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TOPOLOGICAL ENTROPY, SETS OF PERIODS AND TRANSITIVITY FOR GRAPH MAPS

LLUÍS ALSEDÀ, LIANE BORDIGNON, AND JORGE GROISMAN

Abstract. Transitivity, the existence of periodic points and positive topological entropy can be used to characterize complexity in dynamical systems. It is known that for graphs that are not trees, for every $\varepsilon > 0$, there exist (complicate) totally transitive maps (then with cofinite set of periods) such that the topological entropy is smaller than ε (simplicity). We show by means of three examples that for any graph that is not a tree, relatively simple maps (with small entropy) which are totally transitive (and hence robustly complicate) can be constructed so that the set of periods is also relatively simple. To numerically measure the complexity of the set of periods we introduce a notion of a *boundary of cofiniteness*. Larger boundary of cofiniteness means simpler set of periods. With the help of the notion of boundary of cofiniteness we can state precisely what do we mean by extending the entropy simplicity result to the set of periods: *there exist relatively simple maps such that the boundary of cofiniteness is arbitrarily large (simplicity) which are totally transitive (and hence robustly complicate)*. Moreover, we will show that, the above statement holds for arbitrary continuous degree one circle maps.

Contents

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1. Introduction

Transitivity, the existence of infinitely many periods and positive topological entropy often characterize the complexity in dynamical systems. This paper aims at showing that totally transitive maps on graphs, despite of being complicate in the above sense can have somewhat simple sets of periods (simplicity with respect to topological entropy was already known). To be more precise, and to state the main results of the paper we need to introduce some basic notation.

Let X be a topological space and let $f\colon X \longrightarrow X$ be a map. A point $x \in X$ is called a *periodic point of f of period n* if $f^n(x) = x$ and n is the minimum positive integer with this property. The set of all positive integers n such that f has a periodic point of period n is denoted by $\mathsf{Per}(f)$. A set of periods is called *cofinite* (on $\mathbb{N}$)if its complement is finite or, equivalently, it contains all positive integers larger than a given period.

Let X be a compact space and let $f\colon X \longrightarrow X$ be a continuous map. The *topological entropy* of f is defined as in [1] and denoted by $h(f)$.

Definition 1.1. Given a topological space X, a continuous map $f\colon X \longrightarrow X$ is called *transitive* if for every pair of open subsets of X, U and V, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

It is well known that when X has no isolated point, transitivity is equivalent to the existence of a dense orbit (see for instance [17]).

A map f is called *totally transitive* if all its iterates are transitive. ◘

We are interested in totally transitive maps on graphs. A *(topological) graph* is a connected compact Hausdorff space for which there exists a finite non-empty subset whose complement is the disjoint union of a finite number subsets, each of them homeomorphic to an open interval of the real line. A *tree* is a graph without circuits or, equivalently, a uniquely arcwise connected graph. A continuous map from a graph to itself will be called a *graph map*.

A transitive graph map has positive topological entropy and dense set of periodic points [13, 14, 4, 5], with the only exception of an irrational rotation on the circle. Thus, in view of [11] every transitive map on a graph is chaotic in the sense of Devaney (except, again, for an irrational rotation on the circle). Moreover, from [3, Main Theorem] we have

Theorem 1.2. *Let G be a graph and let f be a continuous transitive map from G to itself which has periodic points. Then the following statements are equivalent:*

(a) f *is totally transitive.*
(b) $\mathsf{Per}(f)$ *is cofinite in* $\mathbb{N}$.

Thus, *totally transitive maps on graphs are complicate since they have positive topological entropy, are chaotic in the sense of Devaney and have cofinite set of periods.*

However, from [2] we know that for every graph that is not a tree and for every $\varepsilon > 0$, there exists a totally transitive map such that its topological entropy is positive but smaller than ε. Thus, *transitive maps on graphs may be relatively simple because they may have arbitrarily small positive topological entropy.*

Remark 1.3. This result is valid only for graphs that are not trees since from [7] we know that for any tree T and any transitive map $f\colon T \longrightarrow T$,

$$h(f) \geq \frac{\log 2}{\mathsf{E}(T)},$$

where $\mathsf{E}(T)$ denotes the number of endpoints of T. ◘

The aim of this paper is to study whether the simplicity phenomenon described above, that happens for the topological entropy, can be extended to the set of periods. More precisely, *is it true that when a totally transitive graph map has small positive topological entropy it also has simple set of periods (and in particular small "cofinite part" of the set of periods)?*

To measure the size of the set of periods and, in particular, of its "cofinite part" we introduce the notion of *boundary of cofiniteness*. For clarity we will also introduce three auxiliary notions:

- For every $L \in \mathbb{N}$, we define the *set of successors of* L, denoted by $\mathsf{Succ}\,(L)$, as the set $\{k \in \mathbb{N} : k \geq L\}$.
- Let f be a graph map whose set of periods is cofinite. The *strict boundary of cofiniteness of* f, denoted by $\mathsf{StrBdCof}\,(f)$,is defined as the smallest positive integer n such that $\mathsf{Per}(f) \supset \mathsf{Succ}\,(n)$. Accordingly, the set $\mathsf{Succ}\,(\mathsf{StrBdCof}\,(f))$ will be called the *cofinite part of* $\mathsf{Per}(f)$.
- Given a graph map f whose set of periods is cofinite we define the set

$$\mathsf{sBC}(f) := \Big\{L \in \mathsf{Per}(f) : L > 2,\ L-1 \notin \mathsf{Per}(f) \text{ and } \mathsf{Card}\big(\{1,\dots,L-2\} \cap \mathsf{Per}(f)\big) \leq 2\log_2(L-2)\Big\}.$$

Observe that every $L \in \mathsf{Per}(f)$, $L > 2$, such that $L-1 \notin \mathsf{Per}(f)$ must satisfy $L \leq \mathsf{StrBdCof}\,(f)$. Hence, $\mathsf{sBC}(f) \subset \{1, 2, \dots, \mathsf{StrBdCof}\,(f)\}$ is finite.

Definition 1.4 (Boundary of cofiniteness). Let f be a graph map whose set of periods is cofinite. When $\mathsf{sBC}(f) \neq \emptyset$ we define the *boundary of cofiniteness of* f as the number $\mathsf{BdCof}\,(f) := \max \mathsf{sBC}(f)$. Observe that

- $\mathsf{BdCof}\,(f)$ is not defined if and only if the set $\mathsf{sBC}(f)$ is empty; and
- whenever $\mathsf{BdCof}\,(f)$ is defined we have $\mathsf{BdCof}\,(f) \leq \mathsf{StrBdCof}\,(f)$. Moreover $\mathsf{BdCof}\,(f) < \mathsf{StrBdCof}\,(f)$ if and only if $\mathsf{StrBdCof}\,(f) \notin \mathsf{sBC}(f)$.

□

Now let us see that *larger boundary of cofiniteness implies simpler set of periods*: On the one hand, the facts that $\mathsf{BdCof}(f) \le \mathsf{StrBdCof}(f)$ and $\mathsf{Succ}(\mathsf{StrBdCof}(f))$ is the cofinite part of $\mathsf{Per}(f)$ imply that $\mathsf{BdCof}(f)$ is a lower bound of the start of the cofinite part of $\mathsf{Per}(f)$ (in particular, the larger it is $\mathsf{BdCof}(f)$ the smaller it is the cofinite part of $\mathsf{Per}(f)$). On the other hand,

$$\mathsf{Card}\bigl(\{1,\dots,\mathsf{BdCof}(f)-2\}\cap\mathsf{Per}(f)\bigr) \le 2\log_2(\mathsf{BdCof}(f)-2)$$

is equivalent to

$$\begin{aligned}\mathsf{DensLowPer}_f(\mathsf{BdCof}(f)) &:= \frac{\mathsf{Card}\bigl(\{1,\dots,\mathsf{BdCof}(f)-2\}\cap\mathsf{Per}(f)\bigr)}{\mathsf{BdCof}(f)-2}\\ &\le 2\frac{\log_2(\mathsf{BdCof}(f)-2)}{\mathsf{BdCof}(f)-2},\end{aligned}$$

where for every $L \in \mathsf{sBC}(f)$, $\mathsf{DensLowPer}_f(L)$ denotes the *density of the $L-$low periods of f* which, by definition, are the periods of f which are smaller than L. Consequently, again, the larger it is $\mathsf{BdCof}(f)$ the smaller it is the density of the $\mathsf{BdCof}(f)-$low periods of f because

$$\frac{\log_2(x)}{x} \text{ is decreasing for } x \ge 2 \quad\text{and}\quad \lim_{x\to\infty}\frac{\log_2(x)}{x} = 0.$$

Justifying Remark. The reason for defining $\mathsf{BdCof}(f) := \max\mathsf{sBC}(f)$ is that, among all elements of the set $\mathsf{sBC}(f)$, $\max\mathsf{sBC}(f)$ is the one that gives the largest possible lower bound of the beginning of the cofinite part of the set of periods and *simultaneously* the smallest possible density of the set of $\mathsf{BdCof}(f)-$low periods of f. □

Remark 1.5. The boundary of cofiniteness cannot be replaced by the much simpler notion of strict boundary of cofiniteness. Indeed, since the condition

$$\mathsf{Card}\bigl(\{1,\dots,\mathsf{BdCof}(f)-2\}\cap\mathsf{Per}(f)\bigr) \le 2\log_2(\mathsf{BdCof}(f)-2)$$

need not be verified by $\mathsf{StrBdCof}(f)$ (when $\mathsf{StrBdCof}(f) \notin \mathsf{sBC}(f)$), the density of the $\mathsf{StrBdCof}(f)-$low periods of f could be large (in fact, even, arbitrarily close to one), contradicting the simplicity of the set of periods. □

Now we can state precisely what do we mean by extending the entropy simplicity phenomenon described above to the set of periods: *is it true that there exist totally transitive (and hence dynamically complicate) graph maps with arbitrarily large boundary of cofiniteness?*

We illustrate the above statement with three examples for arbitrary graphs which are not trees.

The rotation interval of a circle map of degree one is a fundamental tool to determine its set of periods. We will define this object in Section 2, where will describe in detail its relation with the set of periods of the map under consideration. In what follows we will denote the set of all liftings of all continuous circle maps of degree one by $\mathcal{L}_1$, and the rotation interval of $F \in \mathcal{L}_1$ by $\mathsf{Rot}(F)$. These notions are well known and, to improve the readability of this rather long introduction, their definitions are written in detail in Subsection 2.2.

In what follows, $\lfloor\cdot\rfloor$denotes the integer part function.

Example 1.6 (the dream example). For every positive integer $n \geq 3$ there exists a totally transitive continuous circle map of degree one, $f_{1,n}$, having a lifting $F_{1,n} \in \mathcal{L}_1$ such that

$$\mathsf{Rot}(F_{1,n}) = \left[\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\right], \quad \mathsf{Per}(f_{1,n}) = \mathsf{Succ}\,(n)\,, \text{ and } \lim_{n\to\infty} h(f_{1,n}) = 0.$$

Hence, $\mathsf{BdCof}\,(f_{1,n}) = \mathsf{StrBdCof}\,(f_{1,n}) = n$. Thus, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_{1,n}) = \infty$.

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{1,n}\}_{n\geq 3}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{\phi_n^{\mathsf{G};1}\}_{n\geq 3}$, such that

$$\mathsf{Per}(\phi_n^{\mathsf{G};1}) = \mathsf{Per}(f_{1,n}) \quad \text{and} \quad \lim_{n\to\infty} h(\phi_n^{\mathsf{G};1}) = 0.$$

Remark 1.7. In this example there are no $\mathsf{BdCof}\left(\phi_n^{\mathsf{G};1}\right)$−low periods. So, $\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};1}\right)$ is enough to control the complexity of the set of periods. ◻

Remark 1.8. Our choice of the rotation interval in this example was influenced by the Farey sequence of order $2n-1$ (which is the ordered sequence of rationals $\frac{p}{q}$ such that $0 \leq p \leq q \leq 2n-1$, $(p,q) = 1$). It follows that two elements $\frac{p}{q} < \frac{r}{s}$ in a Farey sequence are consecutive (called *Farey neighbours*) if and only if $qr - ps = 1$. The endpoints of the rotation interval of Example 1.6 belong to the Farey sequence of order $2n-1$ and the elements of this sequence between them are

$$\tfrac{1}{2n-1} < \tfrac{1}{2n-2} < \tfrac{1}{2n-3} < \tfrac{1}{2n-4} < \cdots < \tfrac{1}{n} < \tfrac{2}{2n-1}.$$

This tells us that $\mathsf{Per}(\phi_n^{\mathsf{G};1}) \cap \{1,2,\dots,2n-2\} = \{n, n+1, \dots, 2n-2\}$ which was the kind of set of periods we were looking for. ◻

Example 1.9 (with persistent fixed low periods). For every $n \in \mathbb{N}$ odd, $n \geq 7$, there exists a totally transitive continuous circle map of degree one, $f_{2,n}$, having a lifting $F_{2,n} \in \mathcal{L}_1$ such that $\mathsf{Rot}(F_{2,n}) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$, $\lim_{n\to\infty} h(f_{2,n}) = 0$,

$$\mathsf{Per}(f_{2,n}) = \{2\} \cup \left\{q \text{ odd} : 2\cdot\left\lfloor\tfrac{n+1}{4}\right\rfloor + 1 \leq q \leq n-2\right\} \cup \mathsf{Succ}\,(n)\,,$$

and $\mathsf{BdCof}\,(f_{2,n})$ exists and verifies $2\cdot\left\lfloor\frac{n+1}{4}\right\rfloor + 1 \leq \mathsf{BdCof}\,(f_{2,n}) \leq n$ (and, hence, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_{2,n}) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{2,n}\}_{n\geq 7, n \text{ odd}}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{\phi_n^{\mathsf{G};2}\}_{n\geq 7, n \text{ odd}}$, such that $\mathsf{Per}(\phi_n^{\mathsf{G};2}) = \mathsf{Per}(f_{2,n})$ and, additionally, $\lim_{n\to\infty} h(\phi_n^{\mathsf{G};2}) = 0$.

Remark 1.10 (A concrete new motivation of the boundary of cofiniteness's definition). The above example is different from the previous one since for every odd n there exist $\mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)$−low periods and, moreover, every map $\phi_n^{\mathsf{G};2}$ has a constant $\mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)$−low period 2.

On the other hand, $\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right) = n \neq \mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)$. In this example this is due to the fact that the set of periods which are smaller than $\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)$ is very large relative to the value of $\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)$. More concretely,

$$\mathsf{DensLowPer}_{\phi_n^{\mathsf{G};2}}\Big(\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)\Big) = \begin{cases} \frac{k+1}{4k-1} & \text{if } n = 4k+1, \\ \frac{k}{4k-3} & \text{if } n = 4k-1. \end{cases}$$

So, for large n,

$$\mathsf{DensLowPer}_{\phi_n^{\mathsf{G};2}}\Big(\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)\Big) \approx \tfrac{1}{4} > \frac{2\log_2(n-2)}{n-2} = \frac{2\log_2\left(\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)-2\right)}{\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)-2}$$

and, hence, the strict boundary of cofiniteness does not belong to $\mathsf{sBC}\big(\phi_n^{\mathsf{G};2}\big)$. Furthermore, the differences $\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right) - \mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)$ are unbounded since, otherwise,

$$\begin{aligned} 0 < \lim_{n\to\infty} \mathsf{DensLowPer}_{\phi_n^{\mathsf{G};2}}\Big(\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)\Big) &= \\ \lim_{n\to\infty} \frac{\mathsf{Card}\Big(\Big\{1,\dots,\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)-2\Big\} \cap \mathsf{Per}\big(\phi_n^{\mathsf{G};2}\big)\Big)}{\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)-2} &= \\ \lim_{n\to\infty} \frac{\mathsf{Card}\Big(\Big\{1,\dots,\mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)-2\Big\} \cap \mathsf{Per}\big(\phi_n^{\mathsf{G};2}\big)\Big)}{\mathsf{StrBdCof}\left(\phi_n^{\mathsf{G};2}\right)-2} &\leq \\ \lim_{n\to\infty} \frac{2\log_2\left(\mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)-2\right)}{\mathsf{BdCof}\left(\phi_n^{\mathsf{G};2}\right)-2} &= 0; \end{aligned}$$

a contradiction. ◘

Example 1.11 (with non-constant low periods). For every $n \in \mathbb{N}$, $n \geq 5$ there exists $f_{3,n}$, a totally transitive continuous circle map of degree one having a lifting $F_{3,n} \in \mathcal{L}_1$ such that

$$\mathsf{Rot}(F_{3,n}) = \left[\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\right] = \left[\tfrac{1}{n} - \tfrac{1}{2n^2}, \tfrac{1}{n} + \tfrac{1}{2n^2}\right],$$

$\lim_{n\to\infty} h(f_{3,n}) = 0$, and

$$\begin{aligned} \mathsf{Per}(f_{3,n}) = \{n, 2n, 2n+1\} \;\cup \Big\{ tn + k : t \in \{3,4,\dots,\nu-1\} \text{ and} \\ k \in \left\{-\left\lfloor\tfrac{t-1}{2}\right\rfloor, -\left\lfloor\tfrac{t-1}{2}\right\rfloor + 1, \dots, 0, 1, \dots, \left\lfloor\tfrac{t}{2}\right\rfloor\right\}\Big\} \cup \\ \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right) \end{aligned}$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, $\mathsf{StrBdCof}\left(f_{3,n}\right) = n\nu + 1 - \frac{\nu}{2}$ and $\mathsf{BdCof}\left(f_{3,n}\right)$ exists and verifies $n \leq \mathsf{BdCof}\left(f_{3,n}\right) \leq n\nu + 1 - \frac{\nu}{2}$ (and hence, $\lim_{n\to\infty} \mathsf{BdCof}\left(f_{3,n}\right) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{3,n}\}_{n=5}^{\infty}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{\phi_n^{\mathsf{G};3}\}_{n=5}^{\infty}$, such that

$$\mathsf{Per}(\phi_n^{\mathsf{G};3}) = \mathsf{Per}(f_{3,n}) \quad \text{and} \quad \lim_{n\to\infty} h(\phi_n^{\mathsf{G};3}) = 0.$$

Remark 1.12. This example is different from the previous two examples since for every n there exist $\mathsf{BdCof}\,(\phi_n^{\mathsf{G};3})$−low periods but there is no constant $\mathsf{BdCof}\,(\phi_n^{\mathsf{G};3})$−low period.

Moreover, as in the previous example, $\mathsf{StrBdCof}\,(\phi_n^{\mathsf{G};3}) \neq \mathsf{BdCof}\,(\phi_n^{\mathsf{G};3})$,

$$\mathsf{DensLowPer}_{\phi_n^{\mathsf{G};3}}\Big(\mathsf{StrBdCof}\,(\phi_n^{\mathsf{G};3})\Big) \approx \tfrac{1}{2},$$

and the differences $\mathsf{StrBdCof}\,(\phi_n^{\mathsf{G};3}) - \mathsf{BdCof}\,(\phi_n^{\mathsf{G};3})$ are unbounded. ◘

Finally, we state the main theorem of the paper that shows that the above examples are not exceptional among circle maps of degree one. On the contrary, a sequence of totally transitive circle maps that unfolds an entropy simplification process also must unfold a set of periods simplification process:

Theorem A. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of totally transitive circle maps of degree one with periodic points such that* $\lim_{n\to\infty} h(f_n) = 0$. *For every* n *let* $F_n \in \mathcal{L}_1$ *be a lifting of* f_n. *Then,*

- $\lim_{n\to\infty} \mathsf{len}\,(\mathsf{Rot}(F_n)) = 0$,
- *there exists* $N \in \mathbb{N}$ *such that* $\mathsf{BdCof}\,(f_n)$ *exists for every* $n \geq N$, *and*
- $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$.

Despite of this very general theorem for totally transitive circle maps of degree one, we emphasize that the examples are strongly relevant and motivated by the following two reasons: from one side they are valid for any general graph that is not a tree (not just the circle) thus showing that the phenomenon described in this paper seems to be much more general than the result obtained in Theorem A. On the other hand, the three examples show that all situations about the density of the $\mathsf{StrBdCof}\,(f)$*−low periods of* f are possible (see Remarks 1.7, 1 and 1.12) and, in particular, they show that the boundary of cofiniteness cannot be replaced by the notion of strict boundary of cofiniteness (see Remark 1.5).

The above three examples are constructed by means of a "factory" of general graph examples whose technology is based in two ingredients: first, the construction of appropriate minimal (in the sense of dynamics and topological entropy) circle maps of degree 1, and second, a general construction that "exports" these minimal maps to arbitrary graphs that are not trees by keeping their basic dynamical properties. With this technology, to construct the above examples we only need to choose appropriate sequences of candidates to rotation intervals and control the corresponding sets of periods for the corresponding circle maps.

This paper is organized as follows. In Section 2 we introduce the definitions and preliminary results for the rest of the paper. Theorem A is proved in Section 3. In Section 4 we show how to "export" certain circle maps of degree one to arbitrary graphs that are not trees while essentially maintaining their dynamical properties. In Section 5 we survey and study the above minimal circle maps of degree 1, and their construction. These two sections constitute what we call a "factory" to construct graph examples. The next Section 6 studies the dynamical properties of the minimalistic extensions of certain families of degree one minimal maps. Examples 1.6, 1.9 and 1.11 are then constructed in Section 7 by using the tools developed in the factory (Sections 4, 5 and 6).

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2. Definitions and preliminary results

In this section we essentially follow the notation and strategy of [9].

2.1. **Basic definitions.** In the rest of the paper we denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ by $\mathbb{S}^1$.

A *topological graph* (or simply a *graph*) is a connected compact Hausdorff space X for which there exists a finite (or empty) set $V(X)$,called the *set of vertices of* X, such that either $X = \mathbb{S}^1$ and $V(X) = \emptyset$ or $X \setminus V(X)$ is the disjoint union of finitely many open subsets of X each of them homeomorphic to an open interval of the real line, called *edges of* X, with the property that the boundary of every edge consists of at most two points which are in $V(X)$. A point $z \in V(X)$ is an *endpoint of* X if there exists an open (in X) neighbourhood U of z such that $X \cap U$ is homeomorphic to the interval $[0, 1)$ being z the preimage of 0. A *circuit* of a graph X is any subset of X homeomorphic to a circle. A *tree* is a graph without circuits.

Let X be a topological space and let $f : X \longrightarrow X$ be a continuous map. When iterating the map f we will use the following notation: f^0 will denote the identity map (in X), and $f^n := f \circ f^{n-1}$ for every $n \in \mathbb{N}$, $n \geq 1$. For a point $x \in X$, we define the *f-orbit of x* (or simply the *orbit of x*), denoted by $\mathsf{Orb}_f(x)$, as the set $\{f^n(x) : n \geq 0\}$. A point $x \in X$ is called a *periodic point of f* if $f^n(x) = x$. In such case $\mathsf{Orb}_f(x)$ is called a *periodic orbit of f* and $\mathsf{Card}\left(\mathsf{Orb}_f(x)\right)$ is called the *period of x* (or *f-period of x* if we need to specify the map). Observe that if x is a periodic point of f of period n, then $f^k(x) \neq x$ for every $1 \leq k < n$ and if P is a periodic orbit of f, then $P = \mathsf{Orb}_f(x)$ for every $x \in P$.

The set of all positive integers n such that f has a periodic point of period n is denoted by $\mathsf{Per}(f)$.

2.2. **Rotation theory and sets of periods for circle maps of degree one.** We start by introducing the key notion to work with circle maps: the liftings. Let $\mathsf{e} : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the natural projection which is defined by $\mathsf{e}(x) := \exp(2\pi i x)$. Given a continuous map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we say that a continuous map $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a *lifting of f* if $\mathsf{e}(F(x)) = f(\mathsf{e}(x))$ for every $x \in \mathbb{R}$. For such F, there exists $d \in \mathbb{Z}$ such that $F(x + 1) = F(x) + d$ for all

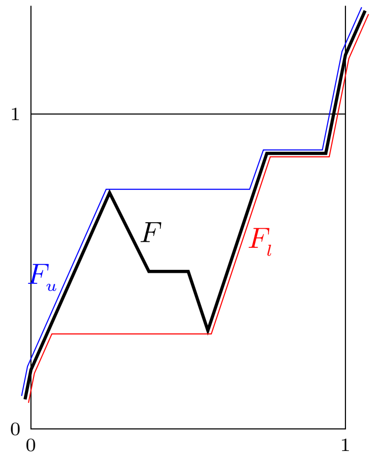


FIGURE 1. An example of a map $F \in \mathcal{L}_1$ with its lower map F_l in red and its upper map F_u in blue.

$x \in \mathbb{R}$, and this integer is called both the *degree of* f and the *degree of* F. If G and F are two liftings of f then $G = F + k$ for some integer k and so F and G have the same degree. We denote by $\mathcal{L}_d$ the set of all liftings of circle maps of degree d.

Next we introduce the important notion of *rotation interval* for maps from $\mathcal{L}_1$. Let $F \in \mathcal{L}_1$ and let $x \in \mathbb{R}$. The number

$$\rho_F(x) := \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

will be called the *rotation number of* x. Moreover, the set

$$\mathsf{Rot}(F) := \{\rho_F(x) : x \in \mathbb{R}\} = \{\rho_F(x) : x \in [0,1]\}$$

will be called the *rotation interval of* F. It is well known that it is a closed interval of the real line [16].

If $F \in \mathcal{L}_1$ is a non-decreasing map, then

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

for every $x \in \mathbb{R}$ and, moreover, it is independent on x (see for instance [9]). Then this number ($\rho_F(x)$ for any $x \in \mathbb{R}$), will be called the *rotation number of* F. For every $F \in \mathcal{L}_1$ we define the *lower map* $F_l : \mathbb{R} \longrightarrow \mathbb{R}$ by (see Figure 1 for a graphical example)

$$F_l(x) = \inf \{F(y) : y \geq x\}$$

and the *upper map* $F_u : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F_u(x) = \sup \{F(y) : y \leq x\}.$$

It is easy to see (see e.g. [9]) that F_l, F_u are non-decreasing maps from $\mathcal{L}_1$.

The next theorem (c.f., [9, Theorem 3.7.20]) gives an effective way to compute the rotation interval from the rotation numbers of the upper and lower maps.

Theorem 2.1. *For every $F \in \mathcal{L}_1$ it follows that* $\mathsf{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

It is well known that the rotation interval of a lifting $F \in \mathcal{L}_1$ can be used to obtain information about the set of periods of the corresponding circle map. To clarify this point we will introduce the notion of *lifted orbit*.

Let f be a continuous circle map of degree d and let $F \in \mathcal{L}_d$ be a lifting of f. A set $P \subset \mathbb{R}$ will be called a *lifted orbit of* F if there exists $z \in \mathbb{S}^1$ such that $P = \mathsf{e}^{-1}\bigl(\mathsf{Orb}_f(z)\bigr)$ and $f(\mathsf{e}(x)) = \mathsf{e}(F(x))$ for every $x \in P$. Whenever z is a periodic point of f of period n, P will be called a *lifted periodic orbit of* F *of period* n. We will denote by $\mathsf{Per}(F)$ the set of periods of all lifted periodic orbits of F. Observe that then, $\mathsf{Per}(F) = \mathsf{Per}(f)$.

Remark 2.2. Let $F \in \mathcal{L}_1$ and let P be a lifted periodic orbit of F of period period n. Set

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

with $x_i < x_j$ if and only if $i < j$. The fact that $P = \mathsf{e}^{-1}\bigl(\mathsf{Orb}_f(z)\bigr)$, in this case, gives

$$\mathsf{Card}\,(P \cap [r, r+1)) = n$$

for every $r \in \mathbb{R}$ and, hence,

$$x_{kn+i} = x_i + k$$

for every $i, k \in \mathbb{Z}$.

Moreover, there exists $m \in \mathbb{Z}$ such that $F^n(x_i) = x_i + m = x_{mn+i}$ for every $x_i \in P$. Consequently,

$$\rho_F(x_i) = \tfrac{m}{n}$$

for every $x_i \in P$. ◘

From the above remark it follows that if P is a lifted periodic orbit of $F \in \mathcal{L}_1$, then all the points of P have the same rotation number. This number will be called the *rotation number of* P (or F*-rotation number of* P if we need to specify the lifting).

A lifted periodic orbit P of $F \in \mathcal{L}_1$ such that $F\big|_P$ is increasing will be called *twist*.

Remark 2.3. Let

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

be a twist lifted periodic orbit of $F \in \mathcal{L}_1$ of period n and rotation number m/n labelled so that $x_i < x_j$ if and only if $i < j$. By [9, Lemma 3.7.4 and Corollary 3.7.6] we have that m and n are co-prime and

$$F(x_i) = x_{i+m},$$

for all $i \in \mathbb{Z}$. ◘

The next theorem due to Misiurewicz (see [18, 9]) already makes the connection between $\mathsf{Rot}(F)$ and $\mathsf{Per}(F)$. To state it, we still need to recall the *Sharkovskiĭ Ordering*.

The *Sharkovskiĭ Ordering* ${}_{\text{Sh}}\!\geq$ (the symbols ${}_{\text{Sh}}\!>$, $<_{\text{Sh}}$ and $\leq_{\text{Sh}}$ will also be used in the natural way) is a linear ordering of $\mathbb{N}_{\text{Sh}} := \mathbb{N} \cup \{2^\infty\}$ (we have to include the symbol $\{2^\infty\}$ in order to ensure the existence of supremum of every subset with respect to the ordering ${}_{\text{Sh}}\!\geq$) defined as follows:

$$3 \;{}_{\text{Sh}}\!> 5 \;{}_{\text{Sh}}\!> 7 \;{}_{\text{Sh}}\!> 9 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!>$$

$$\begin{array}{l} 2\cdot 3 \,_{\mathrm{Sh}}\!> 2\cdot 5 \,_{\mathrm{Sh}}\!> 2\cdot 7 \,_{\mathrm{Sh}}\!> 2\cdot 9 \,_{\mathrm{Sh}}\!> \cdots \,_{\mathrm{Sh}}\!> \\ 4\cdot 3 \,_{\mathrm{Sh}}\!> 4\cdot 5 \,_{\mathrm{Sh}}\!> 4\cdot 7 \,_{\mathrm{Sh}}\!> 4\cdot 9 \,_{\mathrm{Sh}}\!> \cdots \,_{\mathrm{Sh}}\!> \\ \qquad\qquad\vdots \\ 2^n\cdot 3 \,_{\mathrm{Sh}}\!> 2^n\cdot 5 \,_{\mathrm{Sh}}\!> 2^n\cdot 7 \,_{\mathrm{Sh}}\!> 2^n\cdot 9 \,_{\mathrm{Sh}}\!> \cdots \,_{\mathrm{Sh}}\!> \\ \qquad\qquad\vdots \\ 2^\infty \,_{\mathrm{Sh}}\!> \cdots \,_{\mathrm{Sh}}\!> 2^n \,_{\mathrm{Sh}}\!> \cdots \,_{\mathrm{Sh}}\!> 16 \,_{\mathrm{Sh}}\!> 8 \,_{\mathrm{Sh}}\!> 4 \,_{\mathrm{Sh}}\!> 2 \,_{\mathrm{Sh}}\!> 1. \end{array}$$

We introduce the following notation. Given $c, d \in \mathbb{R}$, $c \le d$ we set

$$M(c,d) := \{n \in \mathbb{N} : c < k/n < d \text{ for some integer } k\}.$$

Let $F \in \mathcal{L}_1$ and let c be an endpoint of $\mathsf{Rot}(F)$. We define the set

$$Q_F(c) := \begin{cases} \emptyset & \text{if } c \notin \mathbb{Q} \\ \{sk : k \in \mathbb{N} \text{ and } k \le_{\mathrm{Sh}} s_c\} & \text{if } c = r/s \text{ with } r, s \text{ co-prime} \end{cases}$$

and $s_c \in \mathbb{N}_{\mathrm{Sh}}$ is defined by the Sharkovskiĭ Theorem on the real line. Indeed, since $c = r/s$ and r and s are co-prime, the map $F^s - r$ is a continuous map on the real line with periodic points. Hence, by the Sharkovskiĭ Theorem there exists an $s_c \in \mathbb{N}_{\mathrm{Sh}}$ such that the set of periods (not lifted periods) of $F^s - r$ is precisely $\{t \in \mathbb{N} : t \le_{\mathrm{Sh}} s_c\}$.

Theorem 2.4. *Let f be a continuous circle map of degree one having a lifting $F \in \mathcal{L}_1$. Assume that $\mathsf{Rot}(F) = [c, d]$. Then*

$$\mathsf{Per}(f) = Q_F(c) \cup M(c,d) \cup Q_F(d).$$

Corollary 2.5. *Let f be a continuous transitive circle map of degree one having a lifting $F \in \mathcal{L}_1$. Assume that $\mathsf{Rot}(F) = [c, d]$ is non-degenerate. Then, f is totally transitive.*

Proof. By Theorem 2.4,

$$\mathsf{Succ}\left(\left\lfloor \tfrac{1}{d-c} \right\rfloor + 1\right) \subset M\big(c,d\big) \subset \mathsf{Per}(g_{c,d}).$$

Thus, $\mathsf{Per}(g_{c,d})$ is cofinite and $g_{c,d}$ is totally transitive by Theorem 1.2. □

2.3. **Markov graphs, Markov maps and sets of periods.** Take a finite set $V = \{v_1, v_2, \dots, v_n\}$. The pair $\mathcal{G} = (V, U)$ where $U \subset V \times V$ is called a *combinatorial oriented graph*. The elements of V are called the *vertices of* $\mathcal{G}$ and each element $(v_i, v_j) \in U$ is called an *arrow from v_i to v_j*. An arrow (v_i, v_j) will also be denoted by $v_i \longrightarrow v_j$, which allows us to give a graphical representation of an oriented graph. A *path of length k* is a sequence of $k+1$ vertices $v_0, v_1, \dots, v_k$ with the property that there is an arrow from every vertex to the next one. A path is denoted as $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$. A *loop of length k* is a path of length k where the first and last vertex coincide: $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0$.

Let X be a topological graph. Every subset of X homeomorphic to the interval $[0, 1]$ will in turn be called an *interval of X*. The preimages of 0 and 1 by the homeomorphism will be called the *endpoints of I* and the set of (both) endpoints of I will be denoted by ∂I. The *interior of I*, denoted by $\mathsf{Int}(I)$, is defined to be $I \setminus \partial I$. Observe that when $\mathsf{Int}(I) \cap V(X) \neq \emptyset$, then the interior of I does not coincide with the *topological interior of I*.

Let X be a topological graph and let $f\colon X \longrightarrow X$ be a continuous map. A set $Q \subset X$ will be called *f-invariant* if $f(Q) \subset Q$. A *Markov invariant set* is defined to be a finite f-invariant set $Q \supset V(X)$ such that the closure of each connected component of $X \setminus Q$, called a *Q-basic interval*, is an interval of X. Observe that a Q-basic interval is always non-degenerate, its interior and its topological interior coincide, and that two different Q-basic intervals have disjoint interiors.

The set of all Q-basic intervals will be denoted by $\mathcal{B}(Q)$.

Definition 2.6 (Monotonicity over an interval). Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let I be an interval of X. The map f will be said to be *monotone at I* if the set $\left(f\big|_I\right)^{-1}(y) = \{x \in I : f(x) = y\}$ is connected for every $y \in f(I)$. Clearly, since I is an interval, $\left(f\big|_I\right)^{-1}(y)$ is either a point or an interval for every $y \in f(I)$. Moreover, a simple exercise shows that the subgraph $f(I)$, in turn, must be either a point or an interval. Finally, let $g\colon X \longrightarrow X$ be another continuous map, and let J be another interval of X such that $f(I) \cap \mathsf{Int}(J) \neq \emptyset$ and g is monotone at J. It is easy to see that $\left(f\big|_I\right)^{-1}(J) = \{x \in I : f(x) \in J\}$ is an interval and $g\big|_J \circ f\big|_I$ is monotone at $\left(f\big|_I\right)^{-1}(J)$. ◘

Remark 2.7. The above definition of monotonicity over an interval is equivalent to the usual one: $f(I)$ is a point or an interval and, in the second case, the map $\zeta \circ f\big|_I \circ \xi\colon [0,1] \longrightarrow [0,1]$ is monotone as interval map, where $\xi\colon [0,1] \longrightarrow I$ and $\zeta\colon f(I) \longrightarrow [0,1]$ are homeomorphisms (which exist because I and $f(I)$ are intervals). We prefer the more intrinsic definition given above because it is independent on the choice of the auxiliary homeomorphisms and on whether they are increasing or decreasing. This will be specially helpful when studying compositions of monotone maps over intervals like in the rest of this subsection and Subsection 2.5. ◘

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let $Q \subset X$ be a Markov invariant set. We say that f is *Q-monotone* if f is monotone on each Q-basic interval. In such a case, Q is called a *Markov partition of X with respect to f* and f is called a *Markov map with respect to Q*.

Next we introduce the very important notion of f-covering that allows us to get a combinatorial oriented graph from a Markov partition of a Markov map.

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let Q be a Markov partition of X with respect to f. Given $I, J \in \mathcal{B}(Q)$, we say that I *f-covers* J if $f(I) \supset J$. The *Markov graph of f with respect to Q* (or *f-graph*) is a combinatorial oriented graph whose vertices are all the Q-basic intervals and there is an arrow $I \longrightarrow J$ from the vertex (Q-basic interval) I to the vertex (Q-basic interval) J if and only if I f-covers J. The *Markov matrix of f with respect to Q* is another combinatorial object that describe the dynamical behaviour of a Markov map f and is the *transition matrix*of the Markov graph of f with respect to Q which is, by definition, a

$\mathsf{Card}(\mathcal{B}(Q)) \times \mathsf{Card}(\mathcal{B}(Q))$ matrix $M = (m_{I,J})_{I,J\in\mathcal{B}(Q)}$ such that

$$m_{I,J} = \begin{cases} 1 & \text{if } I \text{ } f\text{-covers } J \\ 0 & \text{otherwise} \end{cases}.$$

The next lemma shows the relation between loops of Markov graphs and periodic points. Essentially it is [9, Corollary 1.2.8] extended to graph maps.

Given $q \in \mathbb{N}$, the congruence classes modulo q will be $\{0, 1, \dots, q-1\}$.

Lemma 2.8. *Let X be a topological graph, let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition Q of X and let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ be a loop in the Markov graph of f with respect to Q. Then, there exists a fixed point $x \in I_0$ of f^n such that $f^i(x) \in I_i$ for $i = 1, 2, \dots, n-1$.*

The next result compiles [9, Lemma 1.2.12 and Theorem 2.6.4] extended to graph maps. To state it we need to introduce some more definitions.

Given two paths $\alpha = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$ and $\beta = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_m$ in a combinatorial graph such that the last vertex of the first path is the first vertex of the second one (i.e., $v_k = w_0$), the *concatenation* α and β is denoted by $\alpha\beta$ and is the path

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_m.$$

Clearly, the length of the concatenated path is the sum of the lengths of the original paths. A loop is an *n-repetition* of a (shorter) loop α if $n \geq 2$ and it is a concatenation of α with itself n times. Such a loop will be called *repetitive*. A loop which is not repetitive will also be called *simple*.

