

LIMIT CYCLES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL HAMILTONIAN SYSTEMS SEPARATED BY A CIRCLE, OR A PARABOLA, OR A HYPERBOLA

JOYCE A. CASIMIRO¹ AND JAUME LLIBRE²

ABSTRACT. Piecewise differential systems in the plane have been extensively studied in the last two decades, due to the vast application of these systems to describe natural phenomena. The knowledge of the existence or not of periodic solutions, in particular, of the limit cycles, is very important for understanding the dynamics of the differential systems. In this article we studied the maximum number of limit cycles of discontinuous piecewise differential systems, formed by two Hamiltonians systems of degree one with three distinct lines of discontinuity either a circle, or a parabola, or a hyperbola.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The study of discontinuity piecewise differential systems started with the works of Andronov, Vitt and Khaikin in the 1920s aiming at applications to some mechanical systems, see [20]. Since then, these equations have been extensively studied due to a new range of applications in others areas such as electronics and economics, to cite a few, for a further discussion see the books of Simpson [21], di Bernardo et al. [2], the survey of Makarenkov and Lamb [19], and the references therein.

We are interested in studying the limit cycles of discontinuity piecewise smooth differential systems. A limit cycle of a system is a periodic orbit of that system for which there is no other periodic orbit in some sufficiently small neighbourhood containing it. Limit cycles of discontinuity piecewise linear differential systems separated by a straight line have been heavily studied over the last decades, see [1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In this work, we study the limit cycles for a class of discontinuous piecewise differential linear Hamiltonian systems separated by conics.

The discontinuity piecewise differential systems here studied are particular Filippov systems, so the properties that we know for differential systems cannot be directly used. Therefore, to clarify the study of this kind of systems we introduce some important notions used in this work.

We recall the Filippov’s convention for discontinuous piecewise smooth vector fields defined on a open set $U \subset \mathbb{R}^2$. We assume that the discontinuities appear in a differentiable submanifold Σ which can be given as $\Sigma = f^{-1}(0) \cap U$, being 0 a

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regular value of the C^k function $f : U \rightarrow \mathbb{R}$ with $k > 0$. Then the curve Σ splits the open set U in two open sets

$$\Sigma^+ := \{(x, y) \in U : f(x, y) \geq 0\} \text{ and } \Sigma^- := \{(x, y) \in U : f(x, y) \leq 0\},$$

In this paper the piecewise smooth differential system is given by the vector fields

$$(1) \quad Z(x, y) = \begin{cases} Z_1(x, y), & \text{if } f(x, y) \leq 0, \\ Z_2(x, y), & \text{if } f(x, y) \geq 0. \end{cases}$$

In [5] Filippov conventioned that the trajectories of (1) correspond to the solutions of the differential inclusion

$$\dot{\mathbf{x}} \in \mathcal{F}_Z(\mathbf{x}),$$

where $\mathbf{x} = (x, y)$ and $\mathcal{F}_Z$ is the following set-valued function

$$\mathcal{F}_Z(\mathbf{x}) = \frac{Z_1(\mathbf{x}) + Z_2(\mathbf{x})}{2} + \text{sign}(f(\mathbf{x})) \frac{Z_1(\mathbf{x}) - Z_2(\mathbf{x})}{2},$$

with

$$\text{sign}(u) = \begin{cases} -1 & \text{if } u < 0, \\ [-1, 1] & \text{if } u = 0, \\ 1 & \text{if } u > 0. \end{cases}$$

The local trajectory through a point $\mathbf{x} \in U \cap \Sigma^+$ is defined by the vector field Z_1 as usual (respectively for a point $\mathbf{x} \in U \cap \Sigma^-$ the local trajectory is defined by the vector field Z_2). To extend the definition of a trajectory through points of Σ , we consider three parts of Σ

- Crossing region: $\Sigma^c = \{\mathbf{x} \in \Sigma : Z_1 f(\mathbf{x}) \cdot Z_2 f(\mathbf{x}) > 0\}$,
- Sliding region: $\Sigma^s = \{\mathbf{x} \in \Sigma : Z_1 f(\mathbf{x}) < 0, Z_2 f(\mathbf{x}) > 0\}$,
- Escaping region: $\Sigma^e = \{\mathbf{x} \in \Sigma : Z_1 f(\mathbf{x}) > 0, Z_2 f(\mathbf{x}) < 0\}$,

where $Z_i f(\mathbf{x}) = Z_i(\mathbf{x}) \cdot \text{grad} f(\mathbf{x})$ is the Lie derivative of f with respect to the vector field X_i , $i = 1, 2$ at the point $\mathbf{x}$.

