a^2 m(C_\ell(m-1) - 2C_{\ell-1}y)$$
$$+2ay(\ell C_\ell + (\ell-1)C_{\ell-1}y^2) + 2aC_\ell(\ell+m)y^2\log y\Big).$$
Since the coefficient of $\log y$ cannot be zero, and $G_{\ell-2}(y)$ must be a polynomial we have a contradiction.

Assume $\ell = 1$. Then recall that $k/a = 1$ and so $F(x,y) = G_0(y) + C_1yx$ and equality (13) becomes
$$C_1y(1-xy^2) - a(y-xy^2)(G_0'(y) + C_1x) = ayx(G_0(y) + C_1yx).$$
Considering the terms without x and we have that $C_1y - ayG_0'(y) = 0$ and so $G_0(y) = (C_1/a)y + C$ with C a constant. Now the terms of x gives $-C_1y^3 - aC_1y + ay^2G_0'(y) = ayG_0(y)$ and substituting the expression of G_0 we obtain $C_1 = 0$, a contradiction. Hence the lemma is proved for $n = 2$.

Now we can assume that (11) holds. Therefore $k_{n-1}(y) = 0$ because $G_\ell(y) \neq 0$. Then the coefficient of of $x^{n+\ell-2}$ in the polynomial (8) must satisfy
$$k_{n-2}(y)G_\ell(y) = 0 \quad \text{if } n+\ell-2 > 1+\ell, \tag{15}$$
or
$$k_{n-2}(y)G_\ell(y) = ay^nG_\ell'(y) \quad \text{if } n+\ell-2 = 1+\ell. \tag{16}$$

If (16) holds, then $n = 3$ and
$$G_\ell(y) = C_\ell e^{\int \frac{k_1(y)}{ay^3}}\, dy,$$
where C_ℓ is a constant. Since $G_\ell(y) \neq 0$ must be a polynomial, we get that $G_\ell(y) = C_\ell \neq 0$ and $k_1(y) = ky^2$. Therefore $G_\ell(y) = C_\ell y^{k/a}$ with $k/a \geq 1$ a non-negative integer and $C_\ell \neq 0$, because y divides $G_\ell(y)$. Therefore from (8) we obtain
$$\begin{aligned}
&(1-xy^3)(G_1(y) + 2G_2(y)x + \cdots + \ell C_\ell y^{k/a}x^{\ell-1})\\
&\quad -a(y-xy^3)\left(G_0'(y) + G_1'(y)x + \cdots + G_{\ell-1}'(y)x^{\ell-1} + \frac{k}{a}C_\ell y^{k/a-1}x^\ell\right)\\
&= (k_0(y) + ky^2x)(G_0(y) + G_1(y)x + \cdots + C_\ell y^{k/a}x^\ell).
\end{aligned} \tag{17}$$
Here $k_0(y) = k_{01}y + k_{02}y^2 + k_{03}y^3$, because $n = 3$ and the cofactor must be divisible by y.

The coefficient of x^0 of the polynomial (17) is
$$G_1(y) - ayG_o'(y) = (k_{01}y + k_{02}y^2 + k_{03}y^3)G_0(y).$$
So
$$G_0(y) = e^{-\frac{1}{a}\left(k_{01}y + \frac{1}{2}k_{02}y^2 + \frac{1}{3}k_{03}y^3\right)}\left(C_0 + \frac{1}{a}\int y^{-1}G_1(y)e^{\left(k_{01}y + \frac{1}{2}k_{02}y^2 + \frac{1}{3}k_{03}y^3\right)}dy\right).$$
where C_0 is a constant. Since $G_0(y)$ and $G_1(y)$ are polynomials we have that $k_{01} = k_{02} = k_{03} = 0$. Therefore
$$G_0(y) = C_0 + \frac{1}{a}\int y^{-1}G_1(y)dy.$$
Consequently $C_0 = f_0 \neq 0$ and we know that $G_1(y)$ is divisible by y.