The *shift of a loop* $\alpha = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0$ is defined to be the loop

$$S(\alpha) := v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0 \longrightarrow v_1.$$

Iteratively, we set $S^0(\alpha) := \alpha$ and, for every $m \in \mathbb{N}$, we define the *m-shift of α*, denoted by $S^m(\alpha)$, as the loop

$$v_{m \pmod k} \longrightarrow v_{m+1 \pmod k} \longrightarrow \cdots \longrightarrow v_{m+k-1 \pmod k} \longrightarrow v_{m \pmod k}.$$

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition, let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{m-1} \longrightarrow I_0$ be a loop in the Markov graph of f and let P be a periodic orbit of period m of f. We say that α and P are *associated* to each other if there exists an $x \in P$ such that $f^k(x) \in I_k$ for $k = 0, 1, \dots, m-1$. Observe that if α is associated to a periodic orbit, then so is $S^k(\alpha)$ for every $k \in \mathbb{N}$ (by replacing x by $f^k(x)$ in the above definition).

From the proof of [9, Lemma 1.2.12] we have

Proposition 2.9. *Let X be a topological graph and let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition Q of X. Assume that P is a periodic orbit of f disjoint from Q. Then, there exists a loop α in the Markov graph of f with respect to Q which is associated to P, and any other loop in the f-graph of Q associated to P is of the form $S^k(\alpha)$ with $k \in \mathbb{N}$.*

2.4. **Markov graphs modulo 1 for circle maps of degree one.** As we have seen, when the topological graph is the circle $\mathbb{S}^1$ and we are dealing with maps of degree one it is better to work with liftings instead of with the original maps, specially to avoid problems with ordering (the circle does not have a linear ordering). In what follows we adapt the definitions related to Markov graphs to this approach.

Let f be a continuous circle map and let $\widetilde{Q} \subset \mathbb{S}^1$ be a finite f-invariant set with at least two elements (in fact, since $V(\mathbb{S}^1) = \emptyset$, $\widetilde{Q}$ is a Markov invariant set). Let $F \in \mathcal{L}_1$ be a lifting of f. Then the set $Q = \mathsf{e}^{-1}\big(\widetilde{Q}\big)$ is F-invariant, is a partition of $\mathbb{R}$ and each interval produced by this partition will be called a *Q-basic interval*. Again the set of all Q-basic intervals will be denoted by $\mathcal{B}(Q)$. If the restriction of F to each Q-basic interval is monotone (as a map from the real line to itself), we say that F is Q-monotone, Q is a *Markov partition with respect to F*, and F is a *Markov map*. Given $I, J \in \mathcal{B}(Q)$, we say I is equivalent to J and denote it by $I \sim J$ if $I = J + k$ for some $k \in \mathbb{Z}$. The equivalence class of I, $\{I + k : k \in \mathbb{Z}\}$, is denoted by $[\![I]\!]$.

Now we define the *Markov graph modulo 1 of F with respect to Q*. It is a combinatorial oriented graph whose vertices are all the equivalence classes of Q-basic intervals and there is an arrow $[\![I]\!] \longrightarrow [\![J]\!]$ from $[\![I]\!]$ to $[\![J]\!]$ if and only if there is a representative $J + k$ of $[\![J]\!]$ such that $F(I) \supset J + k$. Recall that, since $F \in \mathcal{L}_1$, $F(I + \ell) = F(I) + \ell$ for every $\ell \in \mathbb{Z}$. Therefore, the Markov graph modulo 1 of F with respect to Q is well defined. Moreover, two different liftings of f have the same Markov graphs modulo 1 with respect to Q.

Remark 2.10 (On the projection of a Markov graph modulo 1 to the circle). Since the kernel of e is $\mathbb{Z}$ (that is, $\mathsf{e}(x+k) = \mathsf{e}(x)$ for every $x \in \mathbb{R}$ and $k \in \mathbb{Z}$), it follows that $\mathsf{e}(I) = \mathsf{e}(K)$ for every $K \in [\![I]\!]$. So, $\mathsf{e}\big([\![I]\!]\big) := \mathsf{e}(I)$ is well defined. Moreover, since $\widetilde{Q}$ has at least two elements, it follows that every Q-basic interval has length strictly smaller than 1. Hence, if $I \in \mathcal{B}(Q)$, $\mathsf{e}\big([\![I]\!]\big) \in \mathcal{B}(\mathsf{e}(Q))$ (that is, $\mathsf{e}\big([\![I]\!]\big)$ is a $\mathsf{e}(Q)$-basic interval in $\mathbb{S}^1$ — recall that $\mathsf{e}(Q) = \widetilde{Q}$ is a Markov invariant set of $\mathbb{S}^1$ with respect to f). Additionally, since $f(\mathsf{e}(I)) = \mathsf{e}(F(I))$, $\mathsf{e}\big([\![I]\!]\big)$ f-covers $\mathsf{e}\big([\![J]\!]\big)$ if and only if there is an arrow $[\![I]\!] \longrightarrow [\![J]\!]$ in the Markov graph modulo 1 of F.

However, the Q-monotonicity of F implies the $\mathsf{e}(Q)$-monotonicity of f *provided that the length of the F-image of every Q-basic interval is smaller than 1* (otherwise there exists $I \in \mathcal{B}(Q)$ such that $f(\mathsf{e}(I)) = \mathsf{e}(F(I)) = \mathbb{S}^1$ is not an interval). In other words, $\widetilde{Q}$ is a Markov partition of $\mathbb{S}^1$ with respect to f (and f is a Markov map with respect to $\widetilde{Q}$) whenever Q is a Markov partition with respect to F and the length of the F-image of every Q-basic interval is smaller than 1. ◻

This remark motivates the following definition: Let $F \in \mathcal{L}_1$ be a lifting of f and let Q be a Markov partition with respect to F. A Q-basic interval will be called *F-short* if the length of the interval $F(I)$ is strictly smaller than 1. Then, Q will be called a *short Markov partition with respect to F* whenever every Q-basic interval is F-short.

With this definition Remark 2.4 immediately gives the following result that relates Markov partitions in the circle with short Markov partitions with respect to liftings from $\mathcal{L}_1$.

Proposition 2.11. *Let f be a continuous circle map and let $F \in \mathcal{L}_1$ be a lifting of f. Let $Q \subset \mathbb{R}$ be a Markov partition with respect to F. Then, the following statements hold:*

(a) $\mathsf{e}(Q)$ is a Markov partition with respect to f if and only if Q is short.
(b) When Q is short, the Markov graph of f with respect to $\mathsf{e}(Q)$ and the Markov graph modulo 1 of F with respect to Q coincide, provided that we identify $[\![I]\!]$ with $\mathsf{e}\big([\![I]\!]\big)$ for every $I \in \mathcal{B}(Q)$.

2.5. **Transitivity and Markov matrices.** The aim of this subsection is to establish and prove the following result:

Theorem 2.12. *Let X be a topological graph and let $f\colon X \longrightarrow X$ be a Q-expanding Markov map with respect to a Markov partition Q of X. Then f is transitive if and only if the Markov matrix of f with respect to Q is irreducible but not a permutation matrix.*

This result is well known when X is a closed interval of the real line and the map is piecewise affine (see [12, Theorem 3.1]) but we aim at extending it to the general setting of graphs. Its proof in this more general case goes along the lines of the one from [12] for the interval but we will sketch it here for completeness.

In any case we need to recall the definition of irreducibility and establish what we understand by a piecewise expanding graph map.

A $k \times k$ matrix M is called *reducible* if there exists a permutation matrix P such that

$$P^T M P = \left(\begin{array}{c|c} M_{11} & \mathbf{0} \\ \hline M_{21} & M_{22} \end{array}\right)$$

where M_{11}, M_{21} and M_{22} are block matrices, and $\mathbf{0}$ is a zero matrix of the appropriate size. A matrix $P = (p_{ij})_{i,j=1}^k$ is a permutation matrix whenever $p_{ij} \in \{0,1\}$ for all $0 \le i,j \le k$ and in each row and in each column there is exactly one non-zero element. Observe that if P is a permutation matrix, then $P^{-1} = P^T$.

The matrix M is called *irreducible* if it is not reducible or, equivalently (see [15]), if for every $0 \le i,j \le k$ there is a natural number $n = n(i,j)$ such that the i,j-entry of M^n is strictly positive. In the case of a Markov matrix of a Markov partition of X, if we set $M^n = (m_{ij}^{(n)})_{i,j=1}^k$, then $m_{ij}^{(n)}$ is the number of paths of length n in the Markov graph starting at the vertex v_i and ending at the vertex v_j. In this context, M is irreducible if and only if there exists a path from the vertex v_i to the vertex v_j for every $0 \le i,j \le k$. In particular $f(I)$ is a (non-degenerate) interval for every basic interval I.

To define the notion of Q-expanding graph map we need to define a distance on basic intervals of graphs. Let X be a topological graph and let I be an interval of X such that $\mathsf{Int}(I) \cap V(X) = \emptyset$. Every such interval can be endowed in many ways with a distance d_I verifying that the *length of* I, defined as $\max_{x,y\in I} d_I(x,y)$, is 1. For instance, we can fix a homeomorphism

$\mu_I : I \longrightarrow [0,1]$ and set $d_I(x,y) := |\mu_I(x) - \mu_I(y)|$ for every $x, y \in I$. Given a connected set $W \subset I$ we define the *length of* W by

$$\|W\|_I := \max\{d_I(x,y) : x, y \in W\}.$$

From above it follows that $\|I\|_I = 1$ and $\|W\|_I \leq 1$.

Remark 2.13. Given a basic interval I, it is possible to modify the homeomorphism μ_I without modifying the distance d_I. Specifically, we can modify the orientation of the homeomorphism without altering the associated distance d_I. The simplest way of doing this is to define a new homeomorphism $\widetilde{\mu}_I := 1 - \mu_I$ so that

$$d_I(x,y) = |\mu_I(x) - \mu_I(y)| = |(1 - \mu_I(x)) - (1 - \mu_I(y))| = |\widetilde{\mu}_I(x) - \widetilde{\mu}_I(y)|.$$

This allows us to choose the concrete images of the points of the set ∂I. Indeed $\widetilde{\mu}_I\bigl(\partial I\bigr) = \mu_I\bigl(\partial I\bigr) = \{0,1\}$ but $\widetilde{\mu}_I(x) \neq \mu_I(x)$ for every $x \in \partial I$. ◘

Given a tree T (which is uniquely arcwise connected) and a set $A \subset T$, we denote by $\langle A \rangle_T$ the *convex hull of* A *in* T, that is, the smallest closed connected set of T that contains A.

Definition 2.14 (Piecewise expanding). Let X be a (topological) graph and let $f : X \longrightarrow X$ be a continuous map having a Markov invariant set Q.

We say that f is *expanding on* I if $f(I)$ is an interval which is a union of Q-basic intervals and

- **when $f(I) \in \mathcal{B}(Q)$:** f verifies
$$d_{f(I)}(f(x), f(y)) = \lambda_I d_I(x,y) = d_I(x,y)$$
(observe that in this special case we have $\lambda_I = 1$ for every $x, y \in I$);
- **when $f(I)$ contains more than one Q-basic interval:** there exists $\lambda_I > 1$ such that
$$d_J(f(x), f(y)) \geq \lambda_I d_I(x,y)$$
for every $x, y \in I$ such that $\langle f(x), f(y) \rangle_{f(I)} \subset J \in \mathcal{B}(Q)$.

Observe that when f is expanding on I then $f\big|_I$ is one-to-one and, hence, f is monotone at I.

We say that f is Q-*expanding* if it is expanding on every Q-basic interval. In particular, f is a Markov map with respect to Q. ◘

Proof of Theorem 2.12. We start with the simple exercise of proving that if f is transitive then the Markov matrix of f with respect to Q is irreducible but not a permutation matrix. Assume by way of contradiction that the Markov matrix of f with respect to Q is a permutation matrix. This is equivalent to say that we can label the set of all Q-basic intervals as $I_0, I_1, \ldots, I_{m-1}$ so that $f(I_i) = I_{i+1 \pmod m}$, and $f\big|_{I_i}$ is monotone for every $i = 0, 1, \ldots, m-1$. In these conditions and using the fact that f is a Q-expanding Markov map, f cannot have a dense orbit and thus it cannot be transitive.

On the other hand, by transitivity, the image of every Q-basic interval is different from a point. Thus, since f is Markov with respect to a Markov partition Q (in particular Q is f-invariant), it follows that for every basic interval $I \in \mathcal{B}(Q)$, $f(I)$ is an interval which is a union of Q-basic intervals and $f\big|_I$ is monotone. Therefore, it follows inductively that $f^k(I)$ is a union of Q-basic intervals for every $k \geq 1$.

Now we choose two arbitrary intervals $I, J \in \mathcal{B}(Q)$. Since f is transitive there exists a positive integer n such that

$$f^n(I) \cap \mathsf{Int}(J) \supset f^n(\mathsf{Int}(I)) \cap \mathsf{Int}(J) \neq \emptyset.$$

Since $f^n(I)$ is a union of Q-basic intervals and two different Q-basic intervals have disjoint interiors, $f^n(I) \supset J$. This means that there exists a Q-basic interval $J_{n-1} \subset f^{n-1}(I)$ such that J_{n-1} f-covers J. Analogously, there exists a Q-basic interval $J_{n-2} \subset f^{n-2}(I)$ such that J_{n-2} f-covers J_{n-1}. Iterating this argument we obtain a path $I \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \cdots \longrightarrow J_{n-1} \longrightarrow J$ from I to J in the Markov graph of f with respect to Q. Consequently, the Markov matrix of f with respect to Q is irreducible. This ends the proof of the "only if part" of the theorem.

Now we prove the "if part" following the ideas of [12]. So, we assume that the Markov matrix of f with respect to Q is irreducible but not a permutation matrix. The fact that the Markov matrix of f with respect to Q is not a permutation matrix tells us that

$$\{I : I \in \mathcal{B}(Q) \text{ and } I \text{ } f\text{-covers at least two basic intervals}\} \neq \emptyset.$$

Hence,

$$\lambda_f := \min\{\lambda_I : I \in \mathcal{B}(Q) \text{ and } I \text{ } f\text{-covers at least two basic intervals}\} > 1.$$

We claim that $U \cap \left(\cup_{k\geq 0} f^{-k}(Q)\right) \neq \emptyset$ for every connected non-empty open set $U \subset X$. To prove the claim assume by way of contradiction that $f^k(U) \cap Q = \emptyset$ for every $k \geq 0$. Then, there exists a sequence of Q-basic intervals $\{J_k\}_{k=0}^{\infty}$ such that $U \subset \mathsf{Int}(J_0)$ and $f^k(U) \subset \mathsf{Int}(J_k) \subset f(J_{k-1})$ for every $k \geq 1$. Moreover, from Definition 2.14 we have

$$\begin{aligned} \left\|f^{k+1}(U)\right\|_{J_{k+1}} &\geq \lambda_{J_k} \left\|f^k(U)\right\|_{J_k} \geq \lambda_{J_k} \lambda_{J_{k-1}} \left\|f^{k-1}(U)\right\|_{J_{k-1}} \geq \cdots \\ &\geq \|U\|_{J_0} \prod_{i=0}^{k} \lambda_{J_i} \end{aligned} \tag{2.1}$$

for every $k \geq 0$.

Assume that $\lambda_{J_i} \geq \lambda_f > 1$ for infinitely many indices i. Then, since $\lambda_I \geq 1$ for every $I \in \mathcal{B}(Q)$, the sequence

$$\left\{\prod_{i=0}^{k} \lambda_{J_i}\right\}_{k=0}^{\infty}$$

is non-decreasing, and hence

$$\lim_{k\to\infty} \left\|f^{k+1}(U)\right\|_{J_{k+1}} \geq \|U\|_{J_0} \lim_{k\to\infty} \prod_{i=0}^{k} \lambda_{J_i} = \infty.$$

This is a contradiction because, for every $k \geq 0$, J_{k+1} is a Q-basic interval and $f^{k+1}(U) \subset J_{k+1} \subset f(J_k)$; which implies $\left\|f^{k+1}(U)\right\|_{J_{k+1}} \leq \left\|J_{k+1}\right\|_{J_{k+1}} = 1$.

From the part of the claim already proven, there exists $m \in \mathbb{N}$ such that $\lambda_{J_k} = 1$ (that is, $f(J_k) \in \mathcal{B}(Q)$) for every $k \geq m$. Thus, $f(J_k) = J_{k+1}$ for every $k \geq m$ because $J_{k+1} \subset f(J_k)$. Since the number of Q-basic intervals is finite, there exist $\ell \geq m$ and $t \geq 1$ such that $J_\ell = J_{\ell+t}$.

We already know that there exists a basic interval $I \in \mathcal{B}(Q)$ that f-covers at least two basic intervals. So, $I \notin \{J_\ell, J_{\ell+1}, \dots, J_{\ell+t-1}\}$, and in the Markov graph of f with respect to Q there does not exist any path starting in a Q-basic interval from $\{J_\ell, J_{\ell+1}, \dots, J_{\ell+t-1}\}$ and ending at I. This contradicts the irreducibility of the Markov matrix of f with respect to Q and ends the proof of the claim.

Since for every non-empty open set V there exists $I \in \mathcal{B}(Q)$ such that $V \cap \mathsf{Int}(I) \neq \emptyset$, to prove that f is transitive it is enough to show that for every non-empty open set $U \subset X$ and every $I \in \mathcal{B}(Q)$ there exists a positive integer n such that $f^n(U) \supset \mathsf{Int}(I)$.

Let $J \in \mathcal{B}(Q)$ be such that $U \cap \mathsf{Int}(J) \neq \emptyset$. By the above claim with U replaced by a connected component of $U \cap \mathsf{Int}(J)$, it follows that there exists $x \in (U \cap \mathsf{Int}(J)) \cap \left(\cup_{k\geq 0} f^{-k}(Q)\right)$. So, again by the claim for a connected component of $(U \cap \mathsf{Int}(J)) \setminus \{x\}$ instead of U, we obtain that

$$\mathsf{Card}\left((U \cap \mathsf{Int}(J)) \cap \left(\cup_{k\geq 0} f^{-k}(Q)\right)\right) \geq 2.$$

Therefore, there exist $x, y \in U \cap \mathsf{Int}(J)$ with $x \neq y$ and $k_x, k_y \in \mathbb{N}$ such that $\langle x, y\rangle_J \subset U \cap \mathsf{Int}(J)$, $1 \leq k_x \leq k_y$, $f^{k_x}(x), f^{k_y}(y) \in Q$, and, concerning the preimages of Q, $(U \cap \mathsf{Int}(J)) \cap f^{-k}(Q) = \emptyset$ for $k = 0, 1, \dots, k_x - 1$ and $\big((U \cap \mathsf{Int}(J)) \setminus \{x\}\big) \cap f^{-k}(Q) = \emptyset$ for $k = k_x, k_x + 1, \dots, k_y - 1$. Consequently, as in the proof of the above claim and using the fact that f is Q-monotone, it follows inductively that there exist Q-basic intervals $J_0 = J, J_1, \dots, J_{k_y - 1}$ such that

$$\begin{aligned}
&\langle x, y\rangle_{J_0} \subset U \cap \mathsf{Int}(J_0),\\
&f^k\left(\langle x, y\rangle_{J_0}\right) = \langle f^k(x), f^k(y)\rangle_{J_k} \subset \mathsf{Int}(J_k) \text{ for } k = 1, 2, \dots, k_x - 1 \text{ and}\\
&f^k\left(\langle x, y\rangle_{J_0} \setminus \{x\}\right) = \langle f^k(x), f^k(y)\rangle_{J_k} \setminus \{f^k(x)\} \subset \mathsf{Int}(J_k)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } k = k_x, k_x + 1, \dots, k_y - 1
\end{aligned}$$

(recall that $f(Q) \subset Q$ and, hence, $f^k(x) \in Q$ for every $k \geq k_x$). Moreover,

$$f^{k_y}\left(\langle x, y\rangle_{J_0}\right) = \langle f^{k_y}(x), f^{k_y}(y)\rangle_{f(J_{k_y - 1})} \subset f\left(J_{k_y - 1}\right)$$

with $f^{k_y}(x), f^{k_y}(y) \in Q$. On the other hand, from above it follows that $f^k(x), f^k(y) \in J_k$ for $k = 0, 1, \dots, k_y - 1$, and from Definition 2.14, $f\big|_{J_k}$ is one-to-one. Hence, $f^k(x) \neq f^k(y)$ for $k = 0, 1, \dots, k_y$, and consequently there exists $J_{k_y} \in \mathcal{B}(Q)$ such that

$$J_{k_y} \subset \langle f^{k_y}(x), f^{k_y}(y)\rangle_{f(J_{k_y - 1})} = f^{k_y}\left(\langle x, y\rangle_{J_0}\right).$$

Since the Markov matrix of f with respect to Q is irreducible there exists a path of length $r \geq 0$ from J_{k_y} to I in the Markov graph of f with respect to Q. Then, from the definition of path and f-covering it follows that

$f^r\left(J_{k_y}\right) \supset I$. Consequently,

$$f^{r+k_y}(U) \supset f^{r+k_y}\left(\langle x, y\rangle_{J_0}\right) \supset f^r\left(J_{k_y}\right) \supset I.$$

This ends the proof of the theorem. □

3. Proof of Theorem A

In what follows, given $q \in \mathbb{N}$ and $A \subset \mathbb{N}$ we will denote the set $\{q\ell : \ell \in A\}$ by $q \cdot A$.

Proof of Theorem A. Fix $L \in \mathbb{N}$, $L > 8$. Since $\lim_{n\to\infty} h(f_n) = 0$, there exists $N \in \mathbb{N}$ such that

$$h(f_n) < \frac{3 \log \sqrt{2}}{L}.$$

for every $n \geq N$. In the rest of the proof we consider a fixed but arbitrary $n \geq N$ and we denote $\mathsf{Rot}(F_n) = [c_n, d_n]$.

We claim that

$$M(c_n, d_n) \subset \mathsf{Succ}\,(L+1) = \{k \in \mathbb{N} : k \geq L+1\}\,. \tag{3.1}$$

To prove this note that for every $q \in M(c_n, d_n)$ there exists $\frac{r}{s} \in (c_n, d_n)$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ co-prime such that $q = \ell s$ with $\ell \in \mathbb{N}$. In this situation, $h(f_n) \geq \frac{\log 3}{s}$ by [9, Corollary 4.7.7]. Hence,

$$\frac{\log 3}{q} \leq \frac{\log 3}{s} \leq h(f_n) < \frac{3 \log \sqrt{2}}{L} < \frac{\log 3}{L}.$$

Consequently, $q > L$ and the claim holds.

From the claim we get that $\mathsf{Int}\,(\mathsf{Rot}(F_n)) \cap \{k/L : k \in \mathbb{Z}\} = \emptyset$. This implies that $\mathsf{len}\,(\mathsf{Rot}(F_n)) \leq 1/L$ for every $n \geq N$. So,

$$\lim_{n\to\infty} \mathsf{len}\,(\mathsf{Rot}(F_n)) = 0.$$

By Theorem 2.4 and the above claim,

$$\begin{aligned} \mathsf{Per}(f_n) &= Q_{F_n}(c_n) \cup M(c_n, d_n) \cup Q_{F_n}(d_n) \\ &\subset \mathsf{Succ}\,(L+1) \cup Q_{F_n}(c_n) \cup Q_{F_n}(d_n). \end{aligned} \tag{3.2}$$

Thus, we need to study the intersections

$$\{1, 2, \ldots, L\} \cap Q_{F_n}(c_n) \quad \text{and} \quad \{1, 2, \ldots, L\} \cap Q_{F_n}(d_n).$$

We will divide this study in three claims, according to different situations for c_n and d_n.

Claim 1. *If $\alpha \notin \mathbb{Q}$ then $\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) = \emptyset$.*

This claim follows immediately from the definition of $Q_{F_n}(\alpha)$.

Claim 2. *Assume that $\alpha = \frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ co-prime, and $s \geq L$. Then, $\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) \subset \{L\} \cap \{s\}$.*

Again by the definition of $Q_{F_n}(\alpha)$, in this case we have

$$Q_{F_n}(\alpha) = \{sk : k \in \mathbb{N} \text{ and } k \leq_{\text{Sh}} s_\alpha\} \subset s\mathbb{N} = \{sk : k \in \mathbb{N}\}\,.$$

Since $s \geq L$, for every $k \in \mathbb{N}$, $k \geq 2$ we have $sk \geq 2L > L$. Hence,

$$\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) \subset \{1, 2, \ldots, L\} \cap \{sk : k \in \mathbb{N}\} = \{L\} \cap \{s\}.$$

Claim 3. *Assume that $\alpha = \frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in \{1, 2, \dots, L-1\}$ co-prime. Then,* $\mathsf{Card}\Big(\{L-2, L-1, L\} \cap Q_{F_n}(\alpha)\Big) \leq 1$ *and*

$$\{1, 2, \dots, L-1\} \cap Q_{F_n}(\alpha) \subset \left\{s \cdot 2^\ell : \ell \in \left\{0, 1, 2, \dots, \left\lfloor \log_2\left(\tfrac{L-1}{s}\right)\right\rfloor\right\}\right\}.$$

To prove this claim, set $G := F_n^s - r \in \mathcal{L}_1$. In view of the definition of $Q_{F_n}(\alpha)$, we have

$$Q_{F_n}(\alpha) = s \cdot \mathsf{Per}(G),$$

where $\mathsf{Per}(G)$ is the set of periods of all (true) periodic orbits of G as a map of the real line into itself. Assume that G has a (true) periodic orbit P of period $k = t \cdot 2^\ell \in \mathsf{Per}(G)$ for some $t \geq 3$ odd and $\ell \in \mathbb{Z}^+$. Clearly, $G(P) = P \subset \langle P \rangle$, and $\max P \geq G(\min P) > \min P$ because $k > 1$. We set $z := \min P$ (hence, $\langle P \rangle = [z, \max P]$). Suppose that $\max P = z + 1$. Then

$$G(\max P) = G(z+1) = G(z) + 1 > z + 1 = \max P = \max\langle P \rangle;$$

a contradiction. If $\max P - \min P > 1$ then, by [10, Theorem 2.2], the rotation interval of G contains the integer $\frac{0}{1}$ in its interior. So, since G is a lifting of f_n^s, in view of [9, Corollary 4.7.7] (see also [9, Remark 3.9.2]) we get

$$\frac{\log 3}{L} > \frac{3 \log \sqrt{2}}{L} > h(f_n) = \frac{1}{s} h(f_n^s) \geq \frac{\log 3}{s} > \frac{\log 3}{L};$$

again a contradiction. Consequently, the orbit P must verify

$$z < \max P < z + 1.$$

Based on P and G we define a new map $\widetilde{G} \in \mathcal{L}_1$ as follows (see Figure 2):

$$\widetilde{G}(x) := \begin{cases} G(x) & \text{if } x, G(x) \in \langle P \rangle, \\ \max P & \text{if } x \in \langle P \rangle \text{ and } G(x) \geq \max P, \\ z & \text{if } x \in \langle P \rangle \text{ and } G(x) \leq z, \\ G(x) & \text{if } x \in [\max P, z+1], \text{ and} \\ \widetilde{G}(x - \lfloor x - z \rfloor) + \lfloor x - z \rfloor & \text{if } x \notin [z, z+1]. \end{cases}$$

The reason why we need this new lifting $\widetilde{G}$ is twofold: from one hand we have $\widetilde{G}\big|_P = G\big|_P$ (so, P is a periodic orbit of $\widetilde{G}$ of period k) and, moreover, $\widetilde{G}(\langle P \rangle) = \langle P \rangle$ (so, $\langle P \rangle$ is $\widetilde{G}$-invariant). We denote by g the circle map of degree one that has $\widetilde{G}$ as a lifting. Since G is a lifting of f_n^s, $\big\{x \in \mathbb{S}^1 : f_n^s(x) \neq g(x)\big\}$ has finitely many connected components, and g restricted to each of these connected components is constant. Thus, in view of [6, Lemma 2.2] and [9, Lemma 4.7.1],

$$h(f_n^s) \geq h(g) \geq h\Big(\widetilde{G}\big|_{\langle P \rangle}\Big).$$

Useless Observation. All of the above conclusions hold true independently from the fact that k is of the form $t \cdot 2^\ell$ for some $t \geq 3$ odd and $\ell \in \mathbb{Z}^+$ (provided that P is not a fixed point).

By [9, Corollary 4.4.18 and the comment in Page 232],

$$\frac{3}{L} \log \sqrt{2} > h(f_n) = \frac{1}{s} h(f_n^s) \geq \frac{1}{s} h(g) \geq$$

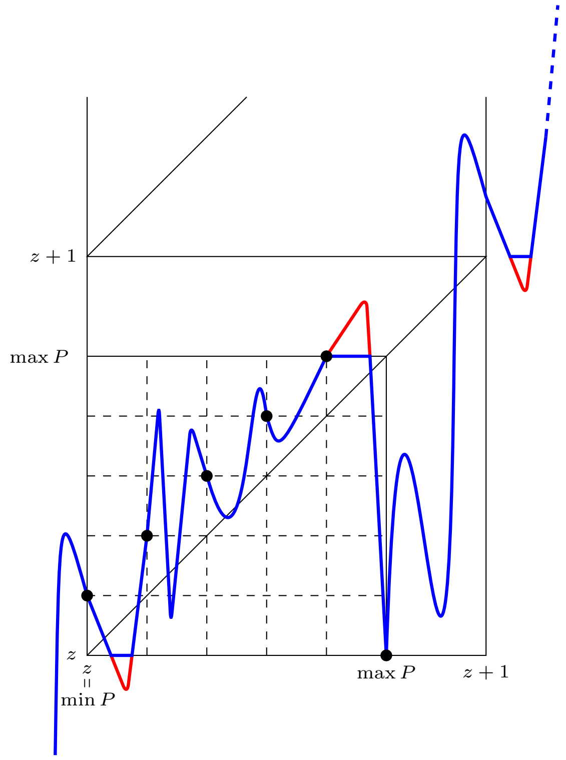


FIGURE 2. A qualitative example of a map $\widetilde{G} \in \mathcal{L}_1$ in blue. The periodic orbit P is drawn with solid black disks: $\bullet$. The part of the original map G whose graph does not coincide with the graph of $\widetilde{G}$ is drawn in red.

$$\frac{1}{s} h\Big(\widetilde{G}\big|_{\langle P\rangle}\Big) \geq \frac{1}{s}\frac{1}{2^\ell}\log\lambda_t > \frac{1}{s\cdot 2^\ell}\log\sqrt{2},$$

where λ_t is the largest root of the polynomial $x^t - 2x^{t-2} - 1$. Hence,

$$sk = s\cdot t\cdot 2^\ell \geq s\cdot 3\cdot 2^\ell > L.$$

Consequently, for every set $\mathsf{A} \subset \{1, 2, \dots, L\}$,

$$\begin{aligned} \mathsf{A} \cap Q_{F_n}(\alpha) &= \mathsf{A} \cap s \cdot \mathsf{Per}(G) \\ &= \mathsf{A} \cap s \cdot \Big(\mathsf{Per}(G) \setminus \Big\{t\cdot 2^\ell : \ell \in \mathbb{Z}^+ \text{ and } t \geq 3 \text{ odd}\Big\}\Big) \\ &= \mathsf{A} \cap s \cdot \Big\{2^\ell : \ell \in \mathbb{Z}^+ \text{ and } 2^\ell \in \mathsf{Per}(G)\Big\} \\ &\subset \mathsf{A} \cap \Big\{s\cdot 2^\ell : \ell \in \mathbb{Z}^+ \text{ and } s\cdot 2^\ell \leq L\Big\} \\ &= \mathsf{A} \cap \Big\{s\cdot 2^\ell : \ell \in \Big\{0, 1, 2, \dots, \Big\lfloor \log_2\big(\tfrac{\max \mathsf{A}}{s}\big)\Big\rfloor\Big\}\Big\}. \end{aligned} \tag{3.3}$$

The second statement of Claim 3 follows from Equations (3.3) by taking $\mathsf{A} = \{1, 2, \dots, L-1\}$.

To prove the first statement of Claim 3 we set $m := \Big\lfloor \log_2\big(\tfrac{L-1}{s}\big)\Big\rfloor$. Clearly,

$$m \leq \log_2\big(\tfrac{L-1}{s}\big) < m+1 \qquad \Longleftrightarrow \qquad s2^m \leq L-1 < s2^{m+1}. \tag{3.4}$$

On the other hand,

$$s \cdot 2^{m+2} = 2\left(s \cdot 2^{m+1}\right) \geq 2L > L,$$

and by (3.4),

$$s2^m \leq L-1 < L + (L-6) = 2(L-3) \qquad \Longleftrightarrow \qquad s2^{m-1} < L-3$$

because $L > 8$. Moreover, if $s \cdot 2^m \geq L-2$ we have,

$$s \cdot 2^{m+1} = 2\left(s \cdot 2^m\right) \geq 2(L-2) = L + (L-4) > L+4$$

again because $L > 8$. So, by (3.3) with $\mathsf{A} = \{L-2, L-1, L\}$,

$$\begin{aligned}
&\{L-2, L-1, L\} \cap Q_{F_n}(\alpha)\\
&\quad\subset \{L-2, L-1, L\} \cap \left\{s \cdot 2^\ell : \ell \in \mathbb{Z}^+ \text{ and } s \cdot 2^\ell \leq L\right\}\\
&\quad\subset \{L-2, L-1, L\} \cap\\
&\qquad\qquad \begin{Bmatrix} \left\{s \cdot 2^\ell : \ell \in \{0,1,2,\ldots,m\}\right\} & \text{if } s \cdot 2^m \geq L-2\\ \left\{s \cdot 2^\ell : \ell \in \{0,1,2,\ldots,m+1\}\right\} & \text{if } s \cdot 2^m \leq L-3 \end{Bmatrix}\\
&\quad\subset \{L-2, L-1, L\} \cap\\
&\qquad\qquad \begin{cases} \{s\} & \text{if } m = 0 \text{ and } s = s \cdot 2^m \geq L-2,\\ \left\{s2^m\right\} & \text{if } m > 0 \text{ and } s \cdot 2^m \geq L-2,\\ \left\{s2^{m+1}\right\} & \text{if } s \cdot 2^m \leq L-3. \end{cases}
\end{aligned}$$

This ends the proof of Claim 3.

From the above three claims we obtain

$$\operatorname{\mathsf{Card}}\Big(\{L-2, L-1, L\} \cap \Big(Q_{F_n}(c_n) \cup Q_{F_n}(d_n)\Big)\Big) \leq 2 \tag{3.5}$$

and, consequently, $\{L-2, L-1, L\} \not\subset Q_{F_n}(c_n) \cup Q_{F_n}(d_n)$. Thus,

$$\{L-2, L-1, L\} \not\subset \mathsf{Per}(f_n)$$

by (3.2). So, for every $n \geq N$ we set

$$\begin{aligned}
\kappa_n &:= \min\left(\{L-2, L-1, L\} \setminus \mathsf{Per}(f_n)\right), \text{ and}\\
\nu_n &:= \min\left(\mathsf{Per}(f_n) \cap \mathsf{Succ}\left(\kappa_n + 1\right)\right).
\end{aligned}$$

The inequality (3.5) is crucial for this proof. It allows us to define κ_n and, hence, ν_n and tells us that $\mathsf{StrBdCof}\left(f_n\right) \geq \nu_n$ (because, as we will see, $\nu_n - 1 \notin \mathsf{Per}(f_n)$). This is implicitly used in the rest of the proof of the theorem.

To end the proof of the theorem it is enough to show that $\nu_n \in \mathsf{sBC}(f_n)$ for every $n \geq N$. Indeed, by Definition 1.4, $\mathsf{BdCof}\left(f_n\right)$ exists and

$$L-1 \leq \kappa_n + 1 \leq \nu_n \leq \mathsf{BdCof}\left(f_n\right) \tag{3.6}$$

for every $n \geq N$. Consequently, $\lim_{n\to\infty} \mathsf{BdCof}\left(f_n\right) = \infty$.

Let us prove that $\nu_n \in \mathsf{sBC}(f_n)$ for every $n \geq N$. By Definition 1.4 we have to show that $\nu_n \in \mathsf{Per}(f_n)$, $\nu_n > 2$, $\nu_n - 1 \notin \mathsf{Per}(f_n)$ and

$$\operatorname{\mathsf{Card}}\big(\{1, \ldots, \nu_n - 2\} \cap \mathsf{Per}(f_n)\big) \leq 2\log_2(\nu_n - 2). \tag{3.7}$$

Since $L > 8$, from the definition of ν_n we get

$$7 < L-1 \leq \nu_n \in \mathsf{Per}(f_n).$$

The following claim will be useful in the rest of the proof. It improves the knowledge of the set $\{1, \ldots, \nu_n - 2\} \cap \mathsf{Per}(f_n)$.

Claim 4. $\{\kappa_n, \kappa_n + 1, \ldots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset$.

When $\nu_n = \kappa_n + 1$, the claim holds because $\nu_n - 1 = \kappa_n \notin \mathsf{Per}(f_n)$ by the definition of κ_n. Now we prove the claim in the case $\nu_n > \kappa_n + 1$. We have $\{\kappa_n + 1, \ldots, \nu_n - 1\} \subset \mathsf{Succ}\,(\kappa_n + 1)$ and, hence,

$$\{\kappa_n + 1, \ldots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset$$

by the minimality of ν_n. Moreover, $\kappa_n \notin \mathsf{Per}(f_n)$ by definition. Hence,

$$\{\kappa_n, \kappa_n + 1, \ldots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset,$$

which ends the proof of Claim 4.

Claim 4, in particular, tells us that $\nu_n - 1 \notin \mathsf{Per}(f_n)$. Hence, to show that $\nu_n \in \mathsf{sBC}(f_n)$, we have to prove inequality (3.7). By Claim 4, (3.2) and (3.1) (notice that $\kappa_n - 1 \le L - 1$ because, by definition, $\kappa_n \le L$),

$$\begin{aligned}
&\mathsf{Card}\,(\{1, \ldots, \nu_n - 2\} \cap \mathsf{Per}(f_n)) = \\
&\qquad \mathsf{Card}\,(\{1, \ldots, \kappa_n - 1\} \cap \mathsf{Per}(f_n)) = \\
&\qquad \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap \left(Q_{F_n}(c_n) \cup Q_{F_n}(d_n)\right)\right) \le \\
&\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\right) + \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\right).
\end{aligned}$$

So, to prove (3.7) it is enough to show that

$$\begin{aligned}
&\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\right) + \\
&\qquad \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\right) \le 2\log_2(\nu_n - 2).
\end{aligned} \tag{3.8}$$

To this end, we have to compute appropriate upper bounds of the two summands in the last expression.