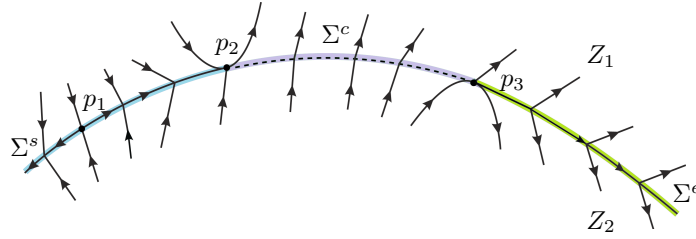


FIGURE 1. Illustration of the sliding, crossing and escaping regions of Σ , in Filippov's convention. The points $p_1 \in \Sigma^s$ represents a pseudo-equilibrium and the points $p_2, p_3 \in \partial \Sigma^c$ represent tangential singularities.

There exist two types of limit cycles in piecewise smooth differential equations. Those of the first type are called sliding limit cycles and are such that some part

of the cycle is contained in the sliding region. On the other hand, those of second type are called crossing limit cycles and correspond to the ones that touch the discontinuity line only on points of the crossing region.

In this paper we provide an upper bound for the maximum number of crossing limit cycles, simple limit cycles in what follows, for the discontinuous system (1), where Z_i for $i \in \{1, 2\}$ are given by,

$$\begin{aligned}\dot{x} &= \frac{\partial}{\partial y} H_i(x, y), \\ \dot{y} &= -\frac{\partial}{\partial x} H_i(x, y).\end{aligned}$$

and H_1, H_2 are Hamiltonians of degree two,

$$\begin{aligned}H_1(x, y) &= a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ H_2(x, y) &= b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2.\end{aligned}\tag{2}$$

So the Hamiltonians vector fields Z_1 and Z_2 of the discontinuous piecewise differential systems (1) that we shall work with in this paper are

$$\begin{aligned}Z_1(x, y) &= (a_2 + a_4x + 2a_5y, -a_1 - 2a_3x - a_4y), \\ Z_2(x, y) &= (b_2 + b_4x + 2b_5y, -b_1 - 2b_3x - b_4y).\end{aligned}\tag{3}$$

We consider three distinct discontinuous lines $f(x, y) = 0$, the circle, the parabola and the hyperbola, which unless of an affine change of variables can be written without loss of generality as

$$\begin{aligned}f(x, y) &= x^2 + y^2 - 1, \\ f(x, y) &= x^2 - y, \\ f(x, y) &= 1 - xy,\end{aligned}$$

respectively. See Figure 2.

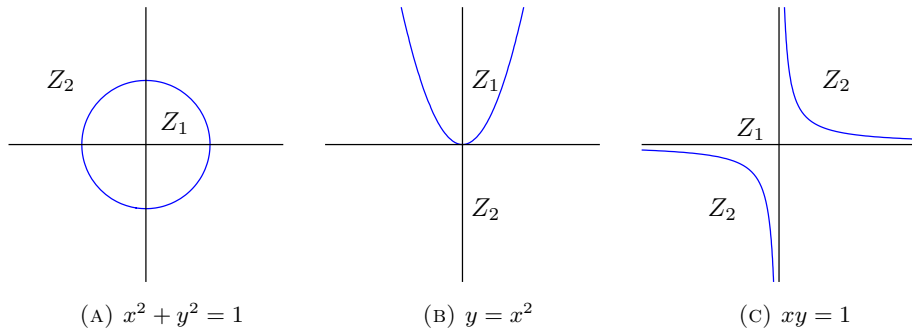


FIGURE 2. Lines of discontinuities

Theorem 1. *The maximum number of limit cycles of the piecewise differential systems (1) with Z_1 and Z_2 given by (3), intersecting in two points the line of discontinuity $f(x, y) = 0$ is*

- (i) 3, for $f(x, y) = x^2 + y^2 - 1$,
- (ii) 3, for $f(x, y) = x^2 - y$,
- (iii) 2, for $f(x, y) = 1 - xy$.

These upper bounds are reached.

Theorem 1 is proved in Section 2.

2. PROOF OF THEOREM 1

Proof of statement (i) of Theorem 1. Notice that $H_1(x, y)$ and $H_2(x, y)$ given in (2) are first integrals of the vectors fields $Z_1(x, y)$ and $Z_2(x, y)$ given in (3), respectively. If the discontinuous piecewise differential system (1) has a limit cycle which intersect the unit circle in the two points

$$p_1 = \left(\frac{2t_1}{t_1^2 + 1}, \frac{1 - t_1^2}{t_1^2 + 1} \right), \quad p_2 = \left(\frac{2t_2}{t_2^2 + 1}, \frac{1 - t_2^2}{t_2^2 + 1} \right),$$

with $t_1 \neq t_2$. These t_1 and t_2 must satisfy the equations

$$(4) \quad \begin{aligned} e_1 &= H_1(p_1) - H_1(p_2) = \frac{2(t_1 - t_2)}{(t_1^2 + 1)^2 (t_2^2 + 1)^2} E_1 = 0, \\ e_2 &= H_2(p_1) - H_2(p_2) = \frac{2(t_1 - t_2)}{(t_1^2 + 1)^2 (t_2^2 + 1)^2} E_2 = 0, \end{aligned}$$

