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Limit cycle bifurcation from a zero-Hopf equilibrium for a class of 3-dimensional Kolmogorov systems

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ABSTRACT

A zero-Hopf equilibrium point p of a 3-dimensional autonomous differential system in $\mathbb{R}^3$ is an equilibrium point such that the eigenvalues of the linear part of the system at p are 0 and $\pm\omega i$ with $\omega \neq 0$. A zero-Hopf bifurcation takes place when from a zero-Hopf equilibrium bifurcate some small-amplitude limit cycles moving the parameters of the system.

Polynomial Kolmogorov systems are the natural extension of the polynomial Lotka–Volterra systems of degree 2 to higher degree. The Kolmogorov systems have been studied intensively due to their applications for modeling many natural phenomena.

In this paper we characterize, up to first order in the averaging theory, the eight distinct zero-Hopf bifurcations which can exhibit the class of 3-dimensional Kolmogorov systems of degree 3, providing an explicit approximation of the bifurcated small-amplitude limit cycles, together with information about their kind of stability.

From each one of this eight different zero-Hopf bifurcations emerges one or two limit cycle using the averaging theory of first order. Moreover we provide an explicit example of each one of these eight zero-Hopf bifurcations.

1. Introduction

As usual a *limit cycle* of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.

A *zero-Hopf equilibrium point* p of a 3–dimensional autonomous differential system in $\mathbb{R}^3$ is an equilibrium such that the eigenvalues of the linear part of the system at p are 0 and $\pm\omega i$ with $\omega \neq 0$.

For an equilibrium point p of a 3–dimensional autonomous differential system in $\mathbb{R}^3$, such that the eigenvalues of the linear part of the system at p are ρ and $\pm\omega i$ with $\rho\omega \neq 0$, there is a general theory (see for instance pages 175–180 of Ref. 1) for studying the limit cycles which can bifurcate from p changing a little the parameters of the system. Such a general theory does not exist for a zero-Hopf equilibrium. But the averaging theory provides an algorithm for solving this problem, see for instance.[2]

The *Kolmogorov systems*[3] in $\mathbb{R}^3$ are of the form

$$\begin{aligned} \dot{x} &= xP(x,y,z),\\ \dot{y} &= yQ(x,y,z),\\ \dot{z} &= zR(x,y,z), \end{aligned} \tag{1}$$

where P, Q and R are polynomials of degree larger than one. In fact, the Kolmogorov systems are a generalization of the Lotka–Volterra systems[4,5] which are the systems (1) when the degree of the polynomials P, Q and R is one.

Kolmogorov systems have been studied intensively because they can model the dynamics of many natural phenomena, see for instance the books.[6–8]

To control the existence of limit cycles and their stability is one of the main objectives of the qualitative theory of differential systems. The zero-Hopf bifurcations allow to study the limit cycles bifurcating from the zero-Hopf equilibrium points. The objective of these paper is to classify all the zero-Hopf equilibria of the Kolmogorov systems of degree 3 in $\mathbb{R}^3$ and the limit cycles bifurcating from themselves.

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Thus, first we study the zero-Hopf equilibrium points of Kolmogorov systems in $\mathbb{R}^3$ of degree 3, and we obtain that there are exactly eight families of such Kolmogorov systems having zero-Hopf equilibria. After using the averaging theory of first order we prove that from such equilibrium points moving the parameters of the system bifurcates one periodic orbit. Finally we also provide for each one of these eight families an example of such zero-Hopf bifurcation, we plot their bifurcated limit cycles, and we study their stability.

This kind of results for the Lotka–Volterra systems in $\mathbb{R}^3$ were studied in Ref. 9.

For other differential systems the zero-Hopf bifurcation has been studied by many authors, for instance in Refs. 1, 10–16. In some cases the existence of a zero-Hopf bifurcation can imply a local birth of "chaos" see for instance the articles (cf. Refs. 16–20).

For their applications the dynamics of the Kolmogorov systems are mainly studied in the positive octant. Let (a,b,c) be an equilibrium point of a Kolmogorov system (1) with all its coordinates positive. Doing the rescaling of the variables $(x,y,z)\to(x/a,y/b,z/c)$, we can assume without loss of generality that such an equilibrium point is the point $(1,1,1)$. Then the general Kolmogorov systems in $\mathbb{R}^3$ of degree 3 having the equilibrium point $(1,1,1)$ are

$$\begin{aligned}\dot{x} &= x(a_1(x-1)+b_1(y-1)+c_1(z-1)+d_1(x-1)^2+e_1(x-1)(y-1)+f_1(x-1)(z-1)+g_1(y-1)^2\\ &\quad +h_1(y-1)(z-1)+k_1(z-1)^2),\\ \dot{y} &= y(a_2(x-1)+b_2(y-1)+c_2(z-1)+d_2(x-1)^2+e_2(x-1)(y-1)+f_2(x-1)(z-1)+g_2(y-1)^2\\ &\quad +h_2(y-1)(z-1)+k_2(z-1)^2),\\ \dot{z} &= z(a_3(x-1)+b_3(y-1)+c_3(z-1)+d_3(x-1)^2+e_3(x-1)(y-1)+f_3(x-1)(z-1)+g_3(y-1)^2\\ &\quad +h_3(y-1)(z-1)+k_3(z-1)^2).\end{aligned} \tag{2}$$

Note that the differential system (2) depends on 27 parameters.

In the next proposition we characterize when the equilibrium point $(1,1,1)$ of the Kolmogorov systems (2) is a zero-Hopf equilibrium.

Proposition 1. *There are eight m–parameter families of Kolmogorov systems (2) for which the equilibrium point $(1,1,1)$ is a zero-Hopf equilibrium. Namely:*

(i) $a_1=b_1=b_2=c_1=c_3=0,\ b_3c_2<0$, *for this family* $m=22$;
(ii) $a_1=-b_2, c_1=c_2=c_3=0,\ a_2b_1+b_2^2<0$, *for this family* $m=23$;
(iii) $a_1=b_1=c_1=0,\ c_3=-b_2,\ b_3c_2+b_2^2<0$, *for this family* $m=23$;
(iv) $a_1=-c_3, b_1=b_2=b_3=0,\ a_3c_1+c_3^2<0$, *for this family* $m=23$;
(v) $a_1=-c_3,\ a_2=-\dfrac{c_2c_3}{c_1},\ b_1=b_2=0,\ a_3c_1+b_3c_2+c_3^2<0$, *for this family* $m=23$;
(vi) $a_1=-b_2-c_3,\ b_3=\dfrac{b_1c_3}{c_1},\ c_2=\dfrac{b_2c_1}{b_1},\ a_2b_1+a_3c_1+b_2^2+2b_2c_3+c_3^2<0$, *for this family* $m=24$;
(vii) $a_1=-b_2-c_3,\ a_2=-\dfrac{b_2(b_2+c_3)}{b_1},\ c_2=\dfrac{b_2c_1}{b_1},\ \dfrac{1}{b_1}(a_3b_1c_1+b_2b_3c_1+b_1c_3^2)<0$, *for this family* $m=24$;
(viii) $a_1=-b_2-c_3,\ a_3=\dfrac{a_2(b_1c_3-b_3c_1)+(b_2+c_3)(b_2c_3-b_3c_2)}{b_1c_2-b_2c_1},\ \dfrac{A}{b_1c_2-b_2c_1}>0$, *for this family* $m=25$,
where $A=-(-b_2^3c_1+b_1b_2^2c_2+b_2(-2b_3c_1c_2+b_1c_2c_3)+c_2(b_1b_3c_2-b_3c_1c_3+b_1c_3^2)+a_2(b_1^2c_2-b_3c_1^2+b_1c_1(c_3-b_2)))$.

Proposition 1 is proved in Section 3.

We define the following eight sets of conditions:

(i) $a_1=a_{11}\varepsilon,\ b_1=b_{11}\varepsilon,\ b_2=b_{21}\varepsilon,\ c_1=c_{11}\varepsilon,\ c_3=c_{31}\varepsilon,\ b_3c_2<0$;
(ii) $a_1=-(b_{20}+b_{21}\varepsilon),\ b_2=b_{20}+b_{21}\varepsilon,\ c_1=c_{11}\varepsilon,\ c_2=c_{21}\varepsilon,\ c_3=c_{31}\varepsilon,\ a_2b_1+b_2^2<0$;
(iii) $a_1=a_{11}\varepsilon,\ b_2=b_{20}+b_{21}\varepsilon,\ c_1=c_{11}\varepsilon,\ b_1=b_{11}\varepsilon,\ c_3=-(b_{20}+b_{21}\varepsilon),\ b_3c_2+b_2^2<0$;
(iv) $a_1=-(c_{30}+c_{31}\varepsilon),\ b_1=b_{11}\varepsilon,\ b_2=b_{21}\varepsilon,\ b_3=b_{31}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ a_3c_1+c_3^2<0$;
(v) $a_1=-(c_{30}+c_{31}\varepsilon),\ b_1=b_{11}\varepsilon,\ b_2=b_{21}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ a_2=-\dfrac{c_2c_{30}}{c_1}-\dfrac{c_2c_{31}}{c_1}\varepsilon,\ a_3c_1+b_3c_2+c_3^2<0$;
(vi) $a_1=-b_{20}-c_{30}-(b_{21}+c_{31})\varepsilon,\ b_2=b_{20}+b_{21}\varepsilon,\ b_3=\dfrac{b_1c_{30}}{c_1}+\dfrac{b_1c_{31}}{c_1}\varepsilon,\ c_2=\dfrac{b_{20}c_1}{b_1}+\dfrac{b_{21}c_1}{b_1}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ a_2b_1+a_3c_1+b_2^2+2b_2c_3+c_3^2<0$;
(vii) $a_1=-b_{20}-c_{30}-(b_{21}+c_{31})\varepsilon,\ b_2=b_{20}+b_{21}\varepsilon,\ a_2=\dfrac{-b_{20}(b_{20}+c_{30})}{b_1}-\dfrac{b_{20}(b_{21}+c_{31})+b_{21}(b_{20}+c_{30})}{b_1}\varepsilon,\ c_2=\dfrac{b_{20}c_1}{b_1}+\dfrac{b_{21}c_1}{b_1}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ \dfrac{1}{b_1}(a_3b_1c_1+b_2b_3c_1+b_1c_3^2)<0$;
(viii) $a_1=-b_2-c_{30}-c_{31}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ \dfrac{A}{b_1c_2-b_2c_1}>0,\ a_3=\dfrac{1}{b_1c_2-b_2c_1}\Big(a_2(b_1c_{30}-b_3c_1)+(b_2+c_{30})(b_2c_{30}-b_3c_2)\Big)+\dfrac{a_2b_1c_{31}+(b_2+c_{30})b_2c_{31}+c_{31}(b_2c_{30}-b_3c_2)}{b_1c_2-b_2c_1}\varepsilon$.

Theorem 2. *Assume that the family of Kolmogorov systems (2) of Proposition 1 is perturbed by the condition (r), for $r\in\{i,ii,iii,iv,v,vi,vii,viii\}$. Then for $\varepsilon\neq 0$ sufficiently small the perturbed system (2) has at least one periodic orbit $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ bifurcating from the zero-Hopf equilibrium $(1,1,1)$ when $\varepsilon=0$.*

The structure of the rest of this paper is the following. In Section 2 we present the basic results of the averaging theory that we need for proving Theorem 2, and in Section 3 we prove Proposition 1 and Theorem 2.

2. The averaging theory for periodic orbits

The averaging theory is a classical and matured tool for studying the dynamics of nonlinear smooth dynamical systems, and in particular for studying their periodic orbits. The averaging theory has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of this theory. The first formalization of this theory is due to Fatou[21] in 1928. Important practical and theoretical contributions to this theory were made by Krylov and Bogoliubov[22] in the 1930s and by Bogoliubov[23] in 1945. The averaging theory of first order for studying periodic orbits can be found in Ref. 24, see also Ref. 12.

Now we present the basic results of the averaging theory of first order that we shall need for proving the results of this paper. The next theorem also provides a first order approximation for the periodic orbits of a periodic differential system and some information about their kind of stability, for a proof see for instance Theorems 11.5 and 11.6 of Verhulst[24].

Consider the differential system

$$\dot{\mathbf{x}} = \varepsilon F(t,\mathbf{x}) + \varepsilon^2 G(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0)=\mathbf{x}_0, \tag{3}$$

with $\mathbf{x}\in D$, where D is an open subset of $\mathbb{R}^n$, $t\geq 0$. Moreover we assume that both $F(t,\mathbf{x})$ and $G(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0)=\mathbf{x}_0, \tag{4}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(t,\mathbf{y})dt. \tag{5}$$

Theorem 3. *Consider the two initial value problems (3) and (4). Suppose that*

(i) F, its Jacobian $\partial F/\partial x$, its Hessian $\partial^2 F/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon\in(0,\varepsilon_0]$.
(ii) F and G are $T-$periodic in t (T independent of ε).

Then the following statements hold.

(a) If p is an equilibrium point of the averaged Eq. (4) and

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\Bigg|_{\mathbf{y}=p} \neq 0, \tag{6}$$

then there exists a limit cycle $\mathbf{x}(t,\varepsilon)$ of period T of Eq. (3) such that $\mathbf{x}(0,\varepsilon)\to p$ as $\varepsilon\to 0$.
(b) The stability or instability of the limit cycle $\mathbf{x}(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system (4). If p is a simple zero of the averaged system (4) the eigenvalues of the Jacobian matrix evaluated at p provides the linear stability, i.e. if some eigenvalue has positive real part then the limit cycle associated to the zero p is unstable; if all the eigenvalues have negative real part then the limit cycle is stable.

3. Proofs

Proof of Proposition 1. The characteristic polynomial of the linear part of the Kolmogorov system (2) at the equilibrium point $(1,1,1)$ is

$$\begin{aligned} p(\lambda) \;=\; & -\lambda^3 + (a_1+b_2+c_3)\lambda^2 + (a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3)\lambda \\ & + a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \end{aligned}$$

Imposing that the characteristic polynomial $p(\lambda) = -\lambda(\lambda^2+\omega^2)$ with $\omega\neq 0$, i.e. imposing that the eigenvalues at the equilibrium $(1,1,1)$ be 0, ωi and $-\omega i$ with $\omega\neq 0$ and consequently the equilibrium point $(1,1,1)$ will be a zero-Hopf equilibrium, we obtain the system

$$\begin{aligned} a_1+b_2+c_3 &= 0, \\ a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3 &= -\omega^2, \\ a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 &= 0. \end{aligned}$$

Tedious computations show that the solutions of the previous system are the eight families of zero-Hopf equilibria described in the statement of Proposition 1. These computations have been also verified using the algebraic manipulators maple and mathematica. This completes the proof. ☐

For $r\in\{i,ii,iii,iv,v,vi,vii,viii\}$ we shall prove that one periodic orbit bifurcates from the zero-Hopf equilibrium point $(1,1,1)$ of the family (r) of the Kolmogorov systems (2) of Proposition 1 satisfying the condition (r) when the parameters of this family are perturbed. We shall provide all the details of the proof for (r)=(i), the proofs for the other families are analogous and we only indicate the main steps of those proofs.

Proof of Theorem 2 under the condition (i)**.** We consider the Kolmogorov system (2) perturbed with the parameters given in the condition (i). After we translate the equilibrium point $(1,1,1)$ of the perturbed system to the origin of coordinates doing the change of variables $x=X+1, y=Y+1, z=Z+1$. Then this perturbed Kolmogorov system in the new variables (X,Y,Z) becomes

$$\begin{aligned} \dot{X} &= (X+1)\left(X^2d_1 + XYe_1 + XZf_1 + \varepsilon Xa_{11} + Y^2g_1 + YZh_1 + \varepsilon Yb_{11} + Z^2k_1 + \varepsilon Zc_{11}\right), \\ \dot{Y} &= (Y+1)\left(X^2d_2 + XYe_2 + XZf_2 + Y^2g_2 + YZh_2 + Y\varepsilon b_{21} + Z^2k_2 + Xa_2 + Zc_2\right), \\ \dot{Z} &= (Z+1)\left(X^2d_3 + XYe_3 + XZf_3 + Y^2g_3 + YZh_3 + Z^2k_3 + Z\varepsilon c_{31} + Xa_3 + Yb_3\right). \end{aligned} \tag{7}$$

In order to facilitate the application of the averaging theory, described in Section 2, for studying the zero-Hopf bifurcation of system (7) at the origin of coordinates, we write the linear part of system (7) with $\varepsilon=0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-b_3c_2} & 0 \\ \sqrt{-b_3c_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{a_3}{\sqrt{-b_3c_2}} & -\sqrt{-\frac{b_3}{c_2}} & 0 \\ \frac{a_2}{c_2} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \text{ with inverse } \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ b_3\sqrt{-\frac{c_2}{b_3^3}} & 0 & -\frac{a_3}{b_3} \\ 0 & 1 & -\frac{a_2}{c_2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{8}$$