Again, let $\alpha \in \{c_n, d_n\}$ denote an arbitrary endpoint of $\mathsf{Rot}(F_n)$. In the assumptions of Claims 1 and 2 we have either

$$\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha) \subset \{1, \ldots, L\} \cap Q_{F_n}(\alpha) = \emptyset,$$

or $\alpha = \frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ co-prime, $s \ge L$, and

$$\begin{aligned}
&\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha) \subset \\
&\qquad \{1, \ldots, L - 1\} \cap \left(\{1, \ldots, L\} \cap Q_{F_n}(\alpha)\right) \subset \\
&\qquad\qquad \{1, \ldots, L - 1\} \cap \left(\{L\} \cap \{s\}\right) = \emptyset.
\end{aligned}$$

In any case,

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha)\right) = 0. \tag{3.9}$$

Now suppose that the assumptions of Claim 3 hold. We want to prove the following estimate:

$$\begin{aligned}
\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha)\right) &\le \log_2\left(\tfrac{\nu_n - 2}{s}\right) + 1 \\
&\le \log_2(\nu_n - 2) + 1.
\end{aligned} \tag{3.10}$$

Assume first that $s \cdot 2^m \leq \nu_n - 2$. Then, by the second statement of Claim 3,

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha)\right) \leq \mathsf{Card}\left(\{1, \ldots, L-1\} \cap Q_{F_n}(\alpha)\right) \leq m + 1 \leq \log_2\left(\tfrac{\nu_n - 2}{s}\right) + 1.$$

Now assume that $\nu_n - 2 < s \cdot 2^m$. By (3.6) and (3.4),

$$L - 3 \leq \kappa_n - 1 \leq \nu_n - 2 < s \cdot 2^m \leq L - 1.$$

Consequently, by (3.3),

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha)\right) \leq \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap \left\{s \cdot 2^\ell : \ell \in \{0, 1, 2, \ldots, m\}\right\}\right) = \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap \left\{s \cdot 2^\ell : \ell \in \{0, 1, 2, \ldots, m-1\}\right\}\right) \leq \mathsf{Card}\left\{s \cdot 2^\ell : \ell \in \{0, 1, 2, \ldots, m-1\}\right\} = m.$$

Moreover, $s \cdot 2^{m-1} \leq \nu_n - 2$ since, otherwise, from the above inequalities and using again the fact that $L > 8$ we obtain

$$L + 2 < L + (L - 6) = 2(L - 3) \leq 2\left(\nu_n - 2\right) < 2 \cdot s \cdot 2^{m-1} = s \cdot 2^m \leq L - 1;$$

a contradiction. Thus, $m - 1 \leq \log_2\left(\frac{\nu_n - 2}{s}\right)$. Putting all together we get,

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(\alpha)\right) \leq m \leq \log_2\left(\tfrac{\nu_n - 2}{s}\right) + 1.$$

This ends the proof of (3.10).

Now we are ready to prove (3.8). First assume that at most one of the endpoints of $\mathsf{Rot}(F_n)$ satisfies the assumptions of Claim 3. By (3.9) and (3.10),

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\right) + \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\right) \leq \log_2\left(\nu_n - 2\right) + 1 < 2\log_2\left(\nu_n - 2\right)$$

because $7 < \nu_n$ implies $\log_2(\nu_n - 2) > 1$.

It remains to consider the case when both endpoints of $\mathsf{Rot}(F_n) = [c_n, d_n]$ satisfy the assumptions of Claim 3. That is, $c_n = \frac{r_n}{s_n}$ with $r_n \in \mathbb{Z}$ and $s_n \in \mathbb{N}$ co-prime, $d_n = \frac{q_n}{t_n}$ with $q_n \in \mathbb{Z}$ and $t_n \in \mathbb{N}$ co-prime, and $s_n, t_n \leq L - 1$. Observe that in this situation we cannot have $s_n = t_n$. Otherwise, $q_n \geq r_n + 1$ and then,

$$\tfrac{1}{L-1} \leq \tfrac{1}{s_n} \leq d_n - c_n = \mathsf{len}\left(\mathsf{Rot}(F_n)\right) \leq \tfrac{1}{L};$$

a contradiction. Therefore, either s_n or t_n is larger than 3. Indeed, if $s_n, t_n \leq 3$,

$$\tfrac{1}{6} \leq \tfrac{1}{t_n s_n} \leq d_n - c_n = \mathsf{len}\left(\mathsf{Rot}(F_n)\right) \leq \tfrac{1}{L} < \tfrac{1}{8};$$

a contradiction.

Assume for definiteness that $s_n \geq 4$. Then, by (3.10),

$$\mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\right) + \mathsf{Card}\left(\{1, \ldots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\right) \leq$$

$$\log_2\left(\tfrac{\nu_n-2}{s_n}\right) + \log_2(\nu_n - 2) + 2 \leq$$
$$\log_2\left(\tfrac{\nu_n-2}{4}\right) + \log_2(\nu_n - 2) + 2 = 2\log_2(\nu_n - 2).$$

This ends the proof of (3.8) and, hence, that $\nu_n \in \mathsf{sBC}(f_n)$. □

4. Exporting circle maps to arbitrary graphs that are not trees: minimalistic extensions

The aim of this section is to perform a general construction to extend certain degree one circle maps to graphs with at least one circuit while keeping their basic dynamical properties such as transitivity and, up to some extent, the topological entropy and the set of periods.

The following lemma is analogous to [2, Lemma 3.6] with the additional assumption that the number of elements of the partition must be even. It will be our main tool to translate the examples from $\mathbb{S}^1$ to any graph that is not a tree (see Figure 3). Due to the additional assumption about the parity of the number of elements of the partition we will include the proof for completeness. However, the proof will be delayed to Subsection 4.1, at the end of this section.

Lemma 4.1. *Let X be a topological graph which is not an interval and let $a, b \in V(X)$ be two different endpoints of X. Then, there exist a partition of the interval $[0, 1]$, $0 = s_0 < s_1 < \cdots < s_m = 1$, with $m = m(X, a, b) \geq 5$ odd, and two continuous surjective maps $\varphi_{a,b}\colon [0, 1] \longrightarrow X$ and $\psi_{a,b}\colon X \longrightarrow [0, 1]$ such that the following statements hold:*

(a) $\varphi_{a,b}^{-1}(W) = \{s_i : i \in \{0, 1, \ldots, m\}\}$, *where*
$$W := \varphi_{a,b}\bigl(\{s_i : i \in \{0, 1, \ldots, m\}\}\bigr) \supset V(X),$$
and $\varphi_{a,b}(0) = a$ and $\varphi_{a,b}(1) = b$.

(b) For every $i = 0, 1, \ldots, m-1$, $\varphi_{a,b}\big|_{[s_i,s_{i+1}]}$ is injective and $\varphi_{a,b}\bigl([s_i, s_{i+1}]\bigr)$ is an interval which is the closure of a connected component of the punctured graph $X \setminus W$.

(c) If $\varphi_{a,b}(s_i) = \varphi_{a,b}(s_j)$ then $i \equiv j \pmod 2$.

(d) $\psi_{a,b}(\varphi_{a,b}(s_i)) = 0$ if i is even and $\psi_{a,b}(\varphi_{a,b}(s_i)) = 1$ if i is odd (in particular, $\psi_{a,b}(a) = 0$ and $\psi_{a,b}(b) = 1$).

(e) The map $\psi_{a,b}\big|_{\varphi_{a,b}([s_i,s_{i+1}])}$ is injective and $\psi_{a,b}\bigl(\varphi_{a,b}([s_i, s_{i+1}])\bigr) = [0, 1]$ for every $i = 0, 1, \ldots, m-1$. In particular, the map $\bigl(\psi_{a,b} \circ \varphi_{a,b}\bigr)\big|_{[s_i,s_{i+1}]}$ is strictly monotone.

In the next definitions we construct the *minimalistic extension* of certain circle maps to graph maps on arbitrary graphs with at least one circuit.

Definition 4.2 (Extendable map). Let g be a Markov circle map of degree one with respect to a Markov partition Q. We say that g is *extendable* whenever there exist pairwise disjoint basic intervals $I, J, K \in \mathcal{B}(Q)$ such that $g(I) = J$, $g(J) = K$, and I is the unique basic interval that g-covers J. The triplet of basic intervals (I, J, K) will be called the *extension platform of g*. ◘

Definition 4.3 (The minimalistic extension to graph maps). Let g be a Markov circle map of degree one with respect to a Markov partition Q. Assume that g is extendable with extension platform (I, J, K). Let G be an arbitrary graph with a circuit C. The *minimalistic extension of* g *to* G *with base at* C, denoted by $g^{G;C}$, is a continuous self map of G defined as follows.

Let $S \subset C$ be an interval such that $S \cap V(G) = \emptyset$ and let $\eta \colon \mathbb{S}^1 \longrightarrow C$ be a homeomorphism such that $C \setminus S = \eta\big(\mathsf{Int}(J)\big)$ (see Figure 4). Clearly, $X := G \setminus \mathsf{Int}(S) \supset \eta(J)$ is a subgraph of G and the two elements of $\partial S = \partial\eta(J)$ are endpoints (and thus vertices) of X but they are not vertices of G because $S \cap V(G) = \emptyset$. Then we have

$$G = X \cup S = X \cup \mathsf{Int}\big(S\big) = \big(X \setminus \{a, b\}\big) \cup S,$$

and

$$V(X) = V(G) \cup \partial S = V(G) \cup \{a, b\}$$

where, for definiteness, we have set $\{a, b\} := \partial S = \partial\eta(J)$.

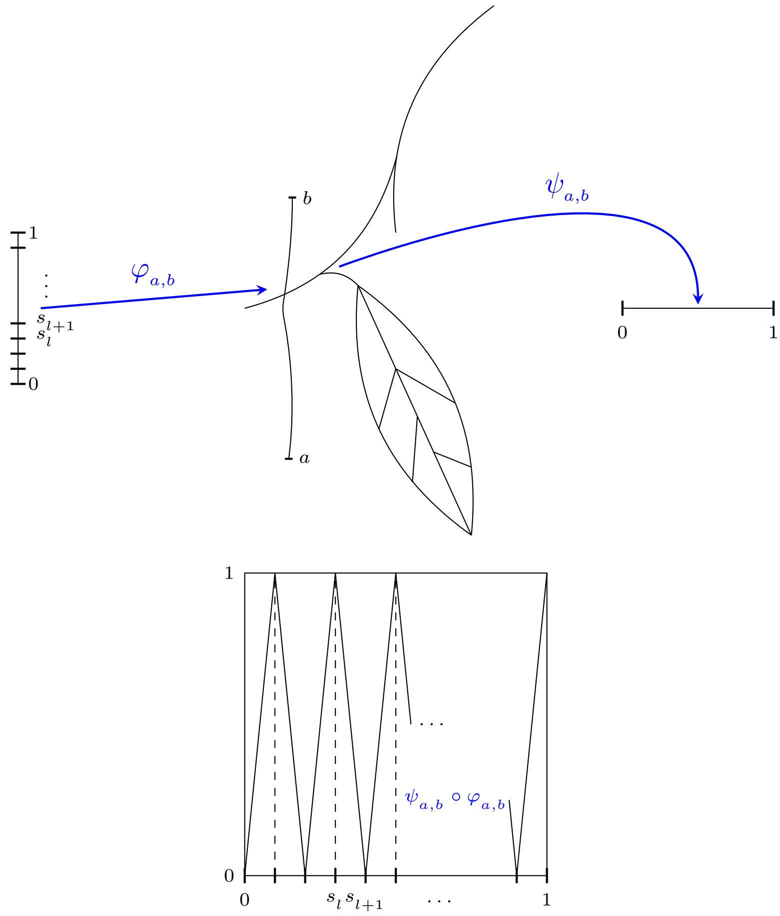


FIGURE 3. A sketch of a topological graph X and the maps from Lemma 4.1 (top picture). The map $\psi_{a,b} \circ \varphi_{a,b}$ is shown in the bottom picture.

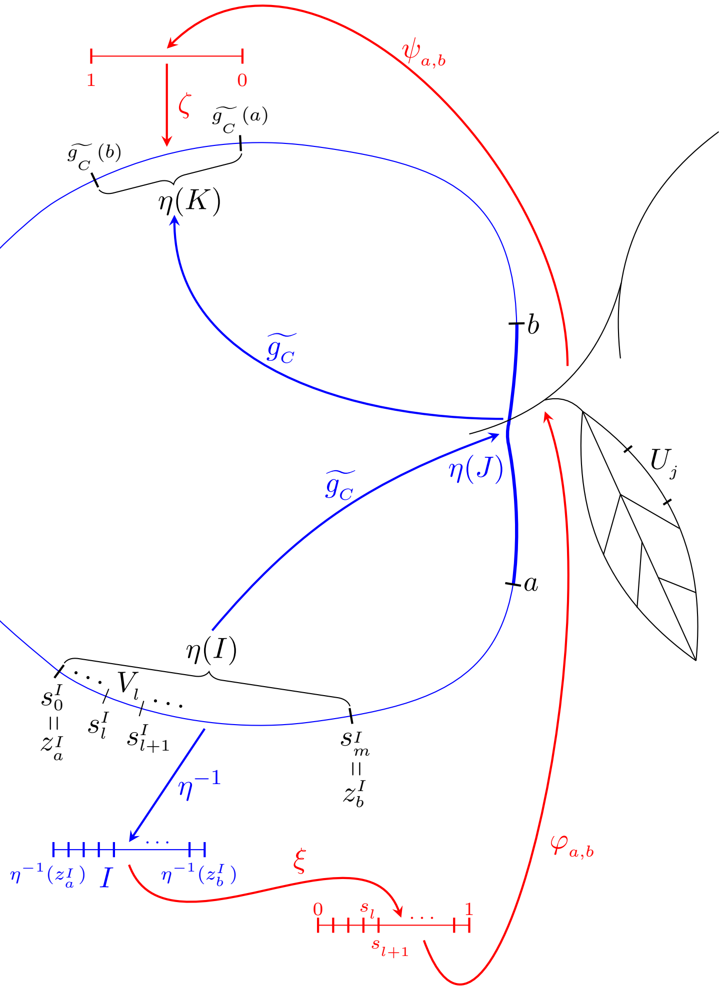


FIGURE 4. A topological graph G and the definition of $g^{\mathsf{G;C}}$. The circuit C (with apple shape) is drawn in blue. Then, the interval S is the (thin) path in C from b to a counterclockwise ($\partial S = \{a, b\}$) and the interval $C \setminus \mathsf{Int}(S) = \eta(J)$ is the **thick** path in C from b to a clockwise.

Moreover, $\{a, b\} \subset \partial\eta(I) \cup \partial\eta(J) \cup \partial\eta(K) \subset \eta(Q) \subset S$, and $\eta(I),\ \eta(K)$ and X are pairwise disjoint because I and K are disjoint and

$$X \cap \eta(I), X \cap \eta(K) \subset X \cap \mathsf{Int}\big(S\big) = \emptyset.$$

Now we define an auxiliary map $\widetilde{g_C} := \eta \circ g \circ \eta^{-1} \colon C \longrightarrow C$. Clearly, $\widetilde{g_C}$ is a graph map which is Markov with respect to $\eta(Q)$, and conjugate to g.

The fact that (I, J, K) is an extension platform of g implies that $\widetilde{g_C}$ sends homeomorphically $\partial\eta(I)$ to $\partial\eta(J)$ and the latter to $\partial\eta(K)$. Hence, we can consider a homeomorphism $\xi \colon I \longrightarrow [0, 1]$ with an orientation such that $\xi\big(\eta^{-1}(z_a^I)\big) = 0$ and $\xi\big(\eta^{-1}(z_b^I)\big) = 1$, where z_a^I (respectively z_b^I) denotes the endpoint (and unique element) of $\eta(I)$ such that $\widetilde{g_C}(z_a^I) = a$

(respectively $\widetilde{g_C}(z_b^I) = b$). Similarly, we consider a second homeomorphism $\zeta\colon [0,1] \longrightarrow \eta(K)$ with an orientation such that $\zeta(0) = \widetilde{g_C}(a)$ and $\zeta(1) = \widetilde{g_C}(b)$.

When $G = C$ we set $g^{G;C} := \widetilde{g_C}$. When $G \neq C$ we define the map $g^{G;C}$ by means of Lemma 4.1 for the subgraph X (see Figure 3). For every $x \in G$ we set (see Figure 4):

$$g^{G;C}(x) := \begin{cases} \varphi_{a,b}\bigl(\xi\bigl(\eta^{-1}(x)\bigr)\bigr) & \text{if } x \in \eta(I); \\ \zeta\bigl(\psi_{a,b}(x)\bigr) & \text{if } x \in X; \\ \widetilde{g_C}(x) & \text{if } x \in S \setminus \mathsf{Int}\bigl(\eta(I)\bigr). \end{cases}$$

Since the minimalistic extensions $g^{G;C}$ will always be defined on a graph G with base at a circuit C we will usually omit the complicate superscript for simplicity. ◻

The following result studies the basic topological and dynamical properties of minimalistic extensions.

Theorem 4.4 (The minimalistic extension factory). *Let g be an extendable Markov circle map of degree one with respect to a Markov partition Q. Let G be an arbitrary graph with a circuit C such that $C \subsetneq G$. Then, the minimalistic extension of g to G with base at C, denoted by f, is a continuous well defined Markov map of G with respect to a Markov partition R. Moreover,*

$$\mathsf{Per}(g) \subset \mathsf{Per}\bigl(f\bigr) \quad \textit{and} \quad h\bigl(f\bigr) \leq h(g) + \frac{\log m}{\rho}$$

where ρ is the minimal length of a loop in the Markov graph of g beginning (and ending) at the first interval of the extension platform of g. Furthermore, if g is Q-expanding and transitive then there exists a choice of the homeomorphisms ζ and ξ such that f is R-expanding and transitive.

Remark 4.5. Notice that the constant m obtained in Lemma 4.1 depends solely on the graph G and not on the extendable map. ◻

4.1. **Proofs of Lemma 4.1 and Theorem 4.4.** In what follows the closure of a set $A \subset X$ will be denoted by $\mathsf{Clos}(A)$.

Proof of Lemma 4.1. The existence of a surjective map $\psi_{a,b}$ which satisfies (d–e) follows easily from the existence of the partition $0 = s_0 < s_1 < \cdots < s_m = 1$ and a map $\varphi_{a,b}$ which satisfy statements (a–c) of the lemma. In particular, (d) can be guaranteed by using (c), and (e) by (b) and (d). So, we only have to show that there exist a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ with $m \geq 5$ odd, and a continuous surjective map $\varphi_{a,b}$ such that (a–c) hold.

Since X is not an interval and a and b are endpoints of X, there exist $v \in V(X) \setminus \{a, b\}$ and an interval $J \subset X$ with endpoints v and b such that $J \cap V(X) = \{v, b\}$.

Let C be an edge of X (i.e. a connected component of $X \setminus V(X)$). Clearly, $\mathsf{Clos}(C)$ is either an interval or a circuit which contains a unique vertex of X. For every C such that $\mathsf{Clos}(C)$ is a circuit we choose a point $v_C \in C$ that will play the role of an artificial vertex. Then we set

$$\widetilde{V}(X) := V(X) \cup \{v_C : C \text{ is an edge of } X \text{ such that } \mathsf{Clos}(C) \text{ is a circuit}\}.$$

Observe that the closure of every connected component of $X \setminus \widetilde{V}(X)$ is an interval and $J \cap \widetilde{V}(X) = J \cap V(X) = \{v, b\}$. Moreover, since $\{a, b, v\} \subset V(X) \subset \widetilde{V}(X)$ and a and b are endpoints of X, $X \setminus \widetilde{V}(X)$ has at least three connected components. So, $\mathsf{Card}\bigl(\widetilde{V}(X)\bigr) \geq 4$.

Since a topological graph is path connected, there exists a path from a to b, denoted $\varphi_{a,b} \colon [0,1] \longrightarrow X$, which is continuous and onto (i.e., visits each point from X going several times trough the same edge, if necessary), and a partition $0 = s^*_0 < s^*_1 < \cdots < s^*_n = 1$ of the interval $[0,1]$ such that the following statements hold:

(i) $\bigl\{s^*_0, s^*_1, \ldots, s^*_n\bigr\} = \varphi^{-1}_{a,b}\bigl(\widetilde{V}(X)\bigr)$ with $\varphi_{a,b}(s^*_0) = \varphi_{a,b}(0) = a$, $\varphi_{a,b}\bigl(s^*_{n-1}\bigr) = v$ and $\varphi_{a,b}(s^*_n) = \varphi_{a,b}(1) = b$,

(ii) for every $j = 0, 1, \ldots, n-1$, $\varphi_{a,b}\big|_{[s^*_j, s^*_{j+1}]}$ is injective and, hence, $\varphi_{a,b}\bigl([s^*_j, s^*_{j+1}]\bigr)$ is an interval which is the closure of a connected component of $X \setminus \widetilde{V}(X)$,

(iii) $\varphi^{-1}_{a,b}(b) = s^*_n$ and $\varphi^{-1}_{a,b}\bigl(J \setminus \{v\}\bigr) = \bigl(s^*_{n-1}, s^*_n\bigr]$.

Note that, by (i–iii),

$$\mathcal{E} := \Bigl\{\varphi_{a,b}\bigl((s^*_j, s^*_{j+1})\bigr) : j \in \{0, 1, \ldots, n-2\}\Bigr\} \tag{4.1}$$

is the set of all connected components of $X \setminus \widetilde{V}(X)$ which are different from $J \setminus \{b, v\}$. For every $C \in \mathcal{E}$ we choose an arbitrary but fixed point $\alpha_C \in C$. Clearly,

$$\varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \subset \bigl(s^*_0, s^*_{n-1}\bigr) \setminus \bigl\{s^*_1, s^*_2, \ldots, s^*_{n-2}\bigr\}.$$

We claim that for every $j = 0, 1, \ldots, n-2$,

$$\mathsf{Card}\Bigl(\varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr)\Bigr) = 1.$$

To prove the claim observe that

$$\emptyset \neq \varphi^{-1}_{a,b}\left(\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)}\right) \cap \bigl(s^*_j, s^*_{j+1}\bigr) \subset \varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr)$$

for every $j = 0, 1, \ldots, n-2$ because, by definition,

$$\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)} \in \varphi_{a,b}\bigl(\bigl(s^*_j, s^*_{j+1}\bigr)\bigr).$$

Assume that, for some $j \in \{0, 1, \ldots, n-2\}$, there exist points

$$s^1_i, s^2_i \in \varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr).$$

Then, in view of (ii) we get $s^1_i = s^2_i$ because

$$\varphi_{a,b}\bigl(s^1_i\bigr), \varphi_{a,b}\bigl(s^2_i\bigr) \in \varphi_{a,b}\bigl((s^*_j, s^*_{j+1})\bigr) \cap \{\alpha_C : C \in \mathcal{E}\} = \Bigl\{\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)}\Bigr\}.$$

This proves the claim.

Now we set $m = m(X, a, b) := 2n - 1$ and by the above claim we define the partition

$$s_0 = s^*_0 = 0 < s_1 < s_2 \cdots < s_{m-2} < s_{m-1} = s_{2(n-1)} = s^*_{n-1} < s_m = s^*_n = 1$$

of the interval $[0,1]$ by:

$s_{2j} := s^*_j$, and

s_{2j+1} is the unique point of $\varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) \cap \big(s^*_j, s^*_{j+1}\big)$

for every $j = 0, 1, 2, \dots, n-2$. With these definitions and (i), the set $\{s_0, s_1, \dots, s_m\}$ is the union of two disjoint sets:

$$\begin{aligned} \{s_0, s_2, \dots, s_{m-1}, s_m\} &= \Big\{s^*_j : j \in \{0,1,\dots,n\}\Big\} = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big), \text{ and} \\ \{s_1, s_3, \dots, s_{m-2}\} &= \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big). \end{aligned} \tag{4.2}$$

By definition $m = m(X, a, b, M)$ is odd, $\varphi_{a,b} \colon [0,1] \longrightarrow X$ is continuous and surjective and, by (i), $\varphi_{a,b}(0) = a$ and $\varphi_{a,b}(1) = b$. Moreover, since the map $\varphi_{a,b}$ is onto,

$$n + 1 = \mathsf{Card}\left(\Big\{s^*_j : j \in \{0,1,\dots,n\}\Big\}\right) = \mathsf{Card}\left(\varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big)\right) \geq \mathsf{Card}\big(\widetilde{V}(X)\big) \geq 4,$$

and hence, $m = 2n - 1 \geq 5$.

On the other hand, by (4.2),

$$\begin{aligned} W &= \varphi_{a,b}\big(\{s_i : i \in \{0,1,\dots,m\}\}\big) \\ &= \varphi_{a,b}\left(\varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big)\right) \cup \varphi_{a,b}\left(\varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big)\right) \\ &= \widetilde{V}(X) \cup \{\alpha_C : C \in \mathcal{E}\} \supset V(X), \end{aligned}$$

and

$$\varphi^{-1}_{a,b}(W) = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) \cup \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) = \{s_i : i \in \{0,1,\dots,m\}\}.$$

Thus, (a) holds.

Statement (b) follows from (ii), Statement (a) and the fact that every interval $[s_i, s_{i+1}]$ is contained in an interval $\big[s^*_j, s^*_{j+1}\big]$.

To end the proof of the lemma it remains to prove (c). Assume that $\varphi_{a,b}(s_i) = \varphi_{a,b}(s_j)$ (or, equivalently, that there exists $\alpha \in W$ such that $s_i, s_j \in \varphi^{-1}_{a,b}(\alpha) \subset \varphi^{-1}_{a,b}(W)$). Since

$$\varphi^{-1}_{a,b}(W) = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) \cup \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) \text{ and } \widetilde{V}(X) \cap \{\alpha_C : C \in \mathcal{E}\} = \emptyset,$$

by (4.2), it follows that either

$$\begin{aligned} s_i, s_j &\in \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) = \{s_0, s_2, \dots, s_{m-1}, s_m\} \text{ or} \\ s_i, s_j &\in \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) = \{s_1, s_3, \dots, s_{m-2}\}. \end{aligned}$$

On the other hand, by (iii), $s_m = s^*_n = \varphi^{-1}_{a,b}(b) \notin \{s_i, s_j\}$. Consequently, either $i, j \in \{0, 2, 4, \dots, m-1\}$ or $i, j \in \{1, 3, 5, \dots, m-2\}$, and (c) holds. □

Proof of Theorem 4.4. We will use all the notation introduced in Definitions 4.2 and 4.3, and Lemma 4.1. By Lemma 4.1 and the orientations established for the homeomorphisms ξ and ζ,

$$\varphi_{a,b}\big(\xi\big(\eta^{-1}(z^I_a)\big)\big) = \varphi_{a,b}(0) = a = \widetilde{g_C}(z^I_a),$$

$$\varphi_{a,b}\big(\xi\big(\eta^{-1}(z^I_b)\big)\big) = \varphi_{a,b}(1) = b = \widetilde{g_C}(z^I_b),$$
$$\zeta\big(\psi_{a,b}(a)\big) = \zeta(0) = \widetilde{g_C}(a), \text{ and}$$
$$\zeta\big(\psi_{a,b}(b)\big) = \zeta(1) = \widetilde{g_C}(b).$$

So, f is well defined and continuous because the maps

$$\big(\varphi_{a,b} \circ \xi \circ \eta^{-1}\big)\big|_{\eta(I)}, \quad \big(\zeta \circ \psi_{a,b}\big)\big|_X, \quad \text{and} \quad \widetilde{g_C}\big|_{S\setminus \mathsf{Int}(\eta(I))}$$

are continuous.

To define the Markov partition R for the map f we introduce the following notation. Set

$$\begin{cases} s^I_i := \eta\big(\xi^{-1}(s_i)\big) \text{ for } i = 0, 1, \dots, m, \text{ and} \\ R := \eta(Q) \cup W \cup \big\{s^I_0, s^I_1, \dots, s^I_m\big\} \end{cases} \tag{4.3}$$

(recall that $W = \big\{\varphi_{a,b}(s_i) : i \in \{0, 1, \dots, m\}\big\}$).

For later purposes it is useful to clarify the relation between the three sets constituting R and the three parts of the graph G considered in the definition of the minimalistic extension of g. We start by noticing that, by Lemma 4.1 and the definition of ξ,

$$\begin{aligned} &W \subset X, \\ &\partial S = \{a, b\} \subset W \\ &s^I_i = \eta\big(\xi^{-1}(s_i)\big) \in \eta(I) \subset \mathsf{Int}(S) \quad \text{for every } i \in \{0, 1, \dots, m\}, \\ &s^I_0 = \eta\big(\xi^{-1}(s_0)\big) = \eta\big(\xi^{-1}(0)\big) = \eta\big(\xi^{-1}\big(\xi\big(\eta^{-1}(z^I_a)\big)\big)\big) = z^I_a, \\ &s^I_m = \eta\big(\xi^{-1}(s_m)\big) = \eta\big(\xi^{-1}(1)\big) = \eta\big(\xi^{-1}\big(\xi\big(\eta^{-1}(z^I_b)\big)\big)\big) = z^I_b, \text{ and} \\ \partial\eta(I) &= \big\{z^I_a, z^I_b\big\} = \big\{s^I_0, s^I_m\big\} \subset \big\{s^I_0, s^I_1, \dots, s^I_m\big\}. \end{aligned}$$

Then we have (see Figure 4):

- $R \cap \eta(I) = \big\{s^I_0, s^I_1, \dots, s^I_m\big\}$,
- $R \cap X = W$, and
- $R \cap \big(S \setminus \mathsf{Int}(\eta(I))\big) = \eta(Q)$.

Additionally, by Lemma 4.1(a),

$$R \supset W \supset V(X) \supset V(G).$$

Hence, R will be a Markov invariant set provided that it is g-invariant, and the closure of each connected component of $G \setminus R$ is an interval in G.

Lemma 4.1(b) and the fact that η is a homeomorphism imply that the closure of each connected component of $G \setminus R$ is an interval in G.

Now we show that R is f-invariant. We will split the proof in three parts in agreement with the three cases of the definition of f and the three sets constituting R:

(a) For every $i \in \{0, 1, \dots, m\}$, $s^I_i = \eta\big(\xi^{-1}(s_i)\big) \in \eta(I)$. Hence,

$$f\big(s^I_i\big) = \varphi_{a,b}\big(\xi\big(\eta^{-1}\big(\eta\big(\xi^{-1}(s_i)\big)\big)\big)\big) = \varphi_{a,b}\big(s_i\big) \in W \subset R.$$

(b) By construction, $\eta(Q) \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$ and, for every $x \in \eta(Q)$,

$$f\big(x\big) = \widetilde{g_C}(x) \in \eta(Q) \subset R$$

because $\widetilde{g_C}$ is a Markov map with respect to $\eta(Q)$.

(c) Since $W = \{\varphi_{a,b}(s_i) : i \in \{0, 1, \ldots, m\}\} \subset X$ and $\{a, b\} = \partial\eta(J) = \eta(\partial J) \subset \eta(Q)$,

$$f(\varphi_{a,b}(s_i)) = \zeta(\psi_{a,b}(\varphi_{a,b}(s_i))) \in \{\zeta(0), \zeta(1)\} = \{\widetilde{g_C}(a), \widetilde{g_C}(b)\} \subset \widetilde{g_C}(\eta(Q)) \subset \eta(Q) \subset R,$$

for every $i \in \{0, 1, \ldots, m\}$.

This ends the proof that R is a Markov invariant set for f.

Next we will show that f is monotone at every basic interval $\mathcal{B}(R)$ and, hence, it is Markov with respect to the Markov partition R. Again we will split this proof in three parts as before. We will consider an arbitrary but fixed basic interval $Y \in \mathcal{B}(R)$.

(A) Assume that $Y \subset S \setminus \mathsf{Int}(\eta(I))$.
In this case, $R \cap S \setminus \mathsf{Int}(\eta(I)) = \eta(Q)$ and, by construction, $Y = \eta(V)$ for some $V \in \mathcal{B}(Q) \setminus \{I, J\}$. Thus, since $\widetilde{g_C}$ is Markov with respect to $\eta(Q)$, Y is an $\eta(Q)$-basic interval of $\widetilde{g_C}$, and $f\big|_Y = \widetilde{g_C}\big|_Y$ is monotone.

(B) Assume that $Y \subset X$.
In this case, $R \cap X = W$ and, since the map $\varphi_{a,b}$ is surjective, by Lemma 4.1 we have $Y = \varphi_{a,b}([s_i, s_{i+1}])$ for some $i \in \{0, 1, \ldots, m-1\}$. Then, again by Lemma 4.1,

$$f\big|_Y = (\zeta \circ \psi_{a,b})\big|_{\varphi_{a,b}([s_i, s_{i+1}])} = \zeta\big|_{[0,1]} \circ \psi_{a,b}\big|_{\varphi_{a,b}([s_i, s_{i+1}])}$$

is a homeomorphism from Y to $\eta(K) \in \mathcal{B}(R)$. So, f is monotone on Y.

(C) Assume that $Y \subset \eta(I)$.
Since $R \cap \eta(I) = \{s^I_0, s^I_1, \ldots, s^I_m\}$ and $\eta(I)$ is an interval,

$$Y = \langle s^I_i, s^I_{i+1} \rangle_{\eta(I)} = \eta(\xi^{-1}([s_i, s_{i+1}]))$$

for some $i \in \{0, 1, \ldots, m-1\}$. Then,

$$f\big|_Y = (\varphi_{a,b} \circ \xi \circ \eta^{-1})\big|_{\eta(\xi^{-1}([s_i, s_{i+1}]))} = \varphi_{a,b}\big|_{[s_i, s_{i+1}]} \circ \xi\big|_{\xi^{-1}([s_i, s_{i+1}])} \circ \eta^{-1}\big|_{\eta(\xi^{-1}([s_i, s_{i+1}]))}$$

which, by Lemma 4.1(b), is a homeomorphism from Y to the basic interval $\varphi_{a,b}([s_i, s_{i+1}]) \in \mathcal{B}(R)$. Hence, f is monotone on Y.

Putting all of the above together we see that f is a Markov map with respect to R.

For the rest of the proof we introduce a *projection* map $\Pi \colon \mathcal{B}(R) \longrightarrow \mathcal{B}(Q)$ as follows:

$$\Pi(Y) := \begin{cases} \eta^{-1}(Y) & \text{if } Y \subset S \setminus \mathsf{Int}(\eta(I)), \\ J & \text{if } Y \subset X, \text{ and} \\ I & \text{if } Y \subset \eta(I). \end{cases} \tag{4.4}$$

Observe that, by (A), for every $Y \subset S \setminus \mathsf{Int}(\eta(I))$ we have

$$\Pi(Y) = \eta^{-1}(Y) \in \mathcal{B}(Q) \setminus \{I, J\}.$$

So, since $S \setminus \mathsf{Int}(\eta(I))$, X and $\eta(I)$ have pairwise disjoint interiors,

$$\Pi^{-1}(I) = \{Y \in \mathcal{B}(R) : Y \subset \eta(I)\},$$

$\Pi^{-1}(J) = \{Y \in \mathcal{B}(R) : Y \subset X\}$, and

$\Pi^{-1}(V) = \{\eta(V)\}$ with $\eta(V) \subset S \setminus \mathsf{Int}\bigl(\eta(I)\bigr)$, for every $V \in \mathcal{B}(Q) \setminus \{I, J\}$.

The basic relation between the f-covering relation in $\mathcal{B}(R)$ and the g-covering relation in $\mathcal{B}(Q)$ is summarized by the following:

Projection Π Properties. *Let $U, V \in \mathcal{B}(Q)$ and let $Y \in \Pi^{-1}(U)$. Then, U g-covers V if and only if there exists $Z \in \Pi^{-1}(V)$ such that Y f-covers Z. Moreover, if $V = I$, U g-covers V if and only if Y f-covers Z for every $Z \in \Pi^{-1}(V)$. In this case we have $U \neq I, J$ and $\Pi^{-1}(U) = \{\eta(U)\}$.*

Now we show that $\mathsf{Per}(g) \subset \mathsf{Per}(f)$. To do this we consider a fixed but arbitrary periodic orbit P of g, and denote the g-period of P by n. We have to show that $n \in \mathsf{Per}(f)$. Assume first that $P \cap \mathsf{Int}(I) = \emptyset$ (observe that this case includes the case $P \subset Q$). Since I is the unique Q-basic interval that g-covers J, P is disjoint from $\mathsf{Int}(I) \cup \mathsf{Int}(J)$ and hence, $\eta(P) \subset S \setminus \mathsf{Int}\bigl(\eta(I)\bigr)$ is a periodic orbit of

$$f\big|_{S\setminus\mathsf{Int}\left(\eta(I)\right)} = \widetilde{g_C}\big|_{S\setminus\mathsf{Int}\left(\eta(I)\right)} = \bigl(\eta \circ g \circ \eta^{-1}\bigr)\big|_{S\setminus\mathsf{Int}\left(\eta(I)\right)}$$

of period n (in fact, $\bigl(f \circ \eta\bigr)\big|_P = \bigl(\eta \circ g\bigr)\big|_P$ which means that that P has the same combinatorial behaviour both under f and under g).

Assume now that $P \cap \mathsf{Int}(I) \neq \emptyset$ (in particular $P \neq Q$). Since $g(I) = J$, $g(J) = K$, and I, J and K are pairwise disjoint, it follows that $n \geq 3$. By Proposition 2.9(a), there exists a loop

$$\begin{array}{ccccccccccc} I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & I_3 & \longrightarrow \cdots \longrightarrow & I_{n-1} & \to & I_0 \\ \shortparallel & & \shortparallel & & \shortparallel & & & & & & \\ I & & J & & K & & & & & & \end{array}$$

of length n in the Markov graph of g with respect to Q which is associated to P. We will show that in the Markov graph of f with respect to R there exists a loop

$$J_0 = \langle s_1^I, s_2^I \rangle_{\eta(I)} \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \to J_0$$

of length n such that $J_j \neq J_0$ for $j = 1, 2, \ldots, n-1$. To see this observe that from the definition of Π and (A–C),

$$\begin{aligned} J_0 = \langle s_1^I, s_2^I \rangle_{\eta(I)} \in \Pi^{-1}(I_0) = \Pi^{-1}(I) = \\ \bigl\{ \langle s_0^I, s_1^I \rangle_{\eta(I)}, \langle s_1^I, s_2^I \rangle_{\eta(I)}, \langle s_2^I, s_3^I \rangle_{\eta(I)}, \langle s_3^I, s_4^I \rangle_{\eta(I)}, \ldots, \langle s_{m-2}^I, s_{m-1}^I \rangle_{\eta(I)} \bigr\} \end{aligned}$$

(recall that $m \geq 5$ by Lemma 4.1). So, since by assumption we know that I_{n-1} g-covers $I_0 = I$, in view of the *Projection* Π *Properties* (with $U = I_{n-1}$), it is enough to show that there is a path

$$J_0 = \langle s_1^I, s_2^I \rangle_{\eta(I)} \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1}$$

in the Markov graph of f with respect to R such that $J_{n-1} \in \Pi^{-1}(I_{n-1})$, and $J_j \neq J_0$ for $j = 1, 2, \ldots, n-1$. We will construct this path recursively starting at J_0. Again by the *Projection* Π *Properties* (with $U = I_0$ and $V = I_1$), there exists an R-basic interval $J_1 \in \Pi^{-1}(J) = \Pi^{-1}(I_1)$ such that J_0 f-covers J_1. Since, $I_0 = I \neq J = I_1$, $\Pi^{-1}(I_0)$ and $\Pi^{-1}(I_1)$ are disjoint, and hence, $J_1 \neq J_0$. Now assume that, for some $\ell \in \{1, 2, \ldots, n-2\}$, we have constructed a path

$$J_0 = \langle s_1^I, s_2^I \rangle_{\eta(I)} \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_\ell$$

in the Markov graph of f with respect to R such that $J_\ell \in \Pi^{-1}(I_\ell)$, and $J_j \neq J_0$ for $j = 1, 2, \dots, \ell$. We are going to prolong this path to a path of length $\ell + 1 \leq n - 1$ verifying analogous properties. Suppose first that $I_{\ell+1} \neq I = I_0$. By the *Projection* Π *Properties* (now with $U = I_\ell$ and $V = I_{\ell+1}$), there exists an R-basic interval $J_{\ell+1} \in \Pi^{-1}(I_{\ell+1})$ such that J_ℓ f-covers $J_{\ell+1}$. Moreover, $\Pi^{-1}(I_0)$ and $\Pi^{-1}(I_{\ell+1})$ are disjoint, and hence, $J_0 \neq J_{\ell+1}$. If $I_{\ell+1} = I = I_0$ then, once more by the *Projection* Π *Properties*, $J_\ell = \eta(I_\ell)$ f-covers Z for every R-basic interval $Z \in \Pi^{-1}(I)$. Thus, we can choose $Z = J_{\ell+1} := \langle s_3^I, s_4^I \rangle_{\eta(I)} \neq \langle s_1^I, s_2^I \rangle_{\eta(I)} = J_0$ so that J_ℓ f-covers $J_{\ell+1}$. This ends the proof of the existence of the prescribed loop.