where $E_1 = a_1 t_1^3 t_2^3 + a_1 t_1^3 t_2 - a_1 t_1^2 t_2^2 - a_1 t_1^2 + a_1 t_1 t_2^3 + a_1 t_1 t_2 - a_1 t_2^2 - a_1 + a_2 t_1^3 t_2^2 + a_2 t_1^3 + a_2 t_1^2 t_2^3 + a_2 t_1^2 t_2 + a_2 t_1 t_2^2 + a_2 t_1 + a_2 t_2^3 + a_2 t_2 + 2a_3 t_1^3 t_2^2 + 2a_3 t_1^2 t_2^3 - 2a_3 t_1 - 2a_3 t_2 - a_4 t_1^3 t_2^3 + a_4 t_1^3 t_2 + 3a_4 t_1^2 t_2^2 + a_4 t_1^2 + a_4 t_1 t_2^3 + 3a_4 t_1 t_2 + a_4 t_2^2 - a_4 - 2a_5 t_1^3 t_2^2 - 2a_5 t_1^2 t_2^3 + 2a_5 t_1 + 2a_5 t_2$, and E_2 has the same expression of E_1 replacing $(a_1, a_2, a_3, a_4, a_5)$ by $(b_1, b_2, b_3, b_4, b_5)$.

Note that if (s_1, s_2) is a solution of (4), then (s_2, s_1) is a solution too, call this property the symmetric property. Of course both solutions give rise to the same possible limit cycle.

On the other hand we know that if (s_1, s_2) is a solution of $E_1 = E_2 = 0$ then s_1 must be a root of the resultant polynomial $R(t_1)$ of E_1 and E_2 respected to the variable t_2 . Similarly s_2 must be a root of the resultant polynomial $R(t_2)$ of E_1 and E_2 respected to the variable t_1 . These resultants $R(t_1)$ and $R(t_2)$ are equal to each other only replacing t_1 by t_2 . We have that

$$R(t_1) = -4 (t_1^2 + 1)^6 \tilde{R}(t_1),$$

the real solutions of $R(t_1)$ are the real zeros of $\tilde{R}(t_1)$, and the degree of the polynomial $\tilde{R}(t_1)$ is six. By the symmetric property, the discontinuous piecewise differential system (1) can have at most three limit limit cycles.

The expression of $\tilde{R}(t_1)$ is huge, so we do not writing here, but it can be computed easily using same algebraic manipulators. $\square$

Proof of statement (ii) of Theorem 1. Notice that $H_1(x, y)$ and $H_2(x, y)$ given in (2) are first integrals of the vector fields $Z_1(x, y)$ and $Z_2(x, y)$ given by (3), respectively. If the discontinuous piecewise differential systems (1) have a limit cycle

which intersect the parabola in the two points

$$p_1 = (t_1, t_1^2), \quad p_2 = (t_2, t_2^2),$$

with $t_1 \neq t_2$. Then these two points must satisfy the equations

$$(5) \quad \begin{aligned} e_1 &= H_1(p_1) - H_1(p_2) = (t_1 - t_2)E_1 = 0, \\ e_2 &= H_2(p_1) - H_2(p_2) = (t_1 - t_2)E_2 = 0, \end{aligned}$$

where $E_1 = a_1 + t_2(a_2 + a_3 + t_1(a_4 + a_5 t_1)) + t_1(a_2 + a_3 + t_1(a_4 + a_5 t_1)) + t_2^2(a_4 + a_5 t_1) + a_5 t_2^3$ and E_2 has the same expression of E_1 replacing $(a_1, a_2, a_3, a_4, a_5)$ by $(b_1, b_2, b_3, b_4, b_5)$. Note that if (s_1, s_2) is a solution of system (5) then (s_2, s_1) is a solution too, as before call this property by the symmetric property. Then both give rise to the same possible limit cycle.

On the other hand we know that if (s_1, s_2) is a solution of $E_1 = E_2 = 0$ then s_1 must be a root of the resultant polynomial $R(t_1)$ of E_1 and E_2 respected to the variable t_2 . Similarly s_2 must be a root of the resultant polynomial $R(t_2)$ of E_1 and E_2 respected to the variable t_1 . These resultants $R(t_1)$ and $R(t_2)$ are equal to each other only replacing t_1 by t_2 .

By the symmetric property the discontinuous piecewise differential systems (1) can have at most three limit limit cycles.