The coefficient of x^ℓ in the polynomial (17) is $-\ell C_\ell y^{k/a+3} - kC_\ell y^{k/a} + ay^3G_{\ell-1}'(y) = ky^2G_{\ell-1}(y)$. Therefore we obtain
$$G_{\ell-1}(y) = \frac{C_\ell}{a}\left(\ell y - \frac{k}{2y^2}\right)y^{\frac{k}{a}} + C_{\ell-1}y^{\frac{k}{a}}$$

Hence if $\ell > 1$ the integer $k/a > 2$ because y divides $G_{\ell-1}(y)$, and if $\ell = 1$ then $k/a = 2$ because $G_0(y) = f_0 + yg(y)$ and consequently $f_0 = -kC_1/(2a) = -C_1$.

Assume $\ell > 1$. The coefficient of $x^{\ell-1}$ in the polynomial (17) is

$$\ell C_\ell y^{k/a} - (\ell - 1)y^3 G_{\ell-1}(y) - ayG'_{\ell-1}(y) + ay^3 G'_{\ell-2}(y) = ky^2 G_{\ell-2}(y).$$

Solving this equation with respect to $G_{\ell-2}(y)$ we obtain

$$\begin{aligned} G_{\ell-2}(y) = \; & \frac{y^{k/a-4}}{8a^2}\Big((-C_\ell(2a-k)k + 4a(\ell C_\ell - kC_{\ell-1})y^2 - 4C_\ell(2a\ell + k + \ell k)y^3 \\ & \qquad +8a^2 C_{\ell-2}y^4 + 8aC_{\ell-1}(-1+\ell)y^5 + 4C_\ell(-1+\ell)\ell y^6)\Big). \end{aligned}$$

Since $k > 0$ then $C_\ell(k + k^2) \neq 0$, and consequently the integer $k/a \geq 4$ because $G_{\ell-2}(y)$ must be a polynomial. We note that $\ell \neq 2$, otherwise we have a contradiction with $G_0(y) = f_0 + yg(y)$. Hence $\ell > 2$.

The coefficient of $x^{\ell-2}$ in the polynomial (17) is

$$(\ell - 1)G_{\ell-1}(y) - (\ell - 2)y^3 G_{\ell-2}(y) - ayG'_{\ell-2}(y) + ay^3 G'_{\ell-3}(y) = ky^2 G_{\ell-3}(y).$$

Solving this differential equation with respect to $G_{\ell-3}(y)$ we obtain

$$G_{\ell-3}(y) = \frac{y^{k/a-6}}{48a^3}\Big(\text{a polynomial in the variable } y\Big) + \frac{1}{a^3}C_\ell(a\ell + k)y^{k/a}\log y.$$

Since $a\ell + k > 0$ we get a contradiction with the fact that $G_{\ell-3}(y)$ is a polynomial.

Assume $\ell = 1$. Then recall that $k/a = 2$ and so $F(x, y) = G_0(y) + C_1 y^2 x$ and equality (17) becomes

$$C_1 y^2(1 - xy^3) - a(y - xy^3)(G'_0(y) + 2C_1 yx) = 2ay^2 x(G_0(y) + C_1 y^2 x).$$

Taking the coefficient withought x we have $C_1 y^2 - ayG'_0(y) = 0$ and so $G_0(y) = (C_1/(2a))y^2 + C_0$ with C_0 a constant. From the coefficients of x we have

$$-C_1 y^5 + ay^3 G'_0(y) - 2aC_1 y^2 = 2ay^2 G_0(y),$$

and substituting the expression of G_0 we have that $C_1 = 0$, a contradiction. Hence the lemma is proved for $n = 3$.

We have $k_{n-1}(y) = 0$ and we can assume that $n > 3$ and that (15) holds. Therefore $k_{n-2}(y) = 0$. Now the coefficient of of $x^{n+\ell-3}$ in the polynomial (8) must satisfy

$$k_{n-3}(y)G_\ell(y) = 0 \quad \text{if } n + \ell - 3 > 1 + \ell, \tag{18}$$

or

$$k_{n-3}(y)G_\ell(y) = ay^n G'_\ell(y) \quad \text{if } n + \ell - 3 = 1 + \ell. \tag{19}$$

