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Global nilpotent reversible centers with cubic nonlinearities symmetric with respect to the x-axis

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Abstract

Let $P_3(x, y)$ and $Q_3(x, y)$ be polynomials of degree three without constant or linear terms. We characterize the global centers of all polynomial differential systems of the form $\dot{x} = y + P_3(x, y)$, $\dot{y} = Q_3(x, y)$ that are reversible and invariant with respect to the x-axis.

Keywords: Global reversible centers, nilpotent centers, cubic polynomial differential systems

MSC Classification: 37D99

1 Introduction and statement of the main results

A planar polynomial differential system of degree three having a nilpotent center at the origin can be written as

$$\begin{aligned} x' &= y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ y' &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{1}$$

We consider systems (1) that are invariant under the symmetry $(x, y, t) \mapsto (x, -y, -t)$. Imposing that systems (1) are invariant under such symmetry we get that $a_{20} = a_{30} = a_{02} = a_{12} = b_{11} = b_{21} = b_{03} = 0$ and they become

$$\begin{aligned} x' &= y(1 + a_{11}x + a_{21}x^2 + a_{03}y^2), \\ y' &= b_{20}x^2 + b_{30}x^3 + b_{02}y^2 + b_{12}xy^2. \end{aligned} \tag{2}$$

Note that $(0,0)$ is a nilpotent singular point. To be isolated we need that the second equation in (2) is not identically zero (which yields $b_{20}^2 + b_{30}^2 + b_{02}^2 + b_{12}^2 > 0$) and that both equations in (2) do not have the common factor y (which gives $b_{20}^2 + b_{30}^2 > 0$). We can prove that if $b_{20}^2 + b_{30}^2 > 0$, then the two equations in (2) cannot have a common factor of the form $ax + by$ with $a \neq 0$ or of the form $ax^2 + bxy + cy^2 + dx + ey$ with $a^2 + b^2 + c^2 > 0$. In short, the singular point $(0,0)$ is isolated if and only if $b_{20}^2 + b_{30}^2 > 0$.

Now we apply [5, Theorem 3.5] to ensure that the singular point is a linear nilpotent center. Since system (3) is reversible, such a linear nilpotent center will be indeed a center. We compute the functions F and G defined in [5, Theorem 3.5] and we get

$$F(x) = b_{20}x^2 + b_{30}x^3 \quad \text{and} \quad G(x) = 0.$$

So the origin is a nilpotent center if and only $b_{20} = 0$ and $b_{30} < 0$. Note that under these conditions the origin is an isolated singular point.

Assume that $b_{20} = 0$ and $b_{30} = -\alpha^2$ with $\alpha \neq 0$. Then system (2) becomes

$$\begin{aligned} x' &= y(1 + a_{11}x + a_{21}x^2 + a_{03}y^2), \\ y' &= -\alpha^2 x^3 + b_{02}y^2 + b_{12}xy^2. \end{aligned} \tag{3}$$

We characterize the planar polynomial differential systems (3) having a global center at the origin, called from now on global nilpotent centers. We recall that a center is a singular point filled up with periodic orbits and that it is global when the period annulus of that center is the plane $\mathbb{R}^2$. The existence of global centers is a key point in a proof of the Jacobian conjecture (see, for instance, [13]).

Global centers are only possible in polynomials with odd degree (see, for instance, [6, 12]). The classification of global centers of planar polynomial vector fields is a very difficult problem mainly due to the fact that until now the complete characterization of centers of planar polynomial differential systems of degree higher than or equal to three has not been done due to its difficulty (and it is even worse for the characterization of global centers). Up to now the classification of global centers has been done for some subclasses of cubic and quintic planar polynomial differential systems for which the existence of centers come automatically from the existence of a linear center. In this scenario we cite [8] where the authors characterized the global nilpotent centers of planar polynomial differential systems of the form: linear + homogeneous cubic polynomials (for which the classification of centers is known) and we also want to cite [3] where the authors provided the global phase portraits in the Poincaré disk of all planar cubic Hamiltonian polynomial differential systems symmetric with respect to the x-axis having a nilpotent center at the origin (the Hamiltonian structure forces that a linear center is indeed a center). For other papers studying global centers in polynomial differential systems of degree three and five see the references [9, 14, 15] and the recent ones [2, 4, 7, 10, 11].

System (3) when $b_{02} = a_{11} = 0$ has been studied in [8] and when $b_{02} = -a_{11}/2$ and $b_{12} = -a_{21}$ has been studied in [3]. From now on we study system (3) assuming that b_{02} and a_{11} are not simultaneously zero, because it has already been studied. To extract from (3) the conditions studied in [3] make the computations much more involved and so we prefer not to exclude them from our system (3).

So we assume $b_{02}^2 + a_{11}^2 > 0$ and we consider the change of variables

$$X = Ax, \qquad Y = By, \quad T = Ct. \tag{4}$$

When $b_{02} = 0$ and $a_{11} \neq 0$, by the change given in (4), system (3) with $A = a_{11}$, $B = a_{11}^2/\alpha$, $C = \alpha/a_{11}$ can be written as

$$X' = Y + XY + aX^2Y + bY^3, \qquad Y' = -X^3 + cXY^2, \tag{5}$$

where $a = a_{21}/a_{11}^2$, $b = \alpha^2 a_{03}/a_{11}^4$, $c = b_{12}/a_{11}^2$.

When $b_{02} \neq 0$, by the change given in (4), system (3) with $A = b_{02}$, $B = b_{02}^2/\alpha$, $C = \alpha/b_{02}$ can be written as

$$X' = Y + aXY + bX^2Y + cY^3, \qquad Y' = -X^3 + Y^2 + dXY^2, \tag{6}$$

where $a = a_{11}/b_{02}$, $b = a_{21}/b_{02}^2$, $c = \alpha^2 a_{03}/b_{02}^4$ and $d = b_{12}/b_{02}^2$.

The main theorems of the paper are the following ones.

Theorem 1. *System (6) has a global nilpotent center at the origin if and only $\{(a,b,c,d)\} \in d_i$ for some $i = 1,\dots,107$ (see Appendix A).*

Theorem 2. *System (5) has a global nilpotent center at the origin if and only $\{(a,b,c)\} \in e_i$ for some $i = 1,\dots,8$ (see Appendix B).*

The proof of Theorem 1 is given in Section 3 and the proof of Theorem 2 is given in Section 2. We have included two appendices with the definition of the sets d_i and e_j for $i = 1,\dots,107$ and $j = 1,\dots,8$ as well as some other sets of conditions that will appear in their proofs. In the appendices we also provide values of the parameters belonging to each one of the sets d_i and e_j, showing that the sets d_1–d_{107} and e_1–e_8 are not empty. From their definitions it is also easy to see that all these sets are disjoint.

2 Proof of Theorem 1

Consider system (6) which, after abuse of notation, can be written as

$$x' = y(1 + a\,x + b\,x^2 + c\,y^2), \qquad y' = -x^3 + y^2 + d\,x\,y^2, \tag{7}$$

for some $a, b, c, d \in \mathbb{R}$.

2.1 Finite singular points of equation (3)

From the first equation of (7) we have that either $y = 0$ or

$$1 + ax + bx^2 + cy^2 = 0. \tag{8}$$

Clearly if $y = 0$ the unique finite singular point of system (7) is the origin. Now we analyze the singular points coming from solutions of equation (8). We distinguish four cases: $c = 0$, $b = 0$ and $a = 0$; $c = 0$, $b = 0$ and $a \neq 0$; $c = 0$, $b \neq 0$; and $c \neq 0$.

When $c = 0$, $b = 0$ and $a = 0$ equation (8) is never satisfied, so the unique singular point of system (7) is the origin. This provides condition c_1 in Appendix A.

If $c = 0$, $b = 0$ and $a \neq 0$, then from (8) we get $x = -1/a$. Substituting it into the second equation of (7) we get

$$\frac{1}{a^3} + \frac{a-d}{a}y^2 = 0.$$

If $d = a$, this last equation is never satisfied. If $d \neq a$, system (7) has no solutions different from the origin when

$$y^2 = -\frac{1}{a^2(a-d)} < 0,$$

and so $a > d$. In short, if $c = 0$, $b = 0$, $a \neq 0$ and $a \geq d$, the unique singular point of (7) is the origin. This provides condition c_2 in Appendix A.

When $c = 0$, $b \neq 0$, from (8) we get

$$x = x^{\pm} = \frac{-a \pm \sqrt{a^2 - 4b}}{2b}.$$

If $a^2 - 4b < 0$ the solutions $x^{\pm}$ are not defined and consequently the unique singular point of (7) is the origin. This gives condition c_3 in Appendix A. Assume now that $a^2 - 4b \geq 0$. Notice that $x^{\pm}$ is never zero. By substituting $x = x^{\pm}$ into the second equation of (7) we get $(1 + dx^{\pm})y^2 - (x^{\pm})^3 = 0$. Let

$$D^{\pm} = \frac{2b}{a \mp \sqrt{a^2 - 4b}} \quad \text{and} \quad k^{\pm} = \frac{(x^{\pm})^3}{1 + d\,x^{\pm}},$$

where $d = D^{\pm}$ is the solution of $1 + dx^{\pm} = 0$. The second equation of (7) has no real solutions either when $d = D^+$ and $k^- < 0$ which is not possible; when $d = D^-$ and $k^+ < 0$ which is satisfied in the set $\{b < 0, d = D^-\} \cup \{0 < b < a^2/4, d = D^-\}$; when $d = D^+ = D^-$ which is satisfied in the set $\{a \neq 0, b = a^2/4, d = a/2\}$; and when $d \neq D^{\pm}$, $k^+ < 0$ and $k^- < 0$ which is satisfied in the set $\{b < 0, d < D^-\} \cup \{0 < b \leq a^2/4, d < D^-\}$. In short, (7) has no real solutions different from the origin in the set

$$\{0 < b \leq a^2/4, d \leq D^-\} \cup \{b < 0, d \leq D^-\}.$$

This set provides, respectively, conditions c_4 and c_5 in Appendix A.

Now we consider the case $c \neq 0$. Isolating y^2 from equation (8) we get

$$y^2 = -\frac{1 + ax + bx^2}{c}. \tag{9}$$

By substituting (9) into the second equation of (7) we obtain the equation

$$(c + bd)x^3 + (b + ad)x^2 + (a + d)x + 1 = 0. \tag{10}$$

Thus when $c \neq 0$ the origin is the unique finite singular point if either (10) has no real solutions or the expression of y^2 in (9) evaluated at the real solutions of (10) is negative. We distinguish four cases: $c + bd = 0$, $b + ad = 0$ and $a + d = 0$; $c + bd = 0$, $b + ad = 0$ and $a + d \neq 0$; $c + bd = 0$ and $b + ad \neq 0$; and $c + bd \neq 0$.