The expression of $\tilde{R}(t_1)$ is huge, so we do not writing here, but it can be computed easily using algebraic manipulators. $\square$

Proof of statement (iii) of Theorem 1. Suppose that exist three limit cycles for the discontinuity piecewise differential system generated by the Hamiltonians (2) and we shall arrive to a contradicton. At least two of these limit cycles intersect the same branch of the hyperbola. Assume without loss of generality that these two limit cycles intersect the branch of the hyperbola contained in the first quadrant of the plane $\mathbb{R}^2$. Consider that the points of intersection of these limit cycles with the hyperbola are given by the points

$$p_1 = \left(t_1, \frac{1}{t_1}\right), \quad p_2 = \left(t_2, \frac{1}{t_2}\right), \quad p_3 = \left(t_3, \frac{1}{t_3}\right) \quad \text{and} \quad p_4 = \left(t_4, \frac{1}{t_4}\right),$$

without loss of generality we can assume that $0 < t_1 < t_2 < t_3 < t_4$ and that if there are other limit cycles intersecting this branch of the hyperbola are contained in the bounded region limited by these two limit cycles. So one of the limit cycles passes through p_1, p_4 , and the other limit cycle passes through points p_2 and p_3 . These points must satisfy the equations

$$(6) \quad \begin{aligned} H_1(p_1) - H_1(p_4) &= 0 \Leftrightarrow a_5(t_1 + t_4) - t_1 t_4(t_1 t_4(a_1 + a_3(t_1 + t_4)) - a_2) = 0, \\ H_2(p_1) - H_2(p_4) &= 0 \Leftrightarrow t_1^2 t_4^2 b_1 - t_1 t_4 b_2 + t_1^3 t_4^2 b_3 + t_1^2 t_4^3 b_3 - t_1 b_5 - t_4 b_5 = 0, \\ H_1(p_2) - H_1(p_3) &= 0 \Leftrightarrow a_5(t_2 + t_3) - t_2 t_3(t_2 t_3(a_1 + a_3(t_2 + t_3)) - a_2) = 0, \\ H_2(p_2) - H_2(p_3) &= 0 \Leftrightarrow t_2^2 t_3^2 b_1 - t_2 t_3 b_2 + t_2^3 t_3^2 b_3 + t_2^2 t_3^3 b_3 - t_2 b_5 - t_3 b_5 = 0, \end{aligned}$$

Solving system (6) in terms of the coefficients of the polynomials $H_1(x, y)$ and $H_2(x, y)$ we obtain

$$\begin{aligned} a_3 &= \frac{t_1 t_2 t_3 (a_2 - a_1 t_2 t_3) - (a_1 t_2^2 t_3^2 - a_2 t_2 t_3 + a_2 t_1 (t_2 + t_3)) t_4 + a_1 t_1^2 (t_2 + t_3) t_4^2}{(t_2 + t_3)(t_1 + t_4)(t_2^2 t_3^2 - t_1^2 t_4^2)}, \\ a_5 &= \frac{t_1 t_2 t_3 t_4 (a_1 t_1 t_2 t_3 (t_2 - t_1 + t_3 - t_4) t_4 + a_2 (-t_2^2 t_3 - t_2 t_3^2 + t_1 t_4 (t_1 + t_4)))}{(t_2 + t_3)(t_1 + t_4)(t_2^2 t_3^2 - t_1^2 t_4^2)}, \\ b_1 &= \frac{b_5 (t_1 t_2 (t_3 - t_4) - t_1 t_3 t_4 + t_2 t_3 t_4) + b_3 t_1 t_2 t_3 t_4 (t_2^2 t_3 + t_2 t_3^2 - t_1 t_4 (t_1 + t_4))}{t_1 t_2 t_3 t_4 (-t_2 t_3 + t_1 t_4)}, \\ b_2 &= \frac{b_3 t_1^2 t_2^2 t_3^2 (t_2 - t_1 + t_3 - t_4) t_4^2 + b_5 (t_1 t_2^2 t_3^2 + t_2^2 t_3^2 t_4 - t_1^2 (t_2 + t_3) t_4^2)}{t_1 t_2 t_3 t_4 (-t_2 t_3 + t_1 t_4)}. \end{aligned}$$

Replacing a_3, a_5, b_1, b_2 in $H_1(x, y)$ we obtain the following expression for $H_1(x, y)$

$$\begin{aligned} (7) \quad & \frac{(t_1 t_2 t_3 (a_2 - a_1 t_2 t_3) - (a_1 t_2^2 t_3^2 - a_2 t_2 t_3 + a_2 t_1 (t_2 + t_3)) t_4 + a_1 t_1^2 (t_2 + t_3) t_4^2) x^2}{(t_2 + t_3)(t_1 + t_4)(t_2 t_3 - t_1 t_4)(t_2 t_3 + t_1 t_4)} \\ & + \frac{t_1 t_2 t_3 t_4 (a_1 t_1 t_2 t_3 (t_2 - t_1 + t_3 - t_4) t_4 - a_2 t_2 t_3 (t_2 + t_3) + a_2 t_1 t_4 (t_1 + t_4)) y^2}{(t_2 + t_3)(t_1 + t_4)(t_2 t_3 - t_1 t_4)(t_2 t_3 + t_1 t_4)} \\ & + a_1 x + a_2 y + a_4 xy, \end{aligned}$$