In a similar way as in the previous cases we get a contradiction. Repeating this process until the iteration $r = n - 1$ we shall arrive to the case $n + \ell - r = 1 + \ell$ with

$$k_{n-r}(y)G_\ell(y) = ay^n G'_\ell(y).$$

and we get again a contradiction. So the lemma is proved. $\square$

Lemma 4. *The invariant line at infinity has multiplicity $n + 1$. The only exponential factors of system* (1) *are $G_i = \exp\big((ax + y)^i\big)$ for $i = 1, \cdots, n$.*

Proof. Applying the definition of the exponential factor (see Appendix) we can see directly that $G_i = \exp\left((ax+y)^i\right)$ are exponential factors for the vector field (2) with cofactors $L_i = -ia(y-1)(ax+y)^{i-1}$ for $i = 1, \cdots, n$.

We claim that the algebraic multiplicity of the line at infinity for the vector field (2) is $n+1$. Consider the expression of the vector field (2) in the chart (U_1, F_1) (see for details chapter 5 of [11])

$$Y = \left(-{z_2}^{n+1} z_1 + {z_1}^{n+1} - {z_2}^{n} a z_1 + a {z_1}^{n}\right) \frac{\partial}{\partial z_1} + z_2 \left({z_1}^{n} - {z_2}^{n+1}\right) \frac{\partial}{\partial z_2}.$$

We set $v_1 = 1$, $v_2 = z_1$ and $v_3 = z_2$. Then the extactic curve $\mathcal{E}_1$ of Y is

$$\begin{aligned} \mathcal{E}_1 &= \begin{vmatrix} v_1 & v_2 & v_3 \\ Y(v_1) & Y(v_2) & Y(v_3) \\ Y(Y(v_1)) & Y(Y(v_2)) & Y(Y(v_3)) \end{vmatrix} \\ &= a z_2^{n+1} \left[z_2\, {z_1}^{2\,n-1} n a + {z_1}^{2\,n+1} + \left(a\,(n-1) - z_2\right) {z_2}^{n} {z_1}^{n+1} \right. \\ &\quad \left. + \left((-n+1)\,a - z_2\right) {z_1}^{2\,n} {z_2}^{2\,n} z_2\, z_1\, a + \left((-n-1)\, z_2\, a {z_2}^{n} + {z_2}^{n+2}\right) {z_1}^{n} \right] = 0. \end{aligned} \tag{20}$$

Note that z_2^{n+1} divides $\mathcal{E}_1$ and z_2^{n+2} does not divide $\mathcal{E}_1$. Then by Proposition 7 of the Appendix, the lemma follows. □

Proof of Theorem 1. Clearly the cofactor of $y = 0$ is $K = a(-1+xy^{n-1})$. Additionally, by Lemma 3 there is no other irreducible invariant algebraic curves of system (1). Since the algebraic multiplicity of $y = 0$ is one it turns out that there no exponential factors associated to the invariant straight line $y = 0$, see Proposition 7 of the Appendix. According to Lemma 4 the straight line at infinity has multiplicity $n+1$. So the infinity provides n exponential factors $G_i = \exp\left((ax+y)^i\right)$ with cofactors $L_i = -ia(y-1)(ax+y)^{i-1}$ for $i = 1, \cdots, n$. System (1) has divergence $\operatorname{div}(X) = anxy^{n-1} - y^n - a$. According to statement (a) of Theorem 8 of the Appendix there is a linear combination between the cofactors K and L_i, if and only if there exists a Darboux first integral, or from its statement (b) there is a linear combination between the cofactors K and L_i and the divergence of the system, if and only if there exists a Darboux integrating factor (24). But we can check easily that there is no such linear combinations. Indeed, considering the linear combination of the cofactors we obtain

$$0 = \lambda K + \sum_{i=1}^{n} \mu_i L_i = \lambda a(-1 + xy^{n-1}) - \sum_{i=1}^{n} \mu_i i a(y-1)(ax+y)^{i-1}.$$

For $n \geq 2$, the coefficient of the monomial x^{n-1} must be zero. So $\mu_n = 0$. Now the coefficient of the monomial x^{n-2} must be zero, so $\mu_{n-1} = 0$. Continuing in this way, at the end we obtain $\mu_n = \mu_{n-1} = \cdots = \mu_2 = 0$. The linear combination now becomes

$$\lambda a(-1 + xy^{n-1}) - \mu_1 a(y-1) = 0,$$

and so $\lambda = \mu_1 = 0$.

