

THE PHASE PORTRAIT OF ALL POLYNOMIAL LIÉNARD ISOCHRONOUS CENTERS

JAUME LLIBRE¹ AND CLAUDIA VALLS²

ABSTRACT. We prove that there is a unique topologically equivalent phase portrait for all Liénard polynomial differential systems of arbitrary degree having an isochronous center

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we deal with the Liénard real polynomial differential equation

$$(1) \quad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where f and g are non-zero polynomials such that $g(x) = x + g_2(x)$ with $g_2(0) = 0$ and $g'_2(0) = 0$. Here one dot denotes the first derivative with respect to the time t , and two dots the second derivative with respect to the time t .

The differential equations (1) have been studied by many authors. Thus for instance in MathSciNet if we put in four different “Anywhere” the four words Liénard, polynomial, differential, equation or system, it appears now 155 references.

Of course the differential equation (1) of second order can be written as the following differential system of first order

$$(2) \quad \dot{x} = -y, \quad \dot{y} = x + g_2(x) - f(x)y.$$

Clearly the origin $O = (0, 0)$ is a singular point of the differential system (2). We say that O is a *center* if there exists a neighborhood U of O such that $U \setminus \{O\}$ is filled of periodic orbits. The maximal connected set of periodic orbits surrounding the center O and having O in its boundary is called the *period annulus* of the center O . Moreover we say that O is an *isochronous center* if O is a center and all the periodic orbits of its period annulus have the same period.

The notion of a center appeared already in the works of Poincaré [9] in 1881 and Dulac [4] in 1908. Informally the notion of an isochronous center goes back to Huygens, who in 1656 invented the pendulum clock, see [7, 8].

The next conjecture was stated by Algaba et al. in [1].

Conjecture 1. *The origin O of the Liénard polynomial differential system (2) with the non-zero polynomials $f(x)$ and $g_2(x)$ is an isochronous center if and only if the*

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polynomial $f(x)$ is an odd function of degree $d \geq 3$ and the polynomial

$$(3) \quad g_2(x) = \frac{1}{x^3} \left(\int_0^x s f(s) ds \right)^2.$$

Doing the change of coordinates $(x, y) \rightarrow (X, Y)$ where

$$(4) \quad X = x, \quad Y = y - xh(x), \quad \text{with} \quad h(x) = \frac{1}{x^2} \int_0^x s f(s) ds,$$

system (2) becomes

$$(5) \quad \dot{X} = -Y - Xh(X), \quad \dot{Y} = X - Yh(X) + g_2(X) - Xh(X)^2.$$

Under condition (3) system (5) writes

$$(6) \quad \dot{x} = -y - xh(x), \quad \dot{y} = x - yh(x),$$

where instead of writting (X, Y) we have written (x, y) . It is well known (see for instance [3]) that the differential systems (6) have a uniform isochronous center at the origin of coordinates, thus the origin is the unique singular point.

The objective of this paper to classify all phase portraits of the Liénard differential systems (2) having an isochronous center, or equivalent of the differential systems (6). In fact, we shall classify the phase portraits of system (6) in the Poincaré disc.

We recall that the *Poincaré disc* $\mathbb{D}^2$ is the unit closed disc at the origin of coordinates with interior diffeomorphic to $\mathbb{R}^2$ and whose boundary $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Any polynomial differential system, as the systems (6), can be extended analytically to $\mathbb{D}^2$. This extension is called the *Poincaré compactification*, see for more details Chapter 5 of [5].

We say that two phase portraits in the Poincaré disc are *topologically equivalent* if there is a diffeomorphism from $\mathbb{D}^2$ into itself such that send orbits of one phase portrait onto the orbits of the other phase portrait preserving or reversing the sense of all the orbits.

The main result of this paper is the following one.

Theorem 2. *The phase portrait in the Poincaré disc of any polynomial differential system (6) is topologically equivalent to the one of Figure 1.*

Theorem 2 is proved in section 2.

2. PROOF OF THEOREM 2

Let $h(x)$ be the odd polynomial

$$h(x) = a_1x + a_3x^3 + \dots + a_dx^d,$$

with $d \geq 3$ odd. Then the differential system (6) becomes

$$(7) \quad \begin{aligned} \dot{x} &= -y - x \left(\frac{a_1}{3}x + \frac{a_3}{5}x^3 + \dots + \frac{a_d}{d+2}x^d \right), \\ \dot{y} &= x - y \left(\frac{a_1}{3}x + \frac{a_3}{5}x^3 + \dots + \frac{a_d}{d+2}x^d \right). \end{aligned}$$

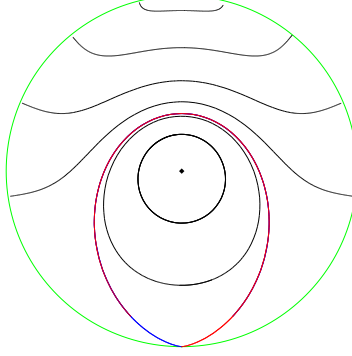


FIGURE 1. The phase portrait in the Poincaré disc of system (6). It has a unique finite singular point, a uniform isochronous center at the origin of coordinates, and the circle of infinity is filled of singular points.