and for $H_2(x, y)$

$$\begin{aligned} (8) \quad & \frac{(b_5 (t_1 t_2 (t_3 - t_4) - t_1 t_3 t_4 + t_2 t_3 t_4) + b_3 t_1 t_2 t_3 t_4 (t_2^2 t_3 + t_2 t_3^2 - t_1 t_4 (t_1 + t_4))) x}{t_1 t_2 t_3 t_4 (-t_2 t_3 + t_1 t_4)} \\ & + \frac{(b_3 t_1^2 t_2^2 t_3^2 (-t_1 + t_2 + t_3 - t_4) t_4^2 + b_5 (t_1 t_2^2 t_3^2 + t_2^2 t_3^2 t_4 - t_1^2 (t_2 + t_3) t_4^2)) y}{t_1 t_2 t_3 t_4 (-t_2 t_3 + t_1 t_4)} \\ & + b_4 xy + b_5 y^2 + b_3 x^2. \end{aligned}$$

The possible limit cycles intersects the hyperbola $xy = 1$ in the points $(x, 1/x)$ and $(X, 1/X)$, their coordinates x, X must satisfy the system

$$\begin{aligned} e_1(x, X) &:= H_1(x, 1/x) - H_1(X, 1/X) = 0, \\ e_2(x, X) &:= H_2(x, 1/x) - H_2(X, 1/X) = 0, \end{aligned}$$

where H_1 and H_2 are given by (7) and (8). This system has three solutions of the form (x, X)

$$(t_1, t_4), \quad (t_2, t_3), \quad (\alpha, \beta)$$

Therefore if the three solutions gave rise to a limit cycle, these limit cycles must pass through the points

$$\begin{aligned} p_1 &= \left(t_1, \frac{1}{t_1} \right) \text{ and } p_4 = \left(t_4, \frac{1}{t_4} \right), \\ p_2 &= \left(t_2, \frac{1}{t_2} \right) \text{ and } p_3 = \left(t_3, \frac{1}{t_3} \right), \\ \alpha &= \left(\alpha, \frac{1}{\alpha} \right) \text{ and } \beta = \left(\beta, \frac{1}{\beta} \right). \end{aligned}$$

Here

$$\alpha = \frac{A - \sqrt{B}}{C}, \quad \beta = \frac{A + \sqrt{B}}{C},$$

where

$$\begin{aligned} A &= (t_2 + t_3 - t_1 - t_4)(t_2 t_3 + t_1 t_4)(t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3)), \\ B &= (t_1 + t_4 - t_2 - t_3)(t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3)) \left(4t_1 t_2 t_3 t_4 (t_2 t_3 - t_1 t_4)^2 \right. \\ &\quad \left. + (t_1 + t_4 - t_2 - t_3)(t_2 t_3 + t_1 t_4)^2 (t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3)) \right), \\ C &= 2(t_2 t_3 - t_1 t_4)(t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3)). \end{aligned}$$

We note that $B \geq 0$, otherwise α and β are non-real.

Note that we can assume that $0 < t_1 < t_2 < t_3 < t_4$ and that if there is another limit cycle intersecting the branch of the hyperbola intersected by the two limit cycles through the points $(t_k, 1/t_k)$ for $k = 1, 2, 3, 4$, such a limit cycle is contained in the bounded region limited by these two limit cycles, it follows that the possible distribution for the coordinates $t_1, t_2, t_3, t_4, \alpha$ and β are

$$\begin{aligned} (9) \quad & 0 < t_1 < t_2 < \alpha < \beta < t_3 < t_4, \\ & 0 < t_1 < t_2 < \beta < \alpha < t_3 < t_4, \\ & \alpha < \beta < 0 < t_1 < t_2 < t_3 < t_4, \\ & \beta < \alpha < 0 < t_1 < t_2 < t_3 < t_4. \end{aligned}$$

The first two cases takes place when the limit cycle through the points $(\alpha, 1/\alpha)$ and $(\beta, 1/\beta)$ intersect the same branch than the two limit cycles through the points $(t_k, 1/t_k)$ for $k = 1, 2, 3, 4$, and the last two cases takes place when the limit cycle through the points $(\alpha, 1/\alpha)$ and $(\beta, 1/\beta)$ intersect the branch of the hyperbola distinct to the branch intersected by the two limit cycles through the points $(t_k, 1/t_k)$ for $k = 1, 2, 3, 4$.