By Lemma 2.8 there exists a fixed point $x \in J_0$ of f^n such that $f^i(x) \in J_i$ for $i = 1, 2, \dots, n - 1$. If x has f-period n then we are done. Otherwise, x has f-period $k < n$ and, hence,

$$x \in \langle s_1^I, s_2^I \rangle_{\eta(I)} \quad \text{and} \quad x = f^k(x) \in J_k \neq \langle s_1^I, s_2^I \rangle_{\eta(I)}.$$

By the characterization of the basic intervals $\mathcal{B}(R)$ obtained in (A–C), either $x = s_1^I \in \eta(I)$ (and $J_k = \langle s_0^I, s_1^I \rangle_{\eta(I)}$), or $x = s_2^I \in \eta(I)$ (and $J_k = \langle s_2^I, s_3^I \rangle_{\eta(I)}$). Therefore, by (a–c),

$$\begin{aligned} f(x) &= f\big(s_i^I\big) = \varphi_{a,b}\big(s_i\big) \in W, \text{ and} \\ f^2(x) &= f\Big(\varphi_{a,b}\big(s_i\big)\Big) \in \eta(Q) \end{aligned}$$

with $i \in \{1, 2\}$. Since $s_1^I, s_2^I \notin \eta(Q)$, $x = s_i^I$ is a pre-periodic point of f which is not periodic; a contradiction. In consequence, x has f-period n and $\mathsf{Per}(g) \subset \mathsf{Per}(f)$.

Now we will explore the relation between the circle and the graph entropies. The fact that g is a Markov circle map of degree one with respect to a Markov partition Q implies that $\mathcal{B}(Q)$ is a *g-mono cover* of $\mathbb{S}^1$ in the sense of the definition in [9, Page 198] (see also [9, Pages 262,263]). In this context, for every $n \geq 1$, we denote $\bigvee_{i=0}^{n-1} g^{-i}\big(\mathcal{B}(Q)\big)$ by $\mathcal{B}(Q)^n$. By [9, Proposition 4.2.3] (additionally, see again [9, Pages 262,263]),

$$h(g) = h\big(g, \mathcal{B}(Q)\big) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\big(\mathcal{B}(Q)^n\big),$$

where, for a cover $\mathcal{A}$, $\mathcal{N}\big(\mathcal{A}\big)$ denotes the minimal possible cardinality of a sub-cover chosen from $\mathcal{A}$.

For every $n \in \mathbb{N}$ we will denote the set $\bigcup_{i=0}^{n-1} g^{-i}(Q)$ by Q^n. Clearly, Q^n is a Markov invariant set for g because g is Q-monotone. Moreover, since $\mathsf{Q}^n \supset Q$, every Q^n-basic interval is contained in a Q-basic interval. Hence, g is Q^n-monotone and Q^n is a Markov partition of $\mathbb{S}^1$ with respect to g.

Claim on the relation between $\mathcal{B}(\mathbf{Q})^{\mathbf{n}}$ and $\mathcal{B}(\mathsf{Q}^{\mathbf{n}})$: *The partition $\mathcal{B}(\mathsf{Q}^n)$ is the set of all elements $I_0 \cap g^{-1}(I_1) \cap \dots \cap g^{-(n-1)}(I_{n-1}) \in \mathcal{B}(Q)^n$ such that $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \in \mathcal{Q}_n$ where $\mathcal{Q}_n$ denotes the set of all paths of length n in the Markov graph of g with respect to Q. Moreover, $\mathsf{Card}\big(\mathcal{B}(\mathsf{Q}^n)\big) = \mathsf{Card}\big(\mathcal{Q}_n\big)$, and $\mathcal{B}(\mathsf{Q}^n)$ is a sub-cover of $\mathcal{B}(Q)^n$ of minimal cardinality.* To prove the claim we start noticing that, by definition,

$$\mathcal{B}(Q)^n = \bigvee_{i=0}^{n-1} g^{-i}\big(\mathcal{B}(Q)\big) =$$

$$\left\{I_0 \cap g^{-1}(I_1) \cap \cdots \cap g^{-(n-1)}(I_{n-1}) : I_0, I_1, \ldots, I_{n-1} \in \mathcal{B}(Q)\right\}.$$

Now we fix an arbitrary $B \in \mathcal{B}(\mathsf{Q}^n)$ and we will prove by induction that $g^i(B) \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for every $i \in \{0, 1, \ldots, n-1\}$. For $i = 0$ the statement holds trivially by assumption. Suppose that $g^i(B) \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for some $0 \le i < n-2$ and prove it for $i+1$. Since $\mathsf{Q}^{n-i} \supset Q$, there exists an interval $I^i(B) \in \mathcal{B}(Q)$ such that $g^i(B) \subset I^i(B)$. On the other hand, the fact that $g^i(B) \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ can be written as

$$\partial g^i(B) \subset \mathsf{Q}^{n-i}, \quad g^i(B) = \langle \partial g^i(B)\rangle_{I^i(B)} \text{ and } \mathsf{Int}\big(g^i(B)\big) \cap \mathsf{Q}^{n-i} = \emptyset.$$

Since $g\big|_{I^i(B)}$ is a homeomorphism,

$$g^{i+1}(B) = g\big|_{I^i(B)}\Big(\langle \partial g^i(B)\rangle_{I^i(B)}\Big) = \big\langle g\big(\partial g^i(B)\big)\big\rangle_{g(I^i(B))} \subset g\big(I^i(B)\big),$$

with $\partial g^{i+1}(B) = g\big(\partial g^i(B)\big) \subset g\big(\mathsf{Q}^{n-i}\big) \subset \mathsf{Q}^{n-(i+1)}$. Furthermore,

$$\begin{aligned}\Big(g\big|_{I^i(B)}\Big)^{-1}&\Big(\mathsf{Int}\big(g^{i+1}(B)\big) \cap \mathsf{Q}^{n-(i+1)}\Big) \subset \\ &\Big(g\big|_{I^i(B)}\Big)^{-1}\Big(\mathsf{Int}\big(g^{i+1}(B)\big)\Big) \cap g^{-1}\big(\mathsf{Q}^{n-(i+1)}\big) \subset \\ &\qquad\qquad\qquad \mathsf{Int}\big(g^i(B)\big) \cap \mathsf{Q}^{n-i} = \emptyset.\end{aligned}$$

Consequently, $\mathsf{Int}\big(g^{i+1}(B)\big) \cap \mathsf{Q}^{n-(i+1)} = \emptyset$ because

$$\mathsf{Int}\big(g^{i+1}(B)\big) \cap \mathsf{Q}^{n-(i+1)} \subset g^{i+1}(B) \subset g\big(I^i(B)\big) = \operatorname{Im}\Big(g\big|_{I^i(B)}\Big).$$

So, we have proved that $g^i(B) \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for every $i \in \{0, 1, \ldots, n-1\}$. Moreover, there exist intervals $I^i(B) \in \mathcal{B}(Q)$ such that $g^i(B) \subset I^i(B)$ for every $i \in \{0, 1, \ldots, n-1\}$. Furthermore, $g^{n-1}(B) \in \mathcal{B}(\mathsf{Q}^1) = \mathcal{B}(Q)$ implies that $g^{n-1}(B) = I^{n-1}(B)$.

Observe that, for every $i = 0, 1, \ldots, n-2$,

$$g\big(I^i(B)\big) \cap I^{i+1}(B) \supset g^{i+1}(B)$$

has non-empty interior because $g^{i+1}(B)$ is a $\mathsf{Q}^{n-(i+1)}$-basic interval. Thus, $g\big(I^i(B)\big) \supset I^{i+1}(B)$ (that is, $I^i(B)$ g-covers $I^{i+1}(B)$) because $I^i(B), I^{i+1}(B)$ are Q-basic intervals and g is Markov with respect to Q. In other words, $I^0(B) \longrightarrow I^1(B) \longrightarrow \cdots \longrightarrow I^{n-1}(B) \in \mathcal{Q}_n$.

Now it is time to show that $B \in \mathcal{B}(Q)^n$. For $i \in \{0, 1, \ldots, n-2\}$ we have $g^i(B) \subset I^i(B)$ and hence,

$$g^i(B) = \Big(g\big|_{I^i(B)}\Big)^{-1}\Big(g\big|_{I^i(B)}\big(g^i(B)\big)\Big) = I^i(B) \cap g^{-1}\big(g^{i+1}(B)\big),$$

because $g\big|_{I^i(B)}$ is a homeomorphism onto its image. Consequently,

$$\begin{aligned}B = g^0(B) &= I^0(B) \cap g^{-1}\big(g(B)\big) = I^0(B) \cap g^{-1}\Big(I^1(B) \cap g^{-1}\big(g^2(B)\big)\Big) \\ &= I^0(B) \cap g^{-1}\big(I^1(B)\big) \cap g^{-2}\big(g^2(B)\big) \\ &= I^0(B) \cap g^{-1}\big(I^1(B)\big) \cap g^{-2}\Big(I^2(B) \cap g^{-1}\big(g^3(B)\big)\Big) \\ &\qquad\vdots\end{aligned}$$

$$= I^0(B) \cap g^{-1}\big(I^1(B)\big) \cap g^{-2}\big(I^2(B)\big) \cap \cdots \cap g^{-(n-3)}\big(I^{n-3}(B)\big) \cap \\ g^{-(n-2)}\Big(I^{n-2}(B) \cap g^{-1}\big(g^{n-1}(B)\big)\Big) \\ = I^0(B) \cap g^{-1}\big(I^1(B)\big) \cap \cdots \cap g^{-(n-2)}\big(I^{n-2}(B)\big) \cap g^{-(n-1)}\big(I^{n-1}(B)\big) \\ \in \mathcal{B}(Q)^n.$$

Towards the end of the proof, we perform the following construction for an arbitrary

$$I_0 \cap g^{-1}(I_1) \cap \cdots \cap g^{-(n-1)}(I_{n-1}) \in \mathcal{B}(Q)^n :$$

$$B_{n-1} := I_{n-1}, \text{ and} \\ B_i := I_i \cap g^{-1}\big(B_{i+1}\big) \quad \text{for } i = 0, 1, \ldots, n-2.$$

Observe that

$$B_0 = I_0 \cap g^{-1}\big(B_1\big) = I_0 \cap g^{-1}\big(I_1 \cap g^{-1}(B_2)\big) = \\ I_0 \cap g^{-1}\big(I_1\big) \cap g^{-2}\big(B_2\big) = \cdots = \\ I_0 \cap g^{-1}(I_1) \cap \cdots \cap g^{-(n-1)}(I_{n-1}) \in \mathcal{B}(Q)^n.$$

For the converse of the first statement of the claim we consider

$$B_0 = I_0 \cap g^{-1}(I_1) \cap \cdots \cap g^{-(n-1)}(I_{n-1}) \in \mathcal{B}(Q)^n$$

such that $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \in \mathcal{Q}_n$. We will prove by induction on i that, in this case, $B_i \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for every $i \in \{0, 1, \ldots, n-1\}$. In particular, $B_0 \in \mathcal{B}\big(\mathsf{Q}^n\big)$ which completes the proof of the first statement of the claim.

By definition, $B_{n-1} = I_{n-1} \in \mathcal{B}(Q) = \mathcal{B}(\mathsf{Q}^1)$. Suppose that $B_i \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for some $i \in \{1, 2, \ldots, n-1\}$. Since I_{i-1} g-covers I_i we have $g\big(I_{i-1}\big) \supset I_i \supset B_i$. Hence, since $B_i \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ and $g\big|_{I_{i-1}}$ is a homeomorphism,

$$B_{i-1} = I_{i-1} \cap g^{-1}\big(B_i\big) = \Big(g\big|_{I_{i-1}}\Big)^{-1}\big(B_i\big) = \\ \Big(g\big|_{I_{i-1}}\Big)^{-1}\Big(\langle \partial B_i \rangle_{I_i}\Big) = \left\langle \Big(g\big|_{I_{i-1}}\Big)^{-1}\big(\partial B_i\big) \right\rangle_{I_{i-1}},$$

and $\partial B_{i-1} = \Big(g\big|_{I_{i-1}}\Big)^{-1}\big(\partial B_i\big) \subset g^{-1}\big(\mathsf{Q}^{n-i}\big) \subset \mathsf{Q}^{n-(i-1)}$. Furthermore,

$$g\big|_{I_{i-1}}\Big(\mathsf{Int}\big(B_{i-1}\big) \cap \mathsf{Q}^{n-(i-1)}\Big) \subset \\ g\big|_{I_{i-1}}\big(\mathsf{Int}\big(B_{i-1}\big)\big) \cap g\big(\mathsf{Q}^{n-(i-1)}\big) \subset \\ \mathsf{Int}\Big(g\big|_{I_{i-1}}\big(B_{i-1}\big)\Big) \cap \mathsf{Q}^{n-i} = \mathsf{Int}\big(B_i\big) \cap \mathsf{Q}^{n-i} = \emptyset.$$

So, $\mathsf{Int}\big(B_{i-1}\big) \cap \mathsf{Q}^{n-(i-1)} = \emptyset$. We have shown that $B_i \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for every $i \in \{0, 1, \ldots, n-1\}$, which ends the proof of the first statement of the claim. In particular we have shown that $\mathcal{B}(\mathsf{Q}^n) \subset \mathcal{B}(Q)^n$. The fact that $\mathsf{Card}\big(\mathcal{B}(\mathsf{Q}^n)\big) = \mathsf{Card}\big(\mathcal{Q}_n\big)$ is a direct consequence of the part of the claim that is already proved.

To deal with the rest of the claim we will show that $Z \subset \mathsf{Q}^n$ for every $Z \in \mathcal{B}(Q)^n \setminus \mathcal{B}(\mathsf{Q}^n)$. By the part of the claim already proved, such Z is of the form $I_0 \cap g^{-1}(I_1) \cap \cdots \cap g^{-(n-1)}(I_{n-1})$ with $I_0, I_1, \ldots, I_{n-1} \in \mathcal{B}(Q)$ such that $I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1}$ is not a path in the Markov graph of g with respect to Q. So, there exists a largest $\ell \in \{0, 1, \ldots, n-2\}$ such that I_ℓ does not g-cover $I_{\ell+1}$. On the other hand, as above, we have $B_i \in \mathcal{B}\big(\mathsf{Q}^{n-i}\big)$ for every $i \in \{\ell+1, \ell+2, \ldots, n-1\}$. Hence,

$$\begin{aligned} &g\big(I_\ell\big) \cap \mathsf{Int}(B_{\ell+1}) \subset g\big(I_\ell\big) \cap \mathsf{Int}(I_{\ell+1}) = \emptyset, \text{ and} \\ &g\big(\mathsf{Int}(I_\ell)\big) \cap B_{\ell+1} = \mathsf{Int}\big(g(I_\ell)\big) \cap B_{\ell+1} = \emptyset \end{aligned}$$

because $I_\ell, I_{\ell+1} \in \mathcal{B}(Q)$ and $g\big|_{I_\ell}$ is a homeomorphism. Therefore,

$$\begin{aligned} B_\ell &= I_\ell \cap g^{-1}\big(B_{\ell+1}\big) \subset \partial I_\ell \cap g^{-1}\big(\partial B_{\ell+1}\big) \subset Q = \mathsf{Q}^1, \\ B_{\ell-1} &= I_{\ell-1} \cap g^{-1}\big(B_\ell\big) \subset g^{-1}\big(B_\ell\big) \subset g^{-1}\big(\mathsf{Q}^1\big) \subset \mathsf{Q}^2, \\ &\ \vdots \\ B_1 &\subset g^{-1}\big(B_2\big) \subset g^{-1}\big(\mathsf{Q}^{\ell-1}\big) \subset \mathsf{Q}^\ell, \text{ and} \\ Z = B_0 &\subset g^{-1}\big(B_1\big) \subset g^{-1}\big(\mathsf{Q}^\ell\big) \subset \mathsf{Q}^{\ell+1} \subset \mathsf{Q}^n. \end{aligned}$$

Thus, $Z \subset \mathsf{Q}^n$ for every $Z \in \mathcal{B}(Q)^n \setminus \mathcal{B}(\mathsf{Q}^n)$.

Observe that $\mathcal{B}(\mathsf{Q}^n)$ is a cover of $\mathbb{S}^1$ because Q^n is a Markov partition of $\mathbb{S}^1$ with respect to g. Moreover, since $\mathcal{B}(\mathsf{Q}^n) \subset \mathcal{B}(Q)^n$, it follows that $\mathcal{B}(\mathsf{Q}^n)$ is a sub-cover of $\mathcal{B}(Q)^n$. Now we will show that any sub-cover $\mathcal{A}$ of $\mathcal{B}(Q)^n$ must contain $\mathcal{B}(\mathsf{Q}^n)$. To do this observe that, for every $Y \in \mathcal{B}(\mathsf{Q}^n)$,

$$\bigcup_{Z \in \mathcal{B}(\mathsf{Q}^n) \setminus \{Y\}} Z = \mathbb{S}^1 \setminus \mathsf{Int}(Y) \supset \mathsf{Q}^n$$

because the elements of $\mathcal{B}(\mathsf{Q}^n)$ have pairwise disjoint interiors. Assume by way of contradiction that there exists $Y \in \mathcal{B}(\mathsf{Q}^n) \setminus \mathcal{A}$. Then,

$$\begin{aligned} \mathbb{S}^1 = \bigcup_{Z \in \mathcal{A}} Z = &\left(\bigcup_{Z \in \mathcal{A} \setminus \mathcal{B}(\mathsf{Q}^n)} Z\right) \cup \left(\bigcup_{Z \in \mathcal{A} \cap \mathcal{B}(\mathsf{Q}^n)} Z\right) \subset \\ &\left(\bigcup_{Z \in \mathcal{B}(Q)^n \setminus \mathcal{B}(\mathsf{Q}^n)} Z\right) \cup \left(\bigcup_{Z \in \mathcal{B}(\mathsf{Q}^n) \setminus \{Y\}} Z\right) \subset \\ &\mathsf{Q}^n \cup \Big(\mathbb{S}^1 \setminus \mathsf{Int}(Y)\Big) = \mathbb{S}^1 \setminus \mathsf{Int}(Y); \end{aligned}$$

a contradiction. Hence, $\mathcal{B}(\mathsf{Q}^n)$ is a sub-cover of $\mathcal{B}(Q)^n$ of minimal cardinality. This ends the proof of the *Claim on the relation between $\mathcal{B}(Q)^n$ and $\mathcal{B}(\mathsf{Q}^n)$*.

As for the circle map g, $\mathcal{B}(R)$ is an f-mono cover of G (see [9, Page 323] and Subsections 2.3 and 2.4), and hence

$$h(f) = h\big(f, \mathcal{B}(R)\big) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\big(\mathcal{B}(R)^n\big),$$

again by [9, Proposition 4.2.3] (additionally, see [9, Pages 262,263,323]). Moreover for every $n \in \mathbb{N}$ we will denote the set $\bigcup_{i=0}^{n-1} f^{-i}(R) \supset R \supset V(G)$

by R^n. Then, following mutatis mutandis the proof of the *Claim on the relation between $\mathcal{B}(Q)^n$ and $\mathcal{B}(\mathsf{Q}^n)$* we get

- f is R^n-monotone and R^n is a Markov partition of G with respect to f,
- $\mathcal{B}(\mathsf{R}^n)$ is the set of all $J_0 \cap f^{-1}(J_1) \cap \cdots \cap f^{-(n-1)}(J_{n-1}) \in \mathcal{B}(R)^n$ such that $J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \in \mathcal{R}_n$, where $\mathcal{R}_n$ denotes the set of all paths of length n in the Markov graph of f with respect to R,
- $\mathsf{Card}\big(\mathcal{B}(\mathsf{R}^n)\big) = \mathsf{Card}\big(\mathcal{R}_n\big)$, and
- $\mathcal{B}(\mathsf{R}^n)$ is a sub-cover of $\mathcal{B}(R)^n$ of minimal cardinality.

The last ingredient of the proof of this theorem is to bound $\mathsf{Card}\big(\mathcal{R}_n\big)$ in terms of $\mathsf{Card}\big(\mathcal{Q}_n\big)$ for n large enough (for instance $n \geq 5$).

Given a path $\alpha \in \mathcal{Q}_n$ we will denote by $\Pi^{-1}(\alpha)$ the set of all paths $J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \in \mathcal{R}_n$ such that

$$\alpha = \Pi(J_0) \longrightarrow \Pi(J_1) \longrightarrow \cdots \longrightarrow \Pi(J_{n-1}).$$

By the *Projection* Π *Properties*, $\Pi^{-1}(\alpha)$ is non-empty for every α, and for every $J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \in \mathcal{R}_n$ it follows that

$$\Pi(J_0) \longrightarrow \Pi(J_1) \longrightarrow \cdots \longrightarrow \Pi(J_{n-1}) \in \mathcal{Q}_n.$$

That is,

$$J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \in \Pi^{-1}\Big(\Pi(J_0) \longrightarrow \Pi(J_1) \longrightarrow \cdots \longrightarrow \Pi(J_{n-1})\Big)$$

and, equivalently, the class of all sets $\Pi^{-1}(\alpha)$ such that $\alpha \in \mathcal{Q}_n$ is a cover of $\mathcal{R}_n$. First we will bound $\mathsf{Card}\big(\Pi^{-1}(\alpha)\big)$ from above.

Let $n \geq 5$ and let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \in \mathcal{Q}_n$. Set

$$\begin{aligned} \mathsf{P}_I(\alpha) &:= \{i \in \{0,1,\ldots,n-1\} : I_i = I\}, \text{ and} \\ \mathsf{P}_J(\alpha) &:= \{i \in \{0,1,\ldots,n-1\} : I_i = J\}. \end{aligned}$$

Clearly, $I_i \neq I, J$ for every $i \in \{0,1,\ldots,n-1\} \setminus \big(\mathsf{P}_I(\alpha) \cup \mathsf{P}_J(\alpha)\big)$. Therefore, $\Pi^{-1}(I_i) = \{\eta(I_i)\}$, and $J_i = \eta(I_i)$ for every path

$$J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{n-1} \in \Pi^{-1}(\alpha).$$

Let us count the number of choices we have for the intervals J_i with $i \in \mathsf{P}_I(\alpha) \cup \mathsf{P}_J(\alpha)$.

- Assume that $0 \in \mathsf{P}_J(\alpha)$ (that is, $I_0 = J$). Since $g(J) = K$, we have $I_1 = K$ and $J_1 = \eta(K)$. Moreover, by the definition of Π and (B),

  $$\begin{aligned} \Pi^{-1}(I_0) = \Pi^{-1}(J) = \{Y \in \mathcal{B}(R) : Y \subset X\} = \\ \big\{\varphi_{a,b}\big([s_j, s_{j+1}]\big) : j \in \{0,1,\ldots,m-1\}\big\}, \end{aligned}$$

  and every basic interval $\varphi_{a,b}\big([s_j, s_{j+1}]\big)$ f-covers $\eta(K)$. So, a path in $\Pi^{-1}(\alpha)$ may start with $J_0 = \varphi_{a,b}\big([s_j, s_{j+1}]\big) \longrightarrow J_1 = \eta(K)$ with $j \in \{0,1,\ldots,m-1\}$, and all these possibilities are realizable. In conclusion there are $\mathsf{Card}\big(\Pi^{-1}(J)\big) \leq m$ possibilities for the initial subpath $J_0 \longrightarrow J_1$ of a path in $\Pi^{-1}(\alpha)$ in the case $I_0 = J$.

- Let $i \in \mathsf{P}_I(\alpha)$, $i \le n-3$. Then, the subpath $I_i \to I_{i+1} \to I_{i+2}$ coincides with $I \to J \to K$ because $g(I) = J$ and $g(J) = K$. By (C) and (B),
$$\Pi^{-1}(I_i) = \Pi^{-1}(I) = \{Y \in \mathcal{B}(R) : Y \subset \eta(I)\} = \Big\{\langle s^I_j, s^I_{j+1}\rangle_{\eta(I)} : j \in \{0, 1, \ldots, m-1\}\Big\}$$
and, for every $j \in \{0, 1, \ldots, m-1\}$,
$$f\big(\langle s^I_j, s^I_{j+1}\rangle_{\eta(I)}\big) = \varphi_{a,b}\big([s_j, s_{j+1}]\big) \in \mathcal{B}(R),$$
which f-covers $\eta(K)$. So, the subpath $J_i \longrightarrow J_{i+1} \longrightarrow J_{i+2}$ is of the form
$$\langle s^I_j, s^I_{j+1}\rangle_{\eta(I)} \longrightarrow \varphi_{a,b}\big([s_j, s_{j+1}]\big) \longrightarrow \eta(K)$$
for some $j \in \{0, 1, \ldots, m-1\}$. Moreover, if $i > 0$, then J_{i-1} f-covers $\langle s^I_j, s^I_{j+1}\rangle_{\eta(I)}$ for every $j \in \{0, 1, \ldots, m-1\}$, by *Projection Π Properties*. In conclusion, we have exactly $\mathsf{Card}\big(\Pi^{-1}(I)\big) = m$ possibilities for the subpath
$$J_{i-1} \longrightarrow J_i = \langle s^I_j, s^I_{j+1}\rangle_{\eta(I)} \longrightarrow J_{i+1} = \varphi_{a,b}\big([s_j, s_{j+1}]\big) \longrightarrow J_{i+2} = \eta(K)$$
when $i > 0$ and, analogously, there are m possibilities for the subpath
$$J_0 = \langle s^I_j, s^I_{j+1}\rangle_{\eta(I)} \longrightarrow J_1 = \varphi_{a,b}\big([s_j, s_{j+1}]\big) \longrightarrow J_2 = \eta(K),$$
if $i = 0$.
- Let $i = n-2 \in \mathsf{P}_I(\alpha)$. This case is analogous to the previous one with the difference that $i > 0$ and I_{i+2} and J_{n+2} do not exist. So, there are m possibilities for the subpath
$$J_{n-3} \longrightarrow J_{n-2} = \langle s^I_j, s^I_{j+1}\rangle_{\eta(I)} \longrightarrow J_{n-1} = \varphi_{a,b}\big([s_j, s_{j+1}]\big).$$
- Assume that $0 < i \in \mathsf{P}_J(\alpha)$. Since $g(I) = J$ and I is the only interval that g-covers J it follows that $i-1 \in \mathsf{P}_I(\alpha)$. So, all positions $0 < i \in \mathsf{P}_J(\alpha)$ ($i \le n-1$) are considered in the previous two cases, where we get that J_i is completely determined by J_{i-1}.
- For $i = n-1 \in \mathsf{P}_I(\alpha)$ there are again m possibilities for the subpath $J_{n-2} \longrightarrow J_{n-1} = \langle s^I_j, s^I_{j+1}\rangle_{\eta(I)}$.

From these last five observations we get
$$\mathsf{Card}\big(\Pi^{-1}(\alpha)\big) \le m^{\mathsf{Card}\left(\mathsf{P}_I(\alpha)\right)+1} \quad \text{if } 0 \in \mathsf{P}_J(\alpha), \text{ and}$$
$$\mathsf{Card}\big(\Pi^{-1}(\alpha)\big) = m^{\mathsf{Card}\left(\mathsf{P}_I(\alpha)\right)} \quad \text{when } 0 \notin \mathsf{P}_J(\alpha).$$

To compute $m^{\mathsf{Card}\left(\mathsf{P}_I(\alpha)\right)+1}$ and $m^{\mathsf{Card}\left(\mathsf{P}_I(\alpha)\right)}$ in terms of n, let us write $\mathsf{P}_I(\alpha) = \{i_1, i_2, \ldots, i_\ell\}$ with $i_1 < i_2 < \cdots < i_\ell$.

- If $0 \in \mathsf{P}_J(\alpha)$, then $i_1 \ge \rho - 1$. Otherwise, $i_1 < \rho - 1$ and there exists a loop
$$I \longrightarrow I_0 = J \longrightarrow I_1 = K \longrightarrow \cdots \longrightarrow I_{i_1} = I$$
in the Markov graph of g of length $i_1 + 1 < \rho$; a contradiction with the fact that ρ is the minimal length of a loop in the Markov graph of g beginning (and ending) at I. By similar reasons, $i_{j+1} - i_j \ge \rho$

for $j = 1, 2, \dots, \ell - 1$ and hence, $i_j \geq j\rho - 1$ for $j = 2, 3, \dots, \ell$. So, $n - 1 \geq i_\ell \geq \ell\rho - 1$, $\frac{n}{\rho} \geq \ell$, and $\mathsf{Card}\big(\mathsf{P}_I(\alpha)\big) + 1 = \ell + 1 \leq \left\lfloor \frac{n}{\rho} \right\rfloor + 1$.

- If $0 \notin \mathsf{P}_J(\alpha)$, then $i_1 \geq 0$ and $i_{j+1} - i_j \geq \rho$ for $j = 1, 2, \dots, \ell - 1$. Hence, $n - 1 \geq i_\ell \geq (\ell - 1)\rho$, $\ell \leq \frac{n-1}{\rho} + 1 < \frac{n}{\rho} + 1$, and
$$\mathsf{Card}\big(\mathsf{P}_I(\alpha)\big) = \ell \leq \left\lfloor \frac{n}{\rho} + 1 \right\rfloor = \left\lfloor \frac{n}{\rho} \right\rfloor + 1.$$

Since the class of all sets $\Pi^{-1}(\alpha)$ such that $\alpha \in \mathcal{Q}_n$ is a cover of $\mathcal{R}_n$,
$$\mathsf{Card}\big(\mathcal{R}_n\big) = \mathsf{Card}\left(\bigcup_{\alpha \in \mathcal{Q}_n} \Pi^{-1}(\alpha)\right) \leq \sum_{\alpha \in \mathcal{Q}_n} \mathsf{Card}\big(\Pi^{-1}(\alpha)\big) \leq \mathsf{Card}\big(\mathcal{Q}_n\big) m^{\left\lfloor \frac{n}{\rho} \right\rfloor + 1}.$$

Therefore, since $\mathcal{B}(\mathsf{Q}^n)$ (respectively $\mathcal{B}(\mathsf{R}^n)$) is a sub-cover of $\mathcal{B}(Q)^n$ (respectively $\mathcal{B}(R)^n$) of minimal cardinality,
$$\begin{aligned} h(f) &= \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\big(\mathcal{B}(R)^n\big) = \lim_{n\to\infty} \frac{1}{n} \log \mathsf{Card}\big(\mathcal{B}(\mathsf{R}^n)\big) = \\ &\lim_{n\to\infty} \frac{1}{n} \log \mathsf{Card}\big(\mathcal{R}_n\big) \leq \lim_{n\to\infty} \frac{1}{n} \log \left(\mathsf{Card}\big(\mathcal{Q}_n\big) m^{\left\lfloor \frac{n}{\rho} \right\rfloor + 1} \right) = \\ &\lim_{n\to\infty} \frac{1}{n} \log \mathsf{Card}\big(\mathcal{Q}_n\big) + \lim_{n\to\infty} \frac{\left\lfloor \frac{n}{\rho} \right\rfloor + 1}{n} \log m = \\ &\lim_{n\to\infty} \frac{1}{n} \log \mathsf{Card}\big(\mathcal{B}(\mathsf{Q}^n)\big) + \frac{1}{\rho} \log m = \\ &\lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\big(\mathcal{B}(Q)^n\big) + \frac{\log m}{\rho} = h(g) + \frac{\log m}{\rho}. \end{aligned}$$

Finally we will show that if g is Q-expanding and transitive then there exists a choice of the homeomorphisms ζ and ξ such that f is R-expanding, and the Markov matrix of f with respect to R is irreducible but not a permutation matrix. In view of Theorem 2.12 this will imply that f is transitive.

R-expandingness of f. Recall that, since g is Q-expanding, for every basic interval $V \in \mathcal{B}(Q)$ there exists a homeomorphism $\mu_V : V \longrightarrow [0,1]$ such that
$$d_V(x, y) := |\mu_V(x) - \mu_V(y)| \quad \text{for every} \quad x, y \in V.$$

Then, since ζ and ξ are arbitrary homeomorphisms (except for its orientation), we can modify them without altering the previous conclusions of the theorem. In view of Remark 2.13 we can choose μ_K so that
$$\mu_K\big(\eta^{-1}(\widetilde{g_C}(a))\big) = 0 \quad \text{and} \quad \mu_K\big(\eta^{-1}(\widetilde{g_C}(b))\big) = 1,$$
without modifying the distance d_K. So we can define $\zeta := \eta \circ \mu_K^{-1}$ without modifying the choices $\zeta(0) = \widetilde{g_C}(a)$ and $\zeta(1) = \widetilde{g_C}(b)$.

Next we define ξ by pieces. To do this we fix points $t_i := \xi^{-1}(s_i) \in I$ for $i = 0, 1, \dots, m$, and define $\xi\big|_{\langle t_i, t_{i+1} \rangle_I}$ for every $i \in \{0, 1, \dots, m - 1\}$.

For definiteness we choose an orientation for μ_I such that μ_I is increasing with respect to the indices of the points t_i. That is, we choose μ_I so that $\mu_I(t_0) = 0$ and $\mu_I(t_m) = 1$, without modifying the distance d_I. The definition of the homeomorphisms $\xi\big|_{\langle t_i, t_{i+1}\rangle_I}$ will be done with the help of auxiliary homeomorphisms $\widetilde{\xi}_i \colon \langle t_i, t_{i+1}\rangle_I \longrightarrow [0,1]$ for every $i \in \{0, 1, \dots, m-1\}$. We define

$$\widetilde{\xi}_i(x) = \begin{cases} \frac{\mu_I(x)-\mu_I(t_i)}{\mu_I(t_{i+1})-\mu_I(t_i)} & \text{if } i \text{ is even, and} \\ \frac{\mu_I(t_{i+1})-\mu_I(x)}{\mu_I(t_{i+1})-\mu_I(t_i)} & \text{if } i \text{ is odd,} \end{cases}$$

for every $x \in \langle t_i, t_{i+1}\rangle_I$. Now we are ready to define

$$\xi\big|_{\langle t_i, t_{i+1}\rangle_I} = \Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \circ \widetilde{\xi}_i$$

for $i = 0, 1, \dots, m-1$. Clearly, $\Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \circ \widetilde{\xi}_i$ is a homeomorphism onto its image. Moreover, when i is even, by Lemma 4.1(d) we have

$$\Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \big(\widetilde{\xi}_i(t_i)\big) = \Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} (0) = s_i, \text{ and}$$
$$\Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \big(\widetilde{\xi}_i(t_{i+1})\big) = \Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} (1) = s_{i+1}.$$

Analogously, when i is odd we have $\Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \big(\widetilde{\xi}_i(t_i)\big) = s_i$, and $\Big((\psi_{a,b} \circ \varphi_{a,b})\big|_{[s_i, s_{i+1}]} \Big)^{-1} \big(\widetilde{\xi}_i(t_{i+1})\big) = s_{i+1}$. So, ξ is a well defined homeomorphism from I to $[0,1]$ such that $\xi(t_i) = s_i$ for every $i = 0, 1, \dots, m$. In particular the orientation of ξ has not been modified and we still have $\xi\big(\eta^{-1}(z_a^I)\big) = \xi(t_0) = s_0 = 0$ and $\xi\big(\eta^{-1}(z_b^I)\big) = \xi(t_m) = s_m = 1$.

Now we define a distance $d_Y(x, y) := |\mu_Y(x) - \mu_Y(y)|$ for every $Y \in \mathcal{B}(R)$ and $x, y \in Y$, where in view of (A–C) and Lemma 4.1, we set

$$\mu_Y = \begin{cases} \mu_{\eta^{-1}(Y)} \circ \eta^{-1}\big|_Y & \text{if } Y \subset S \setminus \mathsf{Int}\big(\eta(I)\big), \\ \widetilde{\xi}_i \circ \eta^{-1}\big|_Y & \text{if } Y = \eta\big(\xi^{-1}([s_i, s_{i+1}])\big) \subset \eta(I) \\ & \quad \text{with } i \in \{0, 1, \dots, m-1\}, \\ \psi_{a,b}\big|_Y & \text{if } Y \subset X. \end{cases}$$

Then,

- If $Y \subset X$, then
$$f\big|_Y = \zeta \circ \psi_{a,b}\big|_Y = \eta \circ \mu_K^{-1} \circ \psi_{a,b}\big|_Y.$$
Hence, $f(Y) = \eta(K) \in \mathcal{B}(R)$, $\eta(K) \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$. Moreover, for every $z \in Y$,
$$\mu_{\eta(K)}(f(z)) = \big(\mu_K \circ \eta^{-1} \circ \eta \circ \mu_K^{-1}\big)\big(\psi_{a,b}(z)\big) = \psi_{a,b}(z) = \mu_Y(z).$$
Thus, for every $x, y \in Y$,
$$d_{\eta(K)}(f(x), f(y)) = \Big|\mu_{\eta(K)}(f(x)) - \mu_{\eta(K)}(f(y))\Big| = |\mu_Y(x) - \mu_Y(y)| = d_Y(x, y).$$

- If $Y = \eta\big(\xi^{-1}([s_i, s_{i+1}])\big) \subset \eta(I)$ with $i \in \{0, 1, \dots, m-1\}$, we have $f\big|_Y = \varphi_{a,b}\big|_{[s_i,s_{i+1}]} \circ \xi\big|_{\xi^{-1}([s_i,s_{i+1}])} \circ \eta^{-1}\big|_Y$. Therefore,
$$f(Y) = \varphi_{a,b}\big(\xi\big(\eta^{-1}(Y)\big)\big) \in \mathcal{B}(R), \quad \varphi_{a,b}\big(\xi\big(\eta^{-1}(Y)\big)\big) \subset X$$
and, for every $z \in Y$,
$$\mu_{f(Y)}(f(z)) = \Big(\big(\psi_{a,b} \circ \varphi_{a,b}\big)\big|_{[s_i,s_{i+1}]} \circ \xi\big|_{\xi^{-1}([s_i,s_{i+1}])}\Big)\,\big(\eta^{-1}(z)\big) = \Big(\big(\psi_{a,b} \circ \varphi_{a,b}\big)\big|_{[s_i,s_{i+1}]} \circ \Big(\big(\psi_{a,b} \circ \varphi_{a,b}\big)\big|_{[s_i,s_{i+1}]}\Big)^{-1} \circ \widetilde{\xi}_i\Big)\,\big(\eta^{-1}(z)\big) = \widetilde{\xi}_i\big(\eta^{-1}(z)\big) = \mu_Y(z).$$
So, for every $x, y \in Y$,
$$d_{f(Y)}(f(x), f(y)) = \Big|\mu_{f(Y)}(f(x)) - \mu_{f(Y)}(f(y))\Big| = |\mu_Y(x) - \mu_Y(y)| = d_Y(x, y).$$
- If $Y = \eta(V) \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$ for some $V \in \mathcal{B}(Q) \setminus \{I, J\}$, then
$$f\big|_Y = \widetilde{g_C}\big|_Y = \eta\big|_{g(V)} \circ g\big|_V \circ \eta^{-1}\big|_Y.$$
On the other hand, since $V \neq I$ and I is the unique interval from $\mathcal{B}(Q)$ that g-covers J, $f(Y) = \eta(g(V)) \subset S$.