We do the rescaling $(x, y) = (\alpha X, \alpha Y)$ with $\alpha = ((d+2)/a_d)^{1/d}$ and system (7) writes

$$(8) \quad \begin{aligned} \dot{X} &= -Y - X \left(\frac{(d+2)^{1/d} a_1}{3a_d^{1/d}} X + \frac{(d+2)^{3/d} a_3}{5a_d^{3/d}} X^3 + \dots + \frac{(d+2)^{(d-2)/d} a_{d-2}}{da_d^{(d-2)/d}} X^{d-2} + X^d \right), \\ \dot{Y} &= X - Y \left(\frac{(d+2)^{1/d} a_1}{3a_d^{1/d}} X + \frac{(d+2)^{3/d} a_3}{5a_d^{3/d}} X^3 + \dots + \frac{(d+2)^{(d-2)/d} a_{d-2}}{da_d^{(d-2)/d}} X^{d-2} + X^d \right). \end{aligned}$$

The differential system (8) can be written as

$$(9) \quad \begin{aligned} \dot{X} &= -Y - X (b_1 X + b_3 X^3 + \dots + b_{d-2} X^{d-2} + X^d), \\ \dot{Y} &= X - Y (b_1 X + b_3 X^3 + \dots + b_{d-2} X^{d-2} + X^d). \end{aligned}$$

This differential system has a unique finite singular point, the origin of coordinates, that is an isochronous center.

Now we shall study the infinite singular points of system (9). Using the notation of Chapter 5 of [5] we must study the infinite singular points of the local chart U_1 and the origin of the local chart U_2 , the remaining infinite singular points are the diametrically opposite singular points in the Poincaré disc. From [5] we know that in order to study the infinite singular points of the extended polynomial differential system

$$(10) \quad \dot{X} = P(X, Y), \quad \dot{Y} = Q(X, Y),$$

of degree n (i.e. n is the maximum degree of the polynomials P and Q) in the local chart U_1 we must study the singular points of the form $(u, 0)$ of the polynomial differential system

$$(11) \quad \begin{aligned} \dot{u} &= v^d \left[-uP \left(\frac{1}{v}, \frac{u}{v} \right) + Q \left(\frac{1}{v}, \frac{u}{v} \right) \right], \\ \dot{v} &= -v^{d+1} P \left(\frac{1}{v}, \frac{u}{v} \right). \end{aligned}$$

For the polynomial differential system (9) the corresponding system (11) becomes

$$(12) \quad \begin{aligned} \dot{u} &= (1 + u^2)v^d, \\ \dot{v} &= v(-1 - b_{d-2}v^2 - b_{d-4}v^4 - \dots - b_3v^{d-3} - b_1v^{d-1} + uv^d). \end{aligned}$$

Clearly all the points $(u, 0)$ are singular points, i.e. the infinity of the local chart U_1 is filled with singular points.

The linear part of the differential system (12) at a singular point $(u^*, 0)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore the continuum of singular points of the infinite line in the local chart U_1 is a normally hyperbolic submanifold of dimension one, and from [6] at every infinite singular point of the local chart U_1 starts an orbit.

In the local chart U_2 the polynomial system (10) writes

$$(13) \quad \begin{aligned} \dot{u} &= v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \\ \dot{v} &= -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned}$$

For the polynomial differential system (9) the corresponding system (13) becomes

$$(14) \quad \begin{aligned} \dot{u} &= -(1 + u^2)v^d, \\ \dot{v} &= -uv(u^{d-1} + b_{d-2}u^{d-3}v^2 + b_{d-4}u^{d-5}v^4 + \dots + b_3u^2v^{d-3} + b_1v^{d-1} + v^d). \end{aligned}$$

Clearly the origin of the chart U_2 is a singular point. Moreover the matrix of the linear part of system (14) is identically zero. So we must study its local phase portraits doing the changes of variables called blow ups, see for more details [2].

First doing the change of the time $dt_1 = vdt$ we eliminate from the differential system (14) the common factor v between the two components of the system. After denoting the variables (u, v) by (u_0, v_0) system (14) becomes

$$(15) \quad \begin{aligned} \dot{u}_0 &= -(1 + u_0^2)v_0^{d-1}, \\ \dot{v}_0 &= -u_0(u_0^{d-1} + b_{d-2}u_0^{d-3}v_0^2 + b_{d-4}u_0^{d-5}v_0^4 + \dots + b_3u_0^2v_0^{d-3} + b_1v_0^{d-1} + v_0^d), \end{aligned}$$

now the dot denotes derivative with respect the new time t_1 .