For $n = 1$, the linear combination becomes $0 = \lambda K + \mu_1 L_1 = \lambda a(-1+x) + \mu_1 a(y-1)$ and therefore $\lambda = \mu_1 = 0$.

Now we consider the linear combination of the cofactors and the divergence of the vector field. We have

$$0 = \lambda K + \sum_{i=1}^{n} \mu_i L_i - \operatorname{div}(X) = \lambda a(-1 + xy^{n-1}) - \sum_{i=1}^{n} \mu_i i a(y-1)(ax+y)^{i-1} + anxy^{n-1} - y^n - a.$$

For $n \geq 2$, similar arguments yields to $\mu_n = \mu_{n-1} = \cdots = \mu_2 = 0$ and so the linear combination now becomes

$$\lambda a(-1 + xy^{n-1}) - \mu_1 a(y-1) + anxy^{n-1} - y^n - a = 0,$$

and for $n \geq 2$ cannot be satisfied. For $n = 1$, the linear combination becomes

$$0 = \lambda K + \mu_1 L_1 + \mathrm{div}(X) = \lambda a(-1 + x) - \mu_1 a(y - 1) + ax - y - a$$

and cannot be satisfied.

In summary, system (1) is not Liouvillian integrable. $\square$

3. Proof of Theorem 2

In order to prove Theorem 2 we need to study the behaviour of the finite and infinite equilibrium points, and the existence of periodic orbits and limit cycles.

3.1. **The finite equilibrium point, periodic orbits and limit cycles.** System (1) has only one finite equilibrium point, namely $P = (1, 1)$. The Jacobian matrix of the system at P is

$$\begin{pmatrix} -1 & -n \\ a & a\,(n-1) \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are

$$\frac{a\,(n-1) - 1 \pm \sqrt{(a(n-1)-1)^2 - 4a}}{2}.$$

For $n = 1$ the point P is a stable node when $a \leq 1/4$ and is a stable focus when $a > 1/4$.

Now we consider that $n > 1$. We set $R(a) = (a\,(n-1) - 1)\,/2$ and $D = (a(n-1) - 1)^2 - 4a)$. Then $R(a) = 0$ gives $a = 1/(n-1)$ and relation $D = 0$ gives

$$a = \left(\frac{\sqrt{n} - 1}{n - 1}\right)^2.$$

Hence for $n > 1$ the equilibrium point $P = (1, 1)$ is (see also [18, 3, 14])

- a stable hyperbolic node if $a \in \left(0,\ \left(\frac{\sqrt{n}-1}{n-1}\right)^2\right]$.
- a stable hyperbolic focus if $a \in \left(\left(\frac{\sqrt{n}-1}{n-1}\right)^2, \frac{1}{n-1}\right)$.
- a stable weak focus if $a = \frac{1}{n-1}$ and there is a Hopf bifurcation.
- an unstable hyperbolic focus if $a \in \left(\frac{1}{n-1},\ \left(\frac{\sqrt{n}+1}{n-1}\right)^2\right)$.
- an unstable hyperbolic node if $a \geq \left(\frac{\sqrt{n}+1}{n-1}\right)^2$.

Note that $R'(a) = (n-1)/2 > 0$ for $n > 1$. In particular, $R'(a) > 0$ for $a = 1/(n-1)$. Then from Theorem 3.4.2 of [12] appears a Hopf bifurcation. So there is a periodic orbit. In order to study the stability of the periodic orbit we need to calculate the first Lyapunov constant at P. By relation (3.4.11) of [12] the first Lyapunov coefficient is equal to $-1/16$. So, there is a supercritical bifurcation for $a \geq 1/(n-1)$ and close to $1/(n-1)$. So the unique bifurcated limit cycle in the Hopf bifurcation must be stable.

The following theorem characterize the existence of periodic orbits and limit cycles, for a proof see [5, 6]. We recall that here we consider $a > 0$.