If $c + bd = 0$, $b + ad = 0$ and $a + d = 0$, then equation (10) is never satisfied. After simplification we get condition c_6 in Appendix A.

When $c + bd = 0$, $b + ad = 0$ and $a + d \neq 0$ the solution of (10) is

$$x = -\frac{1}{a + d}.$$

Substituting this solution into (9) we get

$$y^2 = -\frac{b + ad + d^2}{c(a + d)^2},$$

and since $b + ad = 0$ this last expression is negative when $c > 0$. Therefore the origin is the unique singular point of (7). After simplification we get condition c_7 in Appendix A.

Now we analyze the case $c+bd = 0$ and $b+ad \neq 0$. Under these conditions equation (10) becomes

$$(b + ad)x^2 + (a + d)x + 1 = 0. \tag{11}$$

Solving equation (11) we get

$$x = \tilde{x}^{\pm} = -\frac{a + d \pm \sqrt{(a - d)^2 - 4b}}{2(ad + b)}.$$

If $(a - d)^2 - 4b < 0$, or equivalently if $b > (a - d)^2/4$, the solutions $\tilde{x}^{\pm}$ are not real. Therefore the unique real solution of (7) is the origin. This provides condition c_8 in Appendix A. When $(a - d)^2 - 4b \geq 0$, or equivalently $b \leq (a - d)^2/4$, the solutions $\tilde{x}^{\pm}$ are real and they are never zero. Since we are interested in solutions of (11) that do no provide real solutions of (7) we need that (9) evaluated at $x = \tilde{x}^{\pm}$ and $c = -bd$ be negative. We can see that the Gröbner basis of the polynomials $-(1 + ax + bx^2)/c$ and $(b + ad)x^2 + (a + d)x + 1$ is 1. So there are no solutions of system

$$-\frac{1 + ax + bx^2}{c} = 0, \qquad (b + ad)x^2 + (a + d)x + 1 = 0,$$

and consequently (9) evaluated at $x = \tilde{x}^{\pm}$ and $c = -bd$ is never zero. Then the sign of (9) evaluated at $x = \tilde{x}^{\pm}$ and $c = -bd$ can change only either on the boundaries of the definition domain of $\tilde{x}^{\pm}$ or when $-bd$ changes it sign. We consider the following regions

$$B_1=\{bd>0, b+ad>0, b\le (a-d)^2/4\}, \qquad B_2=\{bd>0, b+ad<0, b\le (a-d)^2/4\},$$
$$B_3=\{bd<0, b+ad>0, b\le (a-d)^2/4\}, \qquad B_4=\{bd<0, b+ad<0, b\le (a-d)^2/4\},$$

whose boundaries are the sets $b=0$, $d=0$ $b=-ad$ and $b=(a-d)^2/4$. Analyzing the intersections of these boundaries we can decompose each region B_i with $i=1,\ldots,4$ as union of several disjoint connected components. In particular the region B_1 can be decomposed as union of the following regions

$$\begin{aligned}
B_{11} &= \{d>0, a>d, 0<b\le (a-d)^2/4\},\\
B_{12} &= \{d>0, a<-d, -ad<b\le (a-d)^2/4\},\\
B_{13} &= \{d<0, a<0, b<0, -ad<b\},\\
B_{14} &= \{d>0, 0<a<d, 0<b\le (a-d)^2/4\},\\
B_{15} &= \{d>0, -d<a\le 0, -ad<b\le (a-d)^2/4\};
\end{aligned}$$

the region B_2 can be decomposed as union of

$$\begin{aligned}
B_{21} &= \{d<0, a>0, b<0\},\\
B_{22} &= \{d<0, a\le 0, b<-ad\},\\
B_{23} &= \{d>0, a<0, 0<b<-ad\};
\end{aligned}$$

the region B_3 can be decomposed as union of

$$\begin{aligned}
B_{31} &= \{d>0, a>0, -ad<b<0\},\\
B_{32} &= \{d<0, a>-d, -ad<b\le (a-d)^2/4\},\\
B_{33} &= \{d<0, a<d, 0<b\le (a-d)^2/4\},\\
B_{34} &= \{d<0, 0<a<-d, -ad<b\le (a-d)^2/4\},\\
B_{35} &= \{d<0, d<a\le 0, 0<b\le (a-d)^2/4\};
\end{aligned}$$

and the region B_4 can be decomposed as union of

$$\begin{aligned}
B_{41} &= \{d>0, a>0, b<-ad\},\\
B_{42} &= \{d>0, b<0, a\le 0\},\\
B_{43} &= \{d<0, a>0, 0<b<-ad\}.
\end{aligned}$$

In order to decompose the regions B_i as union of disjoint connected components we have used the REDUCE function of Mathematica. This will allow us to automate the process in such a way that it can be applied to the remaining cases in this paper. In particular, in the case $c\neq 0$ and $c+bd\neq 0$.

The sign of (9) evaluated at $c=-bd$ and $x=x^{\pm}$ does not change within the same connected component but it could change from one component to the other. We pick up a point in each connected component and we compute the signs of (9) evaluated at $c=-bd$ and the solutions $x=x^{\pm}$ at this point. We see that (9) evaluated at $c=-bd$ and $x=x^{+}$ is negative in B_{11}, B_{32}, B_{34}, B_{35}, B_{43}; and (9) evaluated at $c=-bd$ and $x=x^{-}$ is negative in B_{11}, B_{13}, B_{21}, B_{22}, B_{31}, B_{32}, B_{34}, B_{35}, B_{41}, B_{42} and B_{43}. In order that the unique finite singular point of (7) be the origin we need (9) evaluated at the solutions $x=x^{+}$ and $x=x^{-}$ to be both negative. Thus the parameters must belong to one of the sets B_{11}, B_{32}, B_{34}, B_{35}, or B_{43} and they provide, respectively, conditions c_9, c_{10}, c_{11}, c_{12} and c_{13} in Appendix A.

Finally we consider the case $c\neq 0$ and $c+bd\neq 0$. In this case (10) is a cubic equation of the form

$$\alpha x^3+\beta x^2+\gamma x+1=0, \tag{12}$$

where $\alpha = c + bd$, $\beta = b + ad$ and $\gamma = a + d$. Thus if the discriminant $\Delta = -27\alpha^2 + 18\alpha\beta\gamma - 4\alpha\gamma^3 - 4\beta^3 + \beta^2\gamma^2$ of (12) satisfies $\Delta < 0$ then (12) has one real root and two complex ones, if $\Delta > 0$ it has three distinct real roots, and finally if $\Delta = 0$ it has either a unique real root with multiplicity three when $\beta^2 = 3\alpha\gamma$ or two different reals roots one of them with multiplicity 2 when $\beta^2 \neq 3\alpha\gamma$.

By substituting $\alpha = c + bd$, $\beta = b + ad$ and $\gamma = a + d$ the discriminant Δ becomes

$$\begin{aligned}\Delta = {} & a^4d^2 - 2a^3bd - 4a^3c - 2a^3d^3 + a^2b^2 - 2a^2bd^2 + 6a^2cd + a^2d^4 \\ & + 8ab^2d + 18abc + 8abd^3 + 6acd^2 - 4b^3 - 8b^2d^2 - 36bcd - 4bd^4 - 27c^2 - 4cd^3,\end{aligned}$$

and the equation $\beta^2 = 3\alpha\gamma$ becomes

$$-3c(a + d) - bd(a + 3d) + b^2 + a^2d^2 = 0. \tag{13}$$

We analyze the set where (10) has a unique real root with multiplicity three. The solutions of (13) are

$$\begin{cases} c = \dfrac{a^2d^2 - bd(a + 3d) + b^2}{3(a + d)} & \text{when } a + d \neq 0, \\ b = d^2 & \text{when } a + d = 0. \end{cases} \tag{14}$$

Substituting (14) into equation $\Delta = 0$ we get

$$\begin{cases} -\dfrac{(ad + b)^2\left(a^2 - ad - 3b + d^2\right)^2}{3(a + d)^2} = 0 & \text{when } a + d \neq 0, \\ c = -d^3 & \text{when } a + d = 0. \end{cases}$$

Thus the system formed by the equations $\Delta = 0$ and (13) has the solutions

$$\begin{cases} b = \frac{1}{3}(a^2 - ad + d^2),\ c = \frac{1}{27}(a - 2d)^3 \quad \text{and} \quad b = -ad,\ c = ad^2 & \text{when } a + d \neq 0, \\ b = d^2,\ c = -d^3 & \text{when } a + d = 0. \end{cases} \tag{15}$$

However the second solution in (15) when $a + d \neq 0$ does not satisfy the condition $c + bd \neq 0$ and so it is not possible. Moreover, the solution when $a + d = 0$ coincides with the first solution in (15) when $a + d \neq 0$ with $a = d$. In short equation (10) has a unique real root with multiplicity three in the set

$$\Delta_1 = \Big\{c \neq 0, c + bd \neq 0, b = \frac{1}{3}(a^2 - ad + d^2), c = \frac{1}{27}(a - 2d)^3\Big\}.$$

On the other hand, solving equation $\Delta = 0$ we get the solutions $c = K^{\pm}$ (see Appendix A for the expressions of $K^{\pm}$). Then equation (10) has two different reals roots one of them with multiplicity 2 in the set $\Delta_2 = \Delta_2^+ \cup \Delta_2^-$ with

$$\Delta_2^{\pm} = \Big\{c \neq 0, c + bd \neq 0, c = K^{\pm}, b \neq \frac{1}{3}(a^2 - ad + d^2)\Big\}.$$

Notice that $K^{\pm}|_{b=(a^2-ad+d^2)/3} = (a - 2d)^3/27$.

Since we want that the origin be the unique real solution of (7) we need (9) evaluated at all the real roots of (10) to be negative. To find where this condition is satisfied we proceed as in the previous case. First we see that (9) does not vanish on the solutions of (10). Indeed, the Gröbner basis of the polynomials $-(1 + ax + bx^2)/c$ and $(c + bd)x^3 + (b + ad)x^2 + (a + d)x + 1$ is 1. Therefore (9) evaluated at the solutions of (10) can change its sign only when $c = 0$ or on the boundaries of the definition domain of the solutions of (10). We consider the four regions

$$\begin{aligned} C_1 &= \{c > 0, c + bd > 0\}, & C_2 &= \{c > 0, c + bd < 0\}, \\ C_3 &= \{c < 0, c + bd > 0\}, & C_4 &= \{c < 0, c + bd < 0\}. \end{aligned}$$

When $\Delta < 0$ equation (10) has a unique real solution. Regions C_1, C_2, C_3 and C_4 with the additional constraint $\Delta < 0$ can be decomposed, respectively as union of 56, 51, 51 and 72 disjoint regions where the sign of (9) does not change. We pickup a point in each one of these regions, we compute the real solution of (10) at this point and finally we compute the sign of (9) evaluated at this solution. Doing so we get that (9) is negative in all regions of C_1 and C_2 and positive in all regions of C_3 and C_4. The regions C_1 and C_2 provide, respectively, conditions c_{14} and c_{15} in Appendix A.