The matrix of the linear part of system (15) at the origin O of the local chart U_2 continues being identically zero. Since the characteristic directions at the origin of system (15) are given by $u_0v_0^{d-1} = 0$, the vertical direction $u_0 = 0$ is a characteristic direction. In order to study the local phase portrait of the singular point O doing vertical blow ups we must avoid that the vertical direction $u_0 = 0$ be a characteristic direction. So we rotate the v_0 -axis an angle of $\pi/4$ radians in clockwise sense doing the change of variables $(u_0, v_0) = (u_1 - v_1, v_1)$. In this new variables system (15)

writes

$$\begin{aligned}
 \dot{u}_1 &= u_1^d - du_1^{d-1}v_1 + (nd + b_{d-2})u_1^{d-2}v_1^2 - \dots \\
 &\quad - (nd + 3b_3 + 10b_5 + \dots + (n-1)(d-2)b_{d-2})u_1^2v_1^{d-2} - v_1^{d-1} \\
 &\quad + (d + b_1 + 3b_3 + \dots + (d-2)b_{d-2})u_1v_1^{d-1} - u_1^2v_1^{d-1} \\
 &\quad - (1 + b_1 + b_3 + \dots + b_{d-2} - u_1)v_1^d, \\
 (16) \quad \dot{v}_1 &= (u_1 - v_1) \left(u_1d - 1 - (d-1)u_1^{d-2}v_1 + (n(d-2) \right. \\
 &\quad + b_{d-2})u_1^{d-3}v_1^2 - \dots + (n(d-2) + b_3 + 6b_5 + \dots \\
 &\quad + (d-5)(d-6)b_{d-4}/2 + (d-3)(d-4)b_{d-2}/2)u_1^2v_1^{d-3} \\
 &\quad - 2(n + b_3 + 2b_5 + \dots + (d-3)b_{d-2}/2)u_1v_1^{d-2} \\
 &\quad \left. + (1 + b_1 + b_3 + \dots + b_{d-4} + b_{d-2})v_1^{d-1} - v_1^d \right),
 \end{aligned}$$

where $n = (d-1)/2$.

Now we do the vertical blow $(u_1, v_1) = (u_2, u_2v_2)$ and system (16) becomes

$$\begin{aligned}
 \dot{u}_2 &= u_2^{d-1} \left(u_2 - du_2v_2 + (nd + b_{d-2})u_2v_2^2 - \dots - (nd + 3b_3 \right. \\
 &\quad + 10b_5 + \dots + (n-1)(d-2)b_{d-2})u_2v_2^{d-2} - v_2^{d-1} \\
 &\quad + (d + b_1 + 3b_3 + \dots + (d-2)b_{d-2} + u_2)u_2v_2^{d-1} \\
 &\quad \left. - (1 + b_1 + b_3 + \dots + b_{d-2})u_2v_2^d + u_2^2v_2^9 \right), \\
 (17) \quad \dot{v}_2 &= u_2^{d-2} \left(u_2 - (d+1)u_2v_2 + ((n+1)d + b_{d-2})u_2v_2^2 - \dots \right. \\
 &\quad + ((n+1)d + b_1 + 6b_3 + \dots + n(d-2)b_{d-2})u_2v_2^{d-1} + v_2^d \\
 &\quad - 2((n+1) + b_1 + 2b_3 + \dots + nb_{d-2})u_2v_2^d + \\
 &\quad \left. (1 + b_1 + b_3 + \dots + b_{d-2})u_2v_2^{d+1} \right).
 \end{aligned}$$

Doing the change of the time $dt_2 = vdt_1$ we eliminate from the differential system (16) the common factor u_2^{d-2} between the two components of the system. After denoting the variables (u_2, v_2) by (u_3, v_3) system (14) becomes

$$\begin{aligned}
 \dot{u}_3 &= u_3 \left(u_3 - du_3v_3 + (nd + b_{d-2})u_3v_3^2 - \dots - (nd + 3b_3 \right. \\
 &\quad + 10b_5 + \dots + (n-1)(d-2)b_{d-2})u_3v_3^{d-2} - v_3^{d-1} \\
 &\quad + (d + b_1 + 3b_3 + \dots + (d-2)b_{d-2} + u_3)u_3v_3^{d-1} \\
 &\quad \left. - (1 + b_1 + b_3 + \dots + b_{d-2})u_3v_3^d + u_3^2v_3^9 \right), \\
 (18) \quad \dot{v}_3 &= u_3 - (d+1)u_3v_3 + ((n+1)d + b_{d-2})u_3v_3^2 - \dots \\
 &\quad + ((n+1)d + b_1 + 6b_3 + \dots + n(d-2)b_{d-2})u_3v_3^{d-1} + v_3^d \\
 &\quad - 2((n+1) + b_1 + 2b_3 + \dots + nb_{d-2})u_3v_3^d + \\
 &\quad (1 + b_1 + b_3 + \dots + b_{d-2})u_3v_3^{d+1}.
 \end{aligned}$$

where now the dot denotes derivative with respect the new time t_2 .