In the variables (u, v, w) the differential system (7) writes

$$\begin{aligned}
\dot u = \;& -\frac{1}{\sqrt{-c_2b_3}c_2^2b_3^2}(((u^2b_3g_2 - vb_3^2)c_2^3 + ((-w^2d_2 + (-vf_2 - e_2)w - k_2v^2 - vh_2)b_3^2 - ((e_1-2e_2)w^2 + ((h_1-2h_2) \\
& v + e_1 - 2g_2)w + h_1v)a_3b_3 + a_3^2w((2g_1-3g_2)w+2g_1))c_2^2 + a_2((2vk_2 + wf_2 + h_2)b_3 + ((h_1-2h_2)w+h_1)a_3 \\
&)wb_3c_2 - k_2w^2a_2^2b_3^2)u\sqrt{-c_2b_3} + b_3(-vb_3^2 + ((u^2e_2 + va_3)w + u^2(vh_2+g_2))b_3 + a_3((g_1-3g_2)w+g_1)u^2)c_2^3 + \\
& ((-v^2k_2 - vwf_2 - w^2d_2)b_3^3 + (-a_3(d_1-d_2)w^3 + a_3((-f_1+f_2)v + e_2 - d_1)w^2 + (-v((k_1-k_2)v + f_1 - h_2) \\
& a_3 - u^2h_2a_2)w - k_1v^2a_3)b_3^2 + a_3^2w((e_1-e_2)w^2 + ((h_1-h_2)v + e_1 - g_2)w + h_1v)b_3 - a_3^3w^2((g_1-g_2)w+g_1) \\
&)c_2^2 + a_2((2vk_2 + wf_2)b_3^2 + a_3((f_1-f_2)w^2 + ((2k_1-2k_2)v + f_1 - h_2)w + 2k_1v)b_3 - a_3^2w((h_1-h_2)w+h_1)) \\
& wb_3c_2 - (k_2b_3 + ((k_1-k_2)w+k_1)a_3)a_2^2w^2b_3^2) + \frac{\varepsilon}{c_2^2b_3^2(-c_2b_3)^{\frac{3}{2}}}(c_2^2(-\sqrt{-c_2b_3}c_2u(((b_{11}-2b_{21})w + b_{11})a_3 + \\
& b_{21}b_3) + u^2b_3b_{21}c_2^2 - a_3(((b_{21}-b_{11})w - b_{11})wa_3 - (-w^2a_{11} + (-vc_{11} - a_{11} + b_{21})w - c_{11}v)b_3)c_2 + wa_2a_3b_3 \\
& c_{11}(w+1))b_3^2), \\
\dot v = \;& -\frac{1}{c_2^3b_3^2}(-((v+1)(b_3^2 + (vh_3 + we_3)b_3 - 2g_3wa_3)c_2^2 + a_2(-wb_3^2 + ((e_1-e_3)w^2 + ((h_1-2h_3)v + e_1 - h_3)w + \\
& h_1v)b_3 - 2a_3w((g_1-g_3)w+g_1))c_2 - a_2^2wb_3((h_1-h_3)w+h_1))c_2u\sqrt{-c_2b_3} + u^2g_3b_3(v+1)c_2^4 + (-(v+1)(v^2 \\
& k_3 + vwf_3 + w^2d_3)b_3^2 + (e_3a_3(v+1)w^2 + (u^2(g_1-g_3)a_2 + h_3va_3(v+1))w + u^2g_1a_2)b_3 - g_3w^2a_3^2(v+1))c_2^3 \\
& +(((-d_1+d_3)w^3 + ((-f_1+2f_3)v + f_3 - d_1)w^2 - v((k_1-3k_3)v + f_1 - 2k_3)w - k_1v^2)b_3^2 + a_3w((e_1-e_3)w^2 \\
& +((h_1-2h_3)v + e_1 - h_3)w + h_1v)b_3 - a_3^2w^2((g_1-g_3)w+g_1))a_2c_2^2 + a_2^2wb_3(((f_1-f_3)w^2 + ((2k_1-3k_3)v + \\
& f_1 - k_3)w + 2k_1v)b_3 - a_3w((h_1-h_3)w+h_1))c_2 - a_2^3((k_1-k_3)w+k_1)w^2b_3^2) - \frac{\varepsilon}{c_2^3b_3^2}(-c_2b_3(ua_2c_2b_{11}(w+1) \\
& \sqrt{-c_2b_3} - ((c_{11}-c_{31})w + c_{11})wb_3a_2^2 + c_2((w^2a_{11} + ((-2v-1)c_{31} + c_{11}v + a_{11})w + c_{11}v)b_3 - wa_3b_{11}(w+1 \\
&))a_2 + vb_3c_2^2c_{31}(v+1))), \\
\dot w = \;& -\frac{1}{c_2^2b_3^2}((w+1)(-c_2(((vh_1+we_1)b_3 - 2g_1wa_3)c_2 - h_1wa_2b_3)\sqrt{-c_2b_3}u + u^2b_3c_2^3g_1 + ((-v^2k_1 - vwf_1 - w^2d_1 \\
&)b_3^2 + wa_3(vh_1+we_1)b_3 - g_1w^2a_3^2)c_2^2 + a_2w((2vk_1 + wf_1)b_3 - h_1wa_3)b_3c_2 - w^2a_2^2b_3^2k_1)) - \frac{\varepsilon}{c_2^2b_3^2}(-(w+1)c_2 \\
& b_3(\sqrt{-c_2b_3}ub_{11}c_2 + ((vc_{11} + wa_{11})b_3 - wa_3b_{11})c_2 - a_2wb_3c_{11})).
\end{aligned} \tag{9}$$

Doing the rescaling of the variables $(u, v, w) = (\varepsilon U, \varepsilon V, \varepsilon W)$ system (9) in the new variables (U, V, W) writes

$$\begin{aligned}
\dot U = \;& \frac{b_3c_2V}{\sqrt{-b_3c_2}} - \frac{\varepsilon}{\sqrt{-b_3c_2}b_3^2c_2^2}(((-a_2^2k_2 + a_2c_2f_2 - c_2^2d_2)b_3^3 + (a_3(e_2-d_1)c_2^2 + a_2a_3(f_1-h_2)c_2 - k_1a_2^2a_3)b_3^2 + (a_3 \\
& (e_1-g_2)c_2 - h_1a_2a_3)a_3c_2b_3 - g_1a_3^3c_2^2)W^2 + (-U((-a_2h_2 + c_2e_2)b_3^2 + ((e_1-2g_2)c_2 - h_1a_2)a_3b_3 - 2a_3^2c_2g_1) \\
& c_2\sqrt{-c_2b_3} + (2Va_2c_2k_2 - Vc_2^2f_2)b_3^3 + (Va_3c_2^3 + a_3((-f_1+h_2)V - a_{11} + b_{21})c_2^2 + a_2a_3(2Vk_1 + c_{11})c_2)b_3^2 \\
& +(Vh_1 + b_{11})c_2^2a_3^2b_3)W - U((Vc_2^2 + (Vh_2 + b_{21})c_2)b_3^2 + (Vh_1 + b_{11})c_2a_3b_3)c_2\sqrt{-c_2b_3} - V^2k_2c_2^2b_3^3 + (U^2g_2 \\
& c_2^3 + a_3(-V^2k_1 - Vc_{11})c_2^2)b_3^2 + U^2g_1c_2^3a_3b_3), \\
\dot V = \;& U\sqrt{-c_2b_3} - \frac{\varepsilon}{c_2^3b_3^2}((((-a_3^2g_3 + a_3b_3e_3 - b_3^2d_3)c_2^3 + ((f_3-d_1)b_3^2 + (e_1-h_3)a_3b_3 - g_1a_3^2)a_2c_2^2 + a_2^2b_3((f_1-k_3) \\
& b_3 - h_1a_3)c_2 - a_2^3b_3^2k_1)W^2 - (c_2((-2a_3g_3 + b_3e_3)c_2^2 + (-b_3^2 + (e_1-h_3)b_3 - 2g_1a_3)a_2c_2 - h_1a_2^2b_3)U\sqrt{-c_2b_3} \\
& -(Va_3b_3h_3 - Vb_3^2f_3)c_2^3 - (((-f_1+2k_3)V - a_{11} + c_{31})b_3^2 + (Vh_1 + b_{11})a_3b_3)a_2c_2^2 - a_2^2b_3^2(2Vk_1 + c_{11})c_2)W) \\
& +(-c_2((Vb_3^2 + Vb_3h_3)c_2^2 + (Vh_1 + b_{11})b_3a_2c_2)U\sqrt{-c_2b_3} + U^2b_3c_2^4g_3 + ((-V^2k_3 - Vc_{31})b_3^2 + U^2a_2g_1b_3)c_2^3 \\
& +(-V^2k_1 - Vc_{11})b_3^2a_2c_2^2)), \\
\dot W = \;& -\frac{\varepsilon}{c_2^2b_3^2}(((-a_2^2k_1 + a_2c_2f_1 - c_2^2d_1)b_3^2 - a_3c_2(a_2h_1 - c_2e_1)b_3 - c_2^2g_1a_3^2)W^2 + (((a_2h_1 - c_2e_1)b_3 + 2a_3c_2g_1)U \\
& \sqrt{-c_2b_3} + b_3(((-Vf_1 - a_{11})c_2 + a_2(2Vk_1 + c_{11}))b_3 + (Vh_1 + b_{11})c_2a_3))c_2W + c_2^2b_3(-U(Vh_1 + b_{11})\sqrt{c_2b_3} \\
& +(-V^2k_1 - Vc_{11})b_3 + U^2c_2g_1)).
\end{aligned} \tag{10}$$

Now we pass from the differential system (10) to cylindrical coordinates (r, θ, W) defined by $U = r\cos\theta$ and $V = r\sin\theta$, and we obtain

$$
\begin{aligned}
\dot{r} = {} & \frac{\varepsilon}{\sqrt{-c_2 b_3 b_3^2 c_2^3}}(((-b_3c_2r((g_3-b_3)c_2^2+((-h_2+k_3)b_3+g_1a_2-h_1a_3)c_2+k_1a_2b_3)\sin(\theta)+(((e_2-f_3)W+\\
& b_{21}-c_{31})b_3^2+((e_1-2g_2+h_3)W+b_{11})a_3b_3-2Wg_1a_3^2)c_2^2-a_2b_3^2((f_1+h_2-2k_3)W+c_{11})c_2+2b_3^2Wa_2^2k_1)\\
& c_2r\cos^2(\theta)+(((W^2d_3+r^2k_3)b_3^2-W^2e_3a_3b_3+W^2g_3a_3^2)c_2^3-a_2(((f_3-d_1)W^2+(-a_{11}+c_{31})W-r^2k_1)b_3^2\\
& +W((e_1-h_3)W+b_{11})a_3b_3-W^2g_1a_3^2)c_2^2-Wa_2^2b_3(((f_1-k_3)W+c_{11})b_3-Wh_1a_3)c_2+W^2k_1a_2^3b_3^2)\sin(\theta)\\
& +(((Wf_3+c_{31})b_3-Wh_3a_3)c_2^2+(((f_1-2k_3)W+c_{11})b_3-Wh_1a_3)a_2c_2-2Wk_1a_2^2b_3)b_3c_2r)\sqrt{-b_3c_2}-c_2\\
& \cos(\theta)(((-b_3^2+(g_2-h_3)b_3+g_1a_3)c_2-b_3(a_2h_1-a_3k_1-b_3k_2))b_3c_2^2r^2\cos^2(\theta)+b_3c_2r(((e_3+a_3)b_3-2g_3a_3)\\
& Wc_2^2+(-W(f_2+a_2)b_3^2+(((e_1-h_3)a_2-a_3(f_1-h_2))W+b_{11}a_2-c_{11}a_3)b_3+(-2a_2a_3g_1+a_3^2h_1)W)c_2-\\
& Wa_2b_3(a_2h_1-2a_3k_1-2b_3k_2))\sin(\theta)+r^2b_3^2(h_3+b_3)c_2^3+((-W^2d_2-r^2k_2)b_3^3+(a_3(e_2-d_1)W^2-a_3(a_{11}-\\
& b_{21})W+r^2(a_2h_1-a_3k_1))b_3^2+Wa_3^2((e_1-g_2)W+b_{11})b_3-W^2g_1a_3^3)c_2^2+(Wf_2b_3^2+((f_1-h_2)W+c_{11})a_3b_3\\
& -Wh_1a_3^2)Wa_2b_3c_2-W^2a_2^2b_3^2(a_3k_1+b_3k_2))),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\dot{\theta} = {} & -\frac{b_3c_2}{\sqrt{-b_3c_2}}+\frac{\varepsilon}{\sqrt{-b_3c_2}c_2^3b_3^2r}(\cos\theta(-b_3((g_3-b_3)c_2^2+((-h_2+k_3)b_3+g_1a_2-h_1a_3)c_2+k_1a_2b_3)c_2^2r^2\cos^2\theta-(\\
& (((e_2-f_3)W+b_{21}-c_{31})b_3^2+((e_1-2g_2+h_3)W+b_{11})a_3b_3-2Wg_1a_3^2)c_2^2-a_2b_3^2((f_1+h_2-2k_3)W+c_{11})\\
& c_2+2b_3^2Wa_2^2k_1)c_2r\sin\theta-c_2^4b_3^2r^2+((W^2d_3-r^2(h_2-k_3))b_3^2-a_3(W^2e_3+r^2h_1)b_3+W^2g_3a_3^2)c_2^3-a_2(((f_3\\
& -d_1)W^2+(-a_{11}+c_{31})W-r^2k_1)b_3^2+W((e_1-h_3)W+b_{11})a_3b_3-W^2g_1a_3^2)c_2^2-Wa_2^2b_3(((f_1-k_3)W+c_{11}\\
&)b_3-Wh_1a_3)c_2+W^2k_1a_2^3b_3^2)\sqrt{-b_3c_2}+(-(-c_2((-b_3^2+(g_2-h_3)b_3+g_1a_3)c_2-b_3(a_2h_1-a_3k_1-b_3k_2))r\\
& \sin\theta+((e_3+a_3)b_3-2g_3a_3)Wc_2^2+(-W(f_2+a_2)b_3^2+(((-f_1+h_2)a_3+(e_1-h_3)a_2)W+b_{11}a_2-c_{11}a_3)b_3\\
& +(-2a_2a_3g_1+a_3^2h_1)W)c_2-Wa_2b_3(a_2h_1-2a_3k_1-2b_3k_2))b_3c_2r\cos^2\theta+(((-W^2d_2-r^2k_2)b_3^3+a_3((e_2-\\
& d_1)W^2+(-a_{11}+b_{21})W-r^2k_1)b_3^2+Wa_3^2((e_1-g_2)W+b_{11})b_3-W^2g_1a_3^3)c_2^2+(Wf_2b_3^2+((f_1-h_2)W+\\
& c_{11})a_3b_3-Wh_1a_3^2)Wa_2b_3c_2-W^2a_2^2b_3^2(a_3k_1+b_3k_2))\sin\theta-b_3c_2(-Wa_3c_2^2b_3+(Wf_2b_3^2+((f_1-h_2)W+c_{11}\\
&)a_3b_3-Wh_1a_3^2)c_2-2Wa_2b_3(a_3k_1+b_3k_2))r)c_2),\\
\dot{W} = {} & \frac{\varepsilon}{c_2^2b_3^2}(((a_2^2k_1-a_2c_2f_1+c_2^2d_1)b_3^2+a_3c_2(a_2h_1-c_2e_1)b_3+a_3^2c_2^2g_1)W^2-c_2(r((a_3h_1-b_3f_1)c_2+2k_1a_2b_3)b_3\\
& \sin\theta+r((2a_3g_1-b_3e_1)c_2+h_1a_2b_3)\sqrt{-b_3c_2}\cos\theta+(a_3b_3b_{11}-a_{11}b_3^2)c_2+a_2b_3^2c_{11})W+rb_3c_2^2(-r(b_3k_1+\\
& c_2g_1)\cos^2\theta+\cos\theta(r\sin\theta h_1+b_{11})\sqrt{-b_3c_2}+b_3(rk_1+c_{11}\sin\theta))).
\end{aligned}
$$

We change the independent variable from t to θ, and denoting the derivative with respect to θ by a dot, then the differential system (11) becomes

$$
\begin{aligned}
\dot{r} = {} & -\frac{\varepsilon}{b_3^3c_2^4}(((-b_3c_2r((g_3-b_3)c_2^2+((-h_2+k_3)b_3+g_1a_2-h_1a_3)c_2+k_1a_2b_3)\sin(\theta)+(((e_2-f_3)W+b_{21}-c_{31})\\
& b_3^2+((e_1-2g_2+h_3)W+b_{11})a_3b_3-2Wg_1a_3^2)c_2^2-a_2b_3^2((f_1+h_2-2k_3)W+c_{11})c_2+2b_3^2Wa_2^2k_1)c_2r\cos^2(\theta)\\
& +(((W^2d_3+r^2k_3)b_3^2-W^2e_3a_3b_3+W^2g_3a_3^2)c_2^3-a_2(((f_3-d_1)W^2+(-a_{11}+c_{31})W-r^2k_1)b_3^2+W((e_1-\\
& h_3)W+b_{11})a_3b_3-W^2g_1a_3^2)c_2^2-Wa_2^2b_3(((f_1-k_3)W+c_{11})b_3-Wh_1a_3)c_2+W^2k_1a_2^3b_3^2)\sin(\theta)+(((Wf_3+\\
& c_{31})b_3-Wh_3a_3)c_2^2+(((f_1-2k_3)W+c_{11})b_3-Wh_1a_3)a_2c_2-2Wk_1a_2^2b_3)b_3c_2r)\sqrt{-b_3c_2}-c_2\cos(\theta)(((-b_3^2+\\
& (g_2-h_3)b_3+g_1a_3)c_2-b_3(a_2h_1-a_3k_1-b_3k_2))b_3c_2^2r^2\cos^2(\theta)+b_3c_2r(((e_3+a_3)b_3-2g_3a_3)Wc_2^2+(-W(f_2\\
& +a_2)b_3^2+(((e_1-h_3)a_2-a_3(f_1-h_2))W+b_{11}a_2-c_{11}a_3)b_3+(-2a_2a_3g_1+a_3^2h_1)W)c_2-Wa_2b_3(a_2h_1-2\\
& a_3k_1-2b_3k_2))\sin(\theta)+r^2b_3^2(h_3+b_3)c_2^3+((-W^2d_2-r^2k_2)b_3^3+(a_3(e_2-d_1)W^2-a_3(a_{11}-b_{21})W+r^2(a_2\\
& h_1-a_3k_1))b_3^2+Wa_3^2((e_1-g_2)W+b_{11})b_3-W^2g_1a_3^3)c_2^2+(Wf_2b_3^2+((f_1-h_2)W+c_{11})a_3b_3-Wh_1a_3^2)W\\
& a_2b_3c_2-W^2a_2^2b_3^2(a_3k_1+b_3k_2)))=F_1(\theta,r,W),\\
\dot{W} = {} & -\frac{\varepsilon}{c_2^3b_3^3}(((a_2^2k_1-a_2c_2f_1+c_2^2d_1)b_3^2+a_3c_2(a_2h_1-c_2e_1)b_3+a_3^2c_2^2g_1)W^2-c_2(r((a_3h_1-b_3f_1)c_2+2k_1a_2b_3)b_3\\
& \sin\theta+r((2a_3g_1-b_3e_1)c_2+h_1a_2b_3)\sqrt{-b_3c_2}\cos\theta+(a_3b_3b_{11}-a_{11}b_3^2)c_2+a_2b_3^2c_{11})W+rb_3c_2^2(-r(b_3k_1+\\
& c_2g_1)\cos^2\theta+\cos\theta(r\sin\theta h_1+b_{11})\sqrt{-b_3c_2}+b_3(rk_1+c_{11}\sin\theta)))=F_2(\theta,r,W).
\end{aligned}
\tag{12}
$$

The differential system (12) is writen in the normal form (3) ready for applying to it the averaging theory of Section 2, where $\mathbf{x}=(r,W)$, $t=\theta$, $T=2\pi$,

$$
\begin{pmatrix} \dot{r} \\ \dot{W} \end{pmatrix} = F(\theta,r,W) = \begin{pmatrix} F_1(\theta,r,W) \\ F_2(\theta,r,W) \end{pmatrix}, \quad \text{and} \quad f(r,W) = \begin{pmatrix} f_1(r,W) \\ f_2(r,W) \end{pmatrix}.
\tag{13}
$$

It is immediate to check that system (12) satisfies all the assumptions of Theorem 3. Now we compute the integrals (5), i.e.

$$\begin{aligned} f_1(r,W) = &\ \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,W)d\theta = \frac{r\sqrt{-c_2b_3}}{2c_2^3b_3^3}(T_1W+N_1),\\ f_2(r,W) = &\ \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,W)d\theta = -\frac{\sqrt{-c_2b_3}}{2c_2^3b_3^3}(D_1W^2+R_1r^2+C_1W). \end{aligned}$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_1}\sqrt{\frac{C_1N_1T_1-D_1N_1^2}{R_1}},-\frac{N_1}{T_1}\right),$$

if $T_1>0$, $R_1(C_1N_1T_1-D_1N_1^2)>0$, and the Jacobian (6) at (r^*,W^*) takes the value $J=N_1(N_1D_1-C_1T_1)/(2c_2^5b_3^5T_1)$ $\neq 0$ where

$$\begin{aligned} N_1 = &\ -(((b_{21}+c_{31})c_2+a_2c_{11})b_3+a_3b_{11}c_2)c_2b_3,\\ C_1 = &\ -2c_2((a_3b_{11}-a_{11}b_3)c_2+a_2b_3c_{11})b_3,\\ T_1 = &\ ((-e_2-f_3)c_2^2+a_2(h_2+2k_3-f_1)c_2+2a_2^2k_1)b_3^2+((2g_2+h_3-e_1)c_2+2a_2h_1)c_2a_3b_3+2c_2^2a_3^2g_1,\\ D_1 = &\ (2a_2^2k_1-2a_2c_2f_1+2c_2^2d_1)b_3^2+2a_3c_2(a_2h_1-c_2e_1)b_3+2c_2^2a_3^2g_1,\\ R_1 = &\ b_3c_2^2(b_3k_1-c_2g_1). \end{aligned}$$

Then Theorem 3 guarantees for $\varepsilon>0$ sufficiently small the existence of a periodic solution $(r(\theta,\varepsilon),W(\theta,\varepsilon))$ of system (12) such that $(r(0,\varepsilon),W(0,\varepsilon))\to(r^*,W^*)$ when $\varepsilon\to 0$.