In the first two inequalities of eqrefab we have that

$$\alpha + \beta = \frac{(t_1 + t_4 - t_2 - t_3)(t_2 t_3 + t_1 t_4)}{t_1 t_4 - t_2 t_3} > 0,$$

and consequently

$$0 < t_1 < t_2 < \frac{\alpha + \beta}{2} < t_3 < t_4,$$

i.e.

$$0 < t_1 < t_2 < \frac{(t_1 + t_4 - t_2 - t_3)(t_2 t_3 + t_1 t_4)}{2(t_1 t_4 - t_2 t_3)} < t_3 < t_4.$$

Or equivalently,

$$0 < \frac{t_1}{t_4} < \frac{t_2}{t_4} < \frac{(\frac{t_1}{t_4} + 1 - \frac{t_2}{t_4} - \frac{t_3}{t_4})(t_2 t_3 + t_1 t_4)}{2(t_1 t_4 - t_2 t_3)} < \frac{t_3}{t_4} < 1,$$

i.e.

$$0 < \frac{t_1}{t_4} < \frac{t_2}{t_4} < \frac{(\frac{t_1}{t_4} + 1 - \frac{t_2}{t_4} - \frac{t_3}{t_4})(\frac{t_2}{t_4} \frac{t_3}{t_4} + \frac{t_1}{t_4})}{2(\frac{t_1}{t_4} - \frac{t_2}{t_4} \frac{t_3}{t_4})} < \frac{t_3}{t_4} < 1.$$

Let $s_k = t_k/t_4$ be for $k = 1, 2, 3$. Then the last inequalities become

$$(10) \quad 0 < s_1 < s_2 < \frac{(s_1 + 1 - s_2 - s_3)(s_2 s_3 + s_1)}{2(s_1 - s_2 s_3)} < s_3 < 1.$$

The region R of Figure 3 is defined satisfying the inequalites

$$0 < s_1 < s_2 < \frac{s_1 + 1 - s_2 - s_3)(s_2 s_3 + s_1)}{2(s_1 - s_2 s_3)} \text{ and } s_3 < 1.$$

Hence the region defined by the inequalities (10) is empty. In other words, the possible limit cycle passing through the points $(\alpha, 1/\alpha)$ and $(\beta, 1/\beta)$ intersecting the same branch than the two limit cycles through the points $(t_k, 1/t_k)$ for $k = 1, 2, 3, 4$ does not exist.

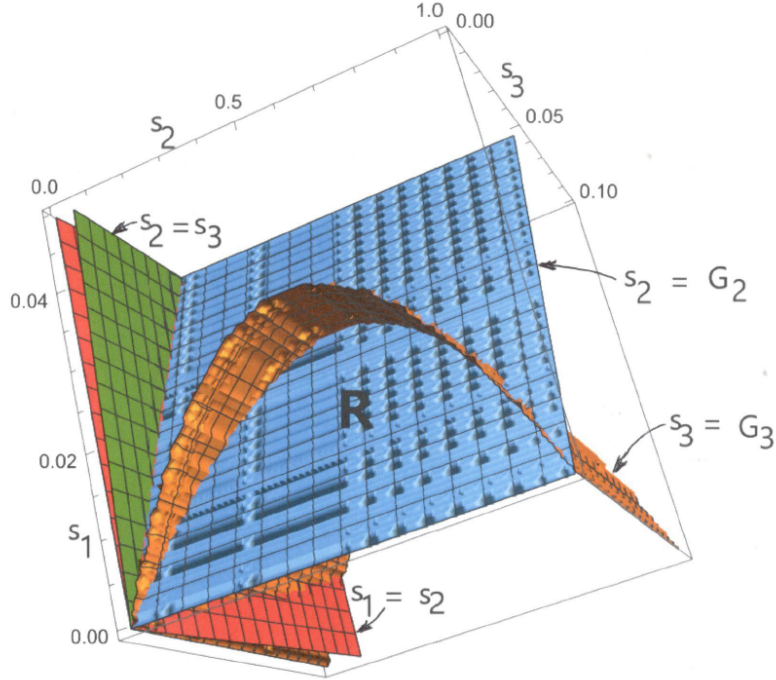


FIGURE 3. The region R defined by the inequalites $0 < s_1 < s_2 < \frac{s_1 + 1 - s_2 - s_3)(s_2 s_3 + s_1)}{2(s_1 - s_2 s_3)}$ and $s_3 > s_2$.