Now we consider the case $\Delta > 0$ with corresponds to the case of the existence of three distinct real roots. Regions C_1, C_2, C_3 and C_4 with the additional constraint $\Delta > 0$ can be decomposed, respectively, as union of 39, 29, 29 and 45 disjoint regions where the signs of (9) evaluated at the solutions do not change. In this case we need (9) to be negative at the three solutions. This is only possible at some subsets of C_1 and C_2. More precisely at 10 subsets of C_1, the sets c_i with $i = 16, \dots, 25$ and 24 subsets of C_2, which provide the sets c_i with $i = 26, \dots, 49$ in Appendix A.

Regions C_1, C_2, C_3 and C_4 with the additional constraint Δ_1 which correspond to a triple real root of (10) can be decomposed, respectively, as union of 2,1,1 and 2 disjoint regions where the signs of (9) evaluated at the solution do not change. In this case (9) evaluated at the solution is negative at the entire C_1 and C_2 which gives, respectively, the sets c_i with $i = 50, \dots, 52$ in Appendix A.

Regions C_1, C_2, C_3 and C_4 with the additional constraint Δ_2 which corresponds to a double real root of (10) can be decomposed, respectively, as union of 30, 24, 24 and 34 disjoint regions where the signs of (9) evaluated at the solutions do not change. In this case we need the sign of (9) evaluated at the two solution to be negative. This happen at 11 subsets of C_1, the subsets c_i with $i = 53, \dots, 63$ in Appendix A, and 22 subsets of C_2, the subsets c_i with $i = 64, \dots, 85$ in Appendix A.

2.2 Infinite singular points in the local chart U_1 of system (7)

In the local chart U_1 system (7) becomes

$$u' = -au^2v - bu^2 - cu^4 + du^2 - u^2v^2 + u^2v - 1, \quad v' = -u(b + cu^2)v - auv^2 - uv^3. \tag{16}$$

The inifinite singular points (that is the ones with $v = 0$) satisfy

$$-cu^4 + u^2(d - b) - 1 = 0. \tag{17}$$

The Jacobian matrix of (16) evaluated at $v = 0$ becomes

$$J = u \begin{pmatrix} -2(2cu^2 + b - d) & (1 - a)u \\ 0 & -(cu^2 + b) \end{pmatrix}.$$

In order the origin to be a global center we need that either there are no infinite singular points on the local chart U_1 or that the singular points, in case they exist, they are all formed by the union of two hyperbolic sectors (and in particular, they are linearly zero). We distinguish three cases: $c = 0$ and $d - b = 0$; $c = 0$ and $d - b \neq 0$; and $c \neq 0$.

If $c = 0$ and $d - b = 0$, equation (17) has no real solutions. This gives the set i_1 in Appendix A.

If $c = 0$ and $d - b \neq 0$, the solutions of (17) are

$$u = u^{\pm} = \pm \frac{1}{\sqrt{d - b}}.$$

Thus if $d - b < 0$ then (17) has no real solutions yielding the set i_2 in Appendix A. When $d - b > 0$ the two solutions $u = u^{\pm}$ exist, so the points $(u^{\pm}, 0)$ must be linearly zero. The Jacobian matrix evaluated at the solutions $u = u^{\pm}$ is

$$\begin{pmatrix} \pm 2\sqrt{d - b} & (a - 1)/(b - d) \\ 0 & \mp b/\sqrt{d - b} \end{pmatrix},$$

which is never identically zero. Therefore the points $(u^{\pm}, 0)$ are never linearly zero.

Finally, when $c \neq 0$ the solutions of (17) are

$$u = \tilde{u}_1^{\pm} = \frac{\sqrt{E^{\pm}}}{\sqrt{2}}, \quad u = \tilde{u}_2^{\pm} = -\frac{\sqrt{E^{\pm}}}{\sqrt{2}},$$

where

$$E^{\pm} = \pm\frac{\sqrt{b^2 - 2bd - 4c + d^2}}{c} - \frac{b}{c} + \frac{d}{c}.$$

If either $b^2 - 4c - 2bd + d^2 < 0$, or $b^2 - 4c - 2bd + d^2 \geq 0$ and E^+ and E^- are both negative, then (17) has no real solutions yielding, respectively, the sets i_3 and i_4 in Appendix A.

Assume now that $b^2 - 4c - 2bd + d^2 \geq 0$ and E^+ or E^- are positive. Since the solutions of (16) are never zero, a solution $(u, 0)$ of (17) is linearly zero if it is a solution of the system of equations

$$\begin{aligned} &f_1 = -cu^4 + u^2(d-b) - 1 = 0, \qquad && f_2 = -2\left(2cu^2 + b - d\right) = 0, \\ &f_3 = 1 - a = 0, && f_4 = -\left(cu^2 + b\right) = 0. \end{aligned} \tag{18}$$

By computing the Gröbner basis of the polynomials f_1, f_2, f_3, f_4 with respect to u we get the set of polynomials

$$\{-1 + a, -c + d^2, b + d, -d + cu^2, -1 + du^2\}.$$

Thus system (18) has solution only when $a = 1$, $b = -d$, $c = d^2$ and $d \neq 0$ and in this case the solutions are $u = \pm 1/\sqrt{d}$, both with multiplicity two. They are real when $d > 0$.

Now we study the linearly zero singular points by doing blow ups (see for instance [1]). Assume that $a = 1$, $b = -d$, $c = d^2$ and $d > 0$. We start studying the singular point $u = 1/\sqrt{d}$. First we do the change of variables $(U, V) = (u - 1/\sqrt{d}, v)$ to move the singular point to the origin and we get

$$\begin{aligned} u' &= -4d^{3/2}u^3 - d^2u^4 - 4du^2 - \frac{2uv^2}{\sqrt{d}} - \frac{v^2}{d} - u^2v^2, \\ v' &= -3d^{3/2}u^2v - d^2u^3v - 2duv - \frac{v^3}{\sqrt{d}} - \frac{v^2}{\sqrt{d}} - uv^3 - uv^2. \end{aligned} \tag{19}$$

Notice that we have renamed the new variables U and V as u and v. The characteristic polynomial of (19) at the origin is $\mathcal{F} = v(2d^2u^2 - \sqrt{d}uv + v^2)/d$. Since $v = 0$ is a simple characteristic direction, we apply the u-directional blow up $(u, v) \to (u, uw)$. Doing so and rescaling the time to eliminate the common factor u, system (19) in the new variables and time becomes

$$\begin{aligned} u' &= -\frac{u}{d}(4d^{5/2}u + d^3u^2 + 4d^2 + du^2w^2 + 2\sqrt{d}uw^2 + w^2), \\ w' &= \frac{w}{d}(d^{5/2}u + 2d^2 + \sqrt{d}uw^2 - duw - \sqrt{d}w + w^2). \end{aligned} \tag{20}$$

The singular points of (20) with $u = 0$ are

$$(0, 0), \qquad (0, \tilde{w}^{\pm}) = \left(0, \frac{\sqrt{d}}{2}(1 \pm \sqrt{1 - 8d})\right).$$

The points $(0, \tilde{w}^{\pm})$ are defined only when $0 < d \leq 1/8$. Moreover $\tilde{w}^+ > \tilde{w}^- > 0$ for all $0 < d < 1/8$ and they coincide at the point $w = \tilde{w}^* = 1/(4\sqrt{2})$ when $d = 1/8$. The point $(0, 0)$ is a saddle with Jacobian matrix $\begin{pmatrix} -4d & 0 \\ 0 & 2d \end{pmatrix}$ (we recall that $d > 0$). The Jacobian matrix at the points $(0, \tilde{w}^{\pm})$ is

$$\begin{pmatrix} \alpha_{11}^{\pm} & 0 \\ \alpha_{21}^{\pm} & \alpha_{22}^{\pm} \end{pmatrix} = \begin{pmatrix} -(4d + 1 \pm \sqrt{1 - 8d})/2 & 0 \\ -d^2(1 \pm \sqrt{1 - 8d})/2 & (-8d + 1 \pm \sqrt{1 - 8d})/2 \end{pmatrix}.$$

It is easy to see that if $0 < d < 1/8$, then $\alpha_{11}^+ < 0$, $\alpha_{22}^+ > 0$, $\alpha_{11}^- < 0$, and $\alpha_{22}^- < 0$ so the point $(0, \tilde{w}^+)$ is a saddle whereas the point $(0, \tilde{w}^-)$ is a stable node.

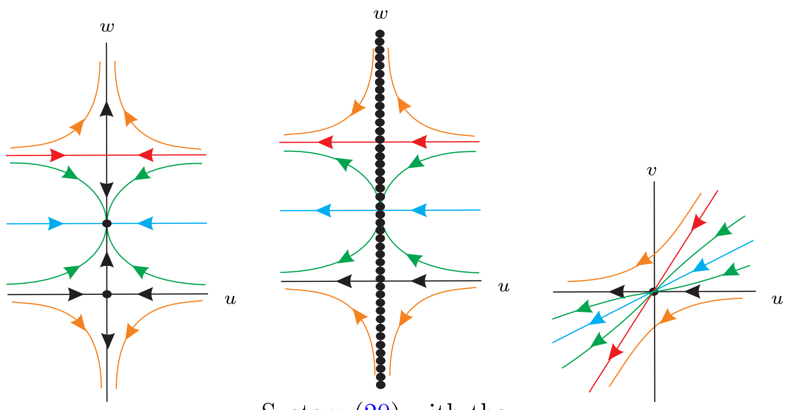

Fig. 1 Sequence of phase portraits near the origin of system (19) with $0 < d < 1/8$.

If $d = 1/8$ then the Jacobian matrix at the singular point $(0, \tilde{w}^*)$ becomes $\begin{pmatrix} -3/4 & 0 \\ -1/128 & 0 \end{pmatrix}$ so the point is semi-hyperbolic. By doing the change of variables $(U, W) = (u - 96(w - 1/(4\sqrt{2})), u)$ we simultaneously move the point $(0, \tilde{w}^*)$ to the origin and we transform the system into the normal form for applying [5, Theorem 2.19]. The resulting system becomes

$$\begin{aligned} u' =& \frac{1}{9216}(-96\sqrt{2}U^2 + 96\sqrt{2}UW - 2304\sqrt{2}W^2 + 8U^3 - 80U^2W + 328UW^2 - 688W^3 \\ &+ 2\sqrt{2}U^3W - 10\sqrt{2}U^2W^2 + 38\sqrt{2}UW^3 - 30\sqrt{2}W^4 - U^2W^3 + 2UW^4 - W^5), \\ w' =& -\frac{W}{9216}(6912 - 192\sqrt{2}U + 2496\sqrt{2}W + 8U^2 - 208UW + 632W^2 + 4\sqrt{2}U^2W \\ &- 32\sqrt{2}UW^2 + 28\sqrt{2}W^3 + U^2W^2 - 2UW^3 + W^4). \end{aligned} \tag{21}$$

Then applying [5, Theorem 2.19] we get that the singular point $(0, \tilde{w}^*)$ is a saddle-node.