So for $\varepsilon>0$ sufficiently small system (11) has the periodic solution $(r(t,\varepsilon),\theta(t,\varepsilon),W(t,\varepsilon))$ with $\theta(t,\varepsilon)=\sqrt{-b_3c_2}\,t+O(\varepsilon)$, such that $(r(0,\varepsilon),\theta(0,\varepsilon),W(0,\varepsilon))\to(r^*,0,W^*)$ when $\varepsilon\to 0$.

Consequently system (10) has the periodic solution

$$(U(t,\varepsilon),V(t,\varepsilon),W(t,\varepsilon))=\left(r^*\cos(\sqrt{-b_3c_2}\,t)+O(\varepsilon),r^*\sin(\sqrt{-b_3c_2}\,t)+O(\varepsilon),W^*+O(\varepsilon)\right),$$

for $\varepsilon>0$ sufficiently small. Therefore system (9) for $\varepsilon>0$ sufficiently small has the periodic solution

$$(u(t,\varepsilon),v(t,\varepsilon),w(t,\varepsilon))=\left(\varepsilon r^*\cos(\sqrt{-b_3c_2}\,t)+O(\varepsilon^2),\varepsilon r^*\sin(\sqrt{-b_3c_2}\,t)+O(\varepsilon^2),\varepsilon W^*+O(\varepsilon^2)\right). \tag{14}$$

Finally for $\varepsilon>0$ sufficiently small system (7) has the periodic solution $(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon))$ obtained from (14) through the change of variables (8). This periodic solution tends to the origin of coordinates when $\varepsilon\to 0$. Therefore there is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon=0$. This completes the proof of Theorem 2 under the condition (i). □

Application 1. Consider the Kolmogorov system

$$\begin{aligned} \dot x = &\ x(x-3+y+z+(x-1)^2+(x-1)(y-1)+(z-1)^2),\\ \dot y = &\ y(x-3+y+z+(x-1)^2+(x-1)(z-1)+(y-1)^2+(z-1)^2),\\ \dot z = &\ z(x-1-y+z+(x-1)^2+(y-1)^2+(y-1)(z-1)+(z-1)^2). \end{aligned} \tag{15}$$

This system in the new variables (X,Y,Z) writes

$$\begin{aligned} &\dot X=(X+1)(X^2+XY+Z^2+(X-Z)\varepsilon),\\ &\dot Y=(Y+1)(X^2+XZ+Y^2+Z^2+X+Z),\\ &\dot Z=(Z+1)(X^2+Y^2+YZ+Z^2+X-Y). \end{aligned}$$

The corresponding system associated to the differential system (13) has its function $F(\theta,r,W)$ with the components

$$\begin{aligned} F_1(\theta,r,W) = &\ -r^2\cos^3\theta+(1+4\,W)r\cos^2\theta+(r(-5\,W-1)\sin\theta+W^2+2\,W)\cos\theta\\ &+(5W^2+2r^2+2\,W)\sin\theta+(-3\,W-1)r,\\ F_2(\theta,r,W) = &\ -r^2\cos^2\theta-\cos\theta Wr-r(2\,W+1)\sin\theta+3W^2+r^2+2\,W. \end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function $f(r,W)=(f_1(r,W),f_2(r,W))$ $=(0,0)$ where

$$f_1(r,W)=\frac{-r(2\,W+1)}{2},\qquad f_2(r,W)=3W^2+\frac{1}{2}r^2+2\,W.$$

This system has four solutions for (r,W) given by $(0,0)$, $(0,-2/3)$, $(\sqrt{2}/2,-1/2)$, $(-\sqrt{2}/2,-1/2)$. The solution $(0,0)$ does not provide any periodic orbit because correspond to the equilibrium point localized at the origin. There are two good solutions provide by the averaing theory of first order the $(0,-2/3)$ and the $(\sqrt{2}/2,-1/2)$. Since the determinants (6) at these two solutions are $-1/3$ and $1/2$ and thus non-zero, respectively, the Kolmogorov system (15) has two limit cycle bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for $\varepsilon=10^{-5}$ in Fig. 1, where we provide the initial conditions of the two limit cycles that we have drawn. The software used for doing all the figures is maple 20.

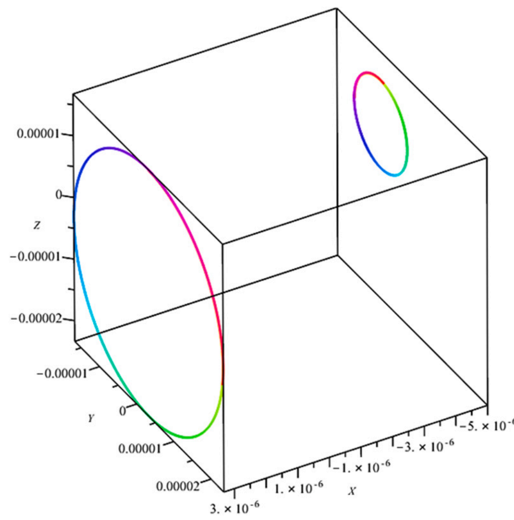

Fig. 1. **1st LC:** $X(0)=\epsilon/3,\ Y(0)=\epsilon/3,\ Z(0)=5\epsilon/3$. **2nd LC:** $X(0)=-\epsilon/2,\ Y(0)=-\epsilon(\sqrt{2}+1)/2,\ Z(0)=\epsilon/2$.

Since the eigenvalues of the Jacobian matrix of (f_1,f_2) at the singular points $(0,-2/3)$ and $(\sqrt{2}/2,-1/2)$ are $(-2,1/6)$ and $(-1\pm i)/2$, respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. Going back through the changes of variables as we did in the proof of Theorem 2 we obtain that the limit cycle bifurcating from the equilibrium point $(1,1,1)$ of system (15) are

$$\begin{aligned}&(x(t,\epsilon),y(t,\epsilon),z(t,\epsilon))=(1-2\epsilon/3+O(\epsilon^2),1-2\epsilon/3+O(\epsilon^2),1+2\epsilon/3+O(\epsilon^2)),\ \text{and}\\&(x(t,\epsilon),y(t,\epsilon),z(t,\epsilon))=(1-\epsilon/2+O(\epsilon^2),1-\epsilon(\sqrt{2}\cos t+1)/2+O(\epsilon^2),1+\epsilon(\sqrt{2}\sin t+1)/2+O(\epsilon^2)),\end{aligned}$$

respectively. This completes Application 1.

Proof of Theorem 2 under the condition (ii)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (ii). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates doing the change of variables $x=X+1, y=Y+1, z=Z+1$. Then system (2) becomes

$$\begin{aligned}\dot X&=(X+1)(d_1X^2+e_1XY+f_1XZ+g_1Y^2+h_1YZ+k_1Z^2-b_{20}X+b_1Y+(-2Xb_{21}+Zc_{11})\epsilon),\\\dot Y&=(Y+1)(d_2X^2+e_2XY+f_2XZ+g_2Y^2+h_2YZ+k_2Z^2+a_2X+b_{20}Y+(Yb_{21}+Zc_{21})\epsilon),\\\dot Z&=(Z+1)(X^2d_3+XYe_3+XZf_3+Y^2g_3+YZh_3+Z^2k_3+Z\epsilon c_{31}+Xa_3+Yb_3).\end{aligned}\tag{16}$$

We write the linear part of system (16) with $\epsilon=0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix}0&-\sqrt{-a_2b_1-b_{20}^2}&0\\\sqrt{-a_2b_1-b_{20}^2}&0&0\\0&0&0\end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}-\dfrac{b_{20}}{b_1\sqrt{-A_2}}&\dfrac{1}{\sqrt{-A_2}}&0\\\dfrac{1}{b_1}&0&0\\\dfrac{S_2}{A_2}&-\dfrac{P_2}{A_2}&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix},\ \text{with inverse}\ \begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}0&b_1&0\\\sqrt{-A_2}&b_{20}&0\\-\dfrac{P_2}{\sqrt{-A_2}}&b_3&1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix},$$

where $A_2=a_2b_1+b_{20}^2,\ P_2=a_3b_1+b_{20}b_3,\ S_2=-a_2b_3+a_3b_{20}$. Now following the same steps as the case of Theorem 2 under the condition (i), we get the system

$$\begin{aligned}
\dot r =\ & \frac{\varepsilon}{A_2^2 b_1}(\sqrt{-A_2}r\cos^2\theta\sin^2\theta(A_2(h_1(b_{20}b_3+P_2)+2b_{20}^2g_1-b_1^2(a_2-b_{20}+e_2)-b_1(b_{20}(2b_{20}-e_1+2g_2)+h_2b_3))\\
&+P_2b_{20}^2h_1+((f_1-h_2)b_1+2b_3k_1)P_2b_{20}-(b_1f_2+2b_3k_2)b_1P_2+A_2^2g_1+P_2^2k_1)+rA_2\cos\theta\sin\theta(W(((f_1-h_2)\\
&b_{20}-2b_3k_2)b_1+(b_{20}^2+A_2)h_1-b_1^2f_2+2k_1(b_{20}b_3+P_2))+(a_3c_{11}-3b_{20}b_{21}-b_3c_{21})b_1+2b_{20}b_3c_{11})+r^2A_2\\
&\cos\theta\sin^2\theta(P_2(b_1f_1+b_{20}h_1+2b_3k_1)+A_2(b_1^2+b_1e_1+2b_{20}g_1+b_3h_1)-b_1^3d_2+((-e_2-a_2+d_1)b_{20}-f_2b_3)b_1^2\\
&+(-b_{20}^3+(e_1-g_2)b_{20}^2-b_3(h_2-f_1)b_{20}-k_2b_3^2)b_1+b_{20}(b_{20}^2g_1+b_{20}b_3h_1+b_3^2k_1))+(-A_2)^{\frac32}\sin\theta W(Wk_1+c_{11}\\
&)+\sin^3\theta(-A_2)^{\frac32}r^2(b_3^2k_1+(b_1f_1+b_{20}h_1)b_3+b_1^2d_1+b_1b_{20}e_1+b_{20}^2g_1)+r(-A_2)^{\frac32}\sin^2\theta(W(b_1f_1+b_{20}h_1+2b_3\\
&k_1)-2b_1b_{21}+b_3c_{11})-r\sqrt{-A_2}\cos^2\theta(((2b_1k_2-2b_{20}k_1)P_2+(b_1h_2-b_{20}h_1)A_2)W+(b_1c_{21}-b_{20}c_{11})P_2+A_2b_1\\
&b_{21})+r^2\cos^3\theta(((b_{20}+g_2)b_1-b_{20}g_1)A_2^2+P_2(b_1h_2-b_{20}h_1)A_2+(b_1k_2-b_{20}k_1)P_2^2)-A_2\cos\theta W((Wk_2+c_{21}\\
&)b_1-b_{20}(Wk_1+c_{11})))=F_1(\theta,r,W),\\
\dot W =\ & -\frac{\varepsilon}{A_2^2\sqrt{-A_2}}(-r\sin^2\theta\cos\theta((e_3b_1+2g_3b_{20}+b_3(h_3+b_3))A_2^2+(((-e_2+f_3-a_2+a_3)b_1-2b_{20}^2+(b_3-2g_2+\\
&h_3)b_{20}-b_3(h_2-2k_3))P_2+S_2(b_1^2+b_1e_1+2b_{20}g_1+b_3h_1))A_2+(S_2(b_1f_1+b_{20}h_1+2b_3k_1)-P_2(b_1f_2+b_{20}h_2\\
&+2b_3k_2))P_2)\sqrt{-A_2}-rA_2\sin\theta(W(((f_3+a_3)b_1+(h_3+b_3)b_{20}+2k_3b_3)A_2+S_2(b_1f_1+b_{20}h_1+2b_3k_1)-P_2(\\
&b_1f_2+b_{20}h_2+2b_3k_2))+A_2b_3c_{31}-S_2(2b_1b_{21}-b_3c_{11})-P_2(b_{20}b_{21}+b_3c_{21}))-r^2A_2\sin^2\theta(A_2((k_3+b_{20})b_3^2+\\
&((f_3+a_3)b_1+b_{20}h_3)b_3+b_1^2d_3+b_1b_{20}e_3+b_{20}^2g_3)+S_2(b_3^2k_1+(b_1f_1+b_{20}h_1)b_3+b_1^2d_1+b_1b_{20}e_1+b_{20}^2g_1)-\\
&P_2(b_{20}^3+b_{20}^2g_2+((a_2+e_2)b_1+h_2b_3)b_{20}+k_2b_3^2+b_1^2d_2+b_1b_3f_2))+r^2\cos^2\theta(A_2^3g_3+((h_3-b_{20}+b_3-g_2)P_2\\
&+S_2g_1)A_2^2+P_2((-h_2+k_3)P_2+h_1S_2)A_2+P_2^2(S_2k_1-P_2k_2))-\cos\theta\sqrt{-A_2}r((A_2^2(h_3+b_3)+A_2(h_1S_2-P_2\\
&(h_2-2k_3))+2P_2(S_2k_1-P_2k_2))W-A_2P_2(b_{21}-c_{31})+P_2(S_2c_{11}-P_2c_{21}))-(A_2^2k_3+A_2(S_2k_1-P_2k_2))W^2\\
&-(A_2^2c_{31}+A_2(S_2c_{11}-P_2c_{21}))W)=F_2(\theta,r,W).
\end{aligned}\tag{17}$$

In this case the integrals (5) are

$$\begin{aligned}
f_1(r,W)=\ & \frac{1}{2\pi}\int_0^{2\pi}F_1(\theta,r,W)d\theta\ =-\frac{r(T_2\ W+N_2)}{2(-A_2)^{\frac32}b_1},\\
f_2(r,W)=\ & \frac{1}{2\pi}\int_0^{2\pi}F_2(\theta,r,W)d\theta\ =-\frac{D_2W^2+R_2r^2+C_2W}{2(-A_2)^{\frac52}}.
\end{aligned}$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_2}\sqrt{\frac{C_2N_2T_2-D_2N_2^2}{R_2}},-\frac{N_2}{T_2}\right),$$

if $T_2>0$, $R_2(C_2N_2T_2-D_2N_2^2)>0$ and the Jacobian (6) at (r^*,W^*) is $-N_2(C_2T_2-N_2D_2)/(2A_2^4T_2b_1)\neq 0$, where

$$\begin{aligned}
N_2 =\ & -A_2(b_1b_{21}-b_3c_{11})+P_2(b_1c_{21}-b_{20}c_{11}),\\
C_2 =\ & -2A_2(A_2c_{31}+S_2c_{11}-P_2c_{21})\\
D_2 =\ & -2A_2(A_2k_3+S_2k_1-P_2k_2),\\
T_2 =\ & A_2(b_1(f_1+h_2)+2b_3k_1)+2P_2b_1k_2-2P_2b_{20}k_1,\\
R_2 =\ & A_2^2((-k_3-b_{20})b_3^2+((-a_3-f_3)b_1-b_{20}h_3)b_3-b_1^2d_3-b_1b_{20}e_3-b_{20}^2g_3+S_2g_1-P_2(b_{20}-b_3+g_2-h_3))+\\
& A_2(-S_2(b_3^2k_1+(b_1f_1+b_{20}h_1)b_3+b_1^2d_1+b_1b_{20}e_1+b_{20}^2g_1)+P_2(b_1(b_{20}(a_2+e_2)+b_3f_2)+b_{20}^2(b_{20}+g_2)+b_1^2\\
& d_2+b_{20}b_3h_2+b_3^2k_2+S_2h_1)-P_2^2(h_2-k_3))+A_2^3g_3+P_2^2(S_2k_1-P_2k_2),
\end{aligned}$$

then, by Theorem 3 for $\varepsilon>0$ sufficiently small system (16) has a periodic solution $(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon))$ tending to $(0,0,0)$ when $\varepsilon\to 0$. Therefore there is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon=0$. This completes the proof of Theorem 2 under the condition (ii). □

Application 2. Consider the Kolmogorov system

$$\begin{aligned}
\dot x =\ & x(x-3+y+z+(x-1)^2+(x-1)(y-1)+(z-1)^2),\\
\dot y =\ & y(-2x+y+z+(x-1)^2+(x-1)(z-1)+(y-1)^2+(z-1)^2),\\
\dot z =\ & z(x-1-y+z+(x-1)^2+(y-1)^2+(y-1)(z-1)+(z-1)^2).
\end{aligned}\tag{18}$$

This system in the variables (X,Y,Z) writes

$$\begin{aligned}
\dot X =\ & (X+1)(X^2+XY+Z^2-Z\varepsilon-X+Y),\\
\dot Y =\ & (Y+1)(X^2+XZ+Y^2+Z^2+Z\varepsilon-2X+Y),\\
\dot Z =\ & (Z+1)(X^2+Y^2+YZ+Z^2-3Z\varepsilon+X-Y).
\end{aligned}$$

The corresponding system (17) is

$$\begin{aligned}
F_1(\theta,r,W)=\ & -r\cos\theta\sin\theta(-W+2)+W\sin\theta(W-1)+3r^2\sin^3\theta+r\sin^2\theta(-2\ W+1)+2r^2\cos^3\theta+2\cos\theta W,\\
F_2(\theta,r,W)=\ & 4r^2\sin\theta\cos\theta-r\sin\theta(3\ W-4)+5r^2\sin^2\theta+r^2\cos^2\theta+2W^2-4\ W.
\end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W) = -\frac{r(2\,W-1)}{2} = 0, \quad f_2(r,W) = 2W^2 + 3r^2 - 4W = 0.$$

This system possesses the solutions (r,W) given by $(0,0)$, $(0,2)$, $(\sqrt{2}/2, 1/2)$, $(-\sqrt{2}/2, 1/2)$. As in Application 1 we have two good solutions, the $(0,2)$ and the $(\sqrt{2}/2, 1/2)$. The determinants (6) at these solutions are -6 and 3, respectively. Hence system (18) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\epsilon = 10^{-5}$ in Fig. 2.