Assume that the two last cases of (9) hold. Then we have that

$$\alpha + \beta = \frac{(t_1 + t_4 - t_2 - t_3)(t_2 t_3 + t_1 t_4)}{t_1 t_4 - t_2 t_3} < 0,$$

$$\alpha \beta = \frac{t_1 t_2 t_3 t_4 (t_2 + t_3 - t_1 - t_4)}{t_2 t_3 (t_1 + t_4) - t_1 t_4 (t_2 + t_3)} > 0.$$

Therefore

$$(11) \quad \frac{t_1 + t_4 - t_2 - t_3}{t_1 t_4 - t_2 t_3} < 0 \text{ and } \frac{t_2 + t_3 - t_1 - t_4}{t_2 t_3 (t_1 + t_4) - t_1 t_4 (t_2 + t_3)} > 0.$$

Assume that $t_1 + t_4 - t_2 - t_3 < 0$, then from the second inequality of (11) we get that $t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3) > 0$. But from the first inequality of (11) we obtain that $t_1 t_4 > t_2 t_3$ and since $t_2 + t_3 > t_1 + t_4$, we get that $t_1 t_4(t_2 + t_3) > t_2 t_3(t_1 + t_4)$ in contradiction with $t_2 t_3(t_1 + t_4) - t_1 t_4(t_2 + t_3) > 0$. If $t_1 + t_4 - t_2 - t_3 > 0$ repeating the same arguments we also arrive to a contradiction. Therefore the last two cases of (9) cannot occur. In summary, the maximum number of limit cycles for the discontinuity piecewise differential system (1) with the line of discontinuity $xy = 1$ is two. $\square$

3. EXAMPLES

To prove that the upper bound is reached we show some examples for Theorem 1.

Example 1. Consider the following two Hamiltonians of degree two, separated by the line of discontinuity $x^2 + y^2 = 1$

$$(12) \quad \begin{aligned} H_1(x, y) &= x^2 - \frac{137xy}{30} - \frac{211x}{100} - \frac{11y^2}{10} + \frac{421y}{100}, \\ H_2(x, y) &= x^2 - \frac{275xy}{303} - \frac{55x}{101} + y^2 + \frac{55y}{101}. \end{aligned}$$

These Hamiltonians generate the Hamiltonian vector fields

$$(13) \quad Z_1(x, y) = \left(-\frac{137x}{30} - \frac{11y}{5} + \frac{421}{100}, -2x + \frac{137y}{30} + \frac{211}{100} \right),$$

$$(14) \quad Z_2(x, y) = \left(-\frac{275x}{303} + 2y + \frac{55}{101}, -2x + \frac{275y}{303} + \frac{55}{101} \right),$$

respectively. Of course $H_1(x, y)$ and $H_2(x, y)$ are first integrals of the vector fields (13) and (14), respectively. Notice that the limit cycles intersecting the circle at the points (x, y) and (X, Y) must satisfy the system

$$(15) \quad \begin{aligned} H_1(x, y) &= H_1(X, Y), \\ H_2(x, y) &= H_2(X, Y), \\ x^2 + y^2 &= 1, \\ X^2 + Y^2 &= 1. \end{aligned}$$

System (15) has the three pairs of solutions formed by the points

$$(16) \quad \begin{aligned} p_1 &= \left(-\frac{4}{5}, -\frac{3}{5} \right), & P_1 &= \left(\frac{3}{5}, -\frac{4}{5} \right), \\ p_2 &= \left(\frac{4}{5}, \frac{3}{5} \right), & P_2 &= (1, 0), \\ p_3 &= (-1, 0), & P_3 &= (0, 1). \end{aligned}$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1, 2, 3\}$ as we can see in Figure 4.

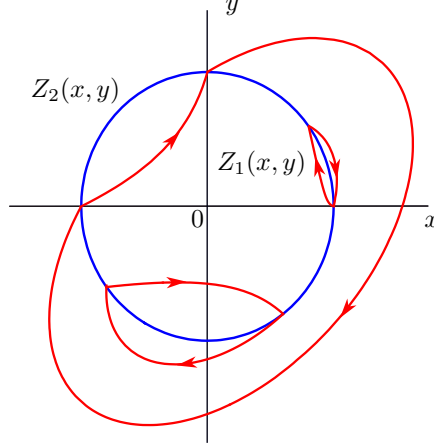


FIGURE 4. The three limit cycles of the discontinuous piecewise differential system generate by Hamiltonians (12) passing through the points p_i and P_i , $i \in \{1, 2, 3\}$ given in (16).

Example 2. This example provided three limits cycles for the discontinuous piecewise differential system

$$(17) \quad Z(x, y) = \begin{cases} Z_1(x, y), & \text{if } y \leq x^2, \\ Z_2(x, y), & \text{if } y \geq x^2, \end{cases}$$

where

$$(18) \quad Z_1(x, y) = \left(-\frac{577x}{15637} + \frac{103y}{15637} - 1, -\frac{30931x}{15637} + \frac{577y}{15637} - 1 \right),$$

$$(19) \quad Z_2(x, y) = \left(\frac{103y}{15637} - \frac{577x}{15637}, \frac{343x}{15637} + \frac{577y}{15637} - 1 \right),$$

The Hamiltonians vector fields Z_1 and Z_2 have, respectively, the Hamiltonians

$$(20) \quad \begin{aligned} H_1(x, y) &= \frac{30931x^2}{31274} - \frac{577xy}{15637} + x + \frac{103y^2}{31274} - y, \\ H_2(x, y) &= -\frac{343x^2}{31274} - \frac{577xy}{15637} + x + \frac{103y^2}{31274}. \end{aligned}$$