Going back trhought the u–directional blow up, undoing the rescaling of time, and taking into account that

$$(\dot{u}, \dot{v})|_{u=0} = (-v^2/\sqrt{d}, -v^2 - v^3)/\sqrt{d} \text{ and } (\dot{u}, \dot{v})|_{v=0} = (-4du^2 - 4d^{3/2}u^3 - d^2u^4, 0),$$

we get the sequence of phase portraits given in Figure 1 when $0 < d < 1/8$, in Figure 2 when $d = 1/8$, and in Figure 3 when $d > 1/8$. Notice that when $d = 1/8$ we have taken into account that in system (21) the separatrices of the saddle are tangent to $U = 0$ and $W = 0$, so in the initial system (20) they are tangent to $u = 96(w - 1/(4\sqrt{2}))$ and $u = 0$, respectively. So the singular point $(1/\sqrt{d}, 0)$ of (16) is formed by the union of two hyperbolic and four parabolic sectors when $0 < d < 1/8$; two hyperbolic and two parabolic sectors when $d = 1/8$; and exactly two hyperbolic sectors when $d > 1/8$.

We proceed in the same way with the linearly zero singular point $(u^-, 0) = (-1/\sqrt{d}, 0)$ and we also get that it is formed by the union of two hyperbolic sectors only when $d > 1/8$. These possible cases are given in condition i_5 in Appendix A.

Notice that when going back through the u–directional blow ups all separatrices transversal to the straight line $u = 0$ persist and they divide different sectors. The parabolic sectors associated to the nodal parts can go back either to parabolic or to elliptic sectors, and the elliptic sectors go back to elliptic sectors. Taking this into account, in order that in the blow down process we could get that the singular point is formed by two degenerated hyperbolic sectors we need that in the last step of the chain of blow ups the origin be a saddle or a linearly zero singular point and the other possible singular points whenever exist be linearly zero. If in the last step of the chain of blow ups we have more than one non linearly zero singular point, then going back through the u–directional blow ups we will obtain either more than two hyperbolic sectors or some parabolic/elliptic sectors.

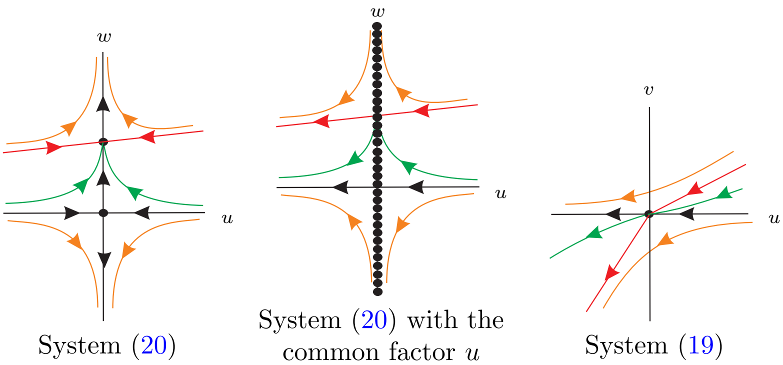

Fig. 2 Sequence of phase portraits near the origin of system (19) with $d = 1/8$.

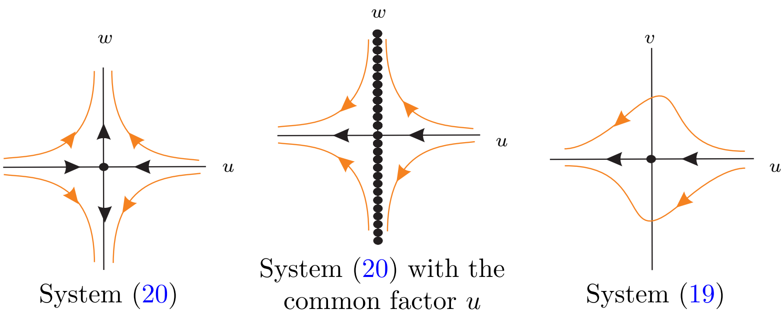

Fig. 3 Sequence of phase portraits near the origin of system (19) with $d > 1/8$.

2.3 Infinite singular points in the local chart U_2 of system (7)

In the local chart U_2 system (7) becomes

$$u' = auv + bu^2 + c - du^2 + u^4 - uv + v^2, \quad v' = -v\left(du - u^3 + v\right). \tag{22}$$

We are interested in the cases where either the origin is not a singular point or if it is a singular point it is formed by the union of two hyperbolic sectors.

If $c \neq 0$ the origin is not a singular point. This gives condition j_1 in Appendix A.

If $c = 0$ then the origin is a linearly zero singular point and we study this point by means of blow ups. The characteristic polynomial of (22) with $c = 0$ at the origin is $\mathcal{F} = -v(auv + bu^2 + v^2)$. Since $v = 0$ is a simple characteristic direction, we apply the u-directional blow up $(u, v) \to (u, uw)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u(aw + b - d + u^2 + w^2 - w), \\ w' &= -w(aw + b + w^2). \end{aligned} \tag{23}$$

The singular points of (23) with $u = 0$ are

$$(0,0) \quad \text{and} \quad (0, w^{\pm}) = \left(0, \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})\right).$$

The points $(0, w^{\pm})$ only exists when $a^2 - 4b \geq 0$. Moreover $w^+ = w^- = 0$ when $a = b = 0$.

The Jacobian matrix at the origin is $\begin{pmatrix} b-d & 0 \\ 0 & -b \end{pmatrix}$ and the Jacobian matrices at the points $(0, w^{\pm})$ are, respectively, $\begin{pmatrix} \lambda_1^{\pm} & 0 \\ 0 & \lambda_2^{\pm} \end{pmatrix}$ where $\lambda_1^{\pm} = (a - 2d \mp \sqrt{a^2 - 4b})/2$ and $\lambda_2^{\pm} = (-a^2 + 4b \pm a\sqrt{a^2 - 4b})/2$.

Now we analyze the cases where the origin $(0,0)$ and $(0, w^{\pm})$ are linearly zero.

We start with the origin. The origin is linearly zero when $b = d = 0$. The characteristic polynomial of (23) at the origin when $b = d = 0$ is $\mathcal{F} = (1 - 2a)uw^2$.

We start analyzing the case $a = 1/2$ where $\mathcal{F} \equiv 0$. System (23) when $a = 1/2$ can be written as

$$u' = -\frac{uw}{2} + h.o.t., \qquad v' = -\frac{w^2}{2} + h.o.t.,$$

so $w = 0$ is a singular direction (see [1]). Hence there exists exactly one semipath tending to the origin in the direction given by the angle θ in forward or backward time for every $\theta \notin \{0, \pi\}$. So in this case the origin of (23) must have parabolic sectors.

Assume now that $a \neq 1/2$. Since $u = 0$ is a simple characteristic direction of (23) at the origin, we apply the u–directional blow up $(u, v) \to (u, uw_1)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u(aw_1 + uw_1^2 + u - w_1), \\ w_1' &= -w_1(2aw_1 + 2uw_1^2 + u - w_1). \end{aligned} \tag{24}$$

The unique singular point of (24) with $u = 0$ is the origin which is again linearly zero. The characteristic polynomial at the origin of (24) is $\mathcal{F} = -uw_1(3aw_1 + 2u - 2w_1)$. Since $u = 0$ is a simple characteristic direction of (24) at the origin, we apply the u–directional blow up $(u, v) \to (u, uw_2)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u(aw_2 + u^2w_2^2 - w_2 + 1), \\ w_2' &= -w_2(3aw_2 + 3u^2w_2^2 - 2w_2 + 2). \end{aligned} \tag{25}$$

The singular points of (25) are the origin and the point $(0, w_2^*) = (0, -2/(3a - 2))$, the last one defined only when $a \neq 2/3$. If $a = 2/3$ the unique singular point is the origin. The origin is always a saddle with Jacobian matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$. The Jacobian matrix at the point $(0, w_2^*)$ is $\begin{pmatrix} a/(3a-2) & 0 \\ 0 & 2 \end{pmatrix}$, so the point is an unstable hyperbolic node when $a \in (-\infty, 0) \cup (2/3, +\infty)$, a hyperbolic saddle when $a \in (0, 2/3)$, and it is semi-hyperbolic when $a = 0$. Applying [5, Theorem 2.19] in a similar way than in Section 2.2 we get that w_2^* is a semi-hyperbolic saddle when $a = 0$.

The singular points $(0, w^{\pm})$ are both linearly zero when $d = a/2$, $b = a^2/4$ and $a \neq 0$ and when $d = 0$, $b = 0$ and $a = 0$ (in this case $w^+ = w^- = 0$ and the origin is the unique linearly zero singular point, so it has already been studied). If $d = 0$, $b = 0$ and $a > 0$, then $(0, w^+) = (0, 0)$ is linearly zero, and $(0, w^-) = (0, -a)$ is a saddle. If $d = 0$, $b = 0$ and $a < 0$, then $(0, w^-) = (0, 0)$ is linearly zero, and $(0, w^+) = (0, -a)$ is a stable node. Now we study the case $d = a/2$, $b = a^2/4$ and $a \neq 0$. In this case $w^+ = w^- = -a/2$. We do the change of variables $(U, W) = (u, w + a/2)$ to move the singular point $(0, -a/2)$ to the origin and we get system

$$u' = u(u^2 + w^2 - w), \qquad w' = \frac{1}{2}w^2(a - 2w). \tag{26}$$

Notice that we have renamed the new variables U and W as u and w. The characteristic polynomial of (26) at the origin is $\mathcal{F} = \frac{1}{2}(a + 2)uw^2$. When $a = -2$ the characteristic polynomial is identically zero with $w = 0$ a singular direction, thus as above the origin of (26) must have parabolic sectors. If $a \neq -2$ then $u = 0$ is a simple characteristic direction so we apply the u–directional blow up

$(u,v) \to (u, uw_1)$ and after eliminating the common factor u we get

$$\begin{aligned} u' &= u(uw_1^2 + u - w_1), \\ w_1' &= -\frac{1}{2} w_1(-aw_1 + 4uw_1^2 + 2u - 2w_1), \end{aligned} \tag{27}$$

The origin is the unique singular point of (27) which is again linearly zero. The characteristic polynomial of (27) at the origin is $\mathcal{F} = -\frac{1}{2}uw_1(-aw_1+4u-4w_1)$, then $u = 0$ is a simple characteristic direction so we apply the u–directional blow up $(u,v) \to (u, uw_2)$ and after eliminating the common factor u we get

$$\begin{aligned} u' &= u(u^2w_2^2 - w_2 + 1), \\ w_2' &= -\frac{1}{2} w_2(-aw_2 + 6u^2w_2^2 - 4w_2 + 4), \end{aligned} \tag{28}$$

The singular points of (28) are the origin which is always a saddle with Jacobian matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$; and the point $(0, \tilde{w}_2^*) = (0, 4/(a+4))$ which is defined for $a \neq -4$ and it is an unstable node when $a \in (-\infty, -4) \cup (0, \infty)$ and a saddle when $a \in (-4, 0)$ and in view of [5, Theorem 2.19] a semi-hyperbolic saddle when $a = 0$.