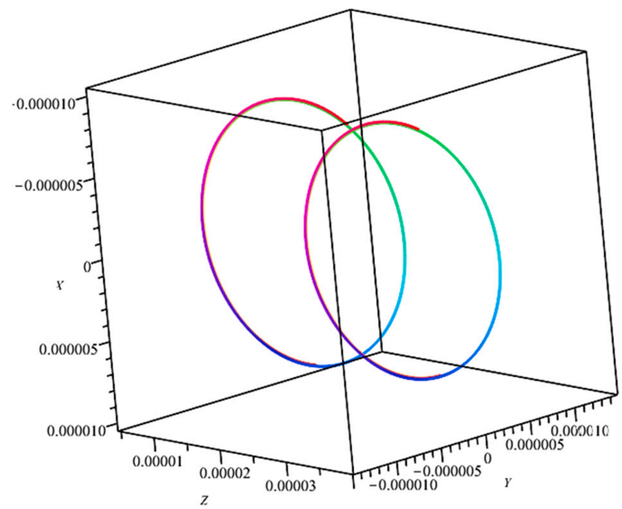

Fig. 2. 1st LC: $X(0)=\epsilon$, $Y(0)=\epsilon$, $Z(0)=2\epsilon$. **2nd LC:** $X(0)=\epsilon$, $Y(0)=\epsilon\sqrt{2}/2$, $Z(0)=\epsilon/2$.

The eigenvalues of the Jacobian matrix of (f_1, f_2) at the singular points $(0,2)$ and $(\sqrt{2}/2, 1/2)$ are $(4, -3/2)$ and $(-1 \pm i\sqrt{2})$, respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. This completes Application 2.

Proof of Theorem 2 under the condition (iii)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (iii). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned}
\dot{X} &= (X+1)((Xa_{11} + Yb_{11} + Zc_{11})\varepsilon + d_1X^2 + e_1XY + f_1XZ + g_1Y^2 + h_1YZ + k_1Z^2),\\
\dot{Y} &= (Y+1)(X^2d_2 + XYe_2 + XZf_2 + Y^2g_2 + YZh_2 + Y\varepsilon b_{21} + Z^2k_2 + Xa_2 + Yb_{20} + Zc_2),\\
\dot{Z} &= (Z+1)(X^2d_3 + XYe_3 + XZf_3 + Y^2g_3 + YZh_3 + Z^2k_3 - Z\varepsilon b_{21} + Xa_3 + Yb_3 - Zb_{20}).
\end{aligned} \tag{19}$$

We write the linear part of system (19) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-b_{20}^2 - b_3c_2} & 0 \\ \sqrt{-b_{20}^2 - b_3c_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{P_3}{A_3} & -1 & 0 \\ \dfrac{a_2}{\sqrt{-A_3}} & \dfrac{b_{20}}{\sqrt{-A_3}} & \dfrac{c_2}{\sqrt{-A_3}} \\ \dfrac{1}{A_3} & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \text{ with inverse } \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & A_3 \\ -1 & 0 & P_3 \\ \dfrac{b_{20}}{c_2} & \dfrac{\sqrt{-A_3}}{c_2} & L_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where $A_3 = b_{20}^2 + b_3 c_2,\ P_3 = -a_2 b_{20} - a_3 c_2,\ L_3 = -a_2 b_3 + a_3 b_{20}$. By following the same steps as the case of Theorem 2 under the condition (i), we get the system

$$
\begin{aligned}
\dot r = {} & \frac{\varepsilon}{A_3 c_2^2\sqrt{-A_3}}\big(- \sqrt{-A_3} r^2\cos^2\theta\sin\theta((-2b_{20}k_1 + c_2h_1)P_3 + (2b_{20}k_2 - c_2^2 - c_2h_2)A_3 + (k_2 - c_2)b_{20}^3 + ((- h_2 + k_3\\
&)c_2 + a_2k_1)b_{20}^2 - c_2((-g_2 + h_3 + b_3)c_2 + a_2h_1)b_{20} + a_2c_2^2g_1 + c_2^3g_3) + A_3 r^2\cos\theta\sin^2\theta((-h_3 - b_{20} - b_3)c_2^2 + (\\
& -2b_{20}^2 + (-h_2 + 2k_3)b_{20} - a_2h_1)c_2 + 2a_2b_{20}k_1 + 2b_{20}^2k_2 + A_3k_2 - P_3k_1) - c_2\sqrt{-A_3} r\cos\theta\sin\theta((A_3^2 f_2 + (- c_2^2\\
& e_3 + ((-e_2 + f_3 - a_2 + a_3)b_{20} - a_2e_1 + P_3)c_2 + b_{20}^2 f_2 + f_1a_2b_{20} + (-f_1 + h_2)P_3 + 2L_3k_2)A_3 + ((- h_3 - b_{20} -\\
& b_3)c_2^2 + (-2b_{20}^2 + (-h_2 + 2k_3)b_{20} - a_2h_1)c_2 + 2a_2b_{20}k_1 + 2b_{20}^2k_2 - 2P_3k_1)L_3 - h_1P_3^2 + ((-c_2 + h_2)b_{20}^2 + ((b_3\\
& -2g_2 + h_3)c_2 + a_2h_1)b_{20} - 2a_2c_2g_1 - 2g_3c_2^2)P_3)W - (a_2b_{11} + 2b_{20}b_{21})c_2 - c_{11}(-a_2b_{20} + P_3)) + c_2 r\cos^2\theta(((a_2\\
& c_2 - b_{20}f_2 + c_2e_2)A_3^2 + (L_3c_2^2 + ((-e_1 + 2g_2 + b_{20})P_3 + h_2L_3)c_2 - b_{20}((-f_1 + h_2)P_3 + 2L_3k_2))A_3 + P_3((2b_{20}\\
& k_1 - c_2h_1)L_3 + P_3(b_{20}h_1 - 2c_2g_1)))W + (b_{20}c_{11} - b_{11}c_2)P_3 + A_3b_{21}c_2) + c_2A_3 r\sin^2\theta((((b_{20} + b_3 + h_3)c_2 + a_2\\
& h_1 + h_2b_{20})P_3 + ((-2c_2 + 2k_2)b_{20} + 2a_2k_1 + 2c_2k_3)L_3 + ((f_3 + a_3)c_2 + f_1a_2 + f_2b_{20})A_3)W + a_2c_{11} - b_{21}c_2)\\
& - r^2\cos^3\theta((b_{20}^2k_2 - b_{20}c_2h_2 + c_2^2g_2)A_3 + (-b_{20}^2k_1 + b_{20}c_2h_1 - c_2^2g_1)P_3) + A_3\sqrt{-A_3}((-b_{20} + k_3)c_2 + a_2k_1 + b_{20}\\
& k_2)r^2\sin^3\theta - c_2^2\sqrt{-A_3}W\sin\theta((A_3^2(a_2d_1 + b_{20}d_2 + c_2d_3) + (((b_{20} + e_1)a_2 + b_{20}e_2 + c_2e_3)P_3 + ((f_3 + a_3)c_2 +\\
& f_1a_2 + f_2b_{20})L_3)A_3 + ((k_2 - c_2)b_{20} + c_2k_3 + a_2k_1)L_3^2 + ((b_{20} + b_3 + h_3)c_2 + a_2h_1 + h_2b_{20})P_3L_3 + (a_2g_1 + b_{20}^2\\
& + b_{20}g_2 + c_2g_3)P_3^2)W + (A_3a_{11} + L_3c_{11} + P_3b_{11})a_2 - b_{21}(L_3c_2 - P_3b_{20})) - c_2^2 W\cos\theta((A_3^3d_2 + ((e_2 + a_2 - d_1\\
&)P_3 + f_2L_3)A_3^2 + (L_3^2k_2 + (c_2 - f_1 + h_2)P_3L_3 + (-e_1 + g_2 + b_{20})P_3^2)A_3 - L_3^2P_3k_1 - L_3P_3^2h_1 - g_1P_3^3)W - P_3\\
& (P_3b_{11} + (a_{11} - b_{21})A_3 + L_3c_{11}))\big) = F_1(\theta, rW),\\
\dot W = {} & \frac{\varepsilon}{c_2^2A_3\sqrt{-A_3}}\big(r^2\cos^2\theta(b_{20}^2k_1 - b_{20}c_2h_1 + c_2^2g_1) - A_3r^2k_1\sin^2\theta + \sin\theta\sqrt{-A_3}c_2 r((A_3f_1 + 2L_3k_1 + P_3h_1)W +\\
& c_{11}) + \cos\theta\sqrt{-A_3}r^2\sin\theta(2b_{20}k_1 - c_2h_1) + c_2 r\cos\theta((A_3(b_{20}f_1 - c_2e_1) + (2b_{20}k_1 - c_2h_1)L_3 + P_3(b_{20}h_1 - 2c_2\\
& g_1))W + b_{20}c_{11} - b_{11}c_2) + c_2^2(P_3^2g_1 + (A_3e_1 + L_3h_1)P_3 + A_3^2d_1 + A_3L_3f_1 + L_3^2k_1)W^2 + (A_3a_{11} + L_3c_{11} + P_3\\
& b_{11})c_2^2\, W\big) = F_2(\theta, rW).
\end{aligned}
\tag{20}
$$

We compute the integrals (5) and we obtain

$$
f_1(r,W) = -\frac{r(T_3\ W + N_3)}{2c_2(-A_3)^{\frac{3}{2}}},\quad f_2(r,W) = -\frac{D_3W^2 + R_3r^2 + c_{30}W}{2c_2^2(-A_3)^{\frac{3}{2}}}.
$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*, W^*) with $r^* > 0$, namely

$$
(r^*, W^*) = \left(\frac{1}{T_3}\sqrt{\frac{c_{30}N_3T_3 - D_3N_3^2}{R_3}}, -\frac{N_3}{T_3}\right),
$$

if $T_3 > 0$, $R_3(c_{30}N_3T_3 - D_3N_3^2) > 0$ and the Jacobian (6) at (r^*, W^*) is $N_3(c_{30}T_3 - N_3D_3)/(2c_2^3A_3^3T_3) \neq 0$, where

$$
\begin{aligned}
D_3 = {} & 2c_2^2(A_3(A_3d_1 + L_3f_1 + P_3e_1) + g_1P_3^2 + L_3(L_3k_1 + P_3h_1)),\\
c_{30} = {} & 2c_2^2(A_3a_{11} + L_3c_{11} + P_3b_{11}),\\
R_3 = {} & -c_2(b_{20}h_1 - c_2g_1) - k_1(-b_{20}^2 + A_3),\\
N_3 = {} & (b_{20}c_{11} - b_{11}c_2)P_3 + A_3a_2c_{11},\\
T_3 = {} & ((a_2 + a_3 + e_2 + f_3)c_2 + a_2f_1)A_3^2 + (((-e_1 + 2g_2 + h_3 + 2b_{20} + b_3)c_2 + a_2h_1 + b_{20}f_1)P_3 + (c_2^2 + (h_2 + 2k_3 -\\
& 2b_{20})c_2 + 2a_2k_1)L_3)A_3 + P_3((2b_{20}k_1 - c_2h_1)L_3 + P_3(b_{20}h_1 - 2c_2g_1)),
\end{aligned}
$$

then, by Theorem 3, for $\varepsilon > 0$ sufficiently small and going back through the changes of variables we obtain that the differential system (2) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon \to 0$. Therefore there is a periodic orbit starting at the zero-Hopf equilibrium point located at $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of Theorem 2 under the condition (iii). □

Application 3. Consider the Kolmogorov system

$$
\begin{aligned}
\dot x = {} & x(x - 3 + y + z + (x-1)^2 + (x-1)(y-1) + (z-1)^2),\\
\dot y = {} & y(-2x + 3 + y - 2z + (x-1)^2 + (x-1)(z-1) + (y-1)^2 + (z-1)^2),\\
\dot z = {} & z(x - 3 + y + z + (x-1)^2 + (y-1)^2 + (y-1)(z-1) + (z-1)^2).
\end{aligned}
\tag{21}
$$

This system in the new variables (X, Y, Z) having the equilibrium point at the origin of coordinates writes

$$
\begin{aligned}
\dot X &= ((X+1)((-3X + Z)\varepsilon + X^2 + XY + Z^2),\\
\dot Y &= (Y+1)(X^2 + XZ + Y^2 + Z^2 - 2X + Y - 2Z),\\
\dot Z &= (Z+1)(X^2 + Y^2 + YZ + Z^2 + X + Y - Z).
\end{aligned}
$$

The two components of the corresponding system (20) are

$$\begin{aligned} F_1(\theta,r,W) = & \frac{1}{2}(r(W(37\cos^2\theta+78\cos\theta\sin\theta+29\sin^2\theta)+4\cos^2\theta+6\cos\theta\sin\theta+2\sin^2\theta))-\frac{1}{4}(r^2(9\cos^3\theta+\\ & 19\cos^2\theta\sin\theta+19\cos\theta\sin^2\theta+\sin^3\theta))+W\sin\theta(-65W-12)+W\cos\theta(-47W-24),\\ F_2(\theta,r,W) = & \frac{1}{2}(r\cos\theta(4\,W+1)-r\sin\theta(r\cos\theta-6\,W-1))-\frac{r^2}{4}-6W^2-6\,W. \end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W)=\frac{r(33\,W+3)}{2}=0,\quad f_2(r,W)=-6W^2-\frac{r^2}{4}-6W=0.$$

This system has the solutions (r,W) given by

$$(0,0),\quad (0,-1),\quad (4\sqrt{15}/11,-1/11),\quad (-4\sqrt{15}/11,-1/11).$$

As in Application 1 we have two good solutions, the $(0,-1)$ and the $(4\sqrt{15}/11,-1/11)$. The determinants (6) at these solutions are -90 and $180/11$, respectively. Hence system (21) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\varepsilon=10^{-5}$ in Fig. 3.

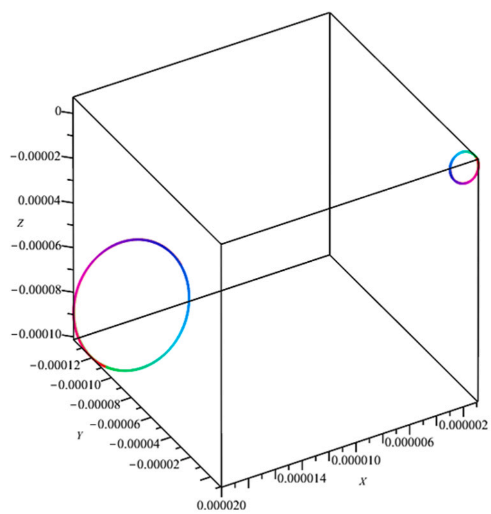

Fig. 3. **1st LC:** $X(0)=2\epsilon,\ Y(0)=-3\epsilon,\ Z(0)=-2\epsilon$. **2nd LC:** $X(0)=\epsilon/11,\ Y(0)=-4(\sqrt{15}+1)\epsilon/11,\ Z(0)=-(2\sqrt{15}+3)\epsilon/11$.

The eigenvalues of the Jacobian matrix of (f_1,f_2) at the singular points $(0,-1)$ and $(4\sqrt{15}/11,-1/11)$ are $(-15,6)$ and $(-27\pm 3i\sqrt{139})/11$, respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. This completes Application 3.

Proof of Theorem 2 under the condition (iv)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (iv). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned} \dot{X} &= (X+1)((-2Xc_{31}+Yb_{11})\varepsilon+d_1X^2+e_1XY+f_1XZ+g_1Y^2+h_1YZ+k_1Z^2-c_{30}X+c_1Z),\\ \dot{Y} &= (Y+1)(X^2d_2+XYe_2+XZf_2+Y^2g_2+YZh_2+Y\varepsilon b_{21}+Z^2k_2+Xa_2+Zc_2),\\ \dot{Z} &= (Z+1)((Yb_{31}+Zc_{31})\varepsilon+d_3X^2+e_3XY+f_3XZ+g_3Y^2+h_3YZ+k_3Z^2+a_3X+c_{30}Z). \end{aligned} \tag{22}$$

We write the linear part of system (22) with $\varepsilon=0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-a_3c_1-c_{30}^2} & 0\\ \sqrt{-a_3c_1-c_{30}^2} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u\\ v\\ w \end{pmatrix}=\begin{pmatrix} -\dfrac{c_{30}}{c_1\sqrt{-A_4}} & 0 & \dfrac{1}{\sqrt{-A_4}}\\ \dfrac{1}{c_1} & 0 & 0\\ K_4 & A_4 & -M_4 \end{pmatrix}\begin{pmatrix} X\\ Y\\ Z \end{pmatrix},\ \text{with inverse}\ \begin{pmatrix} X\\ Y\\ Z \end{pmatrix}=\begin{pmatrix} 0 & c_1 & 0\\ -\dfrac{M_4}{\sqrt{-A_4}} & c_2 & \dfrac{1}{A_4}\\ \sqrt{-A_4} & c_{30} & 0 \end{pmatrix}\begin{pmatrix} u\\ v\\ w \end{pmatrix},$$

where $A_4 = a_3c_1 + c_{30}^2,\ M_4 = a_2c_1 + c_2c_{30},\ K_4 = a_2c_{30} - a_3c_2$. By following the same steps as the case of Theorem 2 under the condition (i) we get the system