  Assume as an easy case that $f(Y) \in \mathcal{B}(R)$. Thus, $g(V) \neq I$ since otherwise, by (C), $f(Y) = \eta(I)$ and Y f-covers all $m \geq 5$ R-basic intervals $\langle s_i^I, s_{i+1}^I\rangle_{\eta(I)} \subset \eta(I)$. Summarizing, $g(V) \in \mathcal{B}(Q) \setminus \{I, J\}$ or, equivalently, $f(Y) = \eta(g(V)) \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$. Thus, by using the fact that g is expanding at V, for every $x, y \in Y$, we have
$$d_{f(Y)}(f(x), f(y)) = \Big|\mu_{f(Y)}(f(x)) - \mu_{f(Y)}(f(y))\Big| = \Big|\big(\mu_{g(V)} \circ \eta^{-1} \circ \eta\big)\Big(g\big(\eta^{-1}(x)\big)\Big) - \big(\mu_{g(V)} \circ \eta^{-1} \circ \eta\big)\Big(g\big(\eta^{-1}(y)\big)\Big)\Big| = \Big|\mu_{g(V)}\Big(g\big(\eta^{-1}(x)\big)\Big) - \mu_{g(V)}\Big(g\big(\eta^{-1}(y)\big)\Big)\Big| = \big|\mu_V\big(\eta^{-1}(x)\big) - \mu_V\big(\eta^{-1}(y)\big)\big| = |\mu_Y(x) - \mu_Y(y)| = d_Y(x, y).$$
  Next suppose that $f(Y) \notin \mathcal{B}(R)$. Then, $f(Y)$ can be written as a union of basic intervals from $\mathcal{B}(R)$. In view of (A–C), every basic interval $Z \in \mathcal{B}(R)$, $Z \subset f(Y) \subset S$ is of one of the following two forms:
  - $Z = \langle s_i^I, s_{i+1}^I\rangle_{\eta(I)} = \eta\big(\xi^{-1}([s_i, s_{i+1}])\big) \subset \eta(I)$ for some index $i \in \{0, 1, \dots, m-1\}$, and $I \subset g(V)$, or
  - $Z = \eta(U)$ with $U \in \mathcal{B}(Q) \setminus \{I, J\}$ and $U \subset g(V)$.

  Let $x, y \in Y$ be such that
$$\langle \eta\big(g\big(\eta^{-1}(x)\big)\big), \eta\big(g\big(\eta^{-1}(y)\big)\big)\rangle_{\eta(g(V))} = \langle f(x), f(y)\rangle_{f(Y)} \subset Z,$$
and of course $\eta^{-1}(x), \eta^{-1}(y) \in V$. We will show that there exists $\lambda_Y := \lambda_V > 1$ such that
$$d_Z(f(x), f(y)) \geq \lambda_Y d_Y(x, y).$$

In the second case we have

$$Z = \eta(U) \subset S \setminus \mathsf{Int}\big(\eta(I)\big), \text{ and}$$
$$\langle g\big(\eta^{-1}(x)\big), g\big(\eta^{-1}(y)\big)\rangle_{g(V)} \subset U.$$

So, again by the expandingness of g at V, there exists $\lambda_V > 1$ such that

$$\begin{aligned}
d_Z(f(x), f(y)) = \Big|\mu_{\eta(U)}(f(x)) - \mu_{\eta(U)}(f(y))\Big| = \\
\Big|\big(\mu_U \circ \eta^{-1} \circ \eta\big)\Big(g\big(\eta^{-1}(x)\big)\Big) - \big(\mu_U \circ \eta^{-1} \circ \eta\big)\Big(g\big(\eta^{-1}(y)\big)\Big)\Big| = \\
\Big|\mu_U\Big(g\big(\eta^{-1}(x)\big)\Big) - \mu_U\Big(g\big(\eta^{-1}(y)\big)\Big)\Big| \geq \\
\lambda_V \left|\mu_V\left(\eta^{-1}(x)\right) - \mu_V\left(\eta^{-1}(y)\right)\right| = \\
\lambda_V \left|\mu_Y(x) - \mu_Y(y)\right| = \lambda_Y d_Y(x, y).
\end{aligned}$$

In the first case,

$$Z = \eta\big(\xi^{-1}([s_i, s_{i+1}])\big) \subset \eta(I), \text{ and}$$
$$\langle g\big(\eta^{-1}(x)\big), g\big(\eta^{-1}(y)\big)\rangle_{g(V)} \subset \xi^{-1}([s_i, s_{i+1}]) \subset I.$$

For $z \in \{x, y\}$ we have

$$\mu_Z(f(z)) = \Big(\widetilde{\xi}_i \circ \eta^{-1}\big|_Z \circ \eta\Big)\big(g\big(\eta^{-1}(z)\big)\big) = \widetilde{\xi}_i\big(g\big(\eta^{-1}(z)\big)\big).$$

On the other hand, when $i > 0$ we have

$$0 = \mu_I(t_0) < \mu_I(t_i) < \mu_I(t_{i+1}) \leq \mu_I(t_m) = 1,$$

and when $i = 0$, since $m \geq 5$, $0 = \mu_I(t_0) < \mu_I(t_1) < \mu_I(t_m) = 1$. So, in both cases, $\mu_I(t_{i+1}) - \mu_I(t_i) < 1$, and thus

$$\frac{\lambda_V}{\mu_I(t_{i+1}) - \mu_I(t_i)} > \lambda_V.$$

We will end the proof of R-expandingness in the case i even. If i is odd, the proof follows in a similar way.

$$\begin{aligned}
d_Z(f(x), f(y)) = |\mu_Z(f(x)) - \mu_Z(f(y))| = \\
\Big|\widetilde{\xi}_i\big(g\big(\eta^{-1}(x)\big)\big) - \widetilde{\xi}_i\big(g\big(\eta^{-1}(y)\big)\big)\Big| = \\
\left|\frac{\mu_I\big(g\big(\eta^{-1}(x)\big)\big) - \mu_I(t_i)}{\mu_I(t_{i+1}) - \mu_I(t_i)} - \frac{\mu_I\big(g\big(\eta^{-1}(y)\big)\big) - \mu_I(t_i)}{\mu_I(t_{i+1}) - \mu_I(t_i)}\right| = \\
\frac{1}{\mu_I(t_{i+1}) - \mu_I(t_i)} \left|\mu_I\big(g\big(\eta^{-1}(x)\big)\big) - \mu_I\big(g\big(\eta^{-1}(y)\big)\big)\right| \geq \\
\frac{\lambda_V}{\mu_I(t_{i+1}) - \mu_I(t_i)} \left|\mu_V\left(\eta^{-1}(x)\right) - \mu_V\left(\eta^{-1}(y)\right)\right| > \\
\lambda_V \left|\mu_Y(x) - \mu_Y(y)\right| = \lambda_Y d_Y(x, y).
\end{aligned}$$

In summary, we have proved that, by choosing ζ and ξ appropriately, f is R-expanding.

The Markov matrix of f with respect to R is not a permutation matrix. Since g is Q-expanding and transitive we know by Theorem 2.12 that the Markov matrix of g with respect to Q is irreducible. This implies that given $V \in \mathcal{B}(Q)$, $V \neq I$, there is a path in the Markov graph of g with respect to Q from V to I. In particular, there is an interval $U \in \mathcal{B}(Q)$ such that U g-covers I. Since $g(I) = J$ and $g(J) = K$, and I, J and K are pairwise disjoint, it follows that $U \neq I, J$. By (A) and (B), $\eta(U) \in \mathcal{B}(R)$, and

$$f\big(\eta(U)\big) \supset \eta(I) \supset \langle s_i^I, s_{i+1}^I \rangle_{\eta(I)}$$

for every $i \in \{0, 1, \ldots, m-1\}$. So, the basic interval $\eta(U) \in \mathcal{B}(R)$ f-covers $m \geq 5$ intervals and, hence, the Markov matrix of f with respect to R is not a permutation matrix.

The Markov matrix of f with respect to R is irreducible. Let $Y, Z \in \mathcal{B}(R)$ be arbitrary basic intervals. Since the Markov matrix of g with respect to Q is irreducible, there is a path

$$\Pi(Y) = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} = \Pi(Z)$$

in the Markov graph of g with respect to Q. Then, the iterative use of the *Projection* Π *Properties* shows that there is a path of the same length in the Markov graph of f with respect to R from Y to Z. So, the Markov matrix of f with respect to R is irreducible.

Putting all together we know that f is R-expanding and that the Markov matrix of f with respect to R is irreducible but it is not a permutation matrix. So, f is transitive by Theorem 2.12. □

5. General construction for minimal circle maps of degree one

We aim at constructing examples of totally transitive graph maps with arbitrarily small topological entropy and an arbitrarily large strict boundary of cofiniteness. To do this we will use the minimum entropy maps depending on the rotation interval. In this discussion (and survey) we will follow the approach from [8, 9].

Definition 5.1 (The lower entropy circle maps of degree one). For $c, d \in \mathbb{R}$, $c < d$ and $z > 1$ we define the series

$$R_{c,d}(z) := \sum_{\left\{(p,q)\,:\, p\in\mathbb{Z},\ q\in\mathbb{N} \text{ and } c < \frac{p}{q} < d\right\}} z^{-q}.$$

One can show that $R_{c,d}(z) = \frac{1}{2}$ has a unique solution $\beta_{c,d} > 1$.

Then we can define a lifting $G_{c,d} \in \mathcal{L}_1$ by (see Figure 5 for a concrete example):

$$G_{c,d}(x) := \begin{cases} \beta_{c,d} x + b_{c,d} & \text{if } 0 \leq x \leq u_{c,d}, \\ \beta_{c,d}(1-x) + b_{c,d} + 1 & \text{if } u_{c,d} \leq x \leq 1, \\ G_{c,d}(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1], \end{cases}$$

with $u_{c,d} := \frac{\beta_{c,d}+1}{2\beta_{c,d}} \in (0,1)$ and

$$b_{c,d} := \frac{(\beta_{c,d}-1)^2}{\beta_{c,d}} \sum_{n=1}^{\infty} \lfloor nc \rfloor \, \beta_{c,d}^{-n}.$$

We will denote by $g_{c,d}$ the circle map of degree 1 which has $G_{c,d}$ as a lifting.

The maps $g_{c,d}$ will be called *c,d-minimal*, or just *minimal* for simplicity. ◻

Useless Remark (On the computation of the numbers $\boldsymbol{\beta_{c,d}}$). For $c \in \mathbb{R}$, $c > 0$, and $z > 1$ we define

$$T_c(z) := \sum_{n=0}^{\infty} z^{-\lfloor \frac{n}{c} \rfloor},$$

and, for definiteness, we set $T_0(z) \equiv 0$. Then, for $c, d \in \mathbb{R}$, $c < d$, $c \in [0,1)$, and $z > 1$ we define

$$Q_{c,d}(z) := z + 1 + 2\left(\frac{z}{z-1} - T_{1-c}(z) - T_d(z)\right)$$

(observe that if $[c,d]$ is a rotation interval then, by replacing the lifting F used to compute the rotation interval by $F - \lfloor c \rfloor$, we get the new rotation interval $\big[c - \lfloor c \rfloor, d - \lfloor c \rfloor\big]$ with $c - \lfloor c \rfloor \in [0,1)$; so the assumption $c \in [0,1)$ above is not restrictive).

One can show that

$$Q_{c,d}(z) = (z-1)\left(1 - 2R_{c,d}(z)\right).$$

Hence, $\beta_{c,d}$ is the largest root of the equation $Q_{c,d}(z) = 0$. This observation gives a much easier way of calculating the numbers $\beta_{c,d}$. ◻

The reason for introducing the above notation and tools is explained by the next result which shows the existence of minimum entropy maps depending on the rotation interval.

Theorem 5.2 ([8]). *Let f be a circle map of degree 1 with rotation interval $[c,d]$ with $c < d$. Then $h(f) \geq \log \beta_{c,d}$. Moreover, for every $c, d \in \mathbb{R}$ with $c < d$, $\mathsf{Rot}(G_{c,d}) = [c,d]$ and $h(g_{c,d}) = \log \beta_{c,d}$.*

The next theorem computes the limit of the topological entropy of certain families of minimal degree one minimal circle maps.

Theorem 5.3. *Let $\{[c_n, d_n]\}_{n \in \mathbb{N}}$ be a sequence of non-degenerate intervals contained in the interval $(0,1)$ for every $n \in \mathbb{N}$. Suppose that*

$$\lim_{n\to\infty} \min M(c_n, d_n) = \infty.$$

Then,

$$\lim_{n\to\infty} h(g_{c_n,d_n}) = \lim_{n\to\infty} \log \beta_{c_n,d_n} = 0.$$

Remark 5.4. The assumption that $\lim_{n\to\infty} \min M(c_n, d_n) = \infty$ in the above theorem implies $\lim_{n\to\infty} d_n - c_n = 0$.

This is a consequence of the definition of limit and the following observation: the inequality $\min M(c_n, d_n) > M$ implies that $M \notin M(c_n, d_n)$ and this is equivalent to $\frac{k}{M} \leq c_n < d_n \leq \frac{k+1}{M}$ for some $k \in \mathbb{Z}$, which implies $d_n - c_n \leq \frac{1}{M}$. ◻

In what follows we will use the auxiliary function $\llfloor \cdot \rrfloor$, similar to the integer part function, defined by:

$$\llfloor x \rrfloor := -1 - \lfloor -x \rfloor = \begin{cases} \lfloor x \rfloor & \text{if } x \notin \mathbb{Z}, \text{ and} \\ x - 1 & \text{otherwise.} \end{cases}$$

Proof of Theorem 5.3. The fact that, for every $n \in \mathbb{N}$, $h(g_{c_n,d_n}) = \log \beta_{c_n,d_n}$ follows from Theorem 5.2. So, we need to show that $\lim_{n\to\infty} \beta_{c_n,d_n} = 1$.

For every $q \in \mathbb{N}$, $\llfloor qd_n \rrfloor - \lfloor qc_n \rfloor$ is equal to the number of integers contained in the interval (qc_n, qd_n) which, in turn, is the cardinality of the set

$$\left\{ p \in \mathbb{Z} : \text{ such that } c_n < \tfrac{p}{q} < d_n \right\}.$$

For every $n \in \mathbb{N}$, let us denote $N_n := \min M(c_n, d_n) - 1$. So, for every $q \leq N_n$, $q \notin M(c_n, d_n)$ which is equivalent to

$$\left\{ p \in \mathbb{Z} : \text{ such that } c_n < \tfrac{p}{q} < d_n \right\} = \emptyset \quad \Longleftrightarrow \quad \llfloor qd_n \rrfloor - \lfloor qc_n \rfloor = 0.$$

Moreover, if $\frac{1}{k} < d_n - c_n$, then $k \in M(c_n, d_n)$ and this implies that $k > N_n$. Hence, $k \leq N_n$ implies $d_n - c_n \leq \frac{1}{k}$, which is equivalent to $(d_n - c_n)k \leq 1$. Consequently, when $z > 1$,

$$R_{c,d}(z) := \sum_{\left\{(p,q) \,:\, p\in\mathbb{Z},\ q\in\mathbb{N} \text{ and } c < \frac{p}{q} < d\right\}} z^{-q} = \sum_{k=N_n+1}^{\infty} \left(\llfloor kd_n \rrfloor - \lfloor kc_n \rfloor\right) z^{-k} \leq$$

$$\sum_{k=N_n+1}^{\infty} \left((d_n - c_n)k + 1\right) z^{-k} = z^{-N_n} \sum_{k=1}^{\infty} \left((d_n - c_n)(N_n + k) + 1\right) z^{-k} \leq$$

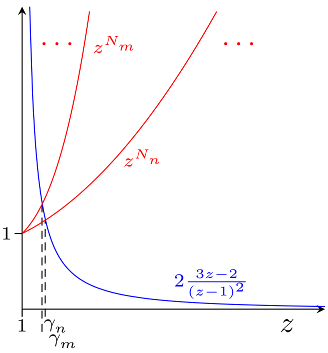
$$z^{-N_n} \sum_{k=1}^{\infty} \left((d_n - c_n)k + 2\right) z^{-k} < z^{-N_n} \sum_{k=1}^{\infty} (k+2) z^{-k} = z^{-N_n} \frac{3z - 2}{(z-1)^2}.$$

Now let us consider the equation $z^{-N_n} \frac{3z-2}{(z-1)^2} = \frac{1}{2}$, which is equivalent to $z^{N_n} = 2\frac{3z-2}{(z-1)^2}$. We note that:

(i) The map $z \mapsto 2\frac{3z-2}{(z-1)^2}$ is strictly decreasing on $(1, +\infty)$, $\lim_{z\to 1^+} 2\frac{3z-2}{(z-1)^2} = +\infty$ and $\lim_{z\to\infty} 2\frac{3z-2}{(z-1)^2} = 0$.

(ii) The map $z \mapsto z^{N_n}$ is strictly increasing and $z^{N_n}\big|_{z=1} = 1$.

(iii) Clearly, $N_n < N_m$ implies $z^{N_n} < z^{N_m}$ for every $z > 1$.



Then, for each n, there exists a unique real number $\gamma_n > 1$ such that $\gamma_n^{N_n} = 2\frac{3\gamma_n - 2}{(\gamma_n - 1)^2}$, $\gamma_m < \gamma_n$ whenever $N_n < N_m$, and $\lim_{n\to\infty} \gamma_n = 1$ because $\lim_{n\to\infty} N_n = \infty$.

Since, $R_{c_n,d_n}(\gamma_n) < \gamma_n^{-N_n} \frac{3\gamma_n - 2}{(\gamma_n - 1)^2} = \frac{1}{2}$, in view of [8, Lemma 2.1(c–e)] we have $1 < \beta_{c_n,d_n} < \gamma_n$. So, $\lim_{n\to\infty} \beta_{c_n,d_n}$ exists and satisfies

$$1 \le \lim_{n\to\infty} \beta_{c_n,d_n} \le \lim_{n\to\infty} \gamma_n = 1,$$

which ends the proof of the theorem. □

The next theorem characterizes completely the dynamics of the c, d-minimal degree one circle maps, and certain of its lifted orbits with rotation number c and d when these numbers are both rational and $\left\lfloor \frac{1}{c} \right\rfloor \neq \left\lfloor \frac{1}{d} \right\rfloor$. In this sense it is a great improvement over Theorem 5.2 and the theory developed in [8, 9] to prove that result. Theorem 5.5 is partially proved in [8, 9], but here we introduce the necessary new tools to prove it. To illustrate

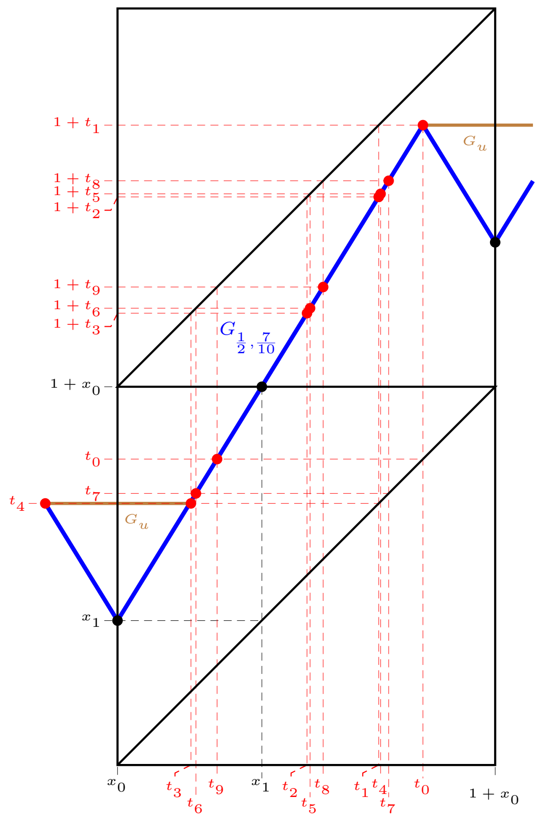


FIGURE 5. The graphs of the maps $G_{\frac{1}{2},\frac{7}{10}}$ in blue and $G_u = \left(G_{\frac{1}{2},\frac{7}{10}}\right)_u$ in brown (note that G_u coincides with $G_{\frac{1}{2},\frac{7}{10}}$ on the interval $[t_3, t_0]$ and hence it is left in blue color for clarity). The orbit of $0 = x_0$ is labelled so that $G_{\frac{1}{2},\frac{7}{10}}(x_0) = x_1$ and $G_{\frac{1}{2},\frac{7}{10}}(x_1) = x_0 + 1$ (in this case the (*temporal*) labelling leaves the orbit sorted in the real line) and the orbit of $u_{\frac{1}{2},\frac{7}{10}} = t_0$ is labelled so that $G_{\frac{1}{2},\frac{7}{10}}(t_i) = 1 + t_{i+1}$ for $i \in \{0, 1, 2, 4, 5, 7, 8\}$, $G_{\frac{1}{2},\frac{7}{10}}(t_i) = t_{i+1}$ for $i \in \{3, 6\}$ and $G_{\frac{1}{2},\frac{7}{10}}(t_9) = t_0$. In this case the (*temporal*) labelling does not sort the orbit; the sorted version of this orbit is obtained by setting $y_9 := t_0$ and

$$y_{k+i} := t_{3i+3} \text{ for } k = 0, 3, 6 \text{ and } i = 0, 1, 2.$$

Theorem 5.5, see Figure 5 where a specific example with $c = \frac{1}{2}$ and $d = \frac{7}{10}$ is shown.

Theorem 5.5. *Let $[c, d]$ be a non-degenerate interval contained in the interval $(0, 1)$ such that $c = \frac{p}{q}$ and $d = \frac{r}{s}$ with $p, q, r, s \in \mathbb{N}$ and $(p, q) = (r, s) = 1$. Then, the following statements hold:*

(a) $g_{c,d}$ is transitive (and, hence, totally transitive by Corollary 2.5).

(b) $Q = \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))\big)$ is a twist periodic orbit of $G_{c,d}$ of period q and rotation number c with the property that $Q \subset \bigcup_{k\in\mathbb{Z}} \Big[k, k + \frac{1}{\beta_{c,d}}\Big)$. Moreover, Q is the unique lifted periodic orbit of $G_{c,d}$ with rotation number c.

(c) $P = \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))\big)$ is a twist periodic orbit of $G_{c,d}$ of period s and rotation number d such that $P \subset \bigcup_{k\in\mathbb{Z}} \Big(k + \frac{\beta_{c,d}-1}{2\beta_{c,d}}, k + u_{c,d}\Big]$. Moreover, P is the unique lifted periodic orbit of $G_{c,d}$ with rotation number d.

(d) Set

$$Q = \{\dots x_{-1}, x_0 = 0, x_1, x_2, \dots, x_{q-1}, x_q, x_{q+1}, \dots\}, \quad \text{and}$$
$$P = \{\dots y_{-1}, y_0, y_1, y_2, \dots, y_{s-1} = u_{c,d}, y_s, y_{s+1}, \dots\}$$

with $x_i < x_j$ and $y_i < y_j$ if and only if $i < j$. Then, for every $i, j \in \mathbb{Z}$,

$$x_{jq+i} = x_i + j, \quad y_{js+i} = y_i + j,$$
$$G_{c,d}(x_i) = x_{i+p} \quad \text{and} \quad G_{c,d}(y_i) = y_{i+r}.$$

Moreover, we also have $0 = x_0 < y_0$, $x_{q-1} < y_{s-1} = u_{c,d}$, and the points $y_{s-1} = u_{c,d} < 1 = x_q$ are consecutive in $Q \cup P$.

(e) Assume that $\ell := \left\lfloor \frac{s}{r} \right\rfloor \neq \left\lfloor \frac{q}{p} \right\rfloor$, and set $t := s - \ell r$, $\widetilde{n} := \Big(\left\lfloor \frac{q}{p} \right\rfloor - \ell\Big)p$ and $n := q - \ell p = q - \left\lfloor \frac{q}{p} \right\rfloor p + \widetilde{n}$. Then we have

(e.1) $1 \leq \ell < \left\lfloor \frac{q}{p} \right\rfloor$.

(e.2) Either $t = 0$, $r = 1$ and $s = \ell > 1$, or $r \geq 2$ and $t \in \{1, 2, \dots, r-1\}$.

(e.3) Either $n = \widetilde{n} = p = 1$ and $q = \ell + 1$, or $n = \widetilde{n} \geq 2$, $p = 1$ and $q = \ell + n$, or $2 \leq p \leq \widetilde{n} < n \leq \widetilde{n} + p - 1$.

Moreover, the points

$$(Q \cup P) \cap [0, 1] = \{x_0, x_1, \dots, x_q\} \cup \{y_0, y_1, \dots, y_{s-1}\}$$

are ordered as follows:

When $t = 0$:

$$\begin{array}{ccccccccccc}
0 = x_0 < x_1 < \cdots < x_{n-1} < & x_n & < \cdots < & x_{n+p-1} & < & y_0 & < & \\
& x_{n+p} & < \cdots < & x_{n+2p-1} & < & y_1 & < & \\
& \vdots & < \cdots < & \vdots & < & \vdots & < & \\
& x_{n+(\ell-2)p} & < \cdots < & x_{n+(\ell-1)p-1} & < & y_{\ell-2} & < & \\
& x_{n+(\ell-1)p} & < \cdots < & x_{n+\ell p-1} & < & y_{\ell-1} & < & x_q. \\
& \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
& x_{q-p} & & x_{q-1} & & y_{s-1} & & 1 \\
& & & & & \shortparallel & & \\
& & & & & u_{c,d} & &
\end{array}$$

When $t \in \{1, 2, \dots, r-1\}$:

$$\begin{array}{llllllllll}
0 = x_0 < x_1 < \cdots < x_{n-1} < y_0 < y_1 < \cdots < y_{t-1} & & & & & & & & & < \\
x_n & < \cdots < & x_{n+p-1} & < & y_t & < \cdots < & y_{t+r-1} & < & & \\
x_{n+p} & < \cdots < & x_{n+2p-1} & < & y_{t+r} & < \cdots < & y_{t+2r-1} & < & & \\
\vdots & < \cdots < & \vdots & < & \vdots & < \cdots < & \vdots & < & & \\
x_{n+(\ell-2)p} & < \cdots < & x_{n+(\ell-1)p-1} & < & y_{t+(\ell-2)r} & < \cdots < & y_{t+(\ell-1)r-1} & < & & \\
x_{n+(\ell-1)p} & < \cdots < & x_{n+\ell p-1} & < & y_{t+(\ell-1)r} & < \cdots < & y_{t+\ell r-1} & < & x_q. & \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel & \\
x_{q-p} & & x_{q-1} & & y_{s-r} & & y_{s-1} & & 1 & \\
 & & & & & & \parallel & & & \\
 & & & & & & u_{c,d} & & &
\end{array}$$

(f) $Q \cup P$ *is a short Markov partition for* $G_{c,d}$*, and* $g_{c,d}$ *is Markov with respect to the Markov partition* $\mathsf{e}(Q \cup P) = \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$.

(g) $g_{c,d}$ *is* $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$*-expanding.*

By Theorem 2.4 and Theorem 5.2 we immediately get the following corollary of Theorem 5.5(b–c):

Corollary 5.6. *Let* $[c, d]$ *be a non-degenerate interval contained in the interval* $(0, 1)$ *such that* $c, d \in \mathbb{Q}$*. Let* q *and* s *be the denominators of* c *and* d *when written in irreducible form, respectively. Then,* $\mathsf{Rot}\big(G_{c,d}\big) = [c, d]$ *and*

$$\mathsf{Per}(g_{c,d}) = \{q, s\} \cup M\big(c, d\big).$$

The next result studies some properties of the Markov graph of the c, d-minimal degree one circle maps in the setting described by Theorem 5.5.

Proposition 5.7. *Let* $\big[c, d\big]$ *be a non-degenerate interval contained in* $(0, 1)$ *such that* $c = \frac{p}{q}$ *and* $d = \frac{r}{s}$ *with* $p, q, r, s \in \mathbb{N}$ *and* $(p, q) = (r, s) = 1$*. Denote by* $Q = \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))\big)$ *(respectively* $P = \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))\big)$*) the unique (twist) periodic orbit of* $G_{c,d}$ *of period* q *(respectively* s*) and rotation number* c *(respectively* d*). Then, every simple loop in the Markov graph of* $g_{c,d}$ *with respect to the partition* $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$ *has a periodic orbit associated to it. Furthermore, let*

$$\alpha = \mathsf{e}\big([a_0, b_0]\big) \longrightarrow \mathsf{e}\big([a_1, b_1]\big) \longrightarrow \cdots \longrightarrow \\ \mathsf{e}\big([a_{j-2}, b_{j-2}]\big) \longrightarrow \mathsf{e}\big([a_{j-1}, b_{j-1}]\big) \longrightarrow \mathsf{e}\big([a_0, b_0]\big)$$

be a loop in the Markov graph of $g_{c,d}$ *with respect to*

$$\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d})).$$

Then α *is associated to* $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$ *if and only if* $j = s$,

$$\big\{\mathsf{e}(b_0), \mathsf{e}(b_1), \dots, \mathsf{e}(b_{s-2}), \mathsf{e}(b_{s-1})\big\} = \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d})),$$

and $g_{c,d}\big(\mathsf{e}(b_i)\big) = \mathsf{e}\big(b_{i+1 \pmod s}\big)$ *for* $i = 0, 1, \dots, s-1$*. Similarly,* α *is associated to* $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))$ *if and only if* $j = q$,

$$\big\{\mathsf{e}(a_0), \mathsf{e}(a_1), \dots, \mathsf{e}(a_{s-2}), \mathsf{e}(a_{s-1})\big\} = \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)),$$

and $g_{c,d}\big(\mathsf{e}(a_i)\big) = \mathsf{e}\big(a_{i+1 \pmod q}\big)$ *for* $i = 0, 1, \dots, q-1$.

Proof. For simplicity we denote the set $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))$ by $\widetilde{Q}$ and $\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$ by $\widetilde{P}$. Let

$$\gamma = I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_{m-1} \longrightarrow I_0$$

be a simple loop in the Markov graph of $g_{c,d}$ with respect to $\widetilde{Q} \cup \widetilde{P}$. By Lemma 2.8, there exists a fixed point $x \in I_0$ of $g_{c,d}^m$ such that $g_{c,d}^i(x) \in I_i$ for $i = 1, 2, \ldots, m-1$.

If the period of x is m then γ and $\mathsf{Orb}_{g_{c,d}}(x)$ are associated. So, let $a \neq m$ be the period of x. Clearly, $m = k \cdot a$ with $k > 1$ and $g_{c,d}^i(x) = g_{c,d}^{i+a}(x) = \cdots = g_{c,d}^{i+(k-1)a}(x)$ for $i = 0, 1, \ldots, a-1$.

Assume that $x \notin \widetilde{Q} \cup \widetilde{P}$. Then, $\mathsf{Orb}_{g_{c,d}}(x) \subset \bigcup_{i=0}^{m-1} \mathsf{Int}(I_i)$, and hence, $I_i = I_{i+a} = \cdots = I_{i+(k-1)a}$ for $i = 0, 1, \ldots, a-1$. This implies that γ is a repetition of the loop $I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_{a-1} \longrightarrow I_0$, which contradicts its simplicity.

So, $x \in \mathsf{Orb}_{g_{c,d}}(x) \subset \widetilde{Q} \cup \widetilde{P}$. We assume that $I_0 = \mathsf{e}\big([\widetilde{x}, z_0]\big)$ with $\mathsf{e}(\widetilde{x}) = x$ (that is, x is the projection of the left endpoints of the intervals from $\mathsf{e}^{-1}(I_0)$). The case $I_0 = \mathsf{e}\big([z_0, \widetilde{x}]\big)$ follows analogously.

In the rest of the proof we will use the notation from Theorem 5.5(d), and Theorem 5.5(e).

We claim that $I_i \neq \mathsf{e}\big([y_{s-1}, x_q = 1]\big)$ for every $i \in \{0, 1, \ldots, m-1\}$. On the contrary, without loss of generality we may assume that $I_0 = \mathsf{e}\big([y_{s-1}, x_q]\big)$ and $\widetilde{x} = y_{s-1}$. Then, $G_{c,d}\big|_{[y_{s-1}, x_q]}$ is strictly decreasing by Theorem 5.5(d). Thus, I_1 is a basic interval contained in $g_{c,d}(I_0) = \mathsf{e}\Big(\big[G_{c,d}(z_0), G_{c,d}(y_{s-1})\big]\Big)$ that contains the point $g_{c,d}(x) = g_{c,d}\big(\mathsf{e}(y_{s-1})\big) = \mathsf{e}\big(G_{c,d}(y_{s-1})\big)$. Consequently, since $\mathsf{e}\big(G_{c,d}(y_{s-1})\big) \in \widetilde{P}$, $\mathsf{e}(x_q) \in \widetilde{Q}$ and $\widetilde{P} \cap \widetilde{Q} = \emptyset$, we have

$$I_1 = \mathsf{e}\Big(\big[z_1, G_{c,d}(y_{s-1})\big]\Big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big).$$

Therefore, by Theorem 5.5(e), $G_{c,d}\big|_{[z_1, G_{c,d}(y_{s-1})]}$ is increasing and

$$I_2 \subset \mathsf{e}\Big(\big[G_{c,d}(z_1), G_{c,d}^2(y_{s-1})\big]\Big)$$

is a basic interval such that $I_2 \ni g_{c,d}^2(x) = g_{c,d}^2\big(\mathsf{e}(y_{s-1})\big) = \mathsf{e}\big(G_{c,d}^2(y_{s-1})\big)$. So, as before, $I_2 = \mathsf{e}\Big(\big[z_2, G_{c,d}^2(y_{s-1})\big]\Big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big)$. By iterating this argument $m-1$ times we get $I_{m-1} = \mathsf{e}\big([z_{m-1}, G_{c,d}^{m-1}(y_{s-1})]\big)$ and then, I_0 must be of the form

$$\mathsf{e}\big([z_m, G_{c,d}^m(y_{s-1})]\big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big) = I_0;$$

a contradiction. This ends the proof of the claim.

By the claim, $I_0 \neq \mathsf{e}\big([y_{s-1}, x_q]\big)$. By Theorem 5.5(e), $G_{c,d}\big|_{[\widetilde{x}, z_0]}$ is strictly increasing, and $I_1 \subset g_{c,d}(I_0) = \mathsf{e}\Big(\big[G_{c,d}(\widetilde{x}), G_{c,d}(z_0)\big]\Big)$ is a basic interval that contains the point $g_{c,d}(x) = g_{c,d}\big(\mathsf{e}(\widetilde{x})\big) = \mathsf{e}\big(G_{c,d}(\widetilde{x})\big)$. So, we can write

$$I_1 = \mathsf{e}\Big(\big[G_{c,d}(\widetilde{x}), z_1\big]\Big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big).$$

By iterating this argument $m-2$ times we get $I_i = \mathsf{e}\Big([G^i_{c,d}(\widetilde{x}), z_i]\Big)$ for $i = 0, 1, \dots, m-1$. On the other hand, we already know that

$$\mathsf{e}\big(G^i_{c,d}(\widetilde{x})\big) = g^i_{c,d}(x) = g^{i+a}_{c,d}(x) = \mathsf{e}\big(G^{i+a}_{c,d}(\widetilde{x})\big) = \cdots = g^{i+(k-1)a}_{c,d}(x) = \mathsf{e}\big(G^{i+(k-1)a}_{c,d}(\widetilde{x})\big)$$

for $i = 0, 1, \dots, a-1$. Hence, $I_i = I_{i+a} = \cdots = I_{i+(k-1)a}$ for $i = 0, 1, \dots, a-1$, because $I_0, I_1, \dots, I_{m-1}$ are basic intervals. So, as before, this contradicts the simplicity of γ.

Now we will prove the second statement of the proposition. That is, that α is associated to $\widetilde{P}$ if and only if $j = s$, $\big\{\mathsf{e}(b_0), \mathsf{e}(b_1), \dots, \mathsf{e}(b_{s-2}), \mathsf{e}(b_{s-1})\big\} = \widetilde{P}$, and $g_{c,d}\big(\mathsf{e}(b_i)\big) = \mathsf{e}\big(b_{i+1 \pmod s}\big)$ for $i = 0, 1, \dots, s-1$. The third statement of the proposition (about $\widetilde{Q}$) follows similarly.

The "if" statement is a direct consequence of the definition of associated loop. Assume now that α is associated to $\widetilde{P}$. Then, again by the definition of associated loop, $j = s$ and there exists points $z_i \in P \cap [a_i, b_i]$ for $i = 0, 1, \dots, s-1$ such that $\widetilde{P} = \big\{\mathsf{e}(z_0), \mathsf{e}(z_1), \dots, \mathsf{e}(z_{s-2}), \mathsf{e}(z_{s-1})\big\}$, and $g_{c,d}\big(\mathsf{e}(z_i)\big) = \mathsf{e}\big(z_{i+1 \pmod s}\big)$ for $i = 0, 1, \dots, s-1$. We have to show that $z_i = b_i$ for $i = 0, 1, \dots, s-1$.

Assume that there exists $i \in \{0, 1, \dots, s-1\}$ such that $\mathsf{e}\big([a_i, b_i]\big) = \mathsf{e}\big([y_{s-1}, x_q = 1]\big)$. Without loos of generality we may assume that $i = 0$. That is, $\mathsf{e}\big([a_0, b_0]\big) = \mathsf{e}\big([y_{s-1}, x_q]\big)$. Then, since $\mathsf{e}\big([a_0, b_0]\big)$ is a $\widetilde{Q} \cup \widetilde{P}$-basic interval,

$$z_0 \in P \cap [a_0, b_0] \subset (Q \cup P) \cap [a_0, b_0] \subset \{a_0, b_0\}.$$

So, $z_0 = a_0 = y_{s-1} + \widetilde{k}$ with $\widetilde{k} \in \mathbb{Z}$. Moreover, as in the proof of the first statement of the proposition, $G_{c,d}\big|_{[a_0, b_0]}$ is strictly decreasing, and

$$\mathsf{e}(z_1) = g_{c,d}\big(\mathsf{e}(z_0)\big) = g_{c,d}\big(\mathsf{e}(y_{s-1})\big) = \mathsf{e}(b_1) \in \widetilde{P}.$$

Consequently, $\mathsf{e}\big([a_1, b_1]\big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big)$, $G_{c,d}\big|_{[a_1, b_1]}$ is strictly increasing, and

$$\mathsf{e}(z_2) = g_{c,d}\big(\mathsf{e}(z_1)\big) = g_{c,d}\big(\mathsf{e}(b_1)\big) = \mathsf{e}(b_2) \in \widetilde{P}.$$

By iterating these arguments $s-2$ times we obtain $\mathsf{e}(z_{s-1}) = \mathsf{e}(b_{s-1}) \in \widetilde{P}$, and

$$\mathsf{e}(a_0) = \mathsf{e}(z_0) = g_{c,d}\big(\mathsf{e}(z_{s-1})\big) = g_{c,d}\big(\mathsf{e}(b_{s-1})\big) = \mathsf{e}(b_0);$$

a contradiction.

Observe that $\mathsf{e}(y_{s-1}) \in \widetilde{P} = \big\{\mathsf{e}(z_0), \mathsf{e}(z_1), \dots, \mathsf{e}(z_{s-2}), \mathsf{e}(z_{s-1})\big\}$. Hence, there exists $i \in \{0, 1, \dots, s-1\}$ such that $\mathsf{e}(y_{s-1}) = \mathsf{e}(z_i)$. Again without loos of generality we may assume that $i = 0$. That is, there exists $\widetilde{k} \in \mathbb{Z}$, such that $y_{s-1} + \widetilde{k} = z_0 \in \{a_0, b_0\}$. Since $\mathsf{e}\big([a_0, b_0]\big) \neq \mathsf{e}\big([y_{s-1}, x_q]\big)$ we must have $y_{s-1} + \widetilde{k} = z_0 = b_0$. Then, as before, we can inductively prove that $G_{c,d}\big|_{[a_i, b_i]}$ is strictly increasing, and

$$\mathsf{e}(z_{i+1}) = g_{c,d}\big(\mathsf{e}(z_i)\big) = g_{c,d}\big(b_i)\big) = \mathsf{e}(b_{i+1}) \in \widetilde{P}$$

for $i = 0, 1, \dots, s-2$. This ends the proof of the proposition. □

Proof of Theorem 5.5. We start by proving statement (c) by using Theorem 5.2 and Lemmas 4.8.5, 4.8.6, 4.8.3 and 4.8.11, and Remark 4.8.4 of [9]. The proof of statement (b) will be omitted since it is analogous (by using Lemma 4.8.7 and its proof, and Remark 4.8.12 of [9] instead of Lemmas 4.8.5, 4.8.6, 4.8.3 and 4.8.11, and Remark 4.8.4 of [9]).