Of course $H_1(x, y)$ and $H_2(x, y)$ are first integrals of systems (18) and (19), respectively. Notice that the limit cycles passing through the points (x, y) and (X, Y) of the parabola must satisfy the system

$$(21) \quad \begin{aligned} H_1(x, y) &= H_1(X, Y), \\ H_2(x, y) &= H_2(X, Y), \\ x^2 &= y, \\ X^2 &= Y. \end{aligned}$$

System (21) has the three pair of solutions formed by the points

$$\begin{aligned}
 p_1 &= (2, 4), \quad P_1 = \left(\frac{1}{206} (227 - \sqrt{1628665}), \frac{840097 - 227\sqrt{1628665}}{21218} \right), \\
 p_2 &= (3, 9), \quad P_2 = \left(\frac{1}{103} (165 - \sqrt{493609}), \frac{2(260417 - 165\sqrt{493609})}{10609} \right), \\
 p_3 &= (-1, 1), \quad P_3 = (-4, 16).
 \end{aligned}
 \tag{22}$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1, 2, 3\}$ as we can see in Figure 4.

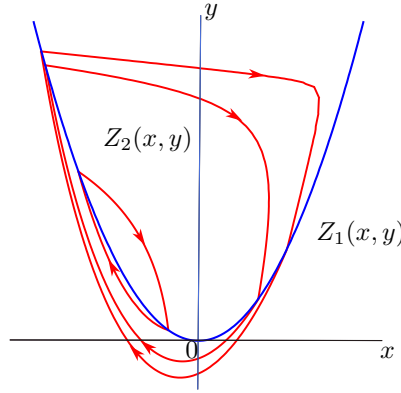


FIGURE 5. The three limit cycles of the discontinuous piecewise differential system generate by Hamiltonians (20) passing through the points p_i and P_i , $i \in \{1, 2, 3\}$ given in (22).

Example 3. Consider the following two Hamiltonians of degree two,

$$\begin{aligned}
 H_1(x, y) &= \left(x - \frac{5}{4} \right)^2 + \left(y - \frac{4}{5} \right)^2, \\
 H_2(x, y) &= -(-x^2 + 1.25507x - 0.335138y^2 - 1.40792y).
 \end{aligned}
 \tag{23}$$

These Hamiltonians generate the Hamiltonian systems

$$Z_1(x, y) = \left(2 \left(y - \frac{4}{5} \right), -2 \left(x - \frac{5}{4} \right) \right),
 \tag{24}$$

$$Z_2(x, y) = (0.670277y + 1.40792, 1.25507 - 2x),
 \tag{25}$$

respectively. Of course $H_1(x, y)$ and $H_2(x, y)$ are first integrals of the vector fields (24) and (25), respectively. Notice that the limit cycles passing through the points (x, y) and (X, Y) of the hyperbola must satisfy the system

$$\begin{aligned}
 H_1(x, y) &= H_1(X, Y), \\
 H_2(x, y) &= H_2(X, Y), \\
 xy &= 1, \\
 XY &= 1.
 \end{aligned}
 \tag{26}$$

System (26) has the two pairs of solutions formed of the points

$$(27) \quad \begin{aligned} p_1 &= (0.9923, 1.007), & P_1 &= (1.546, 0.6467), \\ p_2 &= (1.053, 0.9517), & P_2 &= (1.47, 0.6803). \end{aligned}$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1, 2\}$ as we can see in Figure 4.

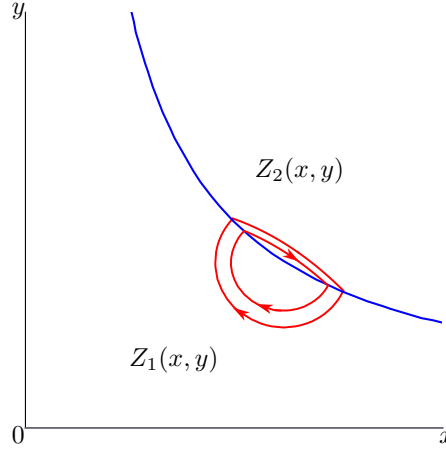


FIGURE 6. The two limit cycles of the discontinuous piecewise differential system generate by Hamiltonians (23) passing through the points p_i and P_i , $i \in \{1, 2\}$ given in (27).

STATEMENTS AND DECLARATIONS

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¹ DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL DE CAMPINAS, RUA SÉRGIO BARUQUE DE HOLANDA, 651, CIDADE UNIVERSITÁRIA ZEFERINO VAZ, 13083–859, CAMPINAS, SP, BRAZIL

Email address: jcasimiro@ime.unicamp.br

² DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BEL-LATERRA, BARCELONA, CATALONIA, SPAIN

Email address: jllibre@mat.uab.cat