$$
\begin{aligned}
\dot r =\ & \frac{\varepsilon}{A_4^4 c_1}(((a_3-d_3)c_1^3 + (c_{30}^2 + (-a_3 - f_3 + d_1)c_{30} + e_1a_2 - e_3c_2 + f_1a_3)c_1^2 + (-c_{30}^3 + (2f_1 - k_3)c_{30}^2 + ((2e_1 - h_3\\
&)c_2 + h_1a_2 + 2k_1a_3)c_{30} - g_3c_2^2 + (2a_2g_1 + a_3h_1)c_2)c_1 + 3c_2^2c_{30}g_1 + 3c_2c_{30}^2h_1 + 3c_{30}^3k_1)A_4^3r^2\sin^2\theta\cos\theta + A_4^2\\
& \sqrt{-A_4}r^2\cos^2\theta\sin\theta(-A_4((a_3 - c_{30} + f_3)c_1^2 + (2c_{30}^2 + (2k_3 - f_1)c_{30} + c_2h_3)c_1 - c_2c_{30}h_1 - 2c_{30}^2k_1) - M_4(c_1^2e_3\\
& + ((h_3 - e_1)c_{30} + 2c_2g_3)c_1 - 2c_2c_{30}g_1 - c_{30}^2h_1) + A_4^2k_1 + A_4M_4h_1 + M_4^2g_1) + A_4^2r\cos\theta\sin\theta((-c_1^2e_3 + ((-h_3\\
& + e_1)c_{30} + 2a_2g_1 + h_1a_3 - 2c_2g_3)c_1 + 2c_{30}^2h_1 + 4c_2c_{30}g_1)W + A_4((a_2b_{11} - b_{31}c_2 - 3c_{30}c_{31})c_1 + 2b_{11}c_2c_{30}))\\
& + A_4^2(-A_4)^{\frac{3}{2}}r^2\sin^3\theta(c_{30}^2k_1 + (c_1f_1 + c_2h_1)c_{30} + c_1^2d_1 + c_1c_2e_1 + c_2^2g_1) + \sin\theta(-A_4)^{\frac{3}{2}}W(A_4b_{11} + Wg_1) + A_4^2r^2\\
& \cos^3\theta(A_4^2(c_1c_{30} + c_1k_3 - c_{30}k_1) + A_4M_4(c_1h_3 - c_{30}h_1) + M_4^2(c_1g_3 - c_{30}g_1)) + A_4W\cos\theta(A_4(b_{11}c_{30} - b_{31}c_1)\\
& - W(c_1g_3 - c_{30}g_1)) - A_4\cos^2\theta\sqrt{-A_4}r((A_4(c_1h_3 - c_{30}h_1) + 2M_4(c_1g_3 - c_{30}g_1))W + A_4^2c_1c_{31} - A_4M_4(b_{11}\\
& c_{30} - b_{31}c_1)) + A_4\sin^2\theta(-A_4)^{\frac{3}{2}}r(A_4(b_{11}c_2 - 2c_1c_{31}) + W(c_1e_1 + 2c_2g_1 + c_{30}h_1))) = F_1(\theta, r, W),\\
\dot W =\ & -\frac{\varepsilon}{A_4^2\sqrt{-A_4}}(-A_4\sqrt{-A_4}r^2\cos\theta\sin\theta(A_4^2(c_1f_2 + c_2^2 + c_2h_2 + 2c_{30}k_2) + A_4(K_4(c_1^2 + c_1f_1 + c_2h_1 + 2c_{30}k_1)\\
& + M_4((a_2 - a_3 + e_2 - f_3)c_1 - 2c_{30}^2 + (-2k_3 + c_2 + h_2)c_{30} - c_2(h_3 - 2g_2))) + K_4M_4(c_1e_1 + 2c_2g_1 + c_{30}h_1) -\\
& M_4^2(c_1e_3 + 2c_2g_3 + c_{30}h_3)) - A_4r\sin\theta((A_4((a_2 + e_2)c_1 + (c_{30} + 2g_2)c_2 + c_{30}h_2) + K_4(c_1e_1 + 2c_2g_1 + c_{30}h_1)\\
& - M_4(c_1e_3 + 2c_2g_3 + c_{30}h_3))W + A_4^2b_{21}c_2 + A_4(K_4(b_{11}c_2 - 2c_1c_{31}) - M_4(b_{31}c_2 + c_{30}c_{31}))) - A_4^2r^2\sin^2\theta(A_4\\
& ((c_{30} + g_2)c_2^2 + ((a_2 + e_2)c_1 + c_{30}h_2)c_2 + c_{30}^2k_2 + c_1^2d_2 + c_1c_{30}f_2) + (c_{30}^2k_1 + (c_1f_1 + c_2h_1)c_{30} + c_1^2d_1 + c_1c_2\\
& e_1 + c_2^2g_1)K_4 - (c_{30}^3 + k_3c_{30}^2 + ((a_3 + f_3)c_1 + c_2h_3)c_{30} + c_1^2d_3 + c_1c_2e_3 + g_3c_2^2)M_4) + A_4r^2\cos^2\theta(A_4^3k_2 + A_4^2\\
& (K_4k_1 + M_4(c_2 - c_{30} + h_2 - k_3)) + M_4A_4(K_4h_1 + M_4(g_2 - h_3)) + K_4M_4^2g_1 - M_4^3g_3) - \sqrt{-A_4}r\cos\theta((A_4^2(c_2\\
& + h_2) + A_4(K_4h_1 + M_4(2g_2 - h_3)) + 2K_4M_4g_1 - 2M_4^2g_3)W + A_4^2M_4(b_{21} - c_{31}) + M_4A_4(K_4b_{11} - M_4b_{31})) +\\
& (-A_4g_2 - K_4g_1 + M_4g_3)W^2 - (A_4^2b_{21} + A_4(K_4b_{11} - M_4b_{31}))W) = F_2(\theta, r, W).
\end{aligned}
\tag{23}
$$

Then

$$
f_1(r,W) = \frac{r(T_4\ W + N_4)}{2(-A_4)^{\frac{5}{2}}c_1},\quad f_2(r,W) = -\frac{D_4W^2 + R_4r^2 + C_4W}{2(-A_4)^{\frac{5}{2}}}.
$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*, W^*) with $r^* > 0$, namely

$$
(r^*, W^*) = \left(\frac{1}{T_4}\sqrt{\frac{C_4N_4T_4 - D_4N_4^2}{R_4}}, -\frac{N_4}{T_4}\right),
$$

if $T_4 > 0,\ R_4(C_4N_4T_4 - D_4N_4^2) > 0$ and the Jacobian (6) at (r^*, W^*) is $-N_4(C_4T_4 - N_4D_4)/(2A_4^5T_4c_1) \neq 0$, where

$$
\begin{aligned}
N_4 =\ & A_4^2(b_{11}c_2 - c_1c_{31}) - A_4M_4(b_{11}c_{30} - b_{31}c_1),\\
C_4 =\ & -2A_4(A_4b_{21} + K_4b_{11} - M_4b_{31}),\\
T_4 =\ & A_4(c_1e_1 + c_1h_3 + 2c_2g_1) + 2M_4(c_1g_3 - c_{30}g_1),\\
D_4 =\ & -2(A_4g_2 + K_4g_1 - M_4g_3),\\
R_4 =\ & A_4^2((c_{30}^3 + k_3c_{30}^2 + ((a_3 + f_3)c_1 + h_3c_2)c_{30} + c_2^2g_3 + c_1^2d_3 + c_1c_2e_3 + K_4h_1)M_4 + M_4^2(g_2 - h_3) - (k_1c_{30}^2 + (c_1\\
& f_1 + c_2h_1)c_{30} + c_1^2d_1 + c_1c_2e_1 + c_2^2g_1)K_4) + A_4^3((-c_{30} - g_2)c_2^2 + ((-a_2 - e_2)c_1 - c_{30}h_2)c_2 - c_{30}^2k_2 - c_1^2d_2 -\\
& c_1c_{30}f_2 + K_4k_1 + M_4(c_2 - c_{30} + h_2 - k_3)) + A_4M_4^2(K_4g_1 - M_4g_3) + A_4^4k_2,
\end{aligned}
$$

then, by Theorem 3 for $\varepsilon > 0$ sufficiently small and going back through the changes of variables system (22) has one periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ which tends to the equilibrium $(0,0,0)$ when $\varepsilon \to 0$. Hence there is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of Theorem 2 under the condition (iv). □

Application 4. Consider the Kolmogorov system

$$
\begin{aligned}
\dot x =\ & x(x + y - 2z + (x-1)^2 + (x-1)(y-1) + (z-1)^2),\\
\dot y =\ & y(-2x + y + z + (x-1)^2 + (x-1)(z-1) + (y-1)^2 + (z-1)^2),\\
\dot z =\ & z(x - 1 - y + z + (x-1)^2 + (y-1)^2 + 10(y-1)(z-1) - (z-1)^2).
\end{aligned}
\tag{24}
$$

This system in the variables (X, Y, Z) writes

$$
\begin{aligned}
\dot X &= (X+1)(X^2 + XY + 6X\varepsilon + Z^2 - X - 2Z),\\
\dot Y &= (Y+1)(X^2 + XZ + Y^2 + Z^2 - 2X + Z),\\
\dot Z &= (Z+1)((Y - 3Z)\varepsilon + X^2 + Y^2 + 10YZ - Z^2 + X + Z).
\end{aligned}
$$

The two components of the corresponding system associated to system (23) are

$$\begin{aligned}F_1(\theta,r,W)=&\ \frac{1}{2}(((-22\ W-16)r\sin\theta+2W^2+13r^2-2\ W)\cos\theta-3r^2\sin\theta)-31r^2\cos^3\theta+r(-43r\sin\theta+W-\\&14)\cos^2\theta-r(W-6),\\F_2(\theta,r,W)=&\ 184r^2\cos^2\theta+(246r^2\sin\theta+r(-9\ W+40))\cos\theta+r\sin\theta(61\ W+46)-6W^2-83r^2+5\ W.\end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W)=-\frac{r(2\ W+4)}{4}=0,\quad f_2(r,W)=-6W^2+9r^2+5W=0.$$

This system has the solutions

$$(0,0),\quad (0,5/6),\quad (\sqrt{34}/3,-2),\quad (-\sqrt{34}/3,-2),$$

As in Application 1 we have two good solutions, the $(0,5/6)$ and the $(\sqrt{34}/3,-2)$. The determinants (6) at these solutions are $85/12$ and 34, respectively. Hence system (24) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\varepsilon=10^{-5}$ in Fig. 4.

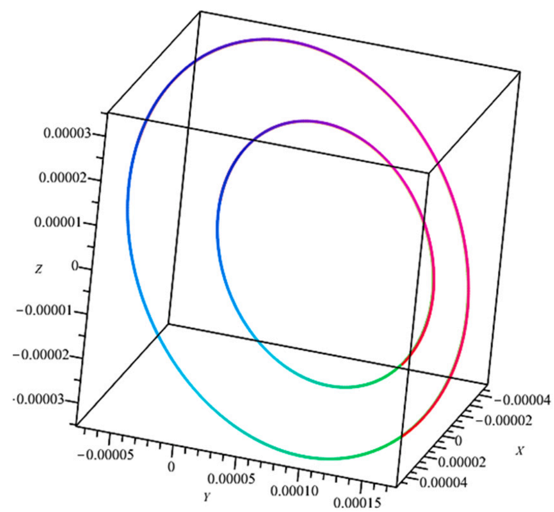

Fig. 4. 1st LC: $X(0)=\epsilon,\ Y(0)=-5\epsilon/6,\ Z(0)=\epsilon$. **2nd LC:** $X(0)=\epsilon,\ Y(0)=-5\epsilon\sqrt{34}/3+2\epsilon,\ Z(0)=\sqrt{34}/3$.

The eigenvalues of the Jacobian matrix of (f_1,f_2) at the singular points $(0,5/6)$ and $(\sqrt{34}/3,-2)$ are $(-5,-17/2)$ and $(29\pm\sqrt{705})/2$, respectively, by Theorem 3 the limit cycles are stable and unstable, respectively. This completes Application 4.

Proof of Theorem 2 under the condition (v)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (v). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates, then system (2) becomes

$$\begin{aligned}\dot X&=(X+1)((-2Xc_{31}+Yb_{11})\varepsilon+d_1X^2+e_1XY+f_1XZ+g_1Y^2+h_1YZ+k_1Z^2-c_{30}X+c_1Z),\\\dot Y&=(Y+1)((-\frac{c_2c_{31}X}{c_1}+b_{21}Y)\varepsilon-\frac{c_2c_{30}X}{c_1}+c_2Z+d_2X^2+e_2XY+f_2XZ+g_2Y^2+h_2YZ+k_2Z^2),\\\dot Z&=(Z+1)(X^2d_3+XYe_3+XZf_3+Y^2g_3+YZh_3+Z^2k_3+Z\varepsilon c_{31}+Xa_3+Yb_3+Zc_{30}).\end{aligned}\tag{25}$$

As usual we write the linear part of system (25) with $\varepsilon=0$ at $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix}0&-\sqrt{-a_3c_1-b_3c_2-c_{30}^2}&0\\\sqrt{-a_3c_1-b_3c_2-c_{30}^2}&0&0\\0&0&0\end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}\frac{M_5}{c_1A_5b_3}&\frac{1}{A_5}&0\\\frac{c_{30}}{c_1\sqrt{-A_5}b_3}&0&-\frac{1}{b_3\sqrt{-A_5}}\\-\frac{c_2}{c_1A_5}&\frac{1}{A_5}&0\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix},$$

with inverse

$$\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}c_1b_3&0&-c_1b_3\\b_3c_2&0&M_5\\c_{30}b_3&-b_3\sqrt{-A_5}&-c_{30}b_3\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix},$$

where $A_5 = a_3c_1 + b_3c_2 + c_{30}^2,\ M_5 = a_3c_1 + c_{30}^2$. By following the same steps as in the proof under the condition (i) we obtain the system

$$\begin{aligned}
\dot{r} = {} & -\frac{\varepsilon}{c_1A_5^2b_3}(\sqrt{-A_5}b_3^2r^2\cos^3\theta(b_3c_1(c_{30}^2k_2 + (c_1f_2 + c_2h_2)c_{30} + c_1^2d_2 + c_1c_2e_2 + c_2^2g_2) + (c_{30}^2k_1 + (c_1f_1 + c_2h_1)\\
& c_{30} + c_1^2d_1 + c_1c_2e_1 + c_2^2g_1)M_5) + \cos\theta\sqrt{-A_5}W((b_3^3c_1(c_1^2d_2 + c_1c_{30}f_2 + c_{30}^2k_2) + ((d_1 - e_2)c_1^2 + c_{30}(f_1 - h_2)\\
& c_1 + c_{30}^2k_1)b_3^2M_5 - b_3((e_1 - g_2)c_1 + h_1c_{30})M_5^2 + M_5^3g_1)W + M_5b_3c_1(b_{21} + 2c_{31}) + b_3^2c_1c_2c_{31} + M_5^2b_{11})+\\
& \sqrt{-A_5}b_3r\cos^2\theta((M_5^2(c_1e_1 + 2c_2g_1 + c_{30}h_1) - b_3^2c_1(2c_{30}^2k_2 + (2c_1f_2 + c_2h_2)c_{30} + 2c_1^2d_2 + c_1c_2e_2) - M_5b_3((2\\
& d_1 - e_2)c_1^2 + ((-h_2 + 2f_1)c_{30} + c_2(e_1 - 2g_2))c_1 + 2c_{30}^2k_1 + c_2c_{30}h_1))W + b_3c_1c_2(b_{21} - c_{31}) + M_5(b_{11}c_2 - 2c_1\\
& c_{31})) + A_5^2b_3^2r^2\sin^3\theta(c_1c_{30} + c_1k_3 - c_{30}k_1) + A_5b_3^2r^2\cos^2\theta\sin\theta(((-a_3 - f_3 + d_1)c_{30} + b_3f_2 - c_2e_3)c_1^2 + (- c_{30}^3\\
& + (f_1 - k_3)c_{30}^2 + ((-h_3 - b_3 + e_1)c_2 + 2k_2b_3)c_{30} - ((g_3 - b_3)c_2 - h_2b_3)c_2)c_1 + c_{30}(c_2^2g_1 + c_2c_{30}h_1 + c_{30}^2k_1)+\\
& M_5(c_1^2 + c_1f_1 + c_2h_1 + 2c_{30}k_1) - c_1^3d_3) + A_5b_3r\cos\theta\sin\theta((b_3(2a_3c_{30} - b_3f_2 + c_2e_3 - 2c_{30}d_1 + 2c_{30}f_3)c_1^2 - b_3\\
& (2(k_1 - c_1)c_{30}^3 + (2(f_1 - k_3)c_1 + c_2h_1)c_{30}^2 + ((-h_3 - b_3 + e_1)c_2 + 2k_2b_3)c_1c_{30} - 2c_1^3d_3) - M_5((b_3 + e_3)c_1^2 + (\\
& (-h_2 - c_2 + c_{30} + f_1)b_3 + (h_3 - e_1)c_{30} + 2c_2g_3)c_1 + c_{30}(2b_3k_1 - 2c_2g_1 - c_{30}h_1)) + M_5^2h_1)W + c_{30}(b_{11}c_2 - 3\\
& c_1c_{31})) - A_5W\sin\theta(((c_1^3d_3 + c_{30}(a_3 + f_3 - d_1)c_1^2 + c_{30}^2(k_3 + c_{30} - f_1)c_1 - k_1c_{30}^3)b_3^2 + M_5^2(c_1g_3 - c_{30}g_1) - (c_1^2\\
& e_3 + c_{30}(b_3 - e_1 + h_3)c_1 - c_{30}^2h_1)b_3M_5)W - c_{30}(3b_3c_1c_{31} + M_5b_{11})) + \cos\theta(-A_5)^{\frac{3}{2}}r^2b_3^2\sin^2\theta((- a_3 + c_{30} -\\
& f_3)c_1^2 + (-2c_{30}^2 + (-2k_3 + f_1)c_{30} + (-h_3 - b_3)c_2 + k_2b_3)c_1 + 2c_{30}^2k_1 + c_2c_{30}h_1 + M_5k_1) - (-A_5)^{\frac{3}{2}}b_3r\sin^2\theta(\\
& (((b_3 + h_3)c_1 - h_1c_{30})M_5 - b_3((a_3 - c_{30} + f_3)c_1^2 + c_{30}(2c_{30} - f_1 + 2k_3)c_1 - 2c_{30}^2k_1))W + c_1c_{31}))\\
= {} & F_1(\theta, r, W),\\
\dot{W} = {} & -\frac{\varepsilon}{c_1A_5\sqrt{-A_5}}(A_5b_3^2r^2\sin^2\theta(c_1k_2 - c_2k_1) - b_3^2r^2\cos^2\theta(c_1^3d_2 + ((-d_1 + e_2)c_2 + c_{30}f_2)c_1^2 + ((g_2 - e_1)c_2^2 + c_{30}(\\
& h_2 - f_1)c_2 + c_{30}^2k_2)c_1 - c_2^3g_1 - c_2^2c_{30}h_1 - c_2c_{30}^2k_1) + \sqrt{-A_5}Wb_3r\sin\theta(((c_2 - f_2)c_1^2 + (c_2f_1 - 2c_{30}k_2)c_1 + 2c_2\\
& c_{30}k_1)b_3 + M_5((c_2 + h_2)c_1 - c_2h_1)) - \sqrt{-A_5}b_3^2r^2\cos\theta\sin\theta((c_2 - f_2)c_1^2 + (-c_2^2 + (f_1 - h_2)c_2 - 2c_{30}k_2)c_1+\\
& c_2^2h_1 + 2c_2c_{30}k_1) - b_3r\cos\theta(((c_1^2e_2 + ((-e_1 + 2g_2)c_2 + h_2c_{30})c_1 - 2c_2^2g_1 - c_2c_{30}h_1)M_5 - b_3(2c_1^3d_2 + ((- 2d_1\\
& + e_2)c_2 + 2c_{30}f_2)c_1^2 + (-c_2^2e_1 + c_{30}(h_2 - 2f_1)c_2 + 2c_{30}^2k_2)c_1 - c_2^2c_{30}h_1 - 2c_2c_{30}^2k_1))W + c_2((b_{21} + c_{31})c_1 -\\
& b_{11}c_2)) + (M_5b_3(c_1^2e_2 - c_1(c_2e_1 - c_{30}h_2) - c_2c_{30}h_1) - M_5^2(c_1g_2 - c_2g_1) - b_3^2(c_1^3d_2 - c_1^2(c_2d_1 - c_{30}f_2) - c_1c_{30}\\
& (c_2f_1 - c_{30}k_2) - c_2c_{30}^2k_1))W^2 + (b_3c_1c_2c_{31} + M_5(b_{11}c_2 - b_{21}c_1))W) = F_2(\theta, r, W),
\end{aligned} \tag{26}$$