From the proof of Theorem 4.8.1 and Lemmas 4.8.5, 4.8.6, 4.8.3 and 4.8.11 of [9] it follows that $P = \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))\big)$ is a lifted periodic orbit of $G_{c,d}$ of period s and rotation number d with the property that $P \subset \bigcup_{k\in\mathbb{Z}} \left[k + \frac{\beta_{c,d}-1}{2\beta_{c,d}}, k + u_{c,d}\right]$. On the other hand,

$$G_{c,d}\left(\tfrac{\beta_{c,d}-1}{2\beta_{c,d}}\right) = \tfrac{\beta_{c,d}-1}{2} + b_{c,d} = \left(\tfrac{\beta_{c,d}+1}{2} + b_{c,d}\right) - 1 = G_{c,d}\left(\tfrac{\beta_{c,d}+1}{2\beta_{c,d}}\right) - 1 = G_{c,d}\big(u_{c,d} - 1\big)$$

Therefore, since $u_{c,d} - 1 \in P$, $u_{c,d} - 1 < 0 < \frac{\beta_{c,d}-1}{2\beta_{c,d}}$, and $G_{c,d}\big|_P$ is bijective, $\frac{\beta_{c,d}-1}{2\beta_{c,d}} \notin P$. Thus, since P is a lifted orbit, $k + \frac{\beta_{c,d}-1}{2\beta_{c,d}} \notin P$ for every $k \in \mathbb{Z}$, and hence $P \subset \bigcup_{k\in\mathbb{Z}} \left(k + \frac{\beta_{c,d}-1}{2\beta_{c,d}}, k + u_{c,d}\right]$.

By [9, Remark 4.8.4] it follows that P is also a lifted orbit of $\big(G_{c,d}\big)_u$ and thus, P is a twist orbit of $G_{c,d}$ because $\big(G_{c,d}\big)_u$ is non-decreasing.

If a point $z \in \mathbb{R}$ belongs to a lifted periodic orbit of $G_{c,d}$ with rotation number d then, by Remark 2.2, $G_{c,d}^{\ell s}(z) = z + \ell r$ for some $\ell \in \mathbb{N}$ (in particular this holds for every $z \in P$). To show that P is the unique lifted periodic orbit of $G_{c,d}$ with rotation number d we will prove that, for every $z \in \mathbb{R} \setminus P$ and every $\ell \in \mathbb{N}$, $G_{c,d}^{\ell s}(z) < z + \ell r$.

To prove this we will use the following

Claim 1. *$G_{c,d}^k(x) \le G_{c,d}^k(y)$ for every $y \in P$, $x \in \mathbb{R}$, $x \le y$ and $k \in \mathbb{Z}^+$.*

We will prove the claim by induction. For $k = 0$ the statement holds by assumption. So assume that we already have proved that $G_{c,d}^k(x) \le G_{c,d}^k(y)$ for some $k \ge 0$. Hence, since $G_{c,d}^k(y) \in P$, $G_{c,d} \le \big(G_{c,d}\big)_u$ and $\big(G_{c,d}\big)_u$ is non-decreasing,

$$G_{c,d}^{k+1}(x) = G_{c,d}\big(G_{c,d}^k(x)\big) \le \big(G_{c,d}\big)_u\big(G_{c,d}^k(x)\big) \le \big(G_{c,d}\big)_u\big(G_{c,d}^k(y)\big) = G_{c,d}^{k+1}(y)$$

by [9, Remark 4.8.4 and Lemma 4.8.5]. So, the claim holds.

Assume that $z \in \mathbb{R} \setminus P$ and let $t, v \in P$ be such that $(t, v) \cap P = \emptyset$ and $t < z < v$. By Claim 1,

$$G_{c,d}^k(z) \le G_{c,d}^k(v) \quad \text{for every} \quad k \in \mathbb{Z}^+.$$

We will also need the following

Claim 2. *Assume that $G_{c,d}^k(t) < G_{c,d}^k(z)$ for every $k = 0, 1, \dots, ms$, for some $m \in \mathbb{N}$. Then, $G_{c,d}^{\ell s}(z) < z + \ell r$ for every $\ell \in \{1, 2, \dots, m\}$.*

Since P is a twist orbit ($G_{c,d}\big|_P$ is bijective and strictly increasing) and $(t,v) \cap P = \emptyset$ it follows that, for every $k \in \mathbb{Z}^+$, $G^k_{c,d}(t), G^k_{c,d}(v) \in P$, and $\big(G^k_{c,d}(t), G^k_{c,d}(v)\big) \cap P = \emptyset$. So, for every $k \in \{0,1,\dots,ms-1\}$ and some $a(k) \in \mathbb{Z}$, either

$$\big[G^k_{c,d}(z), G^k_{c,d}(v)\big] \subset \big[G^k_{c,d}(t), G^k_{c,d}(v)\big] \subset \Big(a(k) + \tfrac{\beta_{c,d}-1}{2\beta_{c,d}}, a(k) + u_{c,d}\Big], \text{ or}$$

$$G^k_{c,d}(t) = u_{c,d} + a(k) - 1 \quad \text{and} \quad G^k_{c,d}(v) \in \Big(a(k) + \tfrac{\beta_{c,d}-1}{2\beta_{c,d}}, a(k) + u_{c,d}\Big].$$

In the second case we have $\big[G^k_{c,d}(z), G^k_{c,d}(v)\big] \subset \big(a(k), a(k)+u_{c,d}\big]$ since otherwise $G^k_{c,d}(z) \in \big[G^k_{c,d}(t), a(k)\big] = \big[u_{c,d} + a(k) - 1, a(k)\big]$ and, by the definition of the map $G_{c,d}$,

$$G^{k+1}_{c,d}(z) \le G_{c,d}\big(u_{c,d} + a(k) - 1\big) = G^{k+1}_{c,d}(t);$$

a contradiction. Thus, in both cases the map $G_{c,d}\Big|_{\big[G^k_{c,d}(z), G^k_{c,d}(v)\big]}$ is affine with slope $\beta_{c,d} > 1$. Consequently, for every $\ell \in \{1,2,\dots,m\}$,

$$G^{\ell s}_{c,d}\big|_{[z,v]} = G_{c,d}\Big|_{\big[G^{\ell s-1}_{c,d}(z), G^{\ell s-1}_{c,d}(v)\big]} \circ G_{c,d}\Big|_{\big[G^{\ell s-2}_{c,d}(z), G^{\ell s-2}_{c,d}(v)\big]} \circ \cdots \circ G_{c,d}\Big|_{\big[G^{2}_{c,d}(z), G^{2}_{c,d}(v)\big]} \circ G_{c,d}\Big|_{\big[G_{c,d}(z), G_{c,d}(v)\big]} \circ G_{c,d}\big|_{[z,v]}$$

is affine with slope $\beta^{\ell s}_{c,d} > 1$. So,

$$G^{\ell s}_{c,d}(z) = \beta^{\ell s}_{c,d}(z - v) + G^{\ell s}_{c,d}(v) < (z - v) + v + \ell r = z + \ell r.$$

This ends the proof of Claim 2.

Now we are ready to prove that $G^{\ell s}_{c,d}(z) < z + \ell r$ for every $\ell \in \mathbb{N}$. When $G^k_{c,d}(t) < G^k_{c,d}(z)$ for every $k \in \mathbb{Z}^+$ this follows from Claim 2. So, we may assume that there exist $m \in \mathbb{Z}^+$ and $n \in \{ms+1, ms+2, \dots, (m+1)s\}$ such that $G^n_{c,d}(z) \le G^n_{c,d}(t)$ and $G^k_{c,d}(t) < G^k_{c,d}(z)$ for $k = 0,1,\dots,n-1$. Then, again by Claim 2, $G^{\ell s}_{c,d}(z) < z + \ell r$ for every $\ell \in \{1,2,\dots,m\}$, and by Claim 1 with $x = G^n_{c,d}(z)$ and $y = G^n_{c,d}(t)$,

$$G^{\ell s}_{c,d}(z) \le G^{\ell s}_{c,d}(t) = t + \ell r < z + \ell r$$

for every $\ell \in \mathbb{N}$ with $\ell > m$. This ends the proof of statement (c).

The first part of statement (d) follows from (b), (c) and Remarks 2.2 and 2.3. Now we prove that $0 = x_0 < y_0$ and $x_{q-1} < y_{s-1} = u_{c,d}$. By (c) and Remark 2.2 we know that the set

$$P \cap [0,1) = P \cap \Big(\tfrac{\beta_{c,d}-1}{2\beta_{c,d}}, u_{c,d}\Big]$$

has cardinality s. Consequently,

$$x_0 = 0 < \tfrac{\beta_{c,d}-1}{2\beta_{c,d}} < y_0 < y_1 < \cdots < y_{s-1} = u_{c,d}$$

because $\beta_{c,d} > 1$. In a similar way, by (b) and Remark 2.2,

$$Q \cap [0,1) = Q \cap \Big[0, \tfrac{1}{\beta_{c,d}}\Big)$$

has cardinality q, and

$$x_0 = 0 < x_1 < \cdots < x_{q-1} < \tfrac{1}{\beta_{c,d}} < \tfrac{\beta_{c,d}+1}{2\beta_{c,d}} = u_{c,d} = y_{s-1}.$$

Observe also that by the part of (d) already proven,

$$x_{q-1} < y_{s-1} = u_{c,d} < 1 = x_q = 1 + x_0 < 1 + y_0 = y_s.$$

So, the points $y_{s-1} = u_{c,d} < 1 = x_q$ are consecutive in $Q \cup P$.

Now we prove statement (f). From statements (b-d) it follows that $Q \cup P$ is a Markov partition for $G_{c,d}$ and every interval of the form $[k + u_{c,d}, k+1]$ with $k \in \mathbb{Z}$ is a $Q \cup P$-basic interval. Next we prove that the partition $Q \cup P$ is short. We have to show that if $I \subset [0,1]$ is a $Q \cup P$-basic interval then $\mathsf{len}\bigl(G_{c,d}(I)\bigr) < 1$.

To this end we need to derive bounds for $\beta_{c,d}$ and $b_{c,d}$. Notice that

$$G_{c,d}(x) < x + 1 \quad \text{for every} \quad x \in \mathbb{R}$$

since, otherwise, there exists $a \in \mathsf{Rot}\bigl(G_{c,d}\bigr)$ with $a \geq 1$, and hence $d \geq 1$ by Theorem 5.2. Thus, $b_{c,d} = G_{c,d}(0) < 1$ and

$$\beta_{c,d} u_{c,d} + b_{c,d} = G_{c,d}\bigl(u_{c,d}\bigr) < u_{c,d} + 1 \iff \bigl(\beta_{c,d} - 1\bigr) u_{c,d} + b_{c,d} < 1.$$

On the other hand, $b_{c,d} > 0$ because $c > 0$ and $\beta_{c,d} > 1$. Consequently,

$$\frac{\beta_{c,d}^2 - 1}{2\beta_{c,d}} = \bigl(\beta_{c,d} - 1\bigr)\frac{\beta_{c,d} + 1}{2\beta_{c,d}} < \bigl(\beta_{c,d} - 1\bigr) u_{c,d} + b_{c,d} < 1,$$

which is equivalent to $\beta_{c,d} < 1 + \sqrt{2}$. In short, the assumption $0 < c < d < 1$ implies that $0 < b_{c,d} < 1$ and $1 < \beta_{c,d} < 1 + \sqrt{2}$.

To prove that $\mathsf{len}\bigl(G_{c,d}(I)\bigr) < 1$ for every $I \in \mathcal{B}(Q \cup P)$, $I \subset [0,1]$ we start with $I = \bigl[u_{c,d}, 1\bigr]$. Since $G_{c,d}\big|_{[u_{c,d},1]}$ is decreasing,

$$\begin{aligned}\mathsf{len}\left(G_{c,d}\left(\bigl[u_{c,d}, 1\bigr]\right)\right) &= G_{c,d}\bigl(u_{c,d}\bigr) - G_{c,d}(1) = \\ G_{c,d}\left(\tfrac{\beta_{c,d}+1}{2\beta_{c,d}}\right) - G_{c,d}(1) &= \frac{\beta_{c,d}+1}{2} + b_{c,d} - \bigl(1 + b_{c,d}\bigr) = \\ &\frac{\beta_{c,d} - 1}{2} < \frac{\sqrt{2}}{2} < 1.\end{aligned}$$

For the other basic intervals we need to clarify the placement of points in the Markov partition $Q \cup P$. Recall that $1 < \beta_{c,d} < 1 + \sqrt{2}$ and $b_{c,d} > 0$. So, we have

$$0 < \frac{\beta_{c,d} - 1}{2\beta_{c,d}} < \frac{1}{\beta_{c,d}} < \frac{\beta_{c,d} + 1}{2\beta_{c,d}} = u_{c,d} < 1.$$

Moreover, we claim that $P \cap \left(\frac{\beta_{c,d}-1}{2\beta_{c,d}}, \frac{1}{\beta_{c,d}}\right) \neq \emptyset$ (in fact, symmetrically, we also have $Q \cap \left(\frac{\beta_{c,d}-1}{2\beta_{c,d}}, \frac{1}{\beta_{c,d}}\right) \neq \emptyset$). Otherwise,

$$P \cap [0,1) = P \cap \left(\tfrac{\beta_{c,d}-1}{2\beta_{c,d}}, u_{c,d}\right] \subset \left[\tfrac{1}{\beta_{c,d}}, u_{c,d}\right].$$

Hence, for every $x \in P \cap [0,1)$,

$$G_{c,d}(x) \geq G_{c,d}\left(\tfrac{1}{\beta_{c,d}}\right) = 1 + b_{c,d} > 1.$$

Thus, by [9, Lemma 3.7.3], $d \geq \rho_{G_{c,d}}(P) \geq 1$; a contradiction. This ends the proof of the claim.

By the above claim there exists a point $z \in \bigl(Q \cup P\bigr) \cap \left(\frac{\beta_{c,d}-1}{2\beta_{c,d}}, \frac{1}{\beta_{c,d}}\right)$. Consequently, if I is a $Q \cup P$-basic interval such that $I \subset \bigl[0, u_{c,d}\bigr]$, then either $I \subset [0, z] \subset \left[0, \frac{1}{\beta_{c,d}}\right)$ or $I \subset \bigl[z, u_{c,d}\bigr] \subset \left(\frac{\beta_{c,d}-1}{2\beta_{c,d}}, u_{c,d}\right]$. In the first case we have

$$\mathsf{len}\bigl(G_{c,d}(I)\bigr) < G_{c,d}\left(\tfrac{1}{\beta_{c,d}}\right) - G_{c,d}(0) = 1 + b_{c,d} - b_{c,d} = 1,$$

whereas in the second one,

$$\mathsf{len}\bigl(G_{c,d}(I)\bigr) < G_{c,d}\bigl(u_{c,d}\bigr) - G_{c,d}\left(\tfrac{\beta_{c,d}-1}{2\beta_{c,d}}\right) = \frac{\beta_{c,d}+1}{2} + b_{c,d} - \frac{\beta_{c,d}-1}{2} - b_{c,d} = 1.$$

This proves that the partition $Q \cup P$ is short.

Then, in view of Proposition 2.11, the map $g_{c,d}$ is Markov with respect to the Markov partition $W := \mathsf{e}(Q \cup P) = \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$. Moreover, $g_{c,d}$ is W-expanding because $G_{c,d}\big|_{[0,1]}$ is piecewise affine in two pieces with the absolute value of the slope equals to $\beta_{c,d} > 1$.

To prove statement (a) we will use Theorem 2.12. Statement (f) and Proposition 2.11 tell us that the transition matrix of the Markov graph of $g_{c,d}$ with respect to W coincides with the transition matrix of the Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$.

Let K be the $Q \cup P$-basic interval whose right endpoint is $G_{c,d}\bigl(u_{c,d}\bigr)$. Since the map $G_{c,d}$ has a local maximum at $u_{c,d}$ (see Figure 5), K is $G_{c,d}$-covered by two $Q \cup P$-basic intervals (namely the two $Q \cup P$-basic intervals which have $u_{c,d}$ as endpoint). This means that the column corresponding to $[\![K]\!]$ in the transition matrix of the Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$ has at least two entries different from zero. So, the Markov matrix of $g_{c,d}$ with respect to W is not a permutation matrix.

Next we will show that the Markov matrix of $g_{c,d}$ with respect to W is irreducible. In the spirit of Proposition 2.11, this amounts showing that for every pair of vertices $[\![I]\!], [\![J]\!]$ in the Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$, there exists a path from $[\![I]\!]$ to $[\![J]\!]$ (equivalently, there exists a representative $J+k$ of $[\![J]\!]$ such that $G^{\ell}_{c,d}(I) \supset J+k$ for some $\ell \in \mathbb{N}$).

Since $c < d$, there exists $\ell \in \mathbb{N}$ such that $\ell(qr - ps) \geq 3$.

Let I be a $Q \cup P$-basic interval with endpoints $x \in Q$ and $y \in P$ (i.e., either $I = [x, y]$ or $I = [y, x]$). We know that $\mathbb{Z} \subset Q$ and $\mathbb{Z} \cap P = \emptyset$. Then, since x and y are consecutive in $Q \cup P$, $\lfloor y \rfloor \in \{\lfloor x \rfloor, \lfloor x \rfloor - 1\}$. In particular, $\lfloor y \rfloor + 1 \geq \lfloor x \rfloor$. Consequently, since Q and P are twist and $G_{c,d} \in \mathcal{L}_1$, by Proposition 3.1.7(c) and Lemma 3.7.4 of [9],

$$G^{\ell qs}_{c,d}(x) = x + \ell sp < \ell sp + \lfloor x \rfloor + 1 < \ell sp + \lfloor x \rfloor + 2 \leq \ell sp + \lfloor y \rfloor + 3 \leq \ell qr + y = G^{\ell qs}_{c,d}(y).$$

So,

$$G_{c,d}^{\ell qs}(I) \supset \left[G_{c,d}^{\ell qs}(x), G_{c,d}^{\ell qs}(y)\right] \supset [k, k+1]$$

with $k = \ell sp + \lfloor x \rfloor + 1 \in \mathbb{Z}$. Since $\mathbb{Z} \subset Q \cup P$, every vertex $[\![J]\!]$ in the Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$ has a representative in the interval $[k, k+1]$. So, there exists a path (of length ℓqs) from $[\![I]\!]$ to every $[\![J]\!]$.

Now assume that $I = [x, y]$ with $x, y \in Q$. Since $\mathbb{Z} \subset Q$ and Q is a lifted periodic orbit, there exists $\ell \in \mathbb{N}$ such that $k := G_{c,d}^{\ell}(y) \in \mathbb{Z}$. Since Q is twist, $G_{c,d}^{\ell}(x) < G_{c,d}^{\ell}(y) = k$ are consecutive in Q. Hence, $G_{c,d}^{\ell}(x) < k - 1 + u_{c,d}$ because $[k - 1 + u_{c,d}, k]$ is a $Q \cup P$-basic interval. Consequently,

$$G_{c,d}^{\ell}(I) \supset \left[G_{c,d}^{\ell}(x), G_{c,d}^{\ell}(y)\right] \supset [k - 1 + u_{c,d}, k],$$

and thus there exists a path (of length ℓ) from $[\![I]\!]$ to $[\![[k-1+u_{c,d}, k]]\!]$. Since $k - 1 + u_{c,d} \in P$ and $k \in Q$, from above it follows that there exists a path from $[\![[k-1+u_{c,d}, k]]\!]$ to every vertex $[\![J]\!]$ in the Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$. By concatenating these two paths we get that there exists a path from $[\![I]\!]$ to every $[\![J]\!]$.

The proof for the case $I = [x, y]$ with $x, y \in P$ goes along the same lines by using that there exists $\ell \in \mathbb{N}$ such that $G_{c,d}^{\ell}(x) = u_{c,d} + k$ with $k \in \mathbb{Z}$.

Putting all together we have shown that the Markov matrix of $g_{c,d}$ with respect to W is irreducible. So, $g_{c,d}$ is transitive by Theorem 2.12.

To end the proof of the theorem we only need to show that statement (e) holds. By assumption we have $0 < c < d < 1$. Therefore, $1 < \frac{s}{r} = \frac{1}{d} < \frac{1}{c} = \frac{q}{p}$ and hence, (e.1) holds because $\ell \neq \left\lfloor \frac{q}{p} \right\rfloor$.

If $r = 1$ we have $\ell = s$ and hence, $t = 0$. Moreover, $\ell > 1$ because $1 > \frac{r}{s} = \frac{1}{\ell}$. If $r > 1$ then $t > 0$ since otherwise $s = \ell r$ which contradicts the fact that r and s are coprime. Thus,

$$0 < t = s - \ell r = s - \left\lfloor \tfrac{s}{r} \right\rfloor r = s \pmod r \in \{1, 2, \ldots, r-1\}.$$

This proves (e.2).

Now we prove (e.3). From (e.1) we have $\widetilde{n} := \left(\left\lfloor \frac{q}{p} \right\rfloor - \ell\right)p \geq p \geq 1$. If $n = p$ we have $q = (\ell+1)p$ which implies $p = 1$ because p and q are coprime. Consequently, $q = \ell + 1$ and $\widetilde{n} := \left(\left\lfloor \frac{q}{p} \right\rfloor - \ell\right)p = q - \ell = 1$.

On the other hand,

$$n = \left(q - \left\lfloor \tfrac{q}{p} \right\rfloor p\right) + \widetilde{n} = q \pmod p + \widetilde{n} \in \left\{\widetilde{n}, \widetilde{n}+1, \ldots, \widetilde{n}+p-1\right\}.$$

In particular, $n \geq \widetilde{n} \geq p$.

If $n = \widetilde{n} > p$ we have $q = \left\lfloor \frac{q}{p} \right\rfloor p$ (that is, q is a multiple of p) and, since q and p are coprime, we must have $p = 1$ and $q = \ell + n$.

The remaining case is $n > \widetilde{n} \geq p$. This implies that $q - \left\lfloor \frac{q}{p} \right\rfloor p \neq 0$ which, in turn, implies that $p \neq 1$. Putting together all the above inequalities we get $2 \leq p \leq \widetilde{n} < n \leq \widetilde{n} + p - 1$.

Now we need to show that the ordering of the points of $\left(Q \cup P\right) \cap [0, 1]$ is the one prescribed in statement (e). To do it we claim that $x_{p-1} < y_0$ and we prove statement (e) assuming that the claim holds.

By statement (d) and the claim we have

$$x_{q-1} < y_{s-1} = u_{c,d} < 1 = x_q < x_{q+1} < \cdots < x_{q+p-1} = x_{p-1} + 1 < y_0 + 1 = y_s.$$

Now we will prove that $x_{q-1} < y_{s-r}$ and $y_{s-r-1} < x_{q-p}$ which, by statement (d), give the last group of the prescribed ordering:

$$y_{t+(\ell-1)r-1} = y_{s-r-1} < x_{q-p} = x_{n+(\ell-1)p} < \cdots < x_{n+\ell p-1} = \\ x_{q-1} < y_{s-r} = y_{t+(\ell-1)r} < \cdots < y_{t+\ell r-1} = y_{s-1} < x_q = 1$$

(by (e.2) we cannot have $t = 0$ and $\ell = 1$ simultaneously).

Suppose that $y_{s-r} < x_{q-1}$. By (d) we have

$$G_{c,d}(x_{q-1}) = x_{q+p-1} < y_s = G_{c,d}(y_{s-r})$$

which contradicts the fact that $G_{c,d}$ is increasing on the interval

$$\big[0, u_{c,d}\big] = [x_0, y_{s-1}] \supset [y_0, y_{s-1}] \supset \big[y_{s-r}, x_{q-1}\big].$$

This shows that $x_{q-1} < y_{s-r}$.

Assume that $x_{q-p} < y_{s-r-1}$. In this case,

$$G_{c,d}(y_{s-r-1}) = y_{s-1} < 1 = x_q = G_{c,d}(x_{q-p}), \text{ and} \\ 0 = x_0 < x_{q-p} < y_{s-r-1} < y_{s-1} = u_{c,d},$$

which again contradicts the fact that $G_{c,d}$ is increasing on $\big[0, u_{c,d}\big]$. So, $y_{s-r-1} < x_{q-p}$.

When $\ell \geq 2$ we have $0 \leq t = s - \ell r \leq s - 2r$, and there is a before last group in the prescribed ordering:

$$y_{t+(\ell-2)r-1} = y_{s-2r-1} < x_{q-2p} = x_{n+(\ell-2)p} < \cdots < x_{n+(\ell-1)p-1} = \\ x_{q-p-1} < y_{s-2r} = y_{t+(\ell-2)r} < \cdots < y_{t+(\ell-1)r-1} = y_{s-r-1} < x_{q-p}$$

(when $t = 0$ and $s = \ell = 2$, $y_{t+(\ell-2)r-1} = y_{-1}$ is not a proper member of the group but, despite of this, it can be used for binding purposes). To fix the ordering of this group, it suffices to show that

$$x_{n+(\ell-1)p-1} = x_{q-p-1} < y_{s-2r} = y_{t+(\ell-2)r}, \text{ and} \\ y_{s-2r-1} = y_{t+(\ell-2)r-1} < x_{q-2p} = x_{n+(\ell-2)p}$$

by using the ordering already obtained for the last group. As before, if $y_{s-2r} < x_{q-p-1}$ we have

$$G_{c,d}(x_{q-p-1}) = x_{q-1} < y_{s-r} = G_{c,d}(y_{s-2r}), \text{ and} \\ 0 = x_0 < y_0 \leq y_{s-2r} < x_{q-p-1} < x_{q-p} < y_{s-1} = u_{c,d},$$

and if $x_{q-2p} < y_{s-2r-1}$ we have

$$G_{c,d}(y_{s-2r-1}) = y_{s-r-1} < x_{q-p} = G_{c,d}(x_{q-2p}), \text{ and} \\ 0 = x_0 < x_p \leq x_{n+(\ell-2)p} = x_{q-2p} < y_{s-2r-1} < y_{s-1} = u_{c,d}.$$

In both cases we obtain again a contradiction with the fact that $G_{c,d}$ is increasing on $\big[0, u_{c,d}\big]$.

In a similar way, by repeating the above arguments backwards, we can fix the ordering of the k-th group for $k = \ell, \ell-1, \ldots, 2, 1:$

$$y_{t+(k-1)r-1} < x_{n+(k-1)p} < \cdots < x_{n+kp-1} < y_{t+(k-1)r} < \cdots < y_{t+kr-1} < x_{n+kp}$$

(again when $k = 1$ and $t = 0$, $y_{t+(k-1)r-1} = y_{-1}$ is not a proper member of the group but again it can be used for binding purposes). So, the ordering of the first group ($k = 1$) reads:

$$y_{t-1} < x_n < \cdots < x_{n+p-1} < y_t < \cdots < y_{t+r-1} < x_{n+p}.$$

When $t = 0$ statement (e) is already proved because, by (d),

$$y_{t-1} = y_{-1} < 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n < \cdots < x_{n+p-1} < y_0 = y_{r-1} < x_{n+p}.$$

So, assume that $t > 0$. It remains to see that

$$0 = x_0 < x_1 < \cdots < x_{n-1} < y_0 < y_1 < \cdots < y_{t-1} < x_n.$$

By statement (d) we have $y_{-1} = y_{s-1} - 1 < 0 = x_0$. So, it is enough to show that $x_{n-1} < y_0$. Assume, again, by way of contradiction that $y_0 < x_{n-1}$. Then, since $t < r$,

$$G_{c,d}(x_{n-1}) = x_{n+p-1} < y_t < y_r = G_{c,d}(y_0);$$

a contradiction with the fact that $G_{c,d}$ is increasing on $[0, u_{c,d}]$. This ends the proof of (e) provided that the claim holds.

Now we prove the claim. When $p = 1$ the claim follows from statement (d): $x_{p-1} = x_0 = 0 < y_0$. So we assume that $p \geq 2$. The claim follows from $x_p \leq \frac{\beta_{c,d}-1}{2\beta_{c,d}}$ because, by statements (c,d),

$$x_{p-1} < x_p \leq \frac{\beta_{c,d} - 1}{2\beta_{c,d}} < y_0.$$

To prove this, we define $\xi_n(x) := \llbracket (n+1)x \rrbracket - \llbracket nx \rrbracket$ for every $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Observe that $\llbracket -c \rrbracket = -1 = \llbracket 0 \rrbracket$ because $0 < c < 1$. Thus, $\xi_0(-c) = \llbracket -c \rrbracket - \llbracket 0 \rrbracket = 0$.

From the definition of $b_{c,d}$ and the proof of [9, Theorem 4.8.1] we get

$$\frac{\beta_{c,d} + 1}{2} = \sum_{k=0}^{\infty} \left(\xi_k(-c) + b_{c,d} - \frac{1-\beta_{c,d}^2}{2\beta_{c,d}} \right) \beta_{c,d}^{-k} = \sum_{k=0}^{\infty} \big(\xi_k(-c) + b_{c,d}\big)\beta_{c,d}^{-k} + \frac{\beta_{c,d}^2 - 1}{2\beta_{c,d}} \sum_{k=0}^{\infty} \beta_{c,d}^{-k} = b_{c,d} + \sum_{k=1}^{\infty} \big(\xi_k(-c) + b_{c,d}\big)\beta_{c,d}^{-k} + \frac{\beta_{c,d} + 1}{2},$$

because $\sum_{k=0}^{\infty} \beta_{c,d}^{-k} = \frac{\beta_{c,d}}{\beta_{c,d}-1}$. Therefore, $0 = b_{c,d} + \sum_{k=1}^{\infty} \big(\xi_k(-c) + b_{c,d}\big)\beta_{c,d}^{-k}$, and by statement (d),

$$x_p = G_{c,d}(x_0) = G_{c,d}(0) = b_{c,d} = b_{c,d} - b_{c,d} - \sum_{k=1}^{\infty} \big(\xi_k(-c) + b_{c,d}\big)\beta_{c,d}^{-k} = \sum_{k=1}^{\infty} \big(-\xi_k(-c) - b_{c,d}\big)\beta_{c,d}^{-k}.$$

On the other hand, from the proof of [9, Lemma 4.8.3],

$$\frac{\beta_{c,d}-1}{2\beta_{c,d}} = \big(\xi_0(d)-1-b_{c,d}\big)\beta_{c,d}^{-1} + \sum_{k=2}^{\infty}\big(\xi_{k-1}(d)-b_{c,d}\big)\beta_{c,d}^{-k} = \\ -\beta_{c,d}^{-1} + \sum_{k=1}^{\infty}\big(\xi_{k-1}(d)-b_{c,d}\big)\beta_{c,d}^{-k}.$$

Consequently, $x_p \leq \frac{\beta_{c,d}-1}{2\beta_{c,d}}$ is equivalent to

$$\sum_{k=1}^{\infty}\big(-\xi_k(-c)-b_{c,d}\big)\beta_{c,d}^{-k} \leq -\beta_{c,d}^{-1} + \sum_{k=1}^{\infty}\big(\xi_{k-1}(d)-b_{c,d}\big)\beta_{c,d}^{-k},$$

and by [9, Lemma 4.8.2], this inequality holds true provided that

$$\sum_{k=1}^{j}\big(-\xi_k(-c)-b_{c,d}\big) \leq -1 + \sum_{k=1}^{j}\big(\xi_{k-1}(d)-b_{c,d}\big)$$

for every $j \geq 1$, which is equivalent to

$$\sum_{k=1}^{j} -\xi_k(-c) \leq -1 + \sum_{k=1}^{j}\xi_{k-1}(d).$$

Thus, in view of the definitions of $\xi_k(\cdot)$ and $\llbracket\cdot\rrbracket$, we have to see that

$$\begin{aligned}\lfloor (j+1)c \rfloor &= -\big(\big(-1-\lfloor (j+1)c\rfloor\big)+1\big) = -\big(\llbracket -(j+1)c\rrbracket - \llbracket -c\rrbracket\big)\\ &= -\sum_{k=1}^{j}\big(\llbracket -(k+1)c\rrbracket - \llbracket -kc\rrbracket\big) = \sum_{k=1}^{j} -\xi_k(-c)\\ &\leq -1 + \sum_{k=1}^{j}\xi_{k-1}(d) = -1 + \sum_{k=1}^{j}\big(\llbracket kd\rrbracket - \llbracket (k-1)d\rrbracket\big)\\ &= \llbracket jd\rrbracket - \llbracket 0\rrbracket - 1 = \llbracket jd\rrbracket,\end{aligned}$$

for every $j \geq 1$.

To show that the above inequalities hold, it is enough to prove the following bound for $(j+1)c$:

$$(j+1)c < \begin{cases} \lfloor jd \rfloor + 1 & \text{if } jd \notin \mathbb{Z}, \text{ and} \\ jd & \text{when } jd \in \mathbb{Z}. \end{cases} \tag{5.1}$$

Indeed, when $jd \in \mathbb{Z}$ we have $(j+1)c < jd$ and, hence,

$$\lfloor (j+1)c \rfloor \leq jd - 1 = \llbracket jd \rrbracket.$$

When $jd \notin \mathbb{Z}$, $(j+1)c < \lfloor jd \rfloor + 1$ and

$$\lfloor (j+1)c \rfloor \leq \lfloor jd \rfloor = \llbracket jd \rrbracket.$$

Now, it remains to show that (5.1) holds. We start with the case $jd \in \mathbb{Z}$. By (e), $s = \ell r + t$ with $0 \leq t \leq r-1$, and $q \geq (\ell+1)p$. Moreover, $p \geq 2$ implies $q > (\ell+1)p$ because p and q are relatively prime. On the other hand, since $\frac{jr}{s} = jd \in \mathbb{N}$ and r and s are relatively prime, it follows that $j = as = a(\ell r + t)$ with $a \in \mathbb{N}$. Then,

$$(j+1)p = (a(\ell r + t) + 1)p \leq a(\ell r + t + 1)p \leq a(\ell + 1)rp < aqr = qas\tfrac{r}{s} = qjd.$$

Hence, $(j+1)c = \frac{(j+1)p}{q} < jd$.

Now we assume that $jd \notin \mathbb{Z}$ for some $j \in \mathbb{N}$. When $(j+1)d \leq \lfloor jd \rfloor + 1$ we clearly have

$$(j+1)c < (j+1)d \leq \lfloor jd \rfloor + 1.$$

So, we may suppose that

$$jd < \lfloor jd \rfloor + 1 < (j+1)d \iff j < \frac{\lfloor jd \rfloor + 1}{d} < j + 1 \iff j = \left\lfloor \tfrac{\lfloor jd \rfloor + 1}{d} \right\rfloor.$$

The assumption $\left\lfloor \frac{1}{d} \right\rfloor = \left\lfloor \frac{s}{r} \right\rfloor < \left\lfloor \frac{q}{p} \right\rfloor = \left\lfloor \frac{1}{c} \right\rfloor$ implies

$$\tfrac{1}{d} < \left\lfloor \tfrac{1}{d} \right\rfloor + 1 \leq \left\lfloor \tfrac{1}{c} \right\rfloor \iff \left\lfloor \tfrac{\lfloor jd \rfloor + 1}{d} \right\rfloor \leq \tfrac{\lfloor jd \rfloor + 1}{d} < \left\lfloor \tfrac{1}{c} \right\rfloor (\lfloor jd \rfloor + 1).$$

Then, since $\left\lfloor \frac{1}{c} \right\rfloor (\lfloor jd \rfloor + 1) \in \mathbb{N}$,

$$(j+1)c = \left(\left\lfloor \tfrac{\lfloor jd \rfloor + 1}{d} \right\rfloor + 1 \right) c \leq \left\lfloor \tfrac{1}{c} \right\rfloor (\lfloor jd \rfloor + 1)c.$$

On the other hand, $\left\lfloor \frac{1}{c} \right\rfloor = \frac{1}{c} = \frac{q}{p}$ implies that $p \geq 2$ is a divisor of q; a contradiction with the fact that p and q are relatively prime. So, $\left\lfloor \frac{1}{c} \right\rfloor < \frac{1}{c}$, and from above,

$$(j+1)c \leq \left\lfloor \tfrac{1}{c} \right\rfloor (\lfloor jd \rfloor + 1)c < \lfloor jd \rfloor + 1.$$

This ends the proof of (5.1) and the theorem. □

6. On the minimalistic extensions of certain families of minimal degree one circle maps: A factory of examples

In this section we will study the dynamical properties of the minimalistic extensions of certain families of degree one minimal maps, thus being a specialization and improvement of Theorem 4.4. As in previous sections, the proofs of the main results of this section are collected at the end in a dedicated subsection.

In what follows $\lceil \cdot \rceil$ will denote the *ceiling function* $\lceil x \rceil$ is by definition the least integer which is greater than or equal to x.

In the next proposition we obtain extension platforms that verify certain useful properties for a large class of cases that cover the three examples stated in the introduction. They allow the automatic minimalistic extension of a very wide range of examples, including the ones stated in the introduction. We do not compute the extension platforms in all cases for simplicity and to avoid enlarging this paper even more.

Proposition 6.1 (Extendable minimal maps)**.** *Let $[c, d] \subset (0, 1)$ be a non-degenerate interval such that $c = \frac{p}{q}$ and $d = \frac{r}{s}$ with $p, q, r, s \in \mathbb{N}$, and $(p, q) = (r, s) = 1$. Assume that $\frac{s}{r} \notin \mathbb{N}$ and $\ell := \left\lfloor \frac{s}{r} \right\rfloor \neq \left\lfloor \frac{q}{p} \right\rfloor$ (here we use the notation from Theorem 5.5(e)). For $\ell \geq 2$ the map $g_{c,d}$ is extendable with extension platform*

$$\Big(I = \mathsf{e}\big([y_{t-1}, x_n]\big), J = \mathsf{e}\big([y_{t+r-1}, x_{n+p}]\big), K = \mathsf{e}\big([y_{t+2r-1}, x_{n+2p}]\big) \Big).$$

For $\ell = 1$ and $t \in \{2, 3, \dots, r-2\}$, $g_{c,d}$ is extendable with extension platform.

$$\Big(I = \mathsf{e}\big([y_{b-r}, y_{b-r+1}]\big),\ J = \mathsf{e}\big([y_b, y_{b+1}]\big),\ K = \mathsf{e}\big([y_{b-t}, y_{b-t+1}]\big)\Big),$$

where

$$b := \begin{cases} s & \text{when } n = p, \\ 2t & \text{when } n > p \text{ and } \frac{r+1}{t} \leq 2, \text{ and} \\ \left\lceil \frac{r+1}{t} \right\rceil t - 1 & \text{when } n > p \text{ and } \frac{r+1}{t} > 2. \end{cases}$$

Proof. We start the proof with the case $\ell \geq 2$. Since $\frac{s}{r} \notin \mathbb{N}$, $r \geq 2$ and $t \in \{1, 2, \dots, r-1\}$ by Theorem 5.5(e) (specially Statement (e.2)). Observe that when $\ell = 2$,

$$K = \mathsf{e}\big(\big[y_{t+2r-1}, x_{n+2p}\big]\big) = \mathsf{e}\big(\big[y_{s-1}, x_q = 1\big]\big) = \mathsf{e}\big(\big[u_{c,d}, 1\big]\big).$$

By the ordering in Theorem 5.5(e), I, J and K are pairwise disjoint $\mathsf{e}(Q \cup P)$-basic intervals. Moreover, by Theorem 5.5(d), $g_{c,d}(I) = J$ and $g_{c,d}(J) = K$. We have to show that I is the unique basic interval that $g_{c,d}$-covers J. To do this we will show that if $U \subset \big[0, y_{t-1}\big]$ (respectively $U \subset \big[x_n, 1\big]$) is a $Q \cup P$-basic interval, then

$$G_{c,d}(U) \quad \text{is disjoint from} \quad \bigcup_{k \in \mathbb{Z}} \big(y_{t+r-1} + k, x_{n+p} + k\big),$$

or equivalently $G_{c,d}(U) \subset \bigcup_{k \in \mathbb{Z}} \big[x_{n+p} + k, y_{t+r-1} + k + 1\big]$.