Now we go back through the u–directional blow ups in a similar way as in Section 2.2. We start going back from blow up (28) to blow up (26). The existence of the point $\tilde{w}_2^*$ would provide either more than two hyperbolic sectors or parabolic sectors. So we are only interested in the case where the origin is the unique singular point of (28). This happens when $a = -4$. In this case going back to the chain of blow ups we have that the origin of (26) is the union of two hyperbolic and two parabolic sectors. Thus the existence of the singular points $(0, w\pm)$ would provide either more than two hyperbolic sectors or parabolic sectors, and so they cannot exist. Thus $a^2 - 4b < 0$.

Now we go back from blow up (25) to blow up (23). The existence of the point w_2^* would provide either more than two hyperbolic sectors or parabolic sectors, so the origin must be the unique singular point of (25) implying that $a = 2/3$. Going back to the chain of blow ups up to blow up (23) we have that the origin of (23) is the union of two hyperbolic and two parabolic sectors. Thus the case where the origin is linearly zero is not possible.

In short, the unique case that could be possible is the case where the origin is the unique singular point of (23) (i.e., $a^2 - 4b < 0$), it is not linearly zero (i.e., $b^2 + d^2 \neq 0$) and it is a saddle. The origin is the unique singular point either when $a^2 - 4b < 0$ (implying that $b > 0$), or when $a = b = 0$. When $a = b = 0$, in wiew of [5, Theorem 2.19] the origin is a semi-hyperbolic saddle when $d < 0$ and an stable node when $d > 0$. When $a^2 - 4b < 0$, the origin is a hyperbolic saddle when $-b(b-d) < 0$, which together with $b > 0$ implies that $b > d$, and it is a semi-hyperbolic singular point when $b = d$. In this last case, since $b = d > 0$, applying [5, Theorem 2.19] we get that the origin is a semi-hyperbolic saddle.

The only possibilities for the origin of the local chart U_2 to be formed by two degenerate hyperbolic sectors are when any of the conditions j_2, j_3 and j_4 in Appendix A hold.

2.4 Proof of Theorem 1

In order to have a global center we need the origin to be the unique finite singular point (corresponding to one of the conditions c_1–c_{85}), that the local chart U_1 has either no infinite singular points or all the infinite singular points are formed by two degenerated hyperbolic sectors (corresponding to one of the conditions i_1–i_5), and that the origin of the local chart U_2 is either not a singular point or it is formed by two degenerated hyperbolic sectors (corresponding to one of the conditions j_1–j_4). Let

$$\tilde{i} = \{(a,b,c,d) \in \mathbb{R}^4 \,:\, i_1 \cup i_2 \cup i_3 \cup i_4 \cup i_5\},$$

be the set where one of the conditions i_1–i_5 is satisfied and

$$\tilde{j} = \{(a,b,c,d) \in \mathbb{R}^4 \,:\, j_1 \cup j_2 \cup j_3 \cup j_4\},$$

be the set where one of the conditions j_1–j_4 is satisfied. Then the set $\tilde{i} \cap \tilde{j}$ can be written as

$$\{(a, b, c, d) \in \mathbb{R}^4 \; : \; ij_1 \cup ij_2 \cup ij_3 \cup ij_4 \cup ij_5 \cup ij_6\},$$

where the sets ij_i for $i = 1, \ldots, 6$ are defined in Appendix A.

In short, system (7) has a global center if (a, b, c, d) belong to one of the sets $c_i \cap ij_j$ for $i = 1, \ldots, 85$, and $j = 1, \ldots, 6$. The sets $c_i \cap ij_j$ that are not empty yield the sets d_1–d_{107} in Appendix A. This completes the proof of Theorem 1.

3 Proof of Theorem 2

Consider system (5) and write it as

$$x' = y(1 + x + ax^2 + by^2), \qquad y' = -x(x^2 - cy^2), \tag{29}$$

for some $a, b, c \in \mathbb{R}$.

3.1 Finite singular points of equation (29)

From the second equation of (29) we have that either $x = 0$ or $y^2 = x^2/c$ whenever $c > 0$. If $x = 0$ then from the first equation of (29) we get $y(1 + by^2) = 0$ which has solutions different from the origin when $b < 0$.

If $c \leq 0$ the origin is the unique finite singular point when $b \geq 0$ providing condition C_1 in Appendix B.

Assume now that $c > 0$, introducing $y^2 = x^2/c$ into the second factor of the first equation in (29) we get an equation equivalent to

$$(ac + b)x^2 + cx + c = 0,$$

whose solutions

$$x = \frac{-c \pm \sqrt{c(-4ac - 4b + c)}}{2(ac + b)},$$

cannot exist. Thus when $c > 0$ the origin is the unique finite singular point if $b \geq 0$ and $a > (c - 4b)/(4c)$, providing condition C_2 in Appendix B.

3.2 Infinite singular points in the local chart U_1 of system (29)

In the local chart U_1 system (29) becomes

$$u' = -au^2 - bu^4 + cu^2 - u^2v^2 - u^2v - 1, \quad v' = -uv\left(a + bu^2 + v^2 + v\right). \tag{30}$$

The infinite singular points satisfy $v = 0$ and

$$-bu^4 + (c - a)u^2 - 1 = 0. \tag{31}$$

The Jacobian matrix of (30) evaluated at $v = 0$ becomes

$$J = \begin{pmatrix} -2u\left(2bu^2 + a - c\right) & -u^2 \\ 0 & -u(bu^2 + a) \end{pmatrix}.$$

Since the solutions of (31) are always different from zero, the infinite singular points on the local chart U_1 whenever exist are never linearly zero, so they cannot exist.

We distinguish three cases: $b = 0$ and $c - a = 0$; $b = 0$ and $c - a \neq 0$; and $b \neq 0$.

If $b = 0$ and $c - a = 0$, equation (31) has no real solutions. This gives the set I_1 in Appendix B.

If $b = 0$ and $c - a \neq 0$, the solutions of (31) are

$$u = u^{\pm} = \pm\frac{1}{\sqrt{c - a}}.$$

Thus (31) has no real solutions when $c-a<0$, yielding the set I_2 in Appendix B.

Finally, when $b \neq 0$ the solutions of (31) are

$$u = \tilde{u}_1^{\pm} = \frac{\sqrt{E^{\pm}}}{\sqrt{2}}, \quad u = \tilde{u}_2^{\pm} = -\frac{\sqrt{E^{\pm}}}{\sqrt{2}},$$

where

$$E^{\pm} = -\frac{a}{b} \pm \frac{c}{b} + \frac{\sqrt{(a-c)^2 - 4b}}{b}.$$

The solutions $\tilde{u}_1^{\pm}$ do not exist either when $(a-c)^2 - 4b < 0$, or when $(a-c)^2 - 4b \geq 0$ and E^+ and E^- are both negative, this provide the sets I_3 and I_4, respectively, in Appendix B.

3.3 Infinite singular points in the local chart U_2 of system (29)

In the local chart U_2 system (29) becomes

$$u' = au^2 + b - cu^2 + u^4 + uv + v^2, \quad v' = v\left(u^3 - cu\right). \tag{32}$$

If $b \neq 0$ the origin is not a singular point. This gives condition J_1 in Appendix B.

If $b = 0$ then the origin is a linearly zero singular point and as in the previous section we study this point by means of blow ups. The characteristic polynomial of (32) with $b = 0$ at the origin is $\mathcal{F} = -v\left(au^2 + uv + v^2\right)$. Since $v = 0$ is a simple characteristic direction, we apply the u-directional blow up $(u, v) \to (u, uw)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u\left(a - c + u^2 + w^2 + w\right), \\ w' &= -w\left(a + w^2 + w\right). \end{aligned} \tag{33}$$

The singular points of (33) with $u = 0$ are

$$(0,0) \quad \text{and} \quad (0, w^{\pm}) = \left(0, \frac{1}{2}\left(-1 \pm \sqrt{1-4a}\right)\right).$$

The points $(0, w^{\pm})$ only exists when $a \leq 1/4$.

The Jacobian matrix at the origin is $\begin{pmatrix} a-c & 0 \\ 0 & -a \end{pmatrix}$ and the Jacobian matrices at the points $(0, w^{\pm})$ are $\begin{pmatrix} -c & 0 \\ 0 & \frac{1}{2}\left(4a - 1 \pm \sqrt{1-4a}\right) \end{pmatrix}$.

Now we analyze the cases where the origin $(0,0)$ and $(0, w^{\pm})$ are linearly zero.

We start with the origin. The origin is linearly zero when $a = c = 0$. The characteristic polynomial of (33) at the origin when $a = c = 0$ is $\mathcal{F} = -2uw^2$. Since $u = 0$ is a simple characteristic direction of (33) at the origin, we apply the u–directional blow up $(u, v) \to (u, uw_1)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u\left(uw_1^2 + u + w_1\right), \\ w_1' &= -w_1\left(2uw_1^2 + u + 2w_1\right). \end{aligned} \tag{34}$$

The unique singular point of (34) with $u = 0$ is the origin which is again linearly zero. The characteristic polynomial at the origin of (24) is $\mathcal{F} = -uw_1(2u + 3w_1)$. Since $u = 0$ is a simple characteristic direction of (24) at the origin, we apply the u–directional blow up $(u, v) \to (u, uw_2)$ and after dividing by the common factor u we get

$$\begin{aligned} u' &= u\left(u^2w_2^2 + w_2 + 1\right), \\ w_2' &= -w_2\left(3u^2w_2^2 + 3w_2 + 2\right). \end{aligned} \tag{35}$$

The singular points of (35) with $u = 0$ are the origin and the point $(0, -2/3)$. The origin is a saddle and the point $(0, -2/3)$ is an unstable node. Going back to the chain of blow ups the point $(0, 2/3)$ would provide parabolic sectors. Therefore this case is not possible.