For this differential system we compute the integrals (5) and we obtain

$$f_1(r,W) = -\frac{r(T_5\ W + N_5)}{2(-A_5)^{\frac{3}{2}}c_1}, \quad f_2(r,W) = \frac{D_5W^2 + R_5r^2 + C_5W}{2(-A_5)^{\frac{3}{2}}c_1}.$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*, W^*) with $r^* > 0$, namely

$$(r^*, W^*) = \left(\frac{1}{T_5}\sqrt{\frac{C_5N_5T_5 - D_5N_5^2}{R_5}}, -\frac{N_5}{T_5}\right),$$

if $T_5 > 0$, $R_5(C_5N_5T_5 - D_5N_5^2) > 0$ and the Jacobian (6) at (r^*, W^*) is $-N_5(C_5T_5 - N_5D_5)/(2T_5c_1^2A_5^3) \neq 0$, where

$$\begin{aligned}
N_5 = {} & b_3c_1c_2b_{21} + M_5(b_{11}c_2 - c_1c_{31}),\\
C_5 = {} & 2b_3c_1c_2c_{31} + 2M_5(b_{11}c_2 - b_{21}c_1),\\
D_5 = {} & 2M_5b_3(c_1^2e_2 - c_1(c_2e_1 - c_{30}h_2) - c_2c_{30}h_1) - 2M_5^2(c_1g_2 - c_2g_1) - 2b_3^2(c_1^3d_2 - c_1^2(c_2d_1 - c_{30}f_2) - c_1c_{30}(c_2f_1 -\\
& c_{30}k_2) - c_2c_{30}^2k_1),\\
R_5 = {} & b_3^2(A_5(c_1k_2 - c_2k_1) - c_1^3d_2 + ((d_1 - e_2)c_2 - c_{30}f_2)c_1^2 + ((-g_2 + e_1)c_2^2 - c_{30}(h_2 - f_1)c_2 - c_{30}^2k_2)c_1 + c_2(c_2^2g_1 +\\
& c_2c_{30}h_1 + c_{30}^2k_1)),\\
T_5 = {} & M_5^2(c_1e_1 + 2c_2g_1 + c_{30}h_1) - b_3^2c_1(2c_{30}^2k_2 + (2c_1f_2 + c_2h_2)c_{30} + 2c_1^2d_2 + c_1c_2e_2) - A_5b_3((a_3 - c_{30} + f_3)c_1^2 +\\
& (2c_{30} - f_1 + 2k_3)c_{30}c_1 - 2c_{30}^2k_1) + M_5(((b_3 + h_3)c_1 - c_{30}h_1)A_5 - ((2d_1 - e_2)c_1^2 + ((-h_2 + 2f_1)c_{30} + c_2(e_1 -\\
& 2g_2))c_1 + c_{30}(c_2h_1 + 2c_{30}k_1))b_3),
\end{aligned}$$

then, from Theorem 3, for $\varepsilon > 0$ sufficiently small and going back through the changes of variables system (25) has a periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ which tends to the equilibrium $(0,0,0)$ when $\varepsilon \to 0$. So there is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of Theorem 2 under the condition (v). □

Application 5. Consider the Kolmogorov system

$$\begin{aligned}
\dot{x} = {} & x(x - 3 + y + z + (x-1)^2 + (x-1)(y-1) + (z-1)^2),\\
\dot{y} = {} & y(-2x + y + z + (x-1)^2 + (x-1)(z-1) - (y-1)^2 - (y-1)(z-1) + (z-1)^2),\\
\dot{z} = {} & z(x + 1 - 3y + z + (x-1)^2 + (y-1)^2 - 6(z-1)^2).
\end{aligned} \tag{27}$$

This system in the variables (X,Y,Z) writes

$$\begin{aligned}
\dot{X} &= (X+1)(X^2+XY-2X\varepsilon+Z^2-X+Z),\\
\dot{Y} &= (Y+1)(X^2+XZ-X\varepsilon-Y^2-YZ+Z^2-X+Z),\\
\dot{Z} &= (Z+1)(X^2+Y^2-6Z^2+Z\varepsilon+X-3Y+Z).
\end{aligned}$$

The corresponding system associated to system (26) is

$$\begin{aligned}
F_1(\theta,r,W) = {} & \frac{1}{3}(W\cos\theta(-3W-3)+W\sin\theta(-65W+9))+r(-33r\cos^3\theta+3r\cos^2\theta\sin\theta+35W\cos^2\theta+37W\\
& \cos\theta\sin\theta-3\cos\theta\sin\theta+42r\cos\theta-18r\sin\theta-42W+1),\\
F_2(\theta,r,W) = {} & 18r^2\cos^2\theta+3r\cos\theta(-8W+1)+7W^2-3W.
\end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W)=-\frac{1}{2}r(49W-2)=0,\quad f_2(r,W)=7W^2+9r^2-3W=0.$$

This system has the solutions

$$(0,0),\quad (0,3/7),\quad (\sqrt{266}/147,2/49),\quad (-\sqrt{266}/147,2/49).$$

As in Application 1 we have two good solutions, the $(0,3/7)$ and the $(\sqrt{266}/147,2/49)$. The determinants (6) at these solutions are $-57/2$ and $38/7$, respectively. Hence system (27) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\varepsilon=10^{-5}$ in Fig. 5.

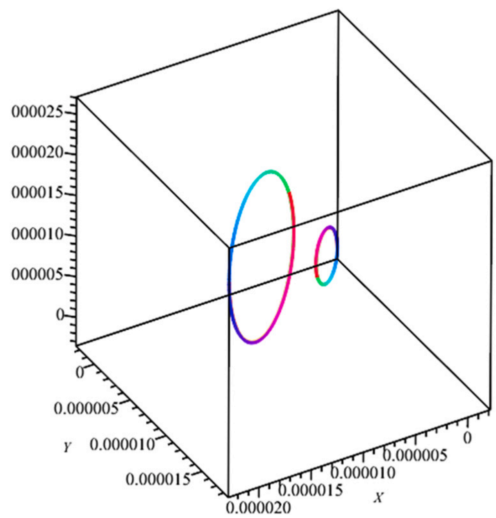

Fig. 5. 1st LC: $X(0)=16\epsilon/7,\ Y(0)=13\epsilon/7,\ Z(0)=16\epsilon/7$. **2nd LC:** $X(0)=\epsilon(6-\sqrt{266})/49,\ Y(0)=\epsilon(4-\sqrt{266})/49,\ Z(0)=\epsilon(6-\sqrt{266})/49$.

The eigenvalues of the Jacobian matrix of (f_1,f_2) at the singular points $(0,3/7)$ and $(\sqrt{266}/147,2/49)$ are $(3,-19/2)$ and $(-17\pm 5\sqrt{31}\,i)/14$, respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. This completes Application 5.

Proof of Theorem 2 under the condition (vi)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (vi). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned}
\dot{X} = {} & (X+1)(-2X(b_{21}+c_{31})\varepsilon+d_1X^2+e_1XY+f_1XZ+g_1Y^2+h_1YZ+k_1Z^2-Xb_{20}-Xc_{30}+b_1Y+c_1Z),\\
\dot{Y} = {} & (Y+1)\left(\left(b_{21}Y+\frac{b_{21}c_1Z}{b_1}\right)\varepsilon+a_2X+Yb_{20}+\frac{Zb_{20}c_1}{b_1}+d_2X^2+e_2XY+f_2XZ+g_2Y^2+h_2YZ+k_2Z^2\right),\\
\dot{Z} = {} & (Z+1)\left(\left(\frac{2b_1c_{31}Y}{c_1}+c_{31}Z\right)\varepsilon+a_3X+\frac{b_1c_{30}Y}{c_1}+c_{30}Z+d_3X^2+e_3XY+f_3XZ+g_3Y^2+h_3YZ+k_3Z^2\right),
\end{aligned}\tag{28}$$

We write the linear part of system (28) with $\varepsilon=0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-A_6} & 0\\ \sqrt{-A_6} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

For this we do the change of variables

$$\begin{pmatrix} u\\ v\\ w \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{A_6}} & 0 & 0\\ \frac{b_{20}+c_{30}}{A_6} & -\frac{b_1}{A_6} & -\frac{c_1}{A_6}\\ -\frac{H_6}{b_1A_6} & -\frac{F_6}{A_6} & \frac{c_1P_6}{b_1A_6} \end{pmatrix}\begin{pmatrix} X\\ Y\\ Z \end{pmatrix},\ \text{with inverse}\ \begin{pmatrix} X\\ Y\\ Z \end{pmatrix}=\begin{pmatrix} \sqrt{A_6} & 0 & 0\\ \frac{b_{20}\sqrt{A_6}}{b_1} & \frac{P_6}{b_1} & 1\\ \frac{c_{30}\sqrt{A_6}}{c_1} & \frac{F_6}{c_1} & -\frac{b_1}{c_1} \end{pmatrix}\begin{pmatrix} u\\ v\\ w \end{pmatrix},$$

where

$$
\begin{aligned}
&A_6=-(a_2b_1+a_3c_1+b_{20}^2+2b_{20}c_{30}+c_{30}^2), && H_6=a_2b_1c_{30}-a_3b_{20}c_1,\\
&P_6=a_2b_1+b_{20}^2+b_{20}c_{30}, && F_6=a_3c_1+b_{20}c_{30}+c_{30}^2,
\end{aligned}
$$

and by following the same steps as in the proof under the condition (i) we get the system

$$
\begin{aligned}
\dot r =\ & \frac{\varepsilon}{A_6^{\frac{3}{2}}b_1^2c_1^2}((W^2\cos\theta(b_1^2k_1-b_1c_1h_1+c_1^2g_1)b_1^2-r\sin\theta\cos\theta(((-c_1f_2-2c_{30}k_2)b_1^3+((e_2-f_3+a_2-a_3)c_1^2+((\\
& f_1-h_2-c_{30})b_{20}+c_{30}(f_1+h_2-2k_3-c_{30}))c_1+2k_1(b_{20}c_{30}+c_{30}^2+F_6))b_1^2-(-e_3c_1^2+(-b_{20}^2+(e_1-2g_2\\
& +h_3-c_{30})b_{20}-c_{30}(h_3-e_1))c_1+h_1(-b_{20}^2+c_{30}^2+F_6-P_6))c_1b_1-2(-b_{20}g_3c_1+g_1(b_{20}^2+b_{20}c_{30}+P_6))c_1^2)\\
& W+b_1c_1^2((3b_{20}+3c_{30})b_{21}+(4b_{20}+3c_{30})c_{31}))b_1+r^2\cos\theta\sin^2\theta(-b_1F_6(((-c_{30}+f_3+a_3-b_{20})b_1+b_{20}(b_{20}\\
& +h_3))c_1^2+(f_2b_1^2+((c_{30}-f_1+h_2)b_{20}+c_{30}(2c_{30}-f_1+2k_3))b_1-b_{20}h_1(b_{20}+c_{30}))c_1-2b_1(-k_2b_1+k_1(b_{20}\\
& +c_{30}))c_{30})-P_6c_1(((a_2-b_{20}-c_{30}+e_2)c_1+c_{30}(c_{30}+h_2))b_1^2+(e_3c_1^2+(2b_{20}^2+(2g_2+c_{30}-e_1)b_{20}+c_{30}(h_3\\
& -e_1))c_1-h_1c_{30}(b_{20}+c_{30}))b_1+2c_1(c_1g_3-(b_{20}+c_{30})g_1)b_{20})+F_6^2b_1^2k_1+F_6P_6b_1c_1h_1+P_6^2c_1^2g_1))\sqrt{A_6}-b_1^2\\
& W\sin\theta((k_2b_1^3+((-h_2+k_3)c_1-k_1(b_{20}+c_{30}))b_1^2-c_1((h_3-g_2)c_1-h_1(b_{20}+c_{30}))b_1-((b_{20}+c_{30})g_1-c_1g_3\\
&)c_1^2)W+b_1c_1^2c_{31})-r^2\sin^3\theta(b_1^2F_6^2(k_2b_1+(k_3+c_{30})c_1-k_1(b_{20}+c_{30}))+((c_{30}+h_2)b_1+(b_{20}+h_3)c_1-h_1(\\
& b_{20}+c_{30}))P_6F_6b_1c_1+((g_2+b_{20})b_1-b_{20}g_1+c_1g_3-c_{30}g_1)c_1^2P_6^2)+b_1r\sin^2\theta((b_1(2b_1^2k_2+((2k_3+c_{30}-h_2)\\
& c_1-2k_1(b_{20}+c_{30}))b_1-c_1((b_{20}+h_3)c_1-h_1(b_{20}+c_{30})))F_6+c_1((c_{30}+h_2)b_1^2+((h_3-b_{20}-2g_2)c_1-h_1(b_{20}\\
& +c_{30}))b_1+2c_1((b_{20}+c_{30})g_1-c_1g_3))P_6)W-((F_6+P_6)b_{21}+c_{31}(F_6+2P_6))c_1^2b_1)+(r^2\cos^3\theta\sqrt{A_6}((c_1^2d_1+\\
& c_1c_{30}f_1+c_{30}^2k_1)b_1^2+b_{20}c_1(c_1e_1+c_{30}h_1)b_1+b_{20}^2c_1^2g_1)+r^2\sin\theta\cos^2\theta((-c_1^2d_2-c_1c_{30}f_2-c_{30}^2k_2)b_1^3+(-d_3c_1^3\\
& +((d_1-e_2-a_2)b_{20}-c_{30}(f_3+a_3-d_1))c_1^2+c_{30}((f_1-h_2-c_{30})b_{20}+c_{30}(f_1-k_3-c_{30}))c_1+c_{30}^2k_1(b_{20}+c_{30}\\
&))b_1^2+b_{20}(-e_3c_1^2+(-b_{20}^2+(-c_{30}+e_1-g_2)b_{20}+c_{30}(e_1-h_3))c_1+h_1c_{30}(b_{20}+c_{30}))c_1b_1+b_{20}^2c_1^2((b_{20}+c_{30}\\
&)g_1-c_1g_3)+b_1F_6((c_1^2+c_1f_1+2c_{30}k_1)b_1+b_{20}h_1c_1)+c_1((b_1^2+b_1e_1+2b_{20}g_1)c_1+b_1c_{30}h_1)P_6)-r\cos^2\theta(((\\
& c_1f_1+2c_{30}k_1)b_1^2+c_1(-c_1e_1+h_1(b_{20}-c_{30}))b_1-2b_{20}c_1^2g_1)W+2c_1^2(b_{21}+c_{31})b_1)b_1)A_6)=F_1(\theta,r,W),\\
\dot W =\ & \frac{\varepsilon}{A_6^{\frac{3}{2}}b_1^3c_1^2}(b_1r\sin\theta(W(F_6^2b_1(2b_1^2k_2-b_1c_1h_2-b_{20}c_1^2)+F_6b_1(H_6(2b_1k_1-c_1h_1)-((b_{20}+2g_2-h_3)c_1+b_1(2k_3\\
& +c_{30}-h_2))P_6c_1)-P_6^2c_1(b_1^2c_{30}+b_1c_1h_3-2c_1^2g_3)+H_6P_6c_1(b_1h_1-2c_1g_1))+A_6F_6b_1c_1^2b_{21}+P_6b_1c_1^2(P_6-A_6\\
&)c_{31})-r^2\sin^2\theta(F_6^3b_1^3k_2-P_6^3c_1^3g_3-c_1b_1((c_{30}-h_2+k_3)b_1-b_{20}c_1)P_6F_6^2-c_1b_1P_6^2F_6((-b_{20}-g_2+h_3)c_1+\\
& b_1c_{30})+H_6(F_6^2b_1^2k_1+F_6P_6b_1c_1h_1+P_6^2c_1^2g_1))-A_6r^2\cos^2\theta(((d_2b_1^2+b_{20}(e_2+a_2)b_1+b_{20}^2(g_2+b_{20}+c_{30}))\\
& c_1^2+b_1c_{30}(b_1f_2+b_{20}h_2)c_1+b_1^2c_{30}^2k_2)b_1F_6-c_1P_6((d_3c_1^2+c_{30}(f_3+a_3)c_1+c_{30}^2(k_3+b_{20}+c_{30}))b_1^2+b_{20}c_1(c_1\\
& e_3+c_{30}h_3)b_1+g_3b_{20}^2c_1^2)+H_6((c_1^2d_1+c_1c_{30}f_1+c_{30}^2k_1)b_1^2+b_{20}c_1(c_1e_1+c_{30}h_1)b_1+b_{20}^2c_1^2g_1))-b_1^2W(W(F_6\\
& b_1(b_1^2k_2-b_1c_1h_2+c_1^2g_2)-P_6c_1(b_1^2k_3-b_1c_1h_3+c_1^2g_3)+H_6(b_1^2k_1-b_1c_1h_1+c_1^2g_1))-P_6b_1c_1^2c_{31})-\sqrt{A_6}(b_1\\
& r\cos\theta(W((((f_3+a_3)c_1+c_{30}(2k_3+b_{20}+c_{30}))b_1^2+c_1(-c_1e_3+h_3(b_{20}-c_{30}))b_1-2b_{20}c_1^2g_3)c_1P_6-H_6((c_1f_1\\
& +2c_{30}k_1)b_1^2+c_1(-c_1e_1+h_1(b_{20}-c_{30}))b_1-2b_{20}c_1^2g_1)-b_1(((-a_2-e_2)b_1-(b_{20}+c_{30}+2g_2)b_{20})c_1^2+(b_1f_2\\
& +h_2(b_{20}-c_{30}))b_1c_1+2b_1^2c_{30}k_2)F_6)+(-2P_6b_1b_{20}c_1^2-P_6b_1c_1^2c_{30}-2H_6b_1c_1^2)c_{31}+(F_6b_1b_{20}c_1^2+F_6b_1c_1^2c_{30}-\\
& 2H_6b_1c_1^2)b_{21})+r^2\cos\theta\sin\theta(F_6^2b_1(b_{20}^2c_1^2+(b_1^2f_2+b_1b_{20}h_2)c_1+2b_1^2c_{30}k_2)+F_6b_1(c_1P_6(((e_2-f_3+a_2-a_3)c_1\\
& +c_{30}(h_2-2k_3-b_{20}-2c_{30}))b_1-b_{20}c_1(h_3-2b_{20}-c_{30}-2g_2))+H_6((c_1^2+c_1f_1+2c_{30}k_1)b_1+b_{20}h_1c_1))-\\
& P_6^2c_1(b_1^2c_{30}^2+b_1c_1^2e_3+b_1c_1c_{30}h_3+2b_{20}c_1^2g_3)+((b_1^2+b_1e_1+2b_{20}g_1)c_1+b_1c_{30}h_1)c_1P_6H_6)))=F_2(\theta,r,W).
\end{aligned}
\tag{29}
$$