Assume first that $U \subset \big[x_n, 1\big]$. Since Q is a twist lifted periodic orbit of $G_{c,d}$ and $n < q$ we have $x_n < x_q = 1$, and $x_{n+p} = G_{c,d}\big(x_n\big) < G_{c,d}(1)$. Then, by the definition of the map $G_{c,d}$ and Theorem 5.5(d),

$$\begin{aligned} G_{c,d}(U) \subset G_{c,d}\big(\big[x_n, 1\big]\big) &= \\ \Big[\min\big\{G_{c,d}\big(x_n\big), G_{c,d}(1)\big\}, G_{c,d}(u_{c,d} = y_{s-1})\Big] &= \\ \big[x_{n+p}, 1 + y_{r-1}\big] &\subsetneq \big[x_{n+p}, 1 + y_{t+r-1}\big]. \end{aligned}$$

When $U \subset \big[0, y_{t-1}\big]$ we have

$$\begin{aligned} G_{c,d}(U) \subset G_{c,d}\big(\big[0, y_{t-1}\big]\big) = \big[G_{c,d}(x_0), G_{c,d}\big(y_{t-1}\big)\big] &= \\ \big[x_p, y_{t+r-1}\big] \subsetneq \big[0, y_{t+r-1}\big] &\subsetneq \big[x_{n+p} - 1, y_{t+r-1}\big]. \end{aligned}$$

Thus, I is the unique basic interval that $g_{c,d}$-covers J.

Now we deal with the case $\ell = 1$ and $t \in \{2, 3, \dots, r-2\}$. We have $r \geq 4$ and, by Theorem 5.5(e), $q = p + n$ and $s = r + t$.

Since there are several extension platforms according to the different values of b we consider first the case $n = p$. By Theorem 5.5(e.3) we have $n = p = 1$ and $q = 2$ (that is, $\frac{p}{q} = \frac{1}{2}$). Moreover, in this case $b = s$. So,

$$\begin{aligned} I &= \mathsf{e}\big([y_{s-r}, y_{s-r+1}]\big) = \mathsf{e}\big([y_t, y_{t+1}]\big), \\ J &= \mathsf{e}\big([y_s, y_{s+1}]\big) = \mathsf{e}\big([y_0, y_1]\big), \text{ and} \\ K &= \mathsf{e}\big([y_{s-t}, y_{s-t+1}]\big) = \mathsf{e}\big([y_r, y_{r+1}]\big). \end{aligned}$$

On the other hand, the ordering in Theorem 5.5(e) becomes

$$\begin{aligned} 0 = x_0 = x_{n-1} < \boxed{y_0 < y_1} \le y_{t-1} < \underset{\overset{\shortparallel}{x_p = x_1}}{x_n} < \\ \boxed{y_t < y_{t+1}} < y_{t+2} \le \boxed{y_r < y_{r+1}} \le \underset{\overset{\shortparallel}{y_{s-1}}}{y_{t+r-1}} < \underset{\overset{\shortparallel}{x_2 = 1}}{x_q}, \end{aligned} \tag{6.1}$$

and thus I, J and K are pairwise disjoint $\mathsf{e}(Q \cup P)$-basic intervals. Furthermore, by Theorem 5.5(d), $g_{c,d}(I) = J$ and $g_{c,d}(J) = K$.

We have to show that I is the unique basic interval that $g_{c,d}$-covers J. To do this we will show that if $U \subset [0, y_t]$ (respectively $U \subset [y_{t+1}, 1]$) is a $Q \cup P$-basic interval, then

$$G_{c,d}(U) \quad \text{is disjoint from} \quad \bigcup_{k\in\mathbb{Z}} (y_0 + k, y_1 + k),$$

or equivalently $G_{c,d}(U) \subset \bigcup_{k\in\mathbb{Z}} \bigl(y_1 + k, y_0 + k + 1\bigr)$. Assume first that $U \subset [y_{t+1}, 1]$. By Theorem 5.5(d) (see also Figure 5),

$$G_{c,d}\bigl(y_{t+1}\bigr) = y_{t+r+1} = 1 + y_1 < 1 + x_1 = G_{c,d}(x_2) = G_{c,d}(1).$$

Then, by the definition of the map $G_{c,d}$,

$$\begin{aligned} G_{c,d}(U) \subset G_{c,d}\bigl([y_{t+1}, 1]\bigr) = \\ \Bigl[\min\{G_{c,d}(y_{t+1}), G_{c,d}(1)\}, G_{c,d}(u_{c,d} = y_{s-1})\Bigr] = \\ [1 + y_1, 1 + y_{r-1}] \subsetneq [1 + y_1, 2]. \end{aligned}$$

When $U \subset [0, y_t]$ we have

$$\begin{aligned} G_{c,d}(U) \subset G_{c,d}\bigl([0, y_t]\bigr) = [G_{c,d}(x_0), G_{c,d}(y_t)] = \\ [x_1, y_{t+r}] = [x_1, 1 + y_0] \subsetneq [y_1, 1 + y_0]. \end{aligned}$$

Consequently, I is the unique basic interval that $g_{c,d}$-covers J.

Now we consider the case $n > p \ge 1$ for which we use the two remaining values of b. The ordering in Theorem 5.5(e) now becomes

$$\begin{aligned} 0 = x_0 < x_1 \le x_p \le x_{n-1} < y_0 < y_1 \le y_{t-1} < \\ x_n \le \underset{\underset{x_{q-1}}{\shortparallel}}{x_{n+p-1}} < y_t < y_{t+1} < \cdots < \underset{\underset{y_{s-1}}{\shortparallel}}{y_{t+r-1}} < \underset{\underset{1}{\shortparallel}}{x_q}. \end{aligned} \tag{6.2}$$

Now we need to identify the intervals of the extension platform in the above array of points of $\bigl(P \cup Q\bigr) \cap [0, 1]$. To this end we claim that $r \le b \le s - 2$. To prove the claim observe that $\frac{r+1}{t} > \frac{r+1}{t+3} \ge 1$. When $\frac{r+1}{t} \le 2$ we have

$$r + 1 \le 2t = b \le t + r - 2 = s - 2.$$

Otherwise, if $\frac{r+1}{t} > 2$ (equivalently, $\lceil\frac{r+1}{t}\rceil \ge 3$) we write $r + 1 = jt + x$ with $j \in \mathbb{N}\setminus\{1\}$ and $x \in \{0, 1, \dots, t-1\}$. If $x = 0$ (that is, $\lceil\frac{r+1}{t}\rceil = \frac{r+1}{t} = j \in \mathbb{N}$),

$$b = \left\lceil\tfrac{r+1}{t}\right\rceil t - 1 = \tfrac{r+1}{t}t - 1 = r \le r + (t-2) = s - 2.$$

If $x = 1$ we have $r + 1 = jt + 1$ which is equivalent to $r = jt$. So, since $s = r + t$, t divides both r and s; a contradiction with the fact that r and s are relatively prime. Therefore, we may assume $x \geq 2$ and hence,

$$\tfrac{r+1}{t} < \left\lceil \tfrac{r+1}{t} \right\rceil = j + 1 = j + \tfrac{x}{t} + \tfrac{t-x}{t} = \tfrac{r+1}{t} + \tfrac{t-x}{t} = \tfrac{s-x+1}{t}$$

because $j < \frac{r+1}{t} = j + \frac{x}{t} < j + 1$. Consequently,

$$r = \tfrac{r+1}{t}t - 1 < \left\lceil \tfrac{r+1}{t} \right\rceil t - 1 = b = \tfrac{s-x+1}{t}t - 1 = s - x \leq s - 2.$$

This ends the proof of the claim.

On the other hand, from the definition of b it follows that b is either $2t$ or, when $\left\lceil \frac{r+1}{t} \right\rceil \geq 3$, $b = \left\lceil \frac{r+1}{t} \right\rceil t - 1 \geq 3t - 1 > 2t$. Thus, by the above claim and the labeling of the points of P,

$$\begin{aligned} y_0 \leq \boxed{y_{b-r} < y_{b-r+1}} \leq \overset{\displaystyle y_{t-1}}{\overset{\shortparallel}{y_{s-r-1}}} < \\ y_t \leq \boxed{y_{b-t} < y_{b-t+1}} < \boxed{y_b < y_{b+1}} \leq y_{s-1}. \end{aligned} \tag{6.3}$$

Hence, in view of the ordering (6.2), we see that I, J and K are pairwise disjoint $\mathsf{e}(Q \cup P)$-basic intervals. Moreover, by Theorem 5.5(d), $g_{c,d}(I) = J$ and $g_{c,d}(J) = K$ (in the last equality we use that $r = s - t$).

Next we have to prove that I is the unique basic interval that $g_{c,d}$-covers J. We will show that if $U \subset [0, y_{b-r}]$ (respectively $U \subset [y_{b-r+1}, 1]$) is a $Q \cup P$-basic interval, then

$$G_{c,d}(U) \quad \text{is disjoint from} \quad \bigcup_{k \in \mathbb{Z}} (y_b + k, y_{b+1} + k).$$

Assume first that $U \subset [y_{b-r+1}, 1]$. By Theorem 5.5(d) (see also Figure 5),

$$G_{c,d}(y_{b-r+1}) = y_{b+1} \leq y_{s-1} < 1 + x_p = G_{c,d}(x_q) = G_{c,d}(1).$$

Then, by the definition of the map $G_{c,d}$ and the above claim,

$$\begin{aligned} G_{c,d}(U) \subset G_{c,d}\big([y_{b-r+1}, 1]\big) = \\ \Big[\min\{G_{c,d}(y_{b-r+1}), G_{c,d}(1)\}, G_{c,d}(u_{c,d} = y_{s-1})\Big] = \\ [y_{b+1}, 1 + y_{r-1}] \subsetneq [y_{b+1}, 1 + y_b]. \end{aligned}$$

When $U \subset [0, y_{b-r}]$ we have

$$G_{c,d}(U) \subset G_{c,d}\big([0, y_{b-r}]\big) = [G_{c,d}(x_0), G_{c,d}(y_{b-r})] = [x_p, y_b] \subsetneq [0, y_b].$$

$\square$

The next result studies the convergence of the entropy of the minimalistic extensions of certain families of degree one circle maps.

Theorem 6.2. *Let* $\{g_{c_k,d_k}\}_{k \in \mathbb{N}}$ *be a family of minimal degree one circle maps, where each* G_{c_k,d_k} *has rotation interval* $[c_k, d_k]$ *with* $0 < c_k < d_k < 1$. *Assume that the following statements hold:*

(a) $c_k = \frac{p_k}{q_k}$ *and* $d_k = \frac{r_k}{s_k}$ *where* $p_k, q_k, r_k, s_k \in \mathbb{N}$, *and* $(p_k, q_k) = (r_k, s_k) = 1$.

(b) $\left\lfloor \frac{s_k}{r_k} \right\rfloor < \frac{s_k}{r_k} < \left\lfloor \frac{q_k}{p_k} \right\rfloor$.

(c) $r_k + 2 \leq s_k \leq 2r_k - 2$ *whenever* $s_k < 2r_k$.

(d) $\lim_{k\to\infty} \min M(c_k, d_k) = \infty$.

Let G be an arbitrary graph with a circuit C. Then, every map g_{c_k,d_k} is extendable to G with base at the circuit C with a transitive minimalistic extension $g^{G;C}_{c_k,d_k}$, and

$$\lim_{k\to\infty} h\bigl(g^{G;C}_{c_k,d_k}\bigr) = \lim_{k\to\infty} h\bigl(g_{c_k,d_k}\bigr) = 0.$$

Proof. We first check that every map of the family $\{g_{c_k,d_k}\}_{k\in\mathbb{N}}$ verifies the assumptions of Proposition 6.1 individually, and hence is extendable. Indeed, from (b) we have that $\frac{s_k}{r_k} \notin \mathbb{N}$ and $\ell_k := \left\lfloor \frac{s_k}{r_k} \right\rfloor \neq \left\lfloor \frac{q_k}{p_k} \right\rfloor$. On the other hand, $s_k < 2r_k$ and $\frac{s_k}{r_k} \notin \mathbb{N}$ is equivalent to $1 < \frac{s_k}{r_k} < 2$ which, with the notation of Theorem 5.5(e) gives $\ell_k = 1$ and $t_k := s_k - r_k$. By (c), when $s_k < 2r_k$ we have $r_k + 2 \le s_k \le 2r_k - 2$ which is equivalent to $2 \le t_k \le r_k - 2$. The other assumptions of Proposition 6.1 hold by (a). Hence, every map g_{c_k,d_k} is extendable with the extension platform (I_k, J_k, K_k) given by Proposition 6.1.

By Theorem 4.4 and Theorem 5.5(a,g) there exists a transitive minimalistic extension $f_{c_k,d_k} := g^{G;C}_{c_k,d_k}$ of g_{c_k,d_k} to G with base at the circuit C. By Theorem 4.4,

$$0 \le h\bigl(f_{c_k,d_k}\bigr) \le h\bigl(g_{c_k,d_k}\bigr) + \frac{\log m}{\rho_k},$$

where m is the constant given by Lemma 4.1, and ρ_k is the minimal length of a loop in the Markov graph of g_{c_k,d_k} beginning (and ending) at I_k. In view of (d), $\lim_{k\to\infty} h\bigl(g_{c_k,d_k}\bigr) = 0$ by Theorem 5.3. So, to end the proof of this result it is enough to show that the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ tends to infinity. In view of (d) it is enough to show that $\rho_k \ge \min M(c_k, d_k)$ for every $k \in \mathbb{N}$.

Now we fix an arbitrary $k \in \mathbb{N}$, and we will show that $\rho_k \ge \min M(c_k, d_k)$. Let α_k be a loop in the Markov graph of g_{c_k,d_k} beginning (and ending) at I_k, whith minimal length ρ_k among all loops in the Markov graph of g_{c_k,d_k} beginning (and ending) at I_k. The minimality of ρ_k implies that α_k is simple. Therefore, by Proposition 5.7, g_{c_k,d_k} has a periodic orbit R_k (of period ρ_k) associated to α_k. By Corollary 5.6 we have

$$\rho_k \in \mathsf{Per}(g_{c_k,d_k}) = \{q_k, s_k\} \cup M(c_k, d_k).$$

More precisely, either $\rho_k \in M(c_k, d_k)$, or $\mathsf{e}(Q_k)$ is associated to α_k (i.e. $R_k = \mathsf{e}(Q_k)$), or $\mathsf{e}(P_k)$ is associated to α_k (i.e. $R_k = \mathsf{e}(P_k)$), where we denote by $Q_k := \mathsf{e}^{-1}\bigl(\mathsf{Orb}_{g_{c_k,d_k}}(\mathsf{e}(0))\bigr)$ the unique twist orbit of G_{c_k,d_k} of period q_k and rotation number $\frac{p_k}{q_k}$, and by $P_k := \mathsf{e}^{-1}\bigl(\mathsf{Orb}_{g_{c_k,d_k}}(\mathsf{e}(u_{c_k,d_k}))\bigr)$ the unique twist orbit of G_{c_k,d_k} of period s_k and rotation number $\frac{r_k}{s_k}$.

If $\ell_k := \left\lfloor \frac{s_k}{r_k} \right\rfloor \ge 2$. Then, by Proposition 6.1, $I_k = \mathsf{e}\bigl([\widetilde{y}, \widetilde{x}]\bigr)$ where $[\widetilde{y}, \widetilde{x}]$ is a $\bigl(P_k \cup Q_k\bigr)$–basic interval such that $\widetilde{y} \in P_k$ and $\widetilde{x} \in Q_k$. Thus, again by Proposition 5.7, α_k is neither associated to $\mathsf{e}(Q_k)$ nor to $\mathsf{e}(P_k)$. Hence, in this case, $\rho_k \in M(c_k, d_k)$, which obviously gives $\rho_k \ge \min M(c_k, d_k)$.

Assume now that $\ell_k = 1$. From above we get $2 \le t_k = s_k - r_k \le r_k - 2$. Again by Proposition 6.1, $I_k \cap \mathsf{e}(Q_k) = \emptyset$. Thus, the periodic orbit

$\mathsf{e}(Q_k)$ cannot be associated to α_k and hence, $\rho_k \in \{s_k\} \cup M(c_k, d_k)$. By Theorem 5.5(e.3) we have either $q_k - p_k = p_k = 1$ and $\frac{p_k}{q_k} = \frac{1}{2}$, or $q_k - p_k \geq p_k + 1$. In the latter case, since $s_k \leq 2r_k - 2$, we have

$$\frac{p_k}{q_k} \leq \frac{p_k}{2p_k + 1} < \frac{1}{2} = \frac{r_k - 1}{2r_k - 2} \leq \frac{r_k - 1}{s_k} < \frac{r_k}{s_k},$$

which implies $s_k \in M(c_k, d_k)$. Thus, $\rho_k \in \{s_k\} \cup M(c_k, d_k) = M(c_k, d_k)$ and hence, $\rho_k \geq \min M(c_k, d_k)$.

When $\frac{p_k}{q_k} = \frac{1}{2}$ the situation is analogous, provided that $t_k < r_k - 2$ (i.e., $s_k \leq 2r_k - 3$). Indeed,

$$\frac{p_k}{q_k} = \frac{1}{2} < \frac{r_k - 1}{2r_k - 3} \leq \frac{r_k - 1}{s_k} < \frac{r_k}{s_k}.$$

Hence, again, $s_k \in M(c_k, d_k)$ and $\rho_k \geq \min M(c_k, d_k)$.

We are left with the case $\frac{p_k}{q_k} = \frac{1}{2}$ and $s_k = 2(r_k - 1)$. Observe that in this case, $2 \leq t_k = s_k - r_k = r_k - 2$ implies $r_k \geq 4 > 2$. The inequality $2 < r_k$ is equivalent to $2(r_k - 1)^2 = 2r_k^2 - 4r_k + 2 < r_k(2r_k - 3)$; which gives

$$\frac{p_k}{q_k} = \frac{1}{2} < \frac{r_k - 1}{s_k - 1} = \frac{r_k - 1}{2r_k - 3} < \frac{r_k}{2(r_k - 1)} = \frac{r_k}{s_k}.$$

In other words, $s_k - 1 \in M(c_k, d_k)$ and hence, $s_k - 1 \geq \min M(c_k, d_k)$.

If $\rho_k \in M(c_k, d_k)$ then, clearly, $\rho_k \geq \min M(c_k, d_k)$ and we are done. Otherwise, since $\rho_k \in \{s_k\} \cup M(c_k, d_k)$, we must have

$$\rho_k = s_k > s_k - 1 \geq \min M(c_k, d_k).$$

□

The next result studies the set of periods of minimalistic extensions of minimal degree one circle maps.

Theorem 6.3. *Let $[c, d]$ be a non-degenerate interval contained in the interval $(0, 1)$ such that $c = \frac{p}{q}$ and $d = \frac{r}{s}$ with $p, q, r, s \in \mathbb{N}$, and $(p, q) = (r, s) = 1$. Assume that $\lfloor \frac{s}{r} \rfloor < \frac{s}{r} < \left\lfloor \frac{q}{p} \right\rfloor$, and $r + 2 \leq s \leq 2r - 3$ whenever $s < 2r$. Let G be an arbitrary graph with a circuit C. Then, the map $g_{c,d}$ is extendable to G with base at the circuit C with a transitive minimalistic extension $g_{c,d}^{\mathsf{G;C}}$, and*

$$\mathsf{Per}(g_{c,d}) = \mathsf{Per}(g_{c,d}^{\mathsf{G;C}}).$$

Proof. From the proof of Theorem 6.2 we see that the map $g_{c,d}$ satisfies the assumptions of Proposition 6.1 and thus it is extendable with the extension platform (I, J, K) given by Proposition 6.1. As before, $g_{c,d}^{\mathsf{G;C}}$ is transitive by Theorem 4.4 and Theorem 5.5(a,g). Moreover, Corollary 5.6 hold, and we have

$$\{q, s\} \cup M(c, d) = \mathsf{Per}(g_{c,d}) \subset \mathsf{Per}(g_{c,d}^{\mathsf{G;C}}).$$

We have to prove that $\mathsf{Per}(g_{c,d}) \supset \mathsf{Per}(g_{c,d}^{\mathsf{G;C}})$.

Let A be a periodic orbit of $g_{c,d}^{\mathsf{G;C}}$ of period k; set $\widetilde{Q} := \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))$ where $Q := \mathsf{e}^{-1}(\widetilde{Q})$ is the unique twist orbit of $G_{c,d}$ of period q and rotation number $\frac{p}{q}$, and set $\widetilde{P} := \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))$ where $P := \mathsf{e}^{-1}(\widetilde{P})$ is the unique twist orbit of $G_{c,d}$ of period s and rotation number $\frac{r}{s}$. From Theorem 4.4 and its proof it follows that $\eta(\widetilde{Q} \cup \widetilde{P})$ is contained in the Markov partition R of $g^{\mathsf{G;C}}_{c,d}$, and

$$\eta \circ g_{c,d}\big|_{\widetilde{Q}\cup\widetilde{P}} = g^{\mathsf{G;C}}_{c,d} \circ \eta\big|_{\widetilde{Q}\cup\widetilde{P}},$$

where $\eta\colon \mathbb{S}^1 \longrightarrow C$ is the homeomorphism from Definition 4.3. In particular, $\eta(\widetilde{Q})$ (respectively $\eta(\widetilde{P})$) is a periodic orbit of $g^{\mathsf{G;C}}_{c,d}$ of period q (respectively s). Moreover, every point in $R \setminus \eta(\widetilde{Q} \cup \widetilde{P})$ is not $g^{\mathsf{G;C}}_{c,d}$−periodic

Since both A and R are $g^{\mathsf{G;C}}_{c,d}$−invariant, either $A \subset R$ or $A \cap R = \emptyset$. In the first case, A must be either $\eta(\widetilde{Q})$ or $\eta(\widetilde{P})$. This implies that $k \in \{q, s\} \subset \mathsf{Per}(g_{c,d})$, and the theorem follows in this case. Thus, we are left with the case $A \cap R = \emptyset$. In view of Proposition 2.9, there exists a loop

$$\alpha = J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{k-1} \longrightarrow J_0$$

(of length k) in the Markov graph of $g^{\mathsf{G;C}}_{c,d}$ with respect to R which is associated to A. Observe that for every $j \in \{0, 1, \ldots, k-1\}$ we have either $J_j \subset \eta(I)$ or $J_j \cap \mathsf{Int}\big(\eta(I)\big) = \emptyset$ because $J_j \in \mathcal{B}(R)$ and $\partial\eta(I) \subset R$.

Assume now that $J_j \cap \mathsf{Int}\big(\eta(I)\big) = \emptyset$ for every $j \in \{0, 1, \ldots, k-1\}$. Then we have (recall that $G = S \cup X$),

$$A \subset \bigcup_{j=0}^{k-1} J_j \subset G \setminus \mathsf{Int}\big(\eta(I)\big) = \Big(S \setminus \mathsf{Int}\big(\eta(I)\big)\Big) \cup X$$

(in particular, $A \cap \mathsf{Int}\big(\eta(I)\big) = \emptyset$). Next we want to see that $A \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$. By Definition 4.3, we get $g^{\mathsf{G;C}}_{c,d}\big(X\big) = \eta(K) \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$. Hence, since A is $g^{\mathsf{G;C}}_{c,d}$−invariant, $A \cap X \neq \emptyset$ implies $A \cap \big(S \setminus \mathsf{Int}\big(\eta(I)\big)\big) \neq \emptyset$. In summary, there exists a point $z \in A \cap \big(S \setminus \mathsf{Int}\big(\eta(I)\big)\big)$. On the other hand, since $g_{c,d}$ is Markov with respect to the partition $\widetilde{P} \cup \widetilde{Q}$, I is the unique basic interval that $g_{c,d}$−covers J, and I, J and K are pairwise disjoint, Definition 4.3 tells us that

$$g^{\mathsf{G;C}}_{c,d}(z) = \widetilde{\big(g_{c,d}\big)}_C(z) = \Big(\eta \circ g_{c,d} \circ \eta^{-1}\Big)(z) \in S.$$

Taking into account that $A \cap \mathsf{Int}\big(\eta(I)\big) = \emptyset$ and by using again the fact that A is $g^{\mathsf{G;C}}_{c,d}$−invariant we get $\big\{z, g^{\mathsf{G;C}}_{c,d}(z)\big\} \subset A \cap \big(S \setminus \mathsf{Int}\big(\eta(I)\big)\big)$. By iterating this argument $k-2$ times with $g^{\mathsf{G;C}}_{c,d}(z), \big(g^{\mathsf{G;C}}_{c,d}\big)^2(z), \ldots, \big(g^{\mathsf{G;C}}_{c,d}\big)^{k-2}(z)$ instead of z we obtain

$$A = \big\{z, g^{\mathsf{G;C}}_{c,d}(z), \ldots, \big(g^{\mathsf{G;C}}_{c,d}\big)^{k-1}(z)\big\} \subset A \cap \big(S \setminus \mathsf{Int}\big(\eta(I)\big)\big) \subset A,$$

which implies $A \subset S \setminus \mathsf{Int}\big(\eta(I)\big)$. Then,

$$g^{\mathsf{G;C}}_{c,d}\big|_A = \widetilde{\big(g_{c,d}\big)}_C\big|_A = \eta \circ g_{c,d} \circ \eta^{-1}\big|_A,$$

and $\eta^{-1}(A)$ is a periodic orbit of $g_{c,d}$ of period $k \in \mathsf{Per}\big(g_{c,d}\big)$, and the theorem is proved in this case.

Now, we may assume without loss of generality that $J_0 \subset \eta(I)$, and we can use the projection $\Pi\colon \mathcal{B}(R) \longrightarrow \mathcal{B}(\widetilde{P} \cup \widetilde{Q})$ introduced in the proof of

Theorem 4.4 (see Formula (4.4)). Clearly,

$$\widetilde{\gamma} = \underset{\substack{\shortparallel \\ I}}{\Pi(J_0)} \longrightarrow \underset{\substack{\shortparallel \\ J}}{\Pi(J_1)} \longrightarrow \Pi(J_2) \longrightarrow \cdots \longrightarrow \Pi(J_{k-1}) \longrightarrow \Pi(J_0)$$

is a loop of length k in the Markov graph of $g_{c,d}$ with respect to $\widetilde{Q} \cup \widetilde{P}$. Furthermore, there exists $\widetilde{k} \le k$ such that

$$\gamma = \underset{\substack{\shortparallel \\ I}}{\Pi(J_0)} \longrightarrow \underset{\substack{\shortparallel \\ J}}{\Pi(J_1)} \longrightarrow \Pi(J_2) \longrightarrow \cdots \longrightarrow \Pi(J_{\widetilde{k}-1}) \longrightarrow \Pi(J_0)$$

is a simple loop of length $\widetilde{k}$ in the Markov graph of $g_{c,d}$ with respect to $\widetilde{Q} \cup \widetilde{P}$, and either $\widetilde{\gamma} = \gamma$ (and then $\widetilde{k} = k$) or $\widetilde{\gamma}$ is a repetition of γ (and then $\widetilde{k}$ is a proper divisor of k).

By Proposition 5.7, γ has a periodic orbit Z of period $\widetilde{k} \in \mathsf{Per}(g_{c,d}) = \{q, s\} \cup M(c,d)$ associated to it. More precisely, either $Z = \widetilde{Q}$, or $Z = \widetilde{P}$ or $\widetilde{k} \in M(c,d)$.

By repeating the arguments at the end of Theorem 6.2 for $g_{c,d}$ instead of g_{c_k,d_k} (notice that here $s < 2r - 2$ when $s < 2r$), we get that $\widetilde{k} \in M(c,d)$. Hence,

$$k \in \{n \cdot m : m \in M(c,d) \text{ and } n \in \mathbb{N}\} \subset M(c,d) \subset \mathsf{Per}(g_{c,d}).$$

□

It turns out that the minimal maps $g_{c,d}$ in the assumptions of Proposition 6.1 have plenty of extension platforms. However, to feed the *minimalistic extension factory* we need to carefully choose appropriate extension platforms such that either the periods of the twist orbits associated to the endpoints of the rotation interval belong to the set $M(c,d)$ or certain loops (for instance a loop of minimal length beginning at the first interval of the extension platform) are not associated to them (see the end of the proofs of Theorems 6.2 and 6.3).

Below we give an example of these facts and of the consequences of choosing a wrong extension platform.

Example 6.4. Let $p = 1$, $q = 6$, $r = 3$, $s = 17$, and let $0 < c = \frac{p}{q} < \frac{r}{s} < 1$. Denote,

$$Q := \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0))\big) = \{\dots x_{-1}, x_0 = 0, x_1, x_2, \dots, x_5, x_6, x_7, \dots\}, \text{ and}$$
$$P := \mathsf{e}^{-1}\big(\mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d}))\big) = \{\dots y_{-1}, y_0, y_1, y_2, \dots, y_{16} = u_{c,d}, y_{17}, y_{18}, \dots\},$$

with $x_i < x_j$ and $y_i < y_j$ if and only if $i < j$. By Theorem 5.5(d,e),

$$x_{6j+i} = x_i + j, \quad y_{17j+i} = y_i + j, \quad G_{c,d}(x_i) = x_{i+1} \quad \text{and} \quad G_{c,d}(y_i) = y_{i+3},$$

for every $i, j \in \mathbb{Z}$.

With the notation of Theorem 5.5(d) we have $\ell = \lfloor \frac{17}{3} \rfloor = 5$, $n = q - lp = 1$, $t = s - lr = 2$, and the spatial distribution of the points of $Q \cup P$ is:

$$\begin{aligned} 0 = x_0 &< y_0 < y_1 < \\ x_1 &< y_2 < y_3 < y_4 < \\ x_2 &< y_5 < y_6 < y_7 < \\ x_3 &< y_8 < y_9 < y_{10} < \\ x_4 &< y_{11} < y_{12} < y_{13} < \\ x_5 &< y_{14} < y_{15} < y_{16} < x_6 = 1 \end{aligned}$$

By Theorem 5.5(f), $Q \cup P$ is a short Markov partition for $G_{c,d}$, and $g_{c,d}$ is Markov with respect to the Markov partition

$$\mathsf{e}(Q \cup P) = \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(0)) \cup \mathsf{Orb}_{g_{c,d}}(\mathsf{e}(u_{c,d})).$$

The Markov graph modulo 1 of $G_{c,d}$ with respect to $Q \cup P$ is shown in Figure 6. On the other hand, by Corollary 5.6, we have

$$\mathsf{Per}(g_{c,d}) = \{6, 17\} \cup M\big(\tfrac{1}{6}, \tfrac{3}{17}\big) = \{6, 17\} \cup \mathsf{Succ}\,(23)$$

because in the Farey sequence of order 23, $\frac{1}{6} < \frac{4}{23} < \frac{3}{17}$ are consecutive elements.

The Markov graph modulo 1 given in Figure 6 shows that the map $g_{\frac{1}{6},\frac{3}{17}}$ is extendable and has, in fact, a lot of extension platforms. In the rest of the example we will compare the extension platform

$$\Big(I = \mathsf{e}\big([y_1, x_1]\big), J = \mathsf{e}\big([y_4, x_2]\big), K = \mathsf{e}\big([y_7, x_3]\big)\Big),$$

versus

$$\Big(\widetilde{I} = \mathsf{e}\big([x_3, y_8]\big), \widetilde{J} = \mathsf{e}\big([x_4, y_{11}]\big), \widetilde{K} = \mathsf{e}\big([x_5, y_{14}]\big)\Big).$$

Observe that the first one of these extension platforms is the one given by Proposition 6.1 for this rotation interval, while the second one, being a standard extension platform, is, of course, different from the one given by Proposition 6.1. Moreover, the shortest loop in the Markov graph of $g_{\frac{1}{6},\frac{3}{17}}$ starting (and ending) at $I = \mathsf{e}\big([y_1, x_1]\big)$ has length $\rho = 23 = \min M\big(\frac{1}{6}, \frac{3}{17}\big)$, and any simple loop starting (and ending) at I cannot be associated neither to $\mathsf{e}(P)$ nor to $\mathsf{e}(Q)$ (see the proofs of Theorems 6.2 and 6.3). On the other hand, the shortest loop in the Markov graph of $g_{\frac{1}{6},\frac{3}{17}}$ starting (and ending) at $\widetilde{I} = \mathsf{e}\big([x_3, y_8]\big)$ has length $\rho = 6 \notin M\big(\frac{1}{6}, \frac{3}{17}\big)$, and is associated to the orbit $\mathsf{e}(Q)$.

The aim of this example is to understand why we need special extension platforms in order that Theorems 6.2 and 6.3 hold, while the fundamental Theorem 4.4 holds for every extension platform. Observe that the map $g_{\frac{1}{6},\frac{3}{17}}$ verifies the assumptions of Theorem 6.3, and the assumptions corresponding to a single member of the family in the statement of Theorem 6.2.

Let G be an arbitrary graph with a circuit C. Then, the map $g_{\frac{1}{6},\frac{3}{17}}$ is extendable to G with base at the circuit C *for both extension platforms* with a transitive minimalistic extension $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ so that Theorem 4.4 holds.

Next we compute the Markov graph of $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ for the first extension platform

$$\Big(I = \mathsf{e}\big([y_1, x_1]\big), J = \mathsf{e}\big([y_4, x_2]\big), K = \mathsf{e}\big([y_7, x_3]\big)\Big).$$

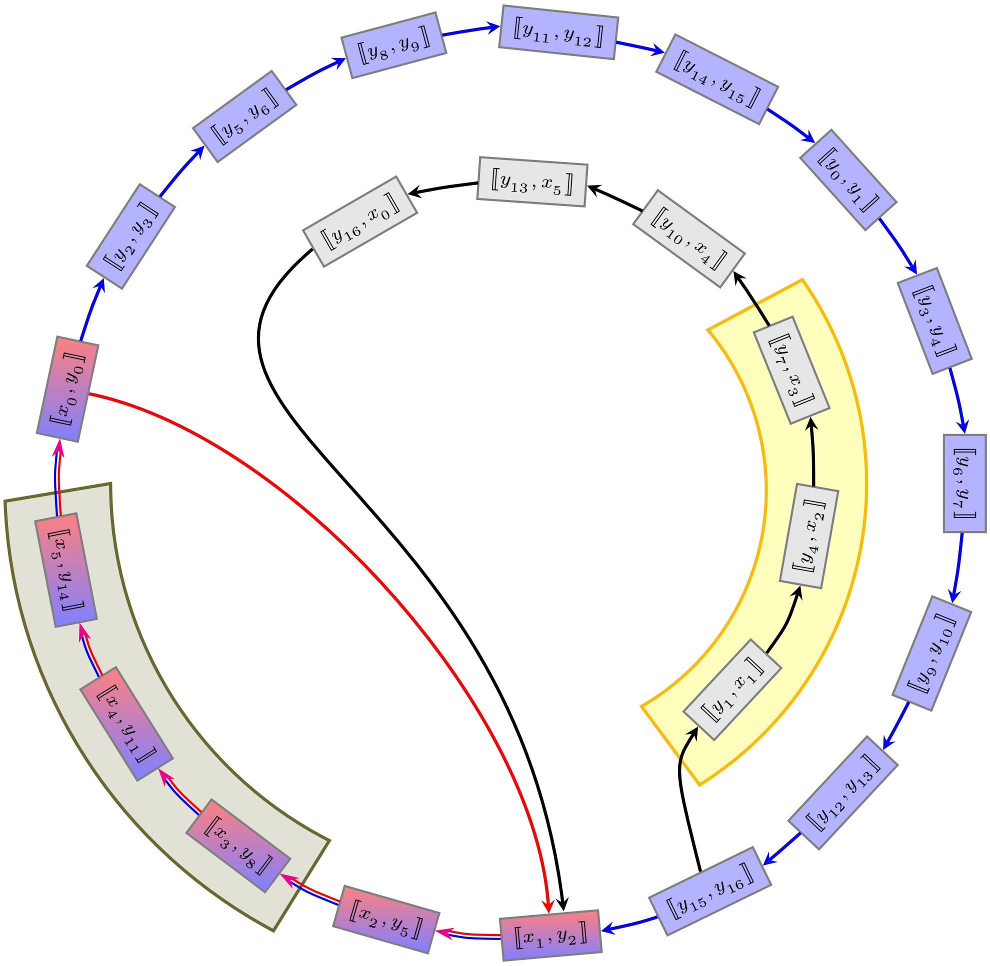


FIGURE 6. The Markov graph modulo 1 of $G_{\frac{1}{6},\frac{3}{17}}$ with respect to $Q \cup P$. The loop with blue vertices and arrows (including the vertices with a gradient color from blue to red and the blue-red arrows) is associated to $\mathsf{e}(P)$, and the loop with red vertices and arrows (including again the vertices with a gradient color from blue to red and the blue-red arrows) is associated to $\mathsf{e}(Q)$. The two coloured arc-like regions contain two extension platforms. The yellow one contains the extension platform given by Proposition 6.1 for this rotation interval. The green one contains a standard extension platform. Recall that if $[a, b]$ is a $Q \cup P-$basic interval, then $[\![a, b]\!]$ is identified with $\mathsf{e}([a, b])$.

To do this we will use the notation from the proof of Theorem 4.4 and, specially, from Figure 4. Indeed, the basic intervals from $\mathcal{B}(R)$ which are contained in $S \setminus \operatorname{Int}\big(\eta(I)\big)$ will be denoted as the corresponding ones in the circle. That is, we will denote $\eta(L)$ by L, for every $L \in \mathcal{B}(\mathsf{e}(Q \cup P)) \setminus \{I, J\}$. The basic intervals contained in X are $U_0, U_1, \dots, U_r$, and the basic intervals contained in $\eta(I)$ are denoted by $V_i := \langle s_i^I, s_{i+1}^I \rangle_{\eta(I)}$ for $i = 0, 1, \dots, m-1$.