The singular points $(0, w^{\pm})$ are both linearly zero when $c = 0$ and $a = 1/4$. Under these assumptions $w^{\pm} = -1/2$. We do the change of variables $(U, W) = (u, w + 1/2)$ to move the singular point $(0, -1/2)$ to the origin and we get system

$$u' = u(u^2 + w^2), \qquad w' = \frac{1}{2}w^2(1 - 2w). \tag{36}$$

Notice that we have renamed the new variables U and W as u and w. The characteristic polynomial of (36) at the origin is $\mathcal{F} = \frac{1}{2}uw^2$. Since $u = 0$ is a simple characteristic direction we apply the u–directional blow up $(u, v) \to (u, uw_1)$ and after eliminating the common factor u we get

$$\begin{aligned} u' &= u^2\left(w_1^2 + 1\right), \\ w_1' &= -\frac{1}{2}w_1\left(4uw_1^2 + 2u - w_1\right), \end{aligned} \tag{37}$$

The origin is the unique singular point of (37) which is again linearly zero. The characteristic polynomial of (27) at the origin is $\mathcal{F} = -\frac{1}{2}uw_1(4u - w_1)$, then $u = 0$ is a simple characteristic direction so we apply the u–directional blow up $(u, v) \to (u, uw_2)$ and after eliminating the common factor u we get

$$\begin{aligned} u' &= u\left(u^2w_2^2 + 1\right), \\ w_2' &= -\frac{1}{2}w_2\left(6u^2w_2^2 - w_2 + 4\right), \end{aligned} \tag{38}$$

The singular points of (38) on $u = 0$ are the origin which is always a saddle, and the point $(0, 4)$ which is an unstable node. Going back to the chain of blow ups the points $w^{\pm}$ would provide more that two hyperbolic sectors and parabolic sectors. Therefore they cannot exist.

Hence, the origin is a singular point in the local chart U_2 formed by two hyperbolic sectors if the points $w^{\pm}$ do no exist and the point $(0, 0)$ is a saddle. The points $w^{\pm}$ do not exist when $a > 1/4$. Assuming that $a > 1/4$, the origin is a hyperbolic saddle when $-a(a - c) < 0$ and in view of [5, Theorem 2.19] it is a semi-hyperbolic saddle when $a = c$. In short we get conditions J_2 and J_3 in Appendix B.

3.4 Proof of Theorem 2

In order to have a global center we need the origin to be the unique finite singular point (corresponding to one of the conditions C_1–C_2), that the local chart U_1 has either no infinite singular points or all of them are formed by two degenerated hyperbolic sectors (corresponding to one of the conditions I_1–I_4), and that the origin of the local chart U_2 is either not a singular point or it is formed by two degenerated hyperbolic sectors (corresponding to one of the conditions J_1–J_3). The set

$$\{(a, b, c) \in \mathbb{R}^3 \;:\; I_1 \cup I_2 \cup I_3 \cup I_4\} \cap \{(a, b, c, d) \in \mathbb{R}^4 \;:\; J_1 \cup J_2 \cup J_3\}$$

can be written as

$$\{(a, b, c) \in \mathbb{R}^3 \;:\; IJ_1 \cup IJ_2 \cup IJ_3 \cup IJ_4 \cup IJ_5\},$$

with the sets IJ_i for $i = 1, \ldots, 5$ being defined in Appendix B.

In short, system (29) has a global center if (a, b, c) belong to one of the sets $C_i \cap IJ_j$ for $i = 1, 2$, and $j = 1, \ldots, 5$. The sets $C_i \cap IJ_j$ that are not empty yield the sets e_1–e_8 in Appendix B. This completes the proof of Theorem 2.

Appendix A Conditions in Theorem 1

Let

$$K^{\pm} = \frac{1}{27}\left(-2a^3 + 3a^2d + 9ab + 3ad^2 - 18bd - 2d^3 \pm 2\sqrt{\left(a^2 - ad - 3b + d^2\right)^3}\right).$$

The conditions in order that the origin is the unique finite singular point are the following ones:

$$\begin{aligned}
c_1 &= \{c = 0, b = 0, a = 0\},\\
c_2 &= \{c = 0, b = 0, a \neq 0, a \geq d\},\\
c_3 &= \{c = 0, b \neq 0, a^2 - 4b < 0\},\\
c_4 &= \{c = 0, 0 < b \leq a^2/4, d \leq D^-\},\\
c_5 &= \{c = 0, b < 0, d \leq D^-\},\\
c_6 &= \{c \neq 0, a = -d, b = d^2, c = -d^3\},\\
c_7 &= \{c > 0, a = c/d^2, b = -c/d, c \neq -d^3, d \neq 0\},\\
c_8 &= \{c \neq 0, c = -bd, b \neq -ad, b > (a-d)^2/4\},\\
c_9 &= \{c \neq 0, c = -bd, a > d, 0 < b \leq (a-d)^2/4, d > 0\},\\
c_{10} &= \{c \neq 0, c = -bd, a > -d, -ad < b \leq (a-d)^2/4, d < 0\},\\
c_{11} &= \{c \neq 0, c = -bd, 0 < a < -d, -ad < b \leq (a-d)^2/4, d < 0\},\\
c_{12} &= \{c \neq 0, c = -bd, d < a \leq 0, 0 < b \leq (a-d)^2/4, d < 0\},\\
c_{13} &= \{c \neq 0, c = -bd, a > 0, 0 < b < -ad, d < 0\}\\
c_{14} &= \{c > 0, c + bd > 0, \Delta < 0\},\\
c_{15} &= \{c > 0, c + bd < 0, \Delta < 0\},\\
c_{16} &= \{a > 0, 0 < b \leq a^2/4, 0 < c < K^+|_{d=0}, d = 0\},\\
c_{17} &= \{a > 0, a^2/4 < b < a^2/3, K^-|_{d=0} < c < K^+|_{d=0}, d = 0\},\\
c_{18} &= \{a > -d, 0 < b < -ad, -bd < c < K^+, d < 0\},\\
c_{19} &= \{a > -d, d(a-d) < b \leq 0, 0 < c < K^+, d < 0\},\\
c_{20} &= \{a > -d, (a-d)^2/4 < b < (a^2 - ad + d^2)/3, K^- < c < K^+, d < 0\},\\
c_{21} &= \{a > -d, -ad < b \leq (a-d)^2/4, -bd < c < K^+, d < 0\},\\
c_{22} &= \{a > 2d, d(a-d) < b \leq a^2/4, 0 < c < K^+, d > 0\},\\
c_{23} &= \{a > 2d, a^2/4 < b < (a^2 - ad + d^2)/3, K^- < c < K^+, d > 0\},\\
c_{24} &= \{d < a \leq -d, 0 < b < (a-d)^2/4, -bd < c < K^+, d < 0\},\\
c_{25} &= \{d < a \leq -d, d(a-d) < b \leq 0, 0 < c < K^+, d < 0\},\\
c_{26} &= \{a = -4d, 0 < b \leq 4d^2, 0 < c < -bd, d < 0\},\\
c_{27} &= \{a = -4d, 4d^2 < b < 25d^2/4, K^-|_{a=-4d} < c < -bd, d < 0\},\\
c_{28} &= \{a = -d, 0 < b \leq d^2/4, 0 < c < -bd, d < 0\},\\
c_{29} &= \{a = -d, d^2/4 < b < d^2, K^-|_{a=-d} < c < -bd, d < 0\},\\
c_{30} &= \{a = d/2, 0 < b \leq d^2/16, 0 < c < -bd, d < 0\},\\
c_{31} &= \{a = d/2, d^2/16 < b < d^2/4, K^-|_{a=d/2} < c < K^+|_{a=d/2}, d < 0\},\\
c_{32} &= \{a = d, 0 < b \leq d^2/4, 0 < c < K^+|_{a=d}, d < 0\},\\
c_{33} &= \{a = d, d^2/4 < b < d^2/3, K^-|_{a=d} < c < K^+|_{a=d}, d < 0\},\\
c_{34} &= \{a > -4d, 0 < b \leq a^2/4, 0 < c < -bd, d < 0\},\\
c_{35} &= \{a > -4d, a^2/4 < b < (a-d)^2/4, K^- < c < -bd, d < 0\},\\
c_{36} &= \{0 < a < -d, 0 < b \leq a^2/4, 0 < c < -bd, d < 0\},\\
c_{37} &= \{0 < a < -d, a^2/4 < b < -ad, K^- < c < -bd, d < 0\},\\
c_{38} &= \{0 < a < -d, (a-d)^2/4 < b < (a^2 - ad + d^2)/3, K^- < c < K^+, d < 0\},
\end{aligned}$$