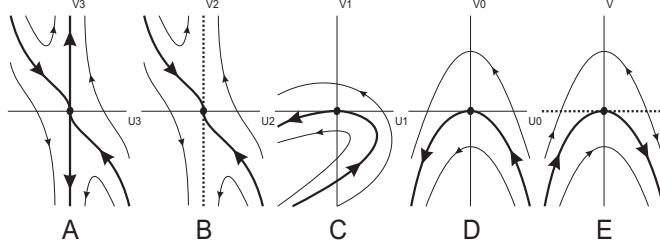


FIGURE 2. The different steps of the blow ups of the origin of the local chart U_2 .

The unique singular point on the v_3 -axis of system (18) is the origin O of coordinates which is a nilpotent singular point. Then using Theorem 3.5 of [5] we obtain that the origin O is a nilpotent saddle (see Figure 3.21(d) of [5]) because using the notation of Theorem 3.5 of [5] we get that $a = 1$ and $m = 2d - 1$. Moreover, since the v_3 -axis is invariant by the flow of system (18), $\dot{v}_3|_{u_3=0} = v_3^d$ and $\dot{v}_3|_{v_3=0} = u_3$, it follows that the local phase portrait of system (18) around the v_3 -axis is the one shown in Figure 2.A.

From Figure 2(a) taking into account the change in the time variable $dt_2 = v dt_1$ and in the variables $(u_2, v_2) = (u_3, v_3)$ it follows immediately the local phase around the v_2 -axis of system (17) given in Figure 2.B.

Going back through the blow up $(u_1, v_1) = (u_2, u_2 v_2)$ we obtain the local phase portrait at the origin of system (16) shown in Figure 2.C.

Undoing the rotation $(u_0, v_0) = (u_1 - v_1, v_1)$ we get the local phase portrait at the origin of system (15) provided in Figure 2.D.

Finally taking into account the change in the time variable $dt_1 = v dt$ and in the variables $(u, v) = (u_0, v_0)$ we obtain the local phase portrait at the origin of the local chart U_2 of system (14) shown in Figure 2.E.

In summary taking into account that at the origin of coordinates of $\mathbb{R}^2$ we have an isochronous center, that at every infinite singular point of the local chart U_1 starts an orbit, that at every infinite singular point of the local chart V_1 ends an orbit (because the degree of system (9) is even, see for more details Chapter 5 of [5]), the local phase portrait at the origin of the local chart U_2 given in Figure 2(e), and consequently at the origin of the local chart V_2 , we obtain the phase portrait in the Poincaré disc of Figure 1. This completes the proof of Theorem 2.

Doing a similar study we could obtain that the phase portrait in the Poincaré disc of the Liénard polynomial differential systems (2) having an isochronous center at the origin of coordinates is topologically equivalent to the one of Figure 3. Note that the phase portraits of Figures 1 and 3 restricted to the interior of the Poincaré disc, i.e. restricted to $\mathbb{R}^2$, are topologically equivalent due to the change of variables (4).

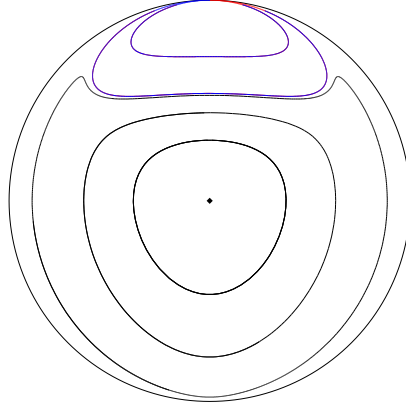


FIGURE 3. The phase portrait in the Poincaré disc of system (2) which has a unique finite singular point, an isochronous center at the origin of coordinates, and two infinite singular points at the origins of the local charts U_2 and V_2 .

3. CONCLUSION

If Conjecture 1 holds we have proved that all Liénard polynomial differential systems of arbitrary degree having an isochronous center, have a unique topologically equivalent phase portrait.

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¹ DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BEL-LATERRA, BARCELONA, CATALONIA, SPAIN

Email address: `jaume.llibre@uab.cat`

² DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS 1049–001, LISBOA, PORTUGAL

Email address: `cvalls@math.tecnico.ulisboa.pt`