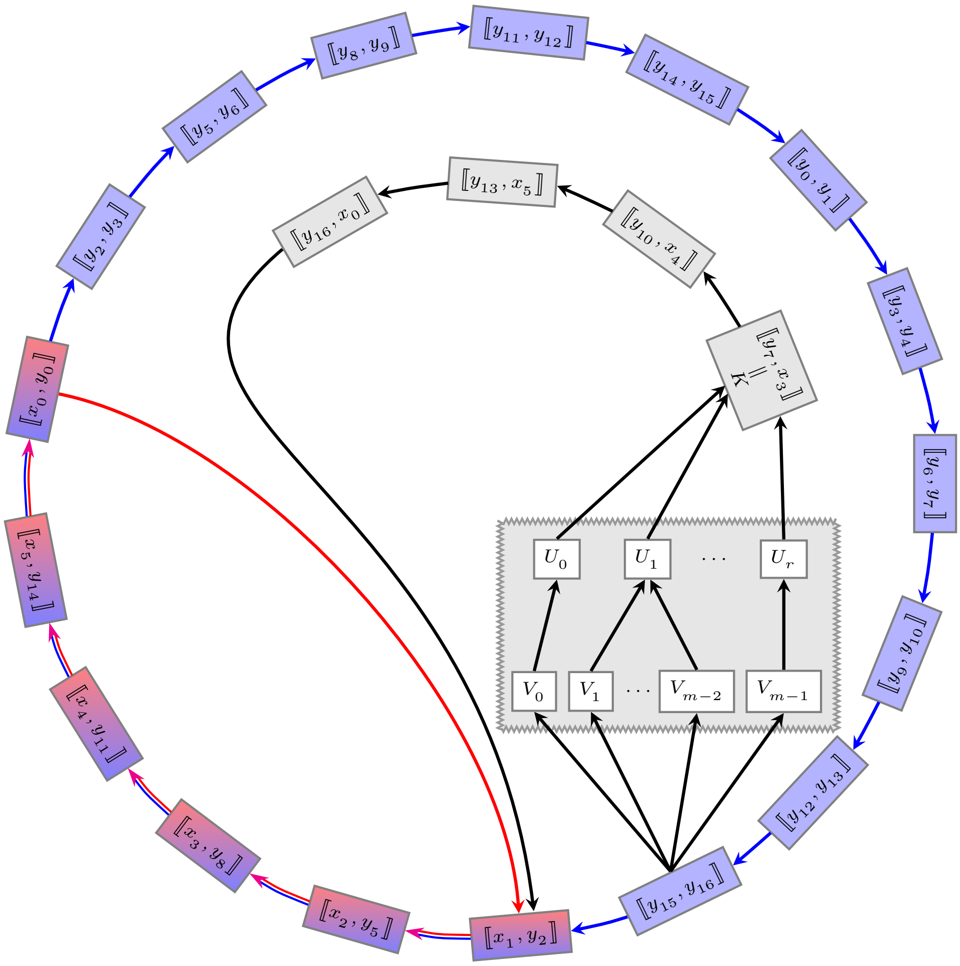


FIGURE 7. The Markov graph of a minimalistic extension of $g_{\frac{1}{6},\frac{3}{17}}$ for the extension platform

$$\Big(I = \mathsf{e}\big([y_1, x_1]\big), J = \mathsf{e}\big([y_4, x_2]\big), K = \mathsf{e}\big([y_7, x_3]\big)\Big).$$

In this graph, for simplicity, we are identifying $[\![a, b]\!]$ with $\eta(\mathsf{e}([a, b]))$ for every $Q \cup P-$basic interval $[a, b]$. The arrows from the vertices V_i to the vertices U_j are symbolic. In reality they depend on the topology and edges of the graph (in fact of the subgraph X) and, specially, on the map $\varphi_{a,b}$ (see Figure 4).

So, the Markov graph of $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ for the first extension platform the one shown in Figure 7. Comparing Figure 7 with Figure 6 we see that the set of lengths of simple loops in both graphs coincide. So, the set of periods of $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ and $g_{\frac{1}{6},\frac{3}{17}}$ coincide, as proven in Theorem 6.3.

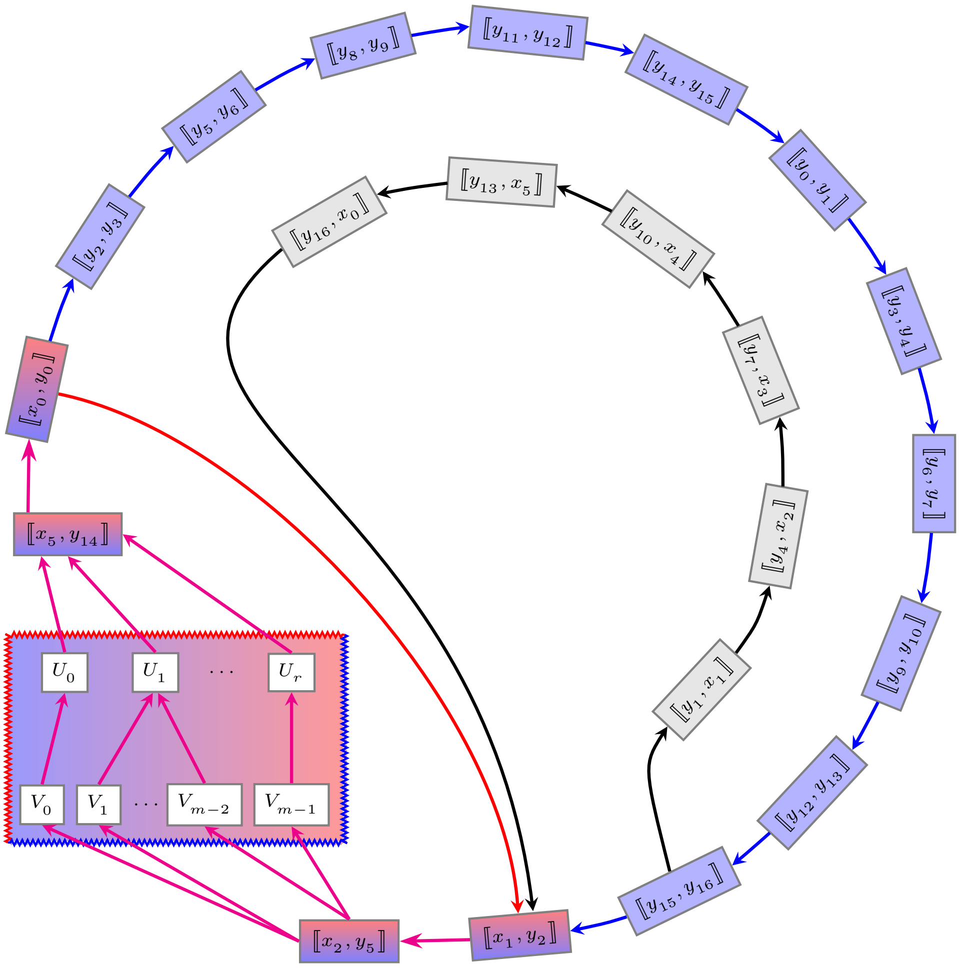


FIGURE 8. The Markov graph of a minimalistic extension of $g_{\frac{1}{6},\frac{3}{17}}$ for the extension platform

$$\Big(\widetilde{I} = \mathsf{e}\big([x_3, y_8]\big), \widetilde{J} = \mathsf{e}\big([x_4, y_{11}]\big), \widetilde{K} = \mathsf{e}\big([x_5, y_{14}]\big)\Big).$$

As in the previous figure, $[\![a, b]\!]$ is identified with $\eta(\mathsf{e}([a, b]))$ for every $Q \cup P-$basic interval $[a, b]$, and the arrows from the vertices V_i to the vertices U_j are symbolic. They depend on the topology and edges of the subgraph X and on the map $\varphi_{a,b}$ (see Figure 4).

.

Now we look at the Markov graph of the minimalistic extension $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ for the second extension platform

$$\Big(\widetilde{I} = \mathsf{e}\big([x_3, y_8]\big), \widetilde{J} = \mathsf{e}\big([x_4, y_{11}]\big), \widetilde{K} = \mathsf{e}\big([x_5, y_{14}]\big)\Big).$$

We use the same notation as before with I, J and K replaced by $\widetilde{I}$, $\widetilde{J}$ and $\widetilde{K}$. In Figure 8 we show the Markov graph of $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ for the second extension platform. From Figure 8 we see that the Markov graph of the minimalistic extension $g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}$ associated to the second extension platform has simple loops of lengths $12, 18, 24, 30, \dots$ associated to the basic intervals V_i contained in $\eta\big(\widetilde{I}\big)$. Hence, $\{6k : k \in \mathbb{N}\} \subset \mathsf{Per}\Big(g^{\mathsf{G;C}}_{\frac{1}{6},\frac{3}{17}}\Big)$. However, the Markov graph of the circle map $g_{\frac{1}{6},\frac{3}{17}}$ has no simple loops of lengths $12, 18, 24, 30, \dots$ associated to the extension platform. In particular, $12 \notin \mathsf{Per}\Big(g_{\frac{1}{6},\frac{3}{17}}\Big)$.

7. Examples

In this section we we will use the *minimalistic extension factory* (Section 4) to construct Examples 1.6, 1.9 and 1.11.

Example 1.6 (the dream example). For every positive integer $n \geq 3$ there exists a totally transitive continuous circle map of degree one, $f_{1,n}$, having a lifting $F_{1,n} \in \mathcal{L}_1$ such that

$$\mathsf{Rot}(F_{1,n}) = \left[\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\right], \quad \mathsf{Per}(f_{1,n}) = \mathsf{Succ}\,(n)\,, \text{ and } \lim_{n\to\infty} h(f_{1,n}) = 0.$$

Hence, $\mathsf{BdCof}\,(f_{1,n}) = \mathsf{StrBdCof}\,(f_{1,n}) = n$. Thus, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_{1,n}) = \infty$.

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{1,n}\}_{n\geq 3}$ can be extended to a sequence of continuous totally transitive self maps of G, $\big\{\phi^{\mathsf{G;1}}_n\big\}_{n\geq 3}$, such that

$$\mathsf{Per}\big(\phi^{\mathsf{G;1}}_n\big) = \mathsf{Per}(f_{1,n}) \quad \text{and} \quad \lim_{n\to\infty} h\big(\phi^{\mathsf{G;1}}_n\big) = 0.$$

Proof. For every $n \geq 3$ we set $c_n = \frac{p_n}{q_n}$ with $p_n = 1$ and $q_n = 2n-1$, $d_n = \frac{r_n}{s_n}$ with $r_n = 2$ and $s_n = 2n-1$, and $F_{1,n} = G_{c_n,d_n} = G_{\frac{1}{2n-1},\frac{2}{2n-1}}$. Clearly, $0 < c_n < d_n < 1$. Then, by Theorems 5.2 and 5.5(a) we know that $f_{1,n}$ is transitive, $\mathsf{Rot}(F_{1,n}) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$, and $h(f_{1,n}) = \log \beta_{c_n,d_n}$.

Now we compute $M\Big(\frac{1}{2n-1}, \frac{2}{2n-1}\Big)$. Observe that

- $M\Big(\frac{1}{2n-1}, \frac{2}{2n-1}\Big) \supset \mathsf{Succ}\,(2n)$ because $\mathsf{len}(\mathsf{Rot}(F_n)) = \frac{1}{2n-1}$,
- $2n-1 \notin M\Big(\frac{1}{2n-1}, \frac{2}{2n-1}\Big)$, and
- $M\Big(\frac{1}{2n-1}, \frac{2}{2n-1}\Big) \cap \{1, 2, \dots, 2n-1\} = \{n, n+1, \dots, 2n-2\}$ by Remark 1.8.

Thus,

$$M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) = \mathsf{Succ}\,(2n) \cup \{n, n+1, \dots, 2n-2\}$$

and hence, $\min M\left(\frac{1}{2n-1}, \frac{2}{2n-1}\right) = n$. So,

$$\lim_{n\to\infty} \min M\left(c_n, d_n\right) = \infty.$$

Then, by Theorem 5.3,

$$\lim_{n\to\infty} h\big(f_{1,n}\big) = \lim_{n\to\infty} \log \beta_{c_n,d_n} = 0.$$

By Corollary 5.6 we have that

$$\mathsf{Per}\big(f_{1,n}\big) = \{2n-1\} \cup M\left(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\right) = \mathsf{Succ}\,(n)\,.$$

Concerning the boundaries of cofiniteness we have $\mathsf{StrBdCof}\big(f_{1,n}\big) = n$ and $\mathsf{sBC}\big(f_{1,n}\big) = \big\{\mathsf{StrBdCof}\big(f_{1,n}\big)\big\}$. So, $\mathsf{BdCof}\big(f_{1,n}\big) = \mathsf{StrBdCof}\big(f_{1,n}\big) = n$, and $\lim_{n\to\infty} \mathsf{BdCof}\big(f_{1,n}\big) = \infty$.

Now we deal with the minimalistic extension of the family $\{f_{1,n}\}_{n\geq 3}$. In this example we have $s_n = 2n-1 > 4 = 2r_n$. Moreover

$$\left\lfloor \tfrac{1}{d_n} \right\rfloor = \left\lfloor \tfrac{s_n}{r_n} \right\rfloor = \left\lfloor \tfrac{2n-1}{2} \right\rfloor = n-1 < \tfrac{2n-1}{2} = \tfrac{s_n}{r_n} < n <$$
$$2n-1 = \tfrac{q_n}{p_n} = \left\lfloor \tfrac{q_n}{p_n} \right\rfloor = \left\lfloor \tfrac{1}{c_n} \right\rfloor.$$

Let G be an arbitrary graph with a circuit C. Then, all the assumptions of Theorems 6.2 and 6.3 are verified and, hence, $f_{1,n}$ is extendable to G with base at the circuit C with a transitive minimalistic extension $\phi_n^{\mathsf{G};1} = f_{1,n}^{\mathsf{G};\mathsf{C}}$, and

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};1}\big) = \mathsf{Per}\big(f_{1,n}\big) \quad \text{and} \quad \lim_{n\to\infty} h\big(\phi_n^{\mathsf{G};1}\big) = \lim_{n\to\infty} h\big(f_{1,n}\big) = 0.$$

Finally, $\phi_n^{\mathsf{G};1}$ is totally transitive by Theorem 1.2 because

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};1}\big) = \mathsf{Per}\big(f_{1,n}\big) = \mathsf{Succ}\,(n)$$

is cofinite. □

Example 1.9 (with persistent fixed low periods). For every $n \in \mathbb{N}$ odd, $n \geq 7$, there exists a totally transitive continuous circle map of degree one, $f_{2,n}$, having a lifting $F_{2,n} \in \mathcal{L}_1$ such that $\mathsf{Rot}(F_{2,n}) = \left[\tfrac{1}{2}, \tfrac{n+2}{2n}\right]$, $\lim_{n\to\infty} h(f_{2,n}) = 0$,

$$\mathsf{Per}(f_{2,n}) = \{2\} \cup \left\{q \text{ odd} : 2\cdot\left\lfloor \tfrac{n+1}{4}\right\rfloor + 1 \leq q \leq n-2\right\} \cup \mathsf{Succ}\,(n)\,,$$

and $\mathsf{BdCof}\,(f_{2,n})$ exists and verifies $2\cdot\left\lfloor \tfrac{n+1}{4}\right\rfloor + 1 \leq \mathsf{BdCof}\,(f_{2,n}) \leq n$ (and, hence, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_{2,n}) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{2,n}\}_{n\geq 7, n \text{ odd}}$ can be extended to a sequence of continuous totally transitive self maps of G, $\big\{\phi_n^{\mathsf{G};2}\big\}_{n\geq 7, n \text{ odd}}$, such that $\mathsf{Per}\big(\phi_n^{\mathsf{G};2}\big) = \mathsf{Per}(f_{2,n})$ and, additionally, $\lim_{n\to\infty} h\big(\phi_n^{\mathsf{G};2}\big) = 0$.

Proof. For every $n \geq 7$ odd we set $c_n = \frac{p_n}{q_n}$ with $p_n = 1$ and $q_n = 2$, $d_n = \frac{r_n}{s_n}$ with $r_n = n+2$ and $s_n = 2n$, and $F_{2,n} = G_{c_n,d_n} = G_{\frac{1}{2},\frac{n+2}{2n}}$. Clearly, $0 < c_n < d_n < 1$. Then, by Theorems 5.2 and 5.5(a) we know that $f_{2,n}$ is transitive, $\mathsf{Rot}(F_{2,n}) = \left[\tfrac{1}{2}, \tfrac{n+2}{2n}\right]$, and $h(f_{2,n}) = \log \beta_{c_n,d_n}$.

Next we will compute the set $M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right)$. Since $\frac{n+2}{2n} - \frac{1}{2} = \frac{1}{n}$, it follows that $\mathsf{Succ}\,(n+1) \subset M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right)$. On the other hand, since n is odd, we have $\frac{n+1}{2} \in \mathbb{Z}$ and $\frac{1}{2} < \frac{(n+1)/2}{n} < \frac{n+2}{2n}$. Hence,

$$M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) = \mathsf{Succ}\,(n) \cup \left\{q \in M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \,:\, q < n\right\},$$

and we need to compute the set $\left\{q \in M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \,:\, q < n\right\}$.

Assume that $\frac{1}{2} < \frac{p}{q} < \frac{n+2}{2n}$ with $p \in \mathbb{Z}$ and $q \in \{1, 2, \dots, n-1\}$. Observe that q is odd for otherwise, $q = 2\ell \leq n-1$ with $\ell \in \mathbb{N}$, and the expression $\frac{1}{2} < \frac{p}{2\ell} < \frac{n+2}{2n}$ is equivalent to

$$\ell < p < \ell + \frac{2\ell}{n} \leq \ell + \frac{n-1}{n} < \ell + 1;$$

a contradiction.

So we can write $q = 2\ell + 1 \leq n - 2$ with $\ell \in \mathbb{N}$ (recall that n is odd). We have,

$$\frac{1}{2} < \frac{p}{2\ell+1} < \frac{n+2}{2n},$$

which is equivalent to

$$\ell + \frac{1}{2} < p < \frac{n(2\ell+1) + 2(2\ell+1)}{2n} \leq \frac{n(2\ell+1) + 2(n-2)}{2n} = (\ell+2)\frac{2n\ell + 3n - 4}{2n\ell + 4n} < \ell + 2.$$

Consequently,

$$\frac{1}{2} < \frac{\ell+1}{2\ell+1} < \frac{n+2}{2n}.$$

The second of these inequalities is equivalent to

$$2n\ell + 2n < (n+2)(2\ell+1) = 2n\ell + n + 2(2\ell+1)$$

which, in turn, is equivalent to $\frac{n}{2} < 2\ell + 1$. Thus, since n is odd, we can write $n + 1 = 4 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + r$ with $r \in \{0, 2\}$, and the last inequality is equivalent to

$$2 \cdot \left\lfloor \tfrac{n+1}{4} \right\rfloor + \frac{r}{2} = \frac{4 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + r}{2} = \frac{n+1}{2} \leq 2\ell + 1.$$

Hence, $q = 2\ell + 1 \geq 2 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + 1$. Summarizing, we have proven that

$$M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) = \mathsf{Succ}\,(n) \cup \left\{q \text{ odd} : 2 \cdot \left\lfloor \tfrac{n+1}{4} \right\rfloor + 1 \leq q \leq n - 2\right\}.$$

Consequently, $\min M\left(\frac{1}{2}, \frac{n+2}{2n}\right) = 2 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + 1$, and hence

$$\lim_{n\to\infty} \min M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) = \infty.$$

Then, by Theorem 5.3 we have

$$\lim_{n\to\infty} h(f_{2,n}) = \lim_{n\to\infty} \log \beta_{c_n, d_n} = 0.$$

By Corollary 5.6,

$$\mathsf{Per}(f_{2,n}) = \{2, 2n\} \cup M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) = \{2\} \cup \left\{q \text{ odd} : 2 \cdot \left\lfloor \tfrac{n+1}{4} \right\rfloor + 1 \leq q \leq n-2\right\} \cup \mathsf{Succ}\,(n)$$

because, obviously, $2n \in \mathsf{Succ}\,(n)$.

Concerning the boundaries of cofiniteness we have $\mathsf{StrBdCof}\,(f_{2,n}) = n$ and $2 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + 1 \in \mathsf{sBC}(f_{2,n})$ (because $n \geq 7$). So, $\mathsf{BdCof}\,(f_{2,n})$ exists and verifies $2 \cdot \left\lfloor \frac{n+1}{4} \right\rfloor + 1 \leq \mathsf{BdCof}\,(f_{2,n}) \leq n$.

Now we deal with the minimalistic extension of $\{f_{2,n}\}_{n \geq 7, n \text{ odd}}$. In this example we have $s_n = 2n < 2(n+2) = 2r_n$ and, since $n \geq 7$,

$$r_n + 2 = n + 4 < 2n = s_n < 2n + 1 = 2(n+2) - 3 = 2r_n - 3.$$

On the other hand,

$$\left\lfloor \tfrac{1}{d_n} \right\rfloor = \left\lfloor \tfrac{s_n}{r_n} \right\rfloor = \left\lfloor \tfrac{2n}{n+2} \right\rfloor = 1 < \tfrac{2n}{n+2} = \tfrac{s_n}{r_n} < 2 = \tfrac{q_n}{p_n} = \left\lfloor \tfrac{q_n}{p_n} \right\rfloor = \left\lfloor \tfrac{1}{c_n} \right\rfloor.$$

Let G be an arbitrary graph with a circuit C. Then, all the assumptions of Theorems 6.2 and 6.3 are verified and, hence, $f_{2,n}$ is extendable to G with base at the circuit C with a transitive minimalistic extension $\phi_n^{\mathsf{G};2} = f_{2,n}^{\mathsf{G;C}}$, and

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};2}\big) = \mathsf{Per}\big(f_{2,n}\big) \quad \text{and} \quad \lim_{n\to\infty} h\big(\phi_n^{\mathsf{G};2}\big) = \lim_{n\to\infty} h\big(f_{2,n}\big) = 0.$$

Finally, $\phi_n^{\mathsf{G};2}$ is totally transitive by Theorem 1.2 because

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};2}\big) = \mathsf{Per}\big(f_{2,n}\big) \supset \mathsf{Succ}\,(n)$$

is cofinite. □

Example 1.11 (Example with low non-constant periods). For every $n \in \mathbb{N}$, $n \geq 5$ there exists $f_{3,n}$, a totally transitive continuous circle map of degree one having a lifting $F_{3,n} \in \mathcal{L}_1$ such that

$$\mathsf{Rot}(F_{3,n}) = \left[\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\right] = \left[\tfrac{1}{n} - \tfrac{1}{2n^2}, \tfrac{1}{n} + \tfrac{1}{2n^2}\right],$$

$\lim_{n\to\infty} h(f_{3,n}) = 0$, and

$$\begin{aligned}\mathsf{Per}(f_{3,n}) = \{n, 2n, 2n+1\} \ \cup \Big\{ & tn + k : t \in \{3, 4, \ldots, \nu - 1\} \text{ and} \\ & k \in \left\{-\left\lfloor \tfrac{t-1}{2} \right\rfloor, -\left\lfloor \tfrac{t-1}{2} \right\rfloor + 1, \ldots, 0, 1, \ldots, \left\lfloor \tfrac{t}{2} \right\rfloor\right\}\Big\} \cup \\ & \qquad \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right)\end{aligned}$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, $\mathsf{StrBdCof}\,(f_{3,n}) = n\nu + 1 - \frac{\nu}{2}$ and $\mathsf{BdCof}\,(f_{3,n})$ exists and verifies $n \leq \mathsf{BdCof}\,(f_{3,n}) \leq n\nu - 1 - \frac{\nu}{2}$ (and hence, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_{3,n}) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_{3,n}\}_{n=5}^{\infty}$ can be extended to a sequence of continuous totally transitive self maps of G, $\big\{\phi_n^{\mathsf{G};3}\big\}_{n=5}^{\infty}$, such that

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};3}\big) = \mathsf{Per}(f_{3,n}) \quad \text{and} \quad \lim_{n\to\infty} h\big(\phi_n^{\mathsf{G};3}\big) = 0.$$

Proof. For every $n \geq 5$ we set $c_n = \frac{p_n}{q_n}$ with $p_n = 2n - 1$ and $q_n = 2n^2$, $d_n = \frac{r_n}{s_n}$ with $r_n = 2n + 1$ and $s_n = 2n^2$, and $F_{3,n} = G_{c_n,d_n} = G_{\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}}$. Clearly, $0 < c_n < d_n < 1$. Then, by Theorems 5.2 and 5.5(a) we know that $f_{3,n}$ is transitive, $\mathsf{Rot}(F_{3,n}) = \left[\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right]$, and $h(f_{3,n}) = \log \beta_{c_n,d_n}$.

Now we will prove that

$$\begin{aligned} M(c_n, d_n) = M\Big(\tfrac{p_n}{q_n}, \tfrac{r_n}{q_n}\Big) = M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big) = \{n, 2n, 2n+1\} \cup \\ \Big\{ tn + k : t \in \{3, 4, \ldots, \nu - 1\} \text{ and} \\ k \in \left\{-\left\lfloor \tfrac{t-1}{2} \right\rfloor, -\left\lfloor \tfrac{t-1}{2} \right\rfloor + 1, \ldots, 0, 1, \ldots, \left\lfloor \tfrac{t}{2} \right\rfloor\right\}\Big\} \cup \end{aligned}$$

$$\mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right),$$

which amounts showing that the three statements below hold:

(1) $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right) \cap \left\{1, 2, \dots, 3n - \frac{\nu}{2}\right\} = \{n, 2n, 2n+1\}$,

(2) $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right) \cap \left\{3n - \frac{\nu}{2} + 1, 3n - \frac{\nu}{2} + 2, \dots, n\nu - \frac{\nu}{2}\right\} =$

$$\Big\{tn + k : t \in \{3, 4, \dots, \nu - 1\} \text{ and } k \in \left\{-\left\lfloor\tfrac{t-1}{2}\right\rfloor, -\left\lfloor\tfrac{t-1}{2}\right\rfloor + 1, \dots, 0, 1, \dots, \left\lfloor\tfrac{t}{2}\right\rfloor\right\}\Big\}, \text{ and}$$

(3) $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right) \supset \mathsf{Succ}\left(n\nu + 1 - \frac{\nu}{2}\right)$.

Note that the elements of $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right)$ are those $q \in \mathbb{N}$ for which there exists $p \in \mathbb{N}$ such that

$$\frac{2n-1}{2n^2} < \frac{p}{q} < \frac{2n+1}{2n^2}. \tag{7.1}$$

Simple computations show that (we are writing in blue the endpoints of the rotation interval for better comprehension):

$$0 < \frac{1}{n+k} \leq \frac{2n-1}{2n^2} < \frac{1}{n} = \frac{2}{2n} < \frac{2n+1}{2n^2} \leq \frac{2}{n+k} < \frac{1}{k}$$

for every $k \in \{1, 2, \dots, n-1\}$. So, $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right) \cap \{1, 2, \dots, 2n\} = \{n, 2n\}$. Moreover, for $k \in \left\{2, 3, \dots, n - \frac{\nu}{2}\right\}$,

$$\frac{2}{2n+k} \leq \frac{2}{2n+2} \leq \frac{2n-1}{2n^2} < \frac{2}{2n+1} < \frac{2n+1}{2n^2} < \frac{6}{5n+1} = \frac{3}{3n - \frac{n-1}{2}} \leq \frac{3}{3n - \frac{\nu}{2}} \leq \frac{3}{2n+k}.$$

Indeed, the non-trivial steps of the above chain of inequalities can be justified as follows (recall that $n \geq 5$ and, equivalently, $2n \geq 10$):

- $\frac{2}{2n+2} \leq \frac{2n-1}{2n^2}$ is equivalent to $4n^2 < 4n^2 + 8 \leq 4n^2 + 2n - 2$.
- $\frac{2}{2n+1} < \frac{2n+1}{2n^2}$ is equivalent to $4n^2 = (2n)^2 < (2n+1)^2$.
- $\frac{2n+1}{2n^2} < \frac{6}{5n+1}$ is equivalent to

$$(2n+1)(5n+1) = 10n^2 + 7n + 1 < 10n^2 + 10n \leq 12n^2 = 6 \cdot 2 \cdot n^2.$$

Thus (1) holds.

Now we prove (2). Clearly we can write

$$\left\{3n - \tfrac{\nu}{2} + 1, 3n - \tfrac{\nu}{2} + 2, \dots, n\nu - \tfrac{\nu}{2}\right\} = \Big\{tn + k : t \in \{3, 4, \dots, \nu - 1\} \text{ and } k \in \left\{1 - \tfrac{\nu}{2}, 2 - \tfrac{\nu}{2}, \dots, 0, 1, \dots, n - \tfrac{\nu}{2}\right\}\Big\}.$$

For $q = tn + k$, inequalities (7.1) are equivalent to

$$(2n-1)(tn+k) < 2n^2 p < (2n+1)(tn+k)$$

which, in turn, are equivalent to

$$(2n-1)k - tn < 2(p-t)n^2 < (2n+1)k + tn. \tag{7.2}$$

Now we have to determine the range of valid values of k for each t. Initially we get

$$\begin{aligned} -2n^2 < -2n^2 + 2n\big(n + \tfrac{3}{2} - \nu\big) + \big(\tfrac{\nu}{2} - 1\big) = \\ (2n-1)\big(1 - \tfrac{\nu}{2}\big) - (\nu-1)n \le (2n-1)k - tn < 2(p-t)n^2 < \\ (2n+1)k + tn \le (2n+1)\big(n - \tfrac{\nu}{2}\big) + (\nu-1)n = 2n^2 - \tfrac{\nu}{2} < 2n^2. \end{aligned}$$

This implies $p = t$ (and $2(p-t)n^2 = 0$). Then, (7.2) becomes

$$-(2n-1)k + tn > 0 > -(2n+1)k - tn,$$

which is equivalent to

$$-\frac{tn}{2n+1} < k < \frac{tn}{2n-1}.$$

Therefore, (2) holds provided that

$$-\left\lfloor \tfrac{t-1}{2} \right\rfloor - 1 < -\frac{tn}{2n+1} < -\left\lfloor \tfrac{t-1}{2} \right\rfloor \le k \le \left\lfloor \tfrac{t}{2} \right\rfloor < \frac{tn}{2n-1} < \left\lfloor \tfrac{t}{2} \right\rfloor + 1. \tag{7.3}$$

If t is even, the above inequalities are equivalent to

$$-\frac{t}{2} < -\frac{tn}{2n+1} < \frac{2-t}{2} = -\frac{t}{2} + 1 < 0 < \frac{t}{2} < \frac{tn}{2n-1} \le \frac{t+2}{2},$$

which are equivalent to

$$\begin{aligned} -t(2n+1) < -2nt < -2nt + 3n + 2 + (n-t) = \\ (2-t)(2n+1) < 0 < t(2n-1) < 2nt < \\ 2nt + 2n + (n-2) + (n-t) = (t+2)(2n-1) \end{aligned}$$

because $3 \le t \le \nu - 1 \le n - 1 < n$ and $n \ge 5$. So, (2) holds when t is even.

Similarly, if t is odd, (7.3) is equivalent to

$$-\frac{t+1}{2} < -\frac{tn}{2n+1} < -\frac{t-1}{2} < 0 < \frac{t-1}{2} < \frac{tn}{2n-1} \le \frac{t+1}{2},$$

which is equivalent to

$$\begin{aligned} -(t+1)(2n+1) < -2nt < -2nt + (n+1) + (n-t) = \\ (1-t)(2n+1) < 0 < (t-1)(2n-1) < 2nt < \\ 2nt + (n-1) + (n-t) = (t+1)(2n-1). \end{aligned}$$

This ends the proof of (2).

To prove (3) observe that

$$n - \tfrac{\nu}{2} = \left\{ \begin{array}{ll} n - \frac{n}{2} = \frac{n}{2} = \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor & \text{if } n \text{ is even} \\ \frac{2(n-1)+2}{2} - \frac{n-1}{2} = \frac{n+1}{2} & \text{if } n \text{ is odd} \end{array} \right\} = \left\lfloor \tfrac{n+1}{2} \right\rfloor.$$

Thus, we can write

$$\begin{aligned} \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right) = \left\{ m \in \mathbb{N} : m \ge n\nu + 1 - \tfrac{\nu}{2} \right\} = \\ \left\{ tn + \left\lfloor \tfrac{n+1}{2} \right\rfloor + k : t \in \mathbb{N},\ t \ge \nu - 1 \text{ and } k \in \{1, 2, \dots, n\} \right\}. \end{aligned}$$

We have to show that there exists $p \in \mathbb{N}$ such that the inequalities (7.1) hold with $q = tn + \left\lfloor \frac{n+1}{2} \right\rfloor + k$, $t \in \mathbb{N}$, $t \geq \nu - 1$ and $k \in \{1, 2, \dots, n\}$. For $q = tn + \left\lfloor \frac{n+1}{2} \right\rfloor + k$, (7.1) is equivalent to

$$(2n-1)\big((t+1)n - \tfrac{\nu}{2} + k\big) = (2n-1)\big(tn + \left\lfloor \tfrac{n+1}{2} \right\rfloor + k\big) < \\ 2n^2 p < (2n+1)\big((t+1)n - \tfrac{\nu}{2} + k\big),$$

which is equivalent to

$$(7.4) \qquad (2n-1)\big(k - \tfrac{\nu}{2}\big) - (t+1)n < 2(p-t-1)n^2 < \\ (2n+1)\big(k - \tfrac{\nu}{2}\big) + (t+1)n.$$

When $k \leq \nu$ we have

$$(2n-1)\big(k - \tfrac{\nu}{2}\big) - (t+1)n \leq (2n-1)\tfrac{\nu}{2} - \nu n = -\frac{\nu}{2}, \text{ and}$$
$$2n + 1 - \frac{\nu}{2} = (2n+1)\big(1 - \tfrac{\nu}{2}\big) + \nu n \leq (2n+1)\big(k - \tfrac{\nu}{2}\big) + (t+1)n;$$

and the inequalities

$$-\frac{\nu}{2} < 2(p-t-1)n^2 < 2n + 1 - \frac{\nu}{2}$$

imply (7.4). Since the last expression holds with $p = t+1$, we get that (7.1) holds with $p = t+1$ and $q = tn + \left\lfloor \frac{n+1}{2} \right\rfloor + k$ (i.e., $q = tn + \left\lfloor \frac{n+1}{2} \right\rfloor + k \in M(c_n, d_n)$) whenever $k \leq \nu$.

Now assume that $k > \nu$. Since $k \leq n$ this implies that n is odd and $k = n > n - 1 = \nu$.

$$(2n-1)\big(k - \tfrac{\nu}{2}\big) - (t+1)n \leq (2n-1)\big(n - \tfrac{n-1}{2}\big) - \nu n = \\ (2n-1)\tfrac{n+1}{2} - (n-1)n = \frac{3n-1}{2}, \text{ and}$$
$$2n^2 + \tfrac{n+1}{2} = 2n^2 + \big(n - \tfrac{\nu}{2}\big) = (2n+1)\big(n - \tfrac{\nu}{2}\big) + \nu n \leq \\ (2n+1)\big(k - \tfrac{\nu}{2}\big) + (t+1)n;$$

and

$$\frac{3n-1}{2} < 2(p-t-1)n^2 < 2n^2 + \tfrac{n+1}{2}$$

implies (7.4). Since the last two inequalities hold with $p = t+2$ we get that $q = tn + \left\lfloor \frac{n+1}{2} \right\rfloor + k \in M(c_n, d_n)$ when $k = n > \nu$. This ends the proof of (3), and the computation of the set $M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right)$.

As a consequence of the above computation we get

$$\min M\left(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\right) = n \quad \text{and, hence,} \quad \lim_{n\to\infty} \min M\left(c_n, d_n\right) = \infty.$$

Then, by Theorem 5.3,

$$\lim_{n\to\infty} h\big(f_{3,n}\big) = \lim_{n\to\infty} \log \beta_{c_n, d_n} = 0.$$

Now we prove that $\mathsf{Per}\big(f_{3,n}\big) = M\left(\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right)$. Observe that

$$n\nu + 1 - \tfrac{\nu}{2} = (\nu-1)n + 1 + \big(n - \tfrac{\nu}{2}\big) < (n-1)n + 1 + (n-1) = n^2 < 2n^2.$$

Consequently,

$$2n^2 \in \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right) \subset M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big).$$

By Corollary 5.6,

$$\mathsf{Per}\big(f_{3,n}\big) = \{2n^2\} \cup M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big) = M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big).$$

Concerning the boundaries of cofiniteness observe that, since ν is always even, $\left\lfloor \frac{\nu-1}{2} \right\rfloor + \frac{\nu}{2} < \frac{\nu-1}{2} + \frac{\nu}{2} < \nu \leq n$. Hence, by (2),

$$\max \mathsf{Per}\big(f_{3,n}\big) \cap \left\{3n - \tfrac{\nu}{2} + 1, 3n - \tfrac{\nu}{2} + 2, \dots, n\nu - \tfrac{\nu}{2}\right\} =$$

$$\max M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big) \cap \left\{3n - \tfrac{\nu}{2} + 1, 3n - \tfrac{\nu}{2} + 2, \dots, n\nu - \tfrac{\nu}{2}\right\} =$$

$$\max\Big\{tn + k : t \in \{3, 4, \dots, \nu - 1\} \text{ and } k \in \left\{-\left\lfloor \tfrac{t-1}{2} \right\rfloor, -\left\lfloor \tfrac{t-1}{2} \right\rfloor + 1, \dots, 0, 1, \dots, \left\lfloor \tfrac{t}{2} \right\rfloor\right\}\Big\} =$$

$$(\nu - 1)n + \left\lfloor \tfrac{\nu-1}{2} \right\rfloor = \nu n + \left\lfloor \tfrac{\nu-1}{2} \right\rfloor - n < \nu n - \tfrac{\nu}{2}.$$

This implies that $\mathsf{StrBdCof}\left(f_{3,n}\right) = n\nu + 1 - \frac{\nu}{2}$. On the other hand, $n \in \mathsf{sBC}\big(f_{3,n}\big)$ and therefore, $\mathsf{BdCof}\left(f_{3,n}\right)$ exists and verifies

$$n \leq \mathsf{BdCof}\left(f_{3,n}\right) \leq n\nu + 1 - \tfrac{\nu}{2}$$

and, hence, $\lim_{n\to\infty} \mathsf{BdCof}\left(f_{3,n}\right) = \infty$.

Now we deal with the minimalistic extension of the family $\{f_{3,n}\}_{n\geq 5}$. In this example (since $n \geq 5$), we have

$$s_n = 2n^2 \geq 5n > 2(2n+1) = 2r_n.$$

On the other hand,

$$\left\lfloor \tfrac{1}{d_n} \right\rfloor = \left\lfloor \tfrac{s_n}{r_n} \right\rfloor = \left\lfloor \tfrac{2n^2}{2n+1} \right\rfloor = n - 1 < \tfrac{2n^2}{2n+1} = \tfrac{s_n}{r_n} < n = \left\lfloor \tfrac{2n^2}{2n-1} \right\rfloor = \left\lfloor \tfrac{q_n}{p_n} \right\rfloor = \left\lfloor \tfrac{1}{c_n} \right\rfloor.$$

Let G be an arbitrary graph with a circuit C. Then, all the assumptions of Theorems 6.2 and 6.3 are verified and, hence, $f_{3,n}$ is extendable to G with base at the circuit C with a transitive minimalistic extension $\phi_n^{\mathsf{G};3} = f_{3,n}^{\mathsf{G};\mathsf{C}}$, and

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};3}\big) = \mathsf{Per}\big(f_{3,n}\big) \quad \text{and} \quad \lim_{n\to\infty} h\big(\phi_n^{\mathsf{G};3}\big) = \lim_{n\to\infty} h\big(f_{3,n}\big) = 0.$$

Finally, $\phi_n^{\mathsf{G};3}$ is totally transitive by Theorem 1.2 because

$$\mathsf{Per}\big(\phi_n^{\mathsf{G};3}\big) = \mathsf{Per}\big(f_{3,n}\big) = M\Big(\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\Big) \supset \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right)$$

is cofinite. □

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Important Definitions

Nomenclature

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