$$
\begin{aligned}
c_{39} &= \{0 < a < -d, -ad < b \leq (a-d)^2/4, K^- < c < -bd, d < 0\},\\
c_{40} &= \{-d < a < -4d, 0 < b \leq a^2/4, 0 < c < -bd, d < 0\},\\
c_{41} &= \{d < a < d/2, 0 < b \leq (a-d)^2/4, 0 < c < -bd, d < 0\},\\
c_{42} &= \{d/2 < a \leq 0, 0 < b \leq a^2/4, 0 < c < -bd, d < 0\},\\
c_{43} &= \{d < a < d/2, (a-d)^2/4 < b \leq a^2/4, 0 < c < K^+, d < 0\},\\
c_{44} &= \{2d < a < d, d(a-d) < b \leq a^2/4, 0 < c < K^+, d < 0\},\\
c_{45} &= \{-d < a < -4d, a^2/4 < b < (a-d)^2/4, K^- < c < -bd, d < 0\},\\
c_{46} &= \{d < a < d/2, a^2/4 < b < (a^2-ad+d^2)/3, K^- < c < K^+, d < 0\},\\
c_{47} &= \{2d < a < d, a^2/4 < b < (a^2-ad+d^2)/3, K^- < c < K^+, d < 0\},\\
c_{48} &= \{d/2 < a \leq 0, a^2/4 < b \leq (a-d)^2/4, K^- < c < -bd, d < 0\},\\
c_{49} &= \{d/2 < a \leq 0, (a-d)^2/4 < b < (a^2-ad+d^2)/3, K^- < c < K^+, d < 0\},\\
c_{50} &= \{a > -d, b = (a^2-ad+d^2)/3, c = (a-2d)^3/27, d \leq 0\},\\
c_{51} &= \{a > 2d, b = (a^2-ad+d^2)/3, c = (a-2d)^3/27, d > 0\},\\
c_{52} &= \{2d < a < -d, b = (a^2-ad+d^2)/3, c = (a-2d)^3/27, d < 0\},\\
c_{53} &= \{a > 0, a^2/4 < b < a^2/3, c = K^-|_{d=0}, d = 0\},\\
c_{54} &= \{a > 0, 0 < b \leq a^2/4, c = K^+|_{d=0}, d = 0\},\\
c_{55} &= \{a > 0, a^2/4 < b < a^2/3, c = K^+|_{d=0}, d = 0\},\\
c_{56} &= \{a > -d, (a-d)^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d < 0\},\\
c_{57} &= \{a > 2d, a^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d > 0\},\\
c_{58} &= \{a > -d, d(a-d) < b < -ad, c = K^+, d < 0\},\\
c_{59} &= \{a > -d, (a-d)^2/4 < b < (a^2-ad+d^2)/3, c = K^+, d < 0\},\\
c_{60} &= \{a > -d, -ad < b \leq (a-d)^2/4, c = K^+, d < 0\},\\
c_{61} &= \{a > 2d, a^2/4 < b < (a^2-ad+d^2)/3, c = K^+, d > 0\},\\
c_{62} &= \{a > 2d, d(a-d) < b \leq a^2/4, c = K^+, d > 0\},\\
c_{63} &= \{d < a \leq -d, d(a-d) < b < (a-d)^2/4, c = K^+, d < 0\},\\
c_{64} &= \{a = -4d, 4d^2 < b < 25d^2/4, c = K^-|_{a=-4d}, d < 0\},\\
c_{65} &= \{a = -d, d^2/4 < b < d^2, c = K^-|_{a=-d}, d < 0\},\\
c_{66} &= \{a = d/2, d^2/16 < b < d^2/4, c = K^-|_{a=d/2}, d < 0\},\\
c_{67} &= \{a = d/2, d^2/16 < b < d^2/4, c = K^+|_{a=d/2}, d < 0\},\\
c_{68} &= \{a = d, d^2/4 < b < d^2/3, c = K^-|_{a=d}, d < 0\},\\
c_{69} &= \{a = d, d^2/4 < b < d^2/3, c = K^+|_{a=d}, d < 0\},\\
c_{70} &= \{a = d, 0 < b \leq d^2/4, c = K^+|_{a=d}, d < 0\},\\
c_{71} &= \{a > -4d, a^2/4 < b < (a-d)^2/4, c = K^-, d < 0\},\\
c_{72} &= \{0 < a < -d, a^2/4 < b < -ad, c = K^-, d < 0\},\\
c_{73} &= \{0 < a < -d, (a-d)^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d < 0\},\\
c_{74} &= \{0 < a < -d, -ad < b \leq (a-d)^2/4, c = K^-, d < 0\},\\
c_{75} &= \{-d < a < -4d, a^2/4 < b < (a-d)^2/4, c = K^-, d < 0\},\\
c_{76} &= \{d < a < d/2, a^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d < 0\},\\
c_{77} &= \{2d < a < d, a^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d < 0\},\\
c_{78} &= \{d/2 < a \leq 0, a^2/4 < b \leq (a-d)^2/4, c = K^-, d < 0\},\\
c_{79} &= \{d/2 < a \leq 0, (a-d)^2/4 < b < (a^2-ad+d^2)/3, c = K^-, d < 0\},\\
c_{80} &= \{0 < a < -d, (a-d)^2/4 < b < (a^2-ad+d^2)/3, c = K^+, d < 0\},\\
c_{81} &= \{d < a < d/2, a^2/4 < b < (a^2-ad+d^2)/3, c = K^+, d < 0\},\\
c_{82} &= \{d < a < d/2, (a-d)^2/4 < b \leq a^2/4, c = K^+, d < 0\},
\end{aligned}
$$

$$
\begin{aligned}
c_{83} &= \{2d < a < d, a^2/4 < b < (a^2 - ad + d^2)/3, c = K^+, d < 0\},\\
c_{84} &= \{2d < a < d, d(a-d) < b \le a^2/4, c = K^+, d < 0\},\\
c_{85} &= \{d/2 < a \le 0, (a-d)^2/4 < b < (a^2 - ad + d^2)/3, c = K^+, d < 0\}.
\end{aligned}
$$

The conditions in order that the local chart U_1 has either no infinite singular points or all the infinite singular points in the local chart U_1 are formed by two degenerated hyperbolic sectors are the following:

$$
\begin{aligned}
i_1 &= \{c = 0, b = d\},\\
i_2 &= \{c = 0, b > d\},\\
i_3 &= \{c \ne 0, b^2 - 4c - 2bd + d^2 < 0\},\\
i_4 &= \{c \ne 0, b > d, 0 < c \le (b-d)^2/4\},\\
i_5 &= \{c \ne 0, a = 1, b = -d, c = d^2, d > 1/8\}.
\end{aligned}
$$

The conditions in order that the origin of the local chart U_2 is either not a singular point or it is formed by two degenerated hyperbolic sectors are:

$$
\begin{aligned}
j_1 &= \{c \ne 0\},\\
j_2 &= \{c = 0, a^2 - 4b < 0, b > d\},\\
j_3 &= \{c = 0, a^2 - 4b < 0, b = d, d > 0\},\\
j_4 &= \{c = 0, a = b = 0, d < 0\}.
\end{aligned}
$$

The sets ij_i for $i = 1, \ldots, 6$ are:

$$
\begin{aligned}
ij_1 &= \{b > a^2/4, b = d, c = 0\},\\
ij_2 &= \{a = 0, b = 0, c = 0, d < 0\},\\
ij_3 &= \{b > a^2/4, c = 0, d < b\},\\
ij_4 &= \{a = 1, b = -d, c = d^2, d > 1/8\},\\
ij_5 &= \{b > d, 0 < c \le (b-d)^2/4\},\\
ij_6 &= \{c > (b-d)^2/4\}.
\end{aligned}
$$

The sets d_i in Theorem 1 are:

$$
\begin{aligned}
d_1 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_3 \cap ij_1\} \ni (0,1,0,1)\,,\\
d_2 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_1 \cap ij_2\} \ni (0,0,0,-1)\,,\\
d_3 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_3 \cap ij_3\} \ni (0,1,0,1/2)\,,\\
d_4 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_7 \cap ij_4\} \ni (1,-1,1,1)\,,\\
d_5 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_6 \cap ij_5\} \ni (2,4,8,-2)\,,\\
d_6 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_7 \cap ij_5\} \ni (1/2,1/8,1/32,-1/4)\,,\\
d_7 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_8 \cap ij_5\} \ni (0,3/2,3,-2)\,,\\
d_8 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{10} \cap ij_5\} \ni (2,17/8,17/8,-1)\,,\\
d_9 &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{11} \cap ij_5\} \ni (1/2,3,15,-5)\,,\\
d_{10} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{12} \cap ij_5\} \ni (-1,5/2,25/2,-5)\,,\\
d_{11} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{13} \cap ij_5\} \ni (1,1/2,1/2,-1)\,,\\
d_{12} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{14} \cap ij_5\} \ni (-3,17/8,161/512,1)\,,\\
d_{13} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{15} \cap ij_5\} \ni (-2333/8,137,280725/8,-269)\,,
\end{aligned}
$$

$$d_{14} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{16} \cap ij_5\} \ni (1/2, 1/32, 1/8192, 0)\,,$$
$$d_{15} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{17} \cap ij_5\} \ni (1/2, 133/2048, 27/32768, 0)\,,$$
$$d_{16} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{18} \cap ij_5\} \ni (3/2, 1/4, 5/16, -1)\,,$$
$$d_{17} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{19} \cap ij_5\} \ni (7/4, -1/2, 1/32, -1)\,,$$
$$d_{18} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{20} \cap ij_5\} \ni (7/4, 245/128, 985/512, -1)\,,$$
$$d_{19} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{21} \cap ij_5\} \ni (3/2, 49/32, 6273/4096, -1)\,,$$
$$d_{20} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{22} \cap ij_5\} \ni (13/4, 39/16, 1/256, 1)\,,$$
$$d_{21} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{23} \cap ij_5\} \ni (13/4, 173/64, 45/1024, 1)\,,$$
$$d_{22} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{24} \cap ij_5\} \ni (-27/32, 3/1024, 9/2048, -1)\,,$$
$$d_{23} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{25} \cap ij_5\} \ni (-1/2, -1/4, 3/512, -1)\,,$$
$$d_{24} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{26} \cap ij_5\} \ni (4, 1/2, 1/4, -1)\,,$$
$$d_{25} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{27} \cap ij_5\} \ni (4, 37/8, 3, -1)\,,$$
$$d_{26} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{28} \cap ij_5\} \ni (5, 3, 29/2, -5)\,,$$
$$d_{27} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{29} \cap ij_5\} \ni (1, 11/32, 11/64, -1)\,,$$
$$d_{28} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{30} \cap ij_5\} \ni (-1/2, 1/32, 1/64, -1)\,,$$
$$d_{29} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{31} \cap ij_5\} \ni (-1/2, 13/128, 5/128, -1)\,,$$
$$d_{30} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{32} \cap ij_5\} \ni (-4, 2, 1/8, -4)\,,$$
$$d_{31} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{33} \cap ij_5\} \ni (-4, 37/8, 23/16, -4)\,,$$
$$d_{32} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{34} \cap ij_5\} \ni (5, 1/2, 1/4, -1)\,,$$
$$d_{33} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{35} \cap ij_5\} \ni (8, 18, 14, -1)\,,$$
$$d_{34} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{36} \cap ij_5\} \ni (5/2, 1, 5/2, -3)\,,$$
$$d_{35} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{37} \cap ij_5\} \ni (1/8, 1/4, 511/512, -4)\,,$$
$$d_{36} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{38} \cap ij_5\} \ni (5/32, 39/8, 73/4, -4)\,,$$
$$d_{37} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{39} \cap ij_5\} \ni (1/8, 9/4, 569/64, -4)\,,$$
$$d_{38} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{40} \cap ij_5\} \ni (5/4, 1/4, 1/8, -1)\,,$$
$$d_{39} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{41} \cap ij_5\} \ni (-12, 7/2, 59, -17)\,,$$
$$d_{40} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{42} \cap ij_5\} \ni (-9, 21/2, 220, -21)\,,$$
$$d_{41} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{43} \cap ij_5\} \ni (-16, 53, 648, -29)\,,$$
$$d_{42} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{44} \cap ij_5\} \ni (-4, 7/2, 1/64, -3)\,,$$
$$d_{43} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{45} \cap ij_5\} \ni (3/2, 25/32, 19/32, -1)\,,$$
$$d_{44} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{46} \cap ij_5\} \ni (-3/4, 13/64, 45/1024, -1)\,,$$
$$d_{45} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{47} \cap ij_5\} \ni (-4, 17/4, 45/128, -13/4)\,,$$
$$d_{46} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{48} \cap ij_5\} \ni (-1/4, 5/64, 17/256, -1)\,,$$
$$d_{47} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{49} \cap ij_5\} \ni (-1/4, 13/64, 163/1024, -1)\,,$$
$$d_{48} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{50} \cap ij_5\} \ni (2, 7/3, 64/27, -1)\,,$$
$$d_{49} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{51} \cap ij_5\} \ni (3, 7/3, 1/27, 1)\,,$$
$$d_{50} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{52} \cap ij_5\} \ni (-2, 4/3, 8/27, -2)\,,$$
$$d_{51} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{53} \cap ij_5\} \ni \Big(1/2, 133/2048, (2768 - 113\sqrt{226})/1769472, 0\Big)\,,$$
$$d_{52} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{54} \cap ij_5\} \ni \Big(35/32, 3/32, (-27755 + 937\sqrt{937})/442368, 0\Big)\,,$$
$$d_{53} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{55} \cap ij_5\} \ni (81/64, 57/128, 49/1024, 0)\,,$$
$$d_{54} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{56} \cap ij_5\} \ni \Big(7/4, 245/128, (1970 - \sqrt{2})/1024, -1\Big)\,,$$
$$d_{55} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{57} \cap ij_5\} \ni \Big(4, 133/32, (616 - 17\sqrt{34})/3456, 1\Big)\,,$$