Theorem 5. *For every positive integer $n \geq 2$, there exists a unique constant $a^* \in \left(\frac{1}{n-1},\ \frac{2^n-1}{2^n-2}\right)$ such that system* (1) *has no periodic orbits when $a \in (-\infty, 1/(n-1)] \bigcup [a^*,\ +\infty)$ and has a unique limit cycle when $a \in (1/(n-1),\ a^*)$, which is stable and hyperbolic. Moreover, when the limit cycle exists, its amplitude increases with a.*

3.2. **The infinite equilibrium points.** The expression of system (1) in the local chart (U_1, F_1) is

$$
\begin{aligned}
\dot{z}_1 &= -{z_2}^{n+1} z_1 + {z_1}^{n+1} - {z_2}^n a z_1 + a{z_1}^n, \\
\dot{z}_2 &= -{z_2}^{n+2} + z_2\, {z_1}^n,
\end{aligned} \tag{21}
$$

and there are two infinite equilibrium points $(0,0)$ and $(-a,0)$. For $n = 1$ the origin is a semi–hyperbolic saddle and the point $(-a,0)$ is a hyperbolic stable node.

For $n > 1$ the Jacobian matrix at the origin is linearly zero and the blow up technique is applied (see [3] and [14]). For n odd the origin of the chart (U_1, F_1) in the Poincaré sphere is the union of one parabolic and four hyperbolic sectors, the line of infinity separates the four hyperbolic sectors two in each side, see the origin of the local chart U_1 in Figure 1. Whereas for n even is the union of two hyperbolic and two parabolic sectors separated by the line of the infinity as it is indicated in Figure 2.

For $n > 1$ the Jacobian matrix at the point $(-a, 0)$ is

$$
\begin{pmatrix} (-a)^n & 0 \\ 0 & (-a)^n \end{pmatrix}.
$$

So the point $(-a,0)$ is a hyperbolic node. For n even is unstable whereas for n odd is stable.

In the local chart (U_2, F_2) system (1) is written

$$
\begin{aligned}
\dot{z}_1 &= -z_1\ (a z_1 + 1) + {z_2}^n a z_1 + {z_2}^{n+1}, \\
\dot{z}_2 &= a z_2\ ({z_2}^n - z_1)\,,
\end{aligned} \tag{22}
$$

and the origin has eigenvalues 0 and -1. Since the origin is a semi–hyperbolic equilibrium point we apply Theorem 2.19 of [11] and for $n = 1$ is a saddle–node. For $n > 1$ and n odd the origin of the chart (U_2, F_2) is a saddle node whereas for $n > 1$ and n even it is a saddle.

Combine all these previous results we are ready to prove Theorem 2.

For $n = 1$ and $a \in (0,\ 1/4]$ the point P is a stable node and the global phase portrait is given in Figure 1(e). Note that for $a > 1/4$ the point P is a stable focus and in this case the global phase portrait is equivalent to Figure 1(e).

Now consider that $n > 1$ and n is odd. Then for $a \in (0,\ (\sqrt{n}-1)^2/(n-1)^2]$ the finite equilibrium point is a stable node. Since the stable node is topological equivalent with a stable focus we obtain the same global phase portrait as the one for $a \in \left((\sqrt{n}-1)^2/(n-1)^2,\ 1/(n-1)\right)$ where the finite equilibrium point is now a stable focus. This corresponds to Figure 1(a). Moreover, the separatrix configuration in Figure 1(a) is different from the separatrix configuration in Figure 1(d). Hence these two phase portraits are not topological equivalent. In addition, the numbers R and S are distinct in the rest of the cases, so we obtain five different global phase portraits for n odd, see Figure 1.

For n even using similar arguments we obtain four distinct global phase portraits, see Figure 2. Moreover, these new four phase portrait have different separatrix configuration than the ones for n odd because of the different behaviour of the origin of the chart (U_1, F_1). This completes the proof of Theorem 2.