Computing the integrals (5) we obtain

$$
f_1(r,W)=-\frac{r(T_6\ W+N_6)}{2b_1A_6^{\frac{3}{2}}c_1^2},\quad f_2(r,W)=-\frac{D_6W^2+R_6r^2+C_6W}{2A_6^{\frac{3}{2}}b_1^3c_1^2}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*)=\left(\frac{1}{T_6}\sqrt{\frac{C_6N_6T_6-D_6N_6^2}{R_6}},-\frac{N_6}{T_6}\right),
$$

if $T_6 > 0$, $R_6(C_6N_6T_6 - D_6N_6^2) > 0$, and the Jacobian (6) at (r^*, W^*) is $-N_6(C_6T_6 - N_6D_6)/(2T_6A_6^3b_1^4c_1^4) \neq 0$, where

$$
\begin{aligned}
D_6 = \;& 2F_6b_1^3(b_1^2k_2 - b_1c_1h_2 + c_1^2g_2) - 2P_6b_1^2c_1(b_1^2k_3 - b_1c_1h_3 + c_1^2g_3) + 2H_6b_1^2(b_1^2k_1 - b_1c_1h_1 + c_1^2g_1),\\
C_6 = \;& -2b_1^3P_6c_1^2c_{31},\\
N_6 = \;& A_6b_1b_{21}c_1^2 - F_6b_1c_1^2c_{31},\\
R_6 = \;& F_6^3b_1^3k_2 + (H_6b_1^2k_1 - P_6c_1b_1((c_{30} - h_2 + k_3)b_1 - b_{20}c_1))F_6^2 + (A_6b_1((d_2b_1^2 + b_{20}(e_2 + a_2)b_1 + b_{20}^2(b_{20} + c_{30}+\\
& g_2))c_1^2 + b_1c_{30}(b_1f_2 + b_{20}h_2)c_1 + k_2b_1^2c_{30}^2) - ((-b_{20} - g_2 + h_3)c_1 + b_1c_{30})P_6^2c_1b_1 + H_6P_6b_1c_1h_1)F_6 + A_6(((c_1^2\\
& d_1 + c_1c_{30}f_1 + c_{30}^2k_1)b_1^2 + b_{20}c_1(c_1e_1 + c_{30}h_1)b_1 + b_{20}^2c_1^2g_1)H_6 - P_6c_1((d_3c_1^2 + c_{30}(f_3 + a_3)c_1 + c_{30}^2(k_3 + b_{20}\\
& + c_{30}))b_1^2 + b_{20}c_1(c_1e_3 + c_{30}h_3)b_1 + g_3b_{20}^2c_1^2)) + P_6^2c_1^2(-P_6c_1g_3 + H_6g_1),\\
T_6 = \;& A_6((c_1f_1 + 2c_{30}k_1)b_1^2 + c_1(-c_1e_1 + h_1(b_{20} - c_{30}))b_1 - 2b_{20}c_1^2g_1) - F_6b_1(2k_2b_1^2 + ((2k_3 + c_{30} - h_2)c_1 - 2k_1(\\
& b_{20} + c_{30}))b_1 - ((b_{20} + h_3)c_1 - h_1(b_{20} + c_{30}))c_1) - P_6c_1((h_2 + c_{30})b_1^2 + ((h_3 - b_{20} - 2g_2)c_1 - h_1(b_{20} + c_{30}))\\
& b_1 + 2(-g_3c_1 + g_1(b_{20} + c_{30}))c_1).
\end{aligned}
$$

then, by Theorem 3, with $\varepsilon > 0$ sufficiently small and going back through the changes of variables system (28) has a periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon)$) which tends to the equilibrium $(0,0,0)$ when $\varepsilon \to 0$. Therefore there is one periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of Theorem 2 under the condition (vi). □

Application 6. Consider the Kolmogorov system

$$
\begin{aligned}
\dot{x} = \;& x(x - 3 + y + z + (x-1)^2 + (x-1)(y-1) + (y-1)^2 + (z-1)^2),\\
\dot{y} = \;& y(-6x + 4 + y + z + (x-1)^2 + (x-1)(z-1) + (y-1)^2 + (y-1)(z-1) + (z-1)^2),\\
\dot{z} = \;& z(x - 1 - y + z + (x-1)^2 + (y-1)^2 + (y-1)(z-1) + (z-1)^2).
\end{aligned}
\tag{30}
$$

This system in the new variables (X, Y, Z) writes

$$
\begin{aligned}
\dot{X} = \;& (X+1)(X^2 + XY - 4X\varepsilon + Y^2 + Z^2 - 2X + Y + Z),\\
\dot{Y} = \;& (Y+1)(X^2 + XZ + Y^2 + YZ + Z^2 - 6X + Y + Z),\\
\dot{Z} = \;& (Z+1)((4Y + 2Z)\varepsilon + X^2 + Y^2 + YZ + Z^2 + X + Y + Z).
\end{aligned}
$$

The corresponding system associated to system (29) is

$$
\begin{aligned}
F_1(\theta, r, W) = \;& 13r^2\cos^3\theta + (-30r^2\sin\theta + r(15\ W - 14))\cos^2\theta + ((-4\ W - 14)r\sin\theta + 2W^2 - 9r^2)\cos\theta + (2W^2\\
& + 23r^2 - 2\ W)\sin\theta + r(-14\ W + 10),\\
F_2(\theta, r, W) = \;& -87r^2\cos^2\theta + (-114r^2\sin\theta + (34\ W - 52)r)\cos\theta + r\sin\theta(-50\ W + 40) + 7W^2 + 84r^2 - 8\ W.
\end{aligned}
$$

To look for the limit cycles we must solve the system

$$
f_1(r, W) = -\frac{r(13W - 6)}{2} = 0, \quad f_2(r, W) = 7W^2 + \frac{81}{2}r^2 - 8W = 0.
$$

This system has the solutions (r, W) given by

$$
(0,0), \quad (0, 8/7), \quad (2\sqrt{186}/117, 6/13), \quad (-2\sqrt{186}/117, 6/13).
$$

As in Application 1 we have two good solutions, the $(0, 8/7)$ and the $(2\sqrt{186}/117, 6/13)$. The determinants (6) at these solutions are $-248/7$ and $372/13$, respectively. Hence system (30) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\varepsilon = 10^{-5}$ in Fig. 6.

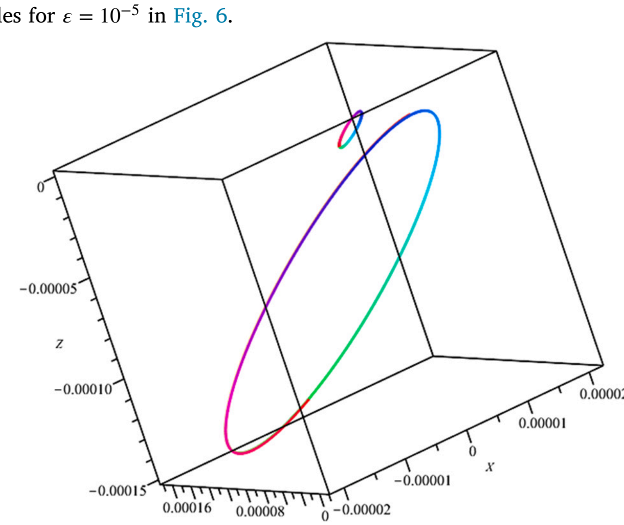

Fig. 6. 1st LC: $X(0) = \epsilon,\ Y(0) = 8\epsilon/7,\ Z(0) = -8\epsilon/7$. **2nd LC:** $X(0) = \frac{2\sqrt{186}\epsilon}{117},\ Y(0) = \frac{\epsilon(2\sqrt{186}+54)}{117},\ Z(0) = \frac{\epsilon(2\sqrt{186}-54)}{117}$.

The eigenvalues of the Jacobian matrix of (f_1, f_2) at singular points $(0, 8/7)$ and $(2\sqrt{186}/117, 6/13)$ are $(8, -31/7)$ and $(-10 \pm 8\,i\sqrt{74})/13$, respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. This completes Application 6.

Proof of Theorem 2 under the condition (vii)**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (vii). We translate the equilibrium point $(1, 1, 1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned}
\dot{X} = {}& (X+1)((-2b_{21}-2c_{31})X\varepsilon + (-b_{20}-c_{30})X + b_1Y + c_1Z + d_1X^2 + e_1XY + f_1XZ + g_1Y^2 + h_1YZ + k_1Z^2),\\
\dot{Y} = {}& (Y+1)\left(\left(\left(-\frac{b_{20}c_{31}}{b_1} - \frac{b_{20}(b_{21}+c_{31}) + b_{21}(b_{20}+c_{30})}{b_1}\right)X + b_{21}Y + \frac{b_{21}c_1Z}{b_1}\right)\varepsilon - \frac{b_{20}(b_{20}+c_{30})X}{b_1} + b_{20}Y\right.\\
& \left. + \frac{b_{20}c_1Z}{b_1} + d_2X^2 + e_2XY + f_2XZ + g_2Y^2 + h_2YZ + k_2Z^2\right),\\
\dot{Z} = {}& (Z+1)(X^2d_3 + XYe_3 + XZf_3 + Y^2g_3 + YZh_3 + Z^2k_3 + Z\varepsilon c_{31} + Xa_3 + Yb_3 + Zc_{30}).
\end{aligned} \tag{31}$$

As usual we write the linear part of system (31) with $\varepsilon = 0$ at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\dfrac{\sqrt{-b_1(a_3b_1c_1 + b_1c_{30}^2 + b_{20}b_3c_1)}}{b_1} & 0 \\ \dfrac{\sqrt{-b_1(a_3b_1c_1 + b_1c_{30}^2 + b_{20}b_3c_1)}}{b_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{B_7}{b_1A_7} & \dfrac{M_7}{b_1A_7} & 0 \\ -\dfrac{b_{20}+c_{30}}{b_1\sqrt{-b_1A_7}} & \dfrac{1}{\sqrt{-b_1A_7}} & \dfrac{c_1}{b_1\sqrt{-b_1A_7}} \\ -\dfrac{b_{20}}{b_1A_7} & \dfrac{1}{A_7} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

with inverse

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -b_1^2 & 0 & b_1M_7 \\ -b_{20}b_1 & 0 & b_1B_7 \\ -\dfrac{b_1^2c_{30}}{c_1} & \dfrac{b_1\sqrt{-b_1A_7}}{c_1} & -b_1P_7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where $A_7 = a_3b_1c_1 + b_1c_{30}^2 + b_{20}b_3c_1$, $M_7 = b_1c_{30} - b_3c_1$, $P_7 = a_3b_1 + b_{20}b_3 + b_3c_{30}$, $B_7 = a_3c_1 + b_{20}c_{30} + c_{30}^2$. By following the same steps as in the proof under the condition (i) we obtain the system

$$\begin{aligned}
\dot{r} = {}& \frac{\varepsilon}{b_1c_1^2A_7^2}(\sqrt{-b_1A_7}r^2b_1^2\cos^3\theta(B_7((c_1^2d_1 + c_1c_{30}f_1 + c_{30}^2k_1)b_1^2 + b_{20}c_1(c_1e_1 + c_{30}h_1)b_1 + b_{20}^2c_1^2g_1) - M_7((c_1^2d_2 + c_1\\
& c_{30}f_2 + c_{30}^2k_2)b_1^2 + b_{20}c_1(c_1e_2 + c_{30}h_2)b_1 + b_{20}^2c_1^2g_2)) + A_7^2b_1^3\sin^3\theta(k_2b_1 + (c_{30}+k_3)c_1 - k_1(b_{20}+c_{30}))r^2 +\\
& A_7b_1^2r^2\cos^2\theta\sin\theta((-d_3c_1^3 + ((-a_3 - f_3 + d_1)c_{30} - b_{20}(e_2 - d_1))c_1^2 + (-c_{30}^2 + (f_1 - k_3)c_{30} + b_{20}(f_1 - h_2))c_{30}\\
& c_1 + c_{30}^2k_1(b_{20}+c_{30}))b_1^2 + b_{20}c_1(-e_3c_1^2 + ((e_1 - h_3 - b_3)c_{30} + b_{20}(e_1 - g_2))c_1 + c_{30}h_1(b_{20}+c_{30}))b_1 + b_{20}^2c_1^2(\\
& -g_3c_1 + g_1(b_{20}+c_{30})) + B_7((c_1^2 + c_1f_1 + 2c_{30}k_1)b_1 + b_{20}c_1h_1)b_1 - (b_{20}^2c_1^2 + (b_1^2f_2 + b_1b_{20}h_2)c_1 + 2b_1^2c_{30}k_2)\\
& M_7 - (c_1^2d_2 + c_1c_{30}f_2 + c_{30}^2k_2)b_1^3) - A_7b_1rc_1\sin\theta\cos\theta((B_7b_1(((b_{20} + c_{30} - e_2)c_1 - h_2c_{30})b_1^2 + (-e_3c_1^2 + (-\\
& b_{20}^2 + (-2g_2 + e_1)b_{20} - c_{30}(h_3 + b_3 - e_1))c_1 + c_{30}h_1(b_{20}+c_{30}))b_1 - 2c_1(g_3c_1 - g_1(b_{20}+c_{30}))b_{20} + M_7((c_1 +\\
& f_1 - h_2)b_1 - b_{20}c_1) - 2P_7b_1k_1) + b_1M_7((-2b_1d_3 - b_{20}e_3)c_1^2 + (-2d_2b_1^2 + (-b_{20}^2 + (-e_2 - 2c_{30} + 2d_1)b_{20} - c_{30}\\
& (f_3 + a_3 + c_{30} - 2d_1))b_1 + b_{20}(b_{20} + e_1)(b_{20} + c_{30}))c_1 + (-f_2b_1 + f_1(b_{20}+c_{30}))b_1c_{30} + 2P_7b_1k_2) + P_7b_1(((\\
& f_3 + a_3 - b_{20} - c_{30})b_1 + b_{20}(h_3 + b_{20} + b_3))c_1^2 + (f_2b_1^2 + ((h_2 - f_1)b_{20} + (2k_3 + 2c_{30} - f_1)c_{30})b_1 - b_{20}h_1(b_{20}\\
& + c_{30}))c_1 - 2b_1c_{30}(-k_2b_1 + k_1(b_{20}+c_{30})))) + b_1^2(B_7^2h_1 - M_7^2f_2))W - (((b_{20} + 3c_{30})b_1 - b_3c_1)b_{21} + 3b_1c_{30}c_{31})\\
& c_1) - A_7b_1Wc_1^2\sin\theta((B_7^2b_1((b_{20} + g_2)b_1 - b_{20}g_1 + g_3c_1 - c_{30}g_1) - B_7b_1(M_7(b_{20}^2 + (b_1 + c_{30} + e_1)b_{20} + (c_{30} -\\
& e_2)b_1 - c_1e_3 + c_{30}e_1) + P_7((h_3 + b_{20} + b_3)c_1 + b_1h_2 - b_{20}h_1 - c_{30}h_1)) + M_7^2(b_{20}^2 + (2c_{30} - d_1)b_{20} + b_1d_2 + c_1\\
& d_3 + c_{30}^2 - c_{30}d_1)b_1 - b_1M_7((f_3 + a_3 - b_{20} - c_{30})c_1 + f_2b_1 - f_1(b_{20}+c_{30}))P_7 + (k_2b_1 + (c_{30}+k_3)c_1 - k_1(b_{20}\\
& + c_{30}))P_7^2b_1)W + B_7b_1b_{21} - P_7c_1(b_{21} + c_{31}) + M_7c_{30}(b_{21} + 2c_{31})) - \sqrt{-b_1A_7}r^2b_1^2A_7\cos\theta\sin^2\theta(((-a_3 + b_{20}\\
& + c_{30} - f_3)b_1 - b_{20}(h_3 + b_{20} + b_3))c_1^2 + ((b_{20}(f_1 - h_2) - (2k_3 + 2c_{30} - f_1)c_{30})b_1 + b_{20}h_1(b_{20}+c_{30}) - f_2b_1^2)c_1\\
& + 2b_1c_{30}(-k_2b_1 + k_1(b_{20}+c_{30})) + b_1(B_7k_1 - M_7k_2)) + \sqrt{-b_1A_7}Wc_1^2\cos\theta((B_7^3b_1^2g_1 + B_7^2((b_1 - b_{20} + e_1 - g_2)
\end{aligned} \tag{32}$$

$$
\begin{aligned}
& M_7 - P_7 h_1) b_1^2 - B_7 b_1 (M_7^2 ((b_{20} + c_{30} - d_1 + e_2) b_1 - b_{20}(b_{20} + c_{30})) + M_7((c_1 + f_1 - h_2) b_1 - b_{20} c_1) P_7 - P_7^2 \\
& b_1 k_1) - M_7 b_1^2 (M_7^2 d_2 - M_7 P_7 f_2 + P_7^2 k_2)) W - B_7 M_7 b_1 (3 b_{21} + 2 c_{31}) + M_7^2 ((2 b_{20} + c_{30}) b_{21} + 2 b_{20} c_{31}) + M_7 P_7 \\
& b_{21} c_1) + r(((-B_7^2 b_1 (b_1^2 c_1 + b_1 (c_1 e_1 + c_{30} h_1) + 2 b_{20} c_1 g_1) + B_7 b_1 (M_7(((b_{20} + c_{30} - 2 d_1 + e_2) b_1 + b_{20}(b_{20} - e_1 \\
& + 2 g_2)) c_1 + b_1 c_{30}(h_2 - f_1)) + P_7((c_1^2 + c_1 f_1 + 2 c_{30} k_1) b_1 + b_{20} c_1 h_1) + A_7((h_3 + b_{20} + b_3) c_1 + b_1 h_2 - b_{20} h_1 - \\
& c_{30} h_1)) + M_7^2 ((2 d_2 b_1^2 + b_1 b_{20} e_2 - b_{20}^2 (b_{20} + c_{30})) c_1 + b_1^2 c_{30} f_2) + M_7 (b_1 A_7 ((f_3 + a_3 - b_{20} - c_{30}) c_1 + f_2 b_1 - f_1 \\
& (b_{20} + c_{30})) - P_7 (b_{20}^2 c_1^2 + (b_1^2 f_2 + b_1 b_{20} h_2) c_1 + 2 b_1^2 c_{30} k_2)) - 2 b_1 A_7 (k_2 b_1 + (c_{30} + k_3) c_1 - k_1 (b_{20} + c_{30})) P_7) W \\
& + 2 B_7 b_1 b_{21} c_1 + 2 B_7 b_1 c_1 c_{31} - M_7 b_{20} b_{21} c_1 - 2 M_7 b_{20} c_1 c_{31} + A_7 b_{21} c_1 + A_7 c_1 c_{31}) \cos^2\theta - A_7((B_7((h_3 + b_{20} + \\
& b_3) c_1 + b_1 h_2 - h_1 (b_{20} + c_{30})) b_1 + b_1 M_7 ((f_3 + a_3 - b_{20} - c_{30}) c_1 + f_2 b_1 - f_1 (b_{20} + c_{30})) - 2(k_2 b_1 + (c_{30} + k_3) \\
& c_1 - k_1 (b_{20} + c_{30})) P_7 b_1) W + c_1 (b_{21} + c_{31}))) b_1 \sqrt{-b_1 A_7 c_1}) = F_1(\theta, r, W),
\end{aligned}
$$

$$
\begin{aligned}
\dot{W} = \; & \frac{\varepsilon\sqrt{-b_1 A_7}}{c_1^2 A_7^2} (A_7 b_1^2 r^2 \sin^2\theta (b_1 k_2 - b_{20} k_1) + \sqrt{-b_1 A_7} b_1 r^2 \cos\theta \sin\theta ((c_1 f_2 + 2 c_{30} k_2) b_1^2 - b_{20}(c_1^2 + (f_1 - h_2) c_1 + \\
& 2 c_{30} k_1) b_1 + b_{20}^2 c_1 (c_1 - h_1)) - \sqrt{-b_1 A_7} c_1 r \sin\theta (W b_1 (B_7((c_1 - h_1) b_{20} + b_1 h_2) + M_7((-c_1 - f_1) b_{20} + f_2 b_1) - \\
& 2 P_7 (b_1 k_2 - b_{20} k_1)) + c_1 b_{21}) - b_1 c_1 r \cos\theta ((B_7((b_{20} c_1 - c_1 e_2 - c_{30} h_2) b_1^2 + b_{20}(-b_{20} c_1 + (-2 g_2 + e_1) c_1 + c_{30} h_1 \\
&) b_1 + 2 b_{20}^2 c_1 g_1) + (b_{20}^3 c_1 - c_1 (b_1 - c_{30} - e_1) b_{20}^2 + b_1((-c_{30} + 2 d_1 - e_2) c_1 + f_1 c_{30}) b_{20} - b_1^2 (2 c_1 d_2 + c_{30} f_2)) M_7 \\
& + P_7((c_1 f_2 + 2 c_{30} k_2) b_1^2 - b_{20}(c_1^2 + (f_1 - h_2) c_1 + 2 c_{30} k_1) b_1 + b_{20}^2 c_1 (c_1 - h_1))) W - b_{20} b_{21} c_1) - b_1 r^2 \cos^2\theta ((c_1^2 \\
& d_2 + c_1 c_{30} f_2 + c_{30}^2 k_2) b_1^3 + ((e_2 - d_1) c_1^2 - c_{30}(f_1 - h_2) c_1 - k_1 c_{30}^2) b_{20} b_1^2 + ((g_2 - e_1) c_1 - c_{30} h_1) b_{20}^2 c_1 b_1 - b_{20}^3 c_1^2 \\
& g_1) + (B_7 b_1 c_1^2 (M_7 (b_{20}^2 + (b_1 + c_{30} + e_1) b_{20} - b_1 e_2) + P_7((c_1 - h_1) b_{20} + b_1 h_2)) - M_7^2 c_1^2 b_1 (b_{20}^2 + (c_{30} - d_1) b_{20} \\
& + b_1 d_2) + b_1 M_7 P_7 (f_2 b_1 - b_{20}(c_1 + f_1)) c_1^2 - P_7^2 b_1 c_1^2 (b_1 k_2 - b_{20} k_1) - B_7^2 ((b_{20} + g_2) b_1 - b_{20} g_1) c_1^2 b_1) W^2 - b_{21} c_1^2 \\
& b_{20} M_7 \; W) = F_2(\theta, r, W),
\end{aligned}
$$