$$d_{56} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{58} \cap ij_5\} \ni \Big(7/4, 25/16, (95+4\sqrt{2})/64, -1\Big)\,,$$
$$d_{57} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{59} \cap ij_5\} \ni \Big(7/4, 245/128, (1970+\sqrt{2})/1024, -1\Big)\,,$$
$$d_{58} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{60} \cap ij_5\} \ni \Big(7/4, 29/16, (1035+4\sqrt{6})/576, -1\Big)\,,$$
$$d_{59} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{61} \cap ij_5\} \ni \Big(4, 133/32, (616+17\sqrt{34})/3456, 1\Big)\,,$$
$$d_{60} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{62} \cap ij_5\} \ni \Big(4, 7/2, (-14+5\sqrt{10})/54, 1\Big)\,,$$
$$d_{61} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{63} \cap ij_5\} \ni \Big(-3/4, -1/8, (-80+19\sqrt{19})/864, -1\Big)\,,$$
$$d_{62} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{64} \cap ij_5\} \ni \Big(4, 5, (36-4\sqrt{6})/9, -1\Big)\,,$$
$$d_{63} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{65} \cap ij_5\} \ni \Big(1/4, 5/128, (10-\sqrt{2})/1024, -1/4\Big)\,,$$
$$d_{64} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{66} \cap ij_5\} \ni \Big(-16, 160, 2560-256\sqrt{2}, -32\Big)\,,$$
$$d_{65} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{67} \cap ij_5\} \ni \Big(-16, 160, 2560+256\sqrt{2}, -32\Big)\,,$$
$$d_{66} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{68} \cap ij_5\} \ni \Big(-1, 37/128, (616-17\sqrt{34})/27648, -1\Big)\,,$$
$$d_{67} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{69} \cap ij_5\} \ni \Big(-1, 37/128, (616+17\sqrt{34})/27648, -1\Big)\,,$$
$$d_{68} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{70} \cap ij_5\} \ni \Big(-1, 1/8, (-14+5\sqrt{10})/432, -1\Big)\,,$$
$$d_{69} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{71} \cap ij_5\} \ni \Big(8, 18, (430-38\sqrt{19})/27, -1\Big)\,,$$
$$d_{70} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{72} \cap ij_5\} \ni \Big(1/8, 1/8, (1881-83\sqrt{249})/2304, -2\Big)\,,$$
$$d_{71} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{73} \cap ij_5\} \ni \Big(5/32, 39/8, (2692737-691\sqrt{2073})/147456, -4\Big)\,,$$
$$d_{72} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{74} \cap ij_5\} \ni (1/8, 9/4, 1125/128, -4)\,,$$
$$d_{73} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{75} \cap ij_5\} \ni \Big(7/4, 17/16, (495-28\sqrt{42})/576, -1\Big)\,,$$
$$d_{74} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{76} \cap ij_5\} \ni \Big(-16, 118, 1080-108\sqrt{2}, -26\Big)\,,$$
$$d_{75} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{77} \cap ij_5\} \ni \Big(-4, 17/4, (305-13\sqrt{13})/864, -13/4\Big)\,,$$
$$d_{76} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{78} \cap ij_5\} \ni \Big(-1/4, 5/64, (595-37\sqrt{37})/6912, -1\Big)\,,$$
$$d_{77} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{79} \cap ij_5\} \ni \Big(-1/4, 13/64, (1099-13\sqrt{13})/6912, -1\Big)\,,$$
$$d_{78} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{80} \cap ij_5\} \ni \Big(7/32, 51/128, (171181+73\sqrt{73})/442368, -1\Big)\,,$$
$$d_{79} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{81} \cap ij_5\} \ni \Big(-16, 118, 1080+108\sqrt{2}, -26\Big)\,,$$
$$d_{80} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{82} \cap ij_5\} \ni \Big(-3/4, 5/64, (-55+37\sqrt{37})/6912, -1\Big)\,,$$
$$d_{81} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{83} \cap ij_5\} \ni \Big(-4, 17/4, (305+13\sqrt{13})/864, -13/4\Big)\,,$$
$$d_{82} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{84} \cap ij_5\} \ni \Big(-4, 7/2, (-14+5\sqrt{10})/54, -3\Big)\,,$$
$$d_{83} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{85} \cap ij_5\} \ni \Big(-1/4, 13/64, (1099+13\sqrt{13})/6912, -1\Big)\,,$$
$$d_{84} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{14} \cap ij_6\} \ni (-5, -70, 1280, 1)\,,$$
$$d_{85} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{16} \cap ij_6\} \ni (1/2, 1/32, 13/32768, 0)\,,$$
$$d_{86} = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ \mathrm{c}_{17} \cap ij_6\} \ni (1/2, 133/2048, 15/8192, 0)\,,$$

$$
\begin{aligned}
d_{87} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{18} \cap ij_6\} \ni (3/2, 1/4, 1/2, -1)\,, \\
d_{88} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{19} \cap ij_6\} \ni (7/4, -21/16, 11/128, -1)\,, \\
d_{89} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{20} \cap ij_6\} \ni (1, 53831/65536, 87485/131072, -13/16)\,, \\
d_{90} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{21} \cap ij_6\} \ni (1, 209/256, 347783/524288, -13/16)\,, \\
d_{91} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{22} \cap ij_6\} \ni (5/16, 19/2048, 1/65536, 3/256)\,, \\
d_{92} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{23} \cap ij_6\} \ni (5/16, 805/32768, 1/4096, 3/256)\,, \\
d_{93} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{24} \cap ij_6\} \ni (29/64, 7/512, 1055/4096, -1)\,, \\
d_{94} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{25} \cap ij_6\} \ni (7/32, -67/64, 1/512, -1)\,, \\
d_{95} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{50} \cap ij_6\} \ni (1/2, 7/64, 1/64, -1/8)\,, \\
d_{96} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{51} \cap ij_6\} \ni (1/2, 19/256, 1/512, 1/16)\,, \\
d_{97} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{53} \cap ij_6\} \ni \Big(1/2, 39/512, (760 - 11\sqrt{22})/221184, 0\Big)\,, \\
d_{98} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{54} \cap ij_6\} \ni \Big(1/2, 1/32, (-14 + 5\sqrt{10})/3456, 0\Big)\,, \\
d_{99} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{55} \cap ij_6\} \ni \Big(1/2, 133/2048, (2768 + 113\sqrt{226})/1769472, 0\Big)\,, \\
d_{100} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{56} \cap ij_6\} \ni \Big(1, 1685/2048, (43820 - \sqrt{2})/65536, -13/16\Big)\,, \\
d_{101} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{57} \cap ij_6\} \ni (1, 37/128, 121/8192, 3/32)\,, \\
d_{102} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{58} \cap ij_6\} \ni \Big(8, 0, (-1280 + 896\sqrt{7})/27, -4\Big)\,, \\
d_{103} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{59} \cap ij_6\} \ni \Big(5/32, 589/65536, (148715 + 217\sqrt{217})/536870912, -25/1024\Big)\,, \\
d_{104} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{60} \cap ij_6\} \ni \Big(1/8, 75/16384, (7652195 + 15577\sqrt{15577})/115964116992, -27/2048\Big)\,, \\
d_{105} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{61} \cap ij_6\} \ni \Big(5/16, 805/32768, (53983 + 1339\sqrt{1339})/226492416, 3/256\Big)\,, \\
d_{106} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{62} \cap ij_6\} \ni \Big(5/16, 1/64, (-141155 + 3097\sqrt{3097})/226492416, 3/256\Big)\,, \\
d_{107} &= \{(a,b,c,d) \in \mathbb{R}^4 \,:\, \mathrm{c}_{63} \cap ij_6\} \ni \Big(-9/2, 0, (-575 + 193\sqrt{193})/108, -8\Big)\,.
\end{aligned}
$$

Appendix B Conditions in Theorem 2

The conditions in order that the origin is the unique finite singular point are:

$$
\begin{aligned}
C_1 &= \{b \geq 0, c \leq 0\}, \\
C_2 &= \{b \geq 0, c > 0, a > \frac{c - 4b}{4c}\}.
\end{aligned}
$$

The conditions in order that the local chart U_1 has either no infinite singular points or all the infinite singular points in the local chart U_1 are formed by two degenerated hyperbolic sectors are the following:

$$
\begin{aligned}
I_1 &= \{b = 0, a = c\}, \\
I_2 &= \{b = 0, a > c\}, \\
I_3 &= \{b \neq 0, b > (a - c)^2/4\}, \\
I_4 &= \{b \neq 0, a > c, 0 < b \leq (a - c)^2/4\}.
\end{aligned}
$$

The conditions in order that the origin of the local chart U_2 is either not a singular point or it is formed by two degenerated hyperbolic sectors are:

$$
\begin{aligned}
J_1 &= \{b \neq 0\}, \\
J_2 &= \{b = 0, a > 1/4, a > c\},
\end{aligned}
$$

$$J_3 = \{b = 0, a = c > 1/4\}.$$

The sets IJ_i for $i = 1, \ldots 5$ are:

$$\begin{aligned}
IJ_1 &= \{b > (a-c)^2/4\}, \\
IJ_2 &= \{b > 0, a > c\}, \\
IJ_3 &= \{b = 0, a = c > 1/4\}, \\
IJ_4 &= \{b = 0, a > 1/4, c \leq 1/4\}, \\
IJ_5 &= \{b = 0, a > c > 1/4\}.
\end{aligned}$$

The conditions in Theorem 2 are:

$$\begin{aligned}
e_1 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_1 \cap IJ_1\} \ni (0, 2, -1)\,, \\
e_2 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_1 \cap IJ_2\} \ni (0, 1, -1)\,, \\
e_3 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_1 \cap IJ_4\} \ni (5/4, 0, 0)\,, \\
e_4 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_2 \cap IJ_1\} \ni (0, 2, 1)\,, \\
e_5 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_2 \cap IJ_2\} \ni (2, 1, 1)\,, \\
e_6 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_2 \cap IJ_3\} \ni (1, 0, 1)\,, \\
e_7 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_2 \cap IJ_4\} \ni (1, 0, 1/8)\,, \\
e_8 &= \{(a,b,c) \in \mathbb{R}^3 \ : \ \mathrm{C}_2 \cap IJ_5\} \ni (2, 0, 1)\,.
\end{aligned}$$

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Statements & Declarations

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

All authors contributed equally to all aspects of the research and preparation of the manuscript.

Data Availability

Not applicable, the study does not report any data.