4. Appendix: Invariant curves, multiplicity and Liouvillian integrability.

Consider the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{23}$$

We say that this system has *degree* n if n is the maximum of the degrees of the polynomials P and Q. The associated vector field to system (23) is

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

and we denote by div(X) the divergence of X, namely, $\mathrm{div(X)} = \partial P/\partial x + \partial Q/\partial y$.

Let $f = f(x, y)$ be a polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of system (23) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

for some polynomial $K = K(x, y)$ called the *cofactor* of the invariant algebraic curve $f = 0$. The polynomial structure of system (23) forces that the degree of the cofactor is at most $n - 1$. Note that the curve $f = 0$ is formed by orbits of the differential system (23), and consequently it is *invariant* under the flow of this system.

The *$n-th$ extactic curve* of X, $\mathcal{E}_m(X)$ is given by the polynomial equation

$$\mathcal{E}_m(X) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_\ell \\ X(v_1) & X(v_2) & \cdots & X(v_\ell) \\ \vdots & \vdots & \cdots & \vdots \\ X^{\ell-1}(v_1) & X^{\ell-1}(v_2) & \cdots & X^{\ell-1}(v_\ell) \end{pmatrix} = 0,$$

where $v_1, \cdots, v_\ell$ is a basis of $\mathbb{C}_m[x, y]$ (the $\mathbb{C}$ vector space of polynomials in $\mathbb{C}[x, y]$ of degree at most m) and consequently $\ell = (m + 1)(m + 2)/2$.

Proposition 6. *Every algebraic curve of degree m invariant by the vector field X is a factor of $\mathcal{E}_m(X)$.*

An invariant algebraic curve f of degree m for the vector field X has *algebraic multiplicity* k when k is the greatest positive integer such that f^k divides $\mathcal{E}_m X$. For more details about the multiplicity of an invariant curve and other properties of the extactic curve see [8].

The algebraic multiplicity of a curve is connected with an object called *exponential factor*. Exponential factors also provide cofactors and appear when invariant algebraic curves collide, i.e. when they have multiplicity larger than one. Let $F(x, y) = \exp(g(x, y)/f(x, y))$ where f, g polynomials. Then F in an *exponential factor* of system (23) if

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = LF,$$

for some polynomial L of degree at most $n-1$ called the *cofactor* of the exponential factor F. The next result is proved in [8].

Proposition 7. *Let $f = 0$ be an irreducible invariant algebraic curve of degree m of the polynomial vector field X with cofactor K. Then the algebraic multiplicity of the curve $f = 0$ is k if and only if X has $k - 1$ exponential factors of the form $\exp(g_i/f^i)$ for $i = 1, \cdots, k - 1$ and the degree of g_i is at most im.*

A *Darboux* function of a vector field X is a function of the form

$$D = \prod f_i^{\lambda_i} F_j^{\mu_j}, \tag{24}$$

where the $f_i = 0$ are invariant algebraic curves and F_j are exponential factors of X.

The first integrals given by functions (24) are called *Darboux first integrals* and the integrating factors given by (24) are called *Darboux integrating factors.*

By definition a *Liouvillian function* is an element in the Liouvillian field extension of the filed of rational functions $\mathbb{C}(x, y)$. For a good review about Darboux and Liouvillian integrability see chapter 3 of [22], and chapter 8 of [11]. In 1992 Singer [19] and later on in 1999 Christopher [7] proved that for a planar polynomial differential system, the existence of Liouvillian first integrals is equivalent to the existence of a Darboux integrating factor. So now we can say that system (1) is *Liouvillian integrable* if has a first integral or an integrating factor given by a Darboux function. The following result started with Darboux [9], for the present version see for instance [11].

Theorem 8. *Suppose that a polynomial system* (23) *admits* p *irreducible invariant algebraic curves* $f_i = 0$ *with cofactors* K_i *and* q *exponential factors* F_j *with cofactors* L_j. *Then the following statements hold.*

(a) *There exist* λ_i*'s and* μ_j*'s in* $\mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$*, if and only if the Darboux function* (24) *is a first integral of system* (23).

(b) *There exist* λ_i*'s and* μ_j*'s in* $\mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(X)$*, if and only if the Darboux function* (24) *is an integrating factor of system* (23).

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