Now we compute the integrals (5) and we get

$$
f_1(r, W) = -\frac{r\sqrt{-A_7 b_1}(T_7 \; W + N_7)}{2 c_1 A_7^2}, \quad f_2(r, W) = \frac{(R_7 r^2 + D_7 W^2 + C_7 \; W)\sqrt{-A_7 b_1}}{2 c_1^2 A_7^2}.
$$

The system $f_1(r, W) = f_2(r, W) = 0$ has a unique solution (r^*, W^*) with $r^* > 0$, namely

$$
(r^*, W^*) = \left(\frac{1}{T_7} \sqrt{\frac{C_7 N_7 T_7 - D_7 N_7^2}{R_7}}, -\frac{N_7}{T_7} \right),
$$

if $T_7 > 0$, $R_7(C_7 N_7 T_7 - D_7 N_7^2) > 0$ and the Jacobian (6) at (r^*, W^*) is $-b_1 N_7 (C_7 T_7 - N_7 D_7)/(2 A_7^3 c_1^3 T_7) \neq 0$, where

$$
\begin{aligned}
N_7 = \; & -((a_3(b_{21} + c_{31}) c_1 + c_{30}((b_{20} + c_{30}) b_{21} + c_{30} c_{31})) b_1 + b_{20} b_3 c_1 c_{31}) c_1, \\
C_7 = \; & -2 b_{21} c_1^2 b_{20} M_7, \\
R_7 = \; & A_7 b_1^2 (b_1 k_2 - b_{20} k_1) - b_1((c_1^2 d_2 + c_1 c_{30} f_2 + c_{30}^2 k_2) b_1^3 + ((-d_1 + e_2) c_1^2 + c_{30}(h_2 - f_1) c_1 - k_1 c_{30}^2) b_{20} b_1^2 + c_1 b_{20}^2 (\\
& (g_2 - e_1) c_1 - c_{30} h_1) b_1 - b_{20}^3 c_1^2 g_1), \\
D_7 = \; & 2 b_1 c_1^2 B_7 (M_7 (b_{20}^2 + (b_1 + c_{30} + e_1) b_{20} - b_1 e_2) + P_7((c_1 - h_1) b_{20} + b_1 h_2)) - 2 M_7^2 (b_{20}^2 + (c_{30} - d_1) b_{20} + b_1 d_2) \\
& c_1^2 b_1 + 2 c_1^2 b_1 (b_1 f_2 - b_{20}(c_1 + f_1)) M_7 P_7 - 2 b_1 c_1^2 P_7^2 (b_1 k_2 - b_{20} k_1) - 2 B_7^2 c_1^2 ((b_{20} + g_2) b_1 - b_{20} g_1) b_1, \\
T_7 = \; & B_7^2 b_1 (b_1^2 c_1 + b_1 (c_1 e_1 + c_{30} h_1) + 2 b_{20} c_1 g_1) + B_7 b_1 (A_7((b_{20} + b_3 + h_3) c_1 + b_1 h_2 - b_{20} h_1 - c_{30} h_1) - M_7(((b_{20} \\
& + c_{30} - 2 d_1 + e_2) b_1 + b_{20}(b_{20} - e_1 + 2 g_2)) c_1 + b_1 c_{30}(h_2 - f_1)) - ((c_1^2 + c_1 f_1 + 2 c_{30} k_1) b_1 + b_{20} c_1 h_1) P_7) - M_7^2 \\
& ((2 b_1^2 d_2 + b_1 b_{20} e_2 - b_{20}^2 (b_{20} + c_{30})) c_1 + b_1^2 c_{30} f_2) + M_7 (P_7 (b_{20}^2 c_1^2 + (b_1^2 f_2 + b_1 b_{20} h_2) c_1 + 2 b_1^2 c_{30} k_2) + b_1 A_7 ((a_3 \\
& - b_{20} - c_{30} + f_3) c_1 + b_1 f_2 - f_1 (b_{20} + c_{30}))) - 2 b_1 A_7 (b_1 k_2 + (c_{30} + k_3) c_1 - k_1 (b_{20} + c_{30})) P_7,
\end{aligned}
$$

then, from Theorem 3 for $\varepsilon > 0$ sufficiently small and going back through the changes of variables system (32) has a periodic solution $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon))$ which tends to the equilibrium $(0, 0, 0)$ when $\varepsilon \to 0$. Therefore there is one periodic solution starting at the zero-Hopf equilibrium $(1, 1, 1)$ when $\varepsilon = 0$. This completes the proof of Theorem 2 under the condition (vii). □

Application 7. Consider the Kolmogorov system

$$
\begin{aligned}
\dot{x} = \; & x(x + y - 2z + (x-1)^2 + (x-1)(y-1) - (y-1)^2 + (z-1)^2), \\
\dot{y} = \; & y(-2x + y + z + (x-1)^2 + (x-1)(z-1) + (y-1)^2 + (z-1)^2), \\
\dot{z} = \; & z(2x - 2 - y + z + (x-1)^2 + (y-1)^2 + (y-1)(z-1) + (z-1)^2).
\end{aligned}
\tag{33}
$$

This system in the variables (X, Y, Z) writes

$$
\begin{aligned}
\dot{X} = \; & (X+1)(X^2 + XY + 6X\varepsilon - Y^2 + Z^2 - 2X + Y - 2Z), \\
\dot{Y} = \; & (Y+1)((2X + 4Y - 8Z)\varepsilon - 2X + Y - 2Z + X^2 + XZ + Y^2 + Z^2), \\
\dot{Z} = \; & (Z+1)(X^2 + Y^2 + YZ + Z^2 - 7Z\varepsilon + 2X - Y + Z).
\end{aligned}
$$

The two components corresponding to system (32) in this case are

$$F_1(\theta,r,W) = -\frac{r^2}{4}(3\cos^3(\theta)+7\cos^2(\theta)\sin(\theta)+5\sin^3(\theta)-7\cos(\theta)\sin^2(\theta)) - \frac{1}{2}(\sin(\theta)\cos(\theta)r(12\,W-26)+r((-10\,W-10)\cos^2(\theta)+3\,W+6)) + \sin(\theta)W(-3W+2)+\cos(\theta)W(7W+2),$$
$$F_2(\theta,r,W) = -\frac{1}{2}(\cos(\theta)\sin(\theta)r^2+\cos^2(\theta)r^2-\sin(\theta)r(W-8)-\cos(\theta)r(-9\,W+8)) - 6W^2+4\,W.$$

To look for the limit cycles we must solve the system

$$f_1(r,W) = \frac{r(4\,W-2)}{4} = 0, \quad f_2(r,W) = -6W^2 - \frac{r^2}{4} + 4W = 0. \tag{34}$$

As in Application 1 we have two good solutions, the $(0,2/3)$ and the $(\sqrt{2},1/2)$. The determinants (6) at these solutions are $-2/3$ and 1, respectively. Hence system (33) has two limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot these two bifurcated limit cycles for $\varepsilon = 10^{-5}$ in Fig. 7.

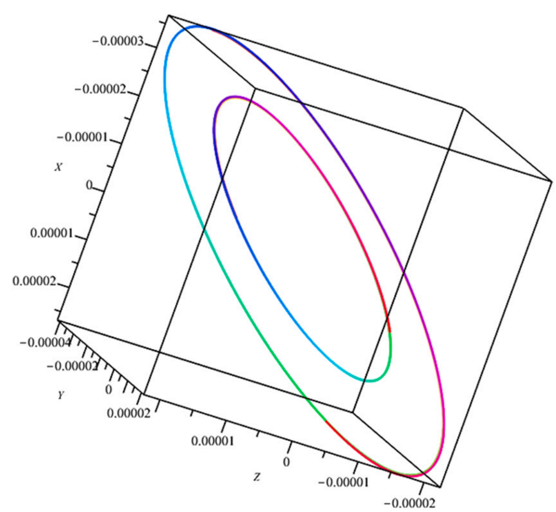

Fig. 7. **1st LC:** $X(0) = -2\varepsilon/3$, $Y(0) = -4\varepsilon/3$, $Z(0) = \varepsilon$. **2nd LC:** $X(0) = -\varepsilon\sqrt{10}-\varepsilon/2$, $Y(0) = -\varepsilon\sqrt{10}-\varepsilon$, $Z(0) = \varepsilon\sqrt{10}/2$.

The eigenvalues of the Jacobian matrix of (f_1,f_2) at the singular points $(0,2/3)$ and $(\sqrt{2},1/2)$ are $(-4,1/6)$ and -1 with multiplicity two,respectively, by Theorem 3 the limit cycles are unstable and stable, respectively. This completes Application 7.

Proof of Theorem 2 under the condition $(viii)$**.** Now we consider the Kolmogorov system (2) perturbed with the parameters given in the condition (viii). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (2) becomes

$$\begin{aligned}
\dot{X} &= (X+1)((-b_2-c_{30})X+b_1Y+c_1Z+d_1X^2+e_1XY+f_1XZ+g_1Y^2+h_1YZ+k_1Z^2-2c_{31}X\varepsilon),\\
\dot{Y} &= (Y+1)(X^2d_2+XYe_2+XZf_2+Y^2g_2+YZh_2+Z^2k_2+Xa_2+Yb_2+Zc_2),\\
\dot{Z} &= (Z+1)\left(\left(\frac{(a_2b_1c_{31}+(b_2+c_{30})b_2c_{31}+c_{31}(b_2c_{30}-b_3c_2))X}{b_1c_2-b_2c_1}+c_{31}Z\right)\varepsilon + \frac{1}{b_1c_2-b_2c_1}((a_2(b_1c_{30}-b_3c_1)+(b_2\right.\\
&\quad \left. +c_{30})(b_2c_{30}-b_3c_2))X)+b_3Y+c_{30}Z+d_3X^2+e_3XY+f_3XZ+g_3Y^2+h_3YZ+k_3Z^2\right).
\end{aligned} \tag{35}$$

We write the linear part of system (35) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\dfrac{\sqrt{(b_1c_2-b_2c_1)A}}{b_1c_2-b_2c_1} & 0 \\ \dfrac{\sqrt{(b_1c_2-b_2c_1)A}}{b_1c_2-b_2c_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $A = -(-b_2^3c_1+b_1b_2^2c_2+b_2(-2b_3c_1c_2+b_1c_2c_{30})+c_2(b_1b_3c_2-b_3c_1c_{30}+b_1c_{30}^2)+a_2(b_1^2c_2-b_3c_1^2+b_1c_1(c_{30}-b_2)))$. For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{FL}{EA} & -\dfrac{FM}{EA} & -\dfrac{F}{A} \\ -\dfrac{K\sqrt{FA}}{EA} & \dfrac{F\sqrt{FA}}{EA} & 0 \\ -\dfrac{G}{A} & \dfrac{H}{A} & -\dfrac{F}{A} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

with inverse

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} F & -\dfrac{c_1 A}{\sqrt{AF}} & -F \\ K & -\dfrac{c_2 A}{\sqrt{AF}} & -K \\ \dfrac{S}{F} & -\dfrac{c_{30} A}{\sqrt{AF}} & P \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$\begin{aligned} E &= a_2c_1^2 - b_1c_2^2 + 2b_2c_1c_2 + c_1c_2c_{30}, \quad P = a_2b_1 + b_2^2 + b_2c_{30}, \quad F = b_1c_2 - b_2c_1, \quad G = b_2c_{30} - b_3c_2, \\ M &= a_2b_1c_1 + b_1c_2c_{30} + b_2^2c_1, \quad K = a_2c_1 + b_2c_2 + c_2c_{30}, \quad H = b_1c_{30} - b_3c_1, \quad A_8 = -A, \\ L &= a_2b_1c_2 + a_2c_1c_{30} + b_2^2c_2 + 2b_2c_2c_{30} + c_2c_{30}^2, \\ S &= a_2b_1c_1c_{30} - a_2b_3c_1^2 + b_1b_3c_2^2 + b_1c_2c_{30}^2 + b_2^2c_1c_{30} - 2b_2b_3c_1c_2 - b_3c_1c_2c_{30}. \end{aligned}$$

By following the same steps as the first case (i) we get the system

$$\begin{aligned} \dot{r} = {} & \frac{\varepsilon}{F^3A_8^2E\sqrt{-FA_8}}(F^4A_8W\cos\theta(((Ed_3 + L(b_2 + c_{30} - d_1) + Md_2)F^3 - KP(E(h_3 + b_3)F - Lh_1F + M(-b_2 \\ & c_1c_2 + Fh_2)) + (E(Ke_3F - P(b_2^2c_{30} + (-b_3c_2 - c_1f_3 + c_{30}^2)b_2 + (a_2b_1 - b_3c_2)c_{30} - a_2b_3c_1 + b_1c_2f_3)) - KF \\ & (L(b_1 + e_1) - M(a_2 + e_2)) + PF(L(c_1 + f_1) - Mf_2))F + F(Eg_3 - Lg_1 + M(b_2 + g_2))K^2 - MKPb_1c_2^2 + P^2 \\ & F(E(k_3 + c_{30}) - Lk_1 + Mk_2))W - c_{31}F(EG + 2FL)) - F^2A_8^2r^2\sin^2\theta(((a_2 + e_2)c_2 + 2c_1d_2 + c_{30}f_2)F^4 + K \\ & F^3((a_2 + 2b_2 + c_{30} - 2d_1 + e_2)c_1 + (2g_2 - b_1 + 2b_2 + c_{30} - e_1)c_2 - c_{30}(f_1 - h_2)) + (E(-b_2c_1^3d_3 + ((b_1d_3 - \\ & b_2e_3)c_2 - c_{30}(a_2b_3 + b_2f_3))c_1^2 + ((b_1e_3 - b_2g_3)c_2^2 + (-b_3c_{30} + f_3b_1 - b_2(h_3 + 2b_3))c_{30}c_2 - c_{30}^2(-a_2b_1 + b_2(k_3 \\ & - b_2)))c_1 + b_1c_2(g_3c_2^2 + c_{30}(h_3 + b_3)c_2 + c_{30}^2(k_3 + c_{30}))) - K^2((b_1 + e_1)c_1 + 2c_2g_1 + c_{30}h_1)F - LF((-b_2 + d_1 \\ &)c_1^2 + ((b_1 + e_1)c_2 + c_{30}f_1)c_1 + c_2^2g_1 + c_2c_{30}h_1 + c_{30}^2k_1) + FM((g_2 + b_2 + c_{30})c_2^2 + ((a_2 + e_2)c_1 + h_2c_{30})c_2 + \\ & c_1c_{30}f_2 + c_{30}^2k_2 + c_1^2d_2) + S(c_1f_2 + c_2^2 + c_2h_2 + 2c_{30}k_2)F)F - KS(c_1^2 + c_1f_1 + c_2h_1 + 2c_{30}k_1)F)\cos\theta + F^2 \\ & A_8r^2\cos^3\theta((F(Ed_3 + Md_2) + L(b_2 + c_{30} - d_1)F + (Ee_3 - L(b_1 + e_1) + M(a_2 + e_2))K)F^4 + (F(Eg_3 - Lg_1 \\ & + M(b_2 + g_2))K^2 + S(E(Ff_3 + G(b_2 + c_{30}) + Ha_2) - F(L(c_1 + f_1) - Mf_2)) + KS(E(h_3 + b_3) - Lh_1 + M \\ & (c_2 + h_2)))F^2 + S^2F(E(k_3 + c_{30}) - Lk_1 + Mk_2)) - F^3A_8r\cos^2\theta(((2Ed_3 + 2L(b_2 + c_{30} - d_1) + 2Md_2)F^4 + \\ & (E(2Ke_3F - P(Ff_3 - Kb_3 + Pc_{30})) + 2F(M(a_2 + e_2) - L(b_1 + e_1))K + PF(L(c_1 + f_1) - Mf_2))F^2 + (2F \\ & (Eg_3 - Lg_1 + M(b_2 + g_2))K^2 - PF(E(h_3 + b_3) - Lh_1 + M(c_2 + h_2))K + S(E(Ff_3 - Kb_3 + Pc_{30}) - L(c_1 \\ & + f_1)F + Mf_2F))F + SF(E(K(h_3 + b_3) - 2P(k_3 + c_{30})) - K(Lh_1 - M(c_2 + h_2)) + 2P(Lk_1 - Mk_2)))W - \\ & F^2c_{31}(E(P + G) + 2FL) - ESc_{31}F) + (-FA_8)^{\frac{3}{2}}F^3W\sin\theta((F^3d_2 + (K(a_2 + b_2 + c_{30} - d_1 + e_2) - Pf_2)F^2 \\ & + (KP(c_1 - c_2 + f_1 - h_2) + P^2k_2 - K^2(b_1 - b_2 + e_1 - g_2))F - K^3g_1 + K^2Ph_1 - KP^2k_1)W - 2FKc_{31}) + \\ & F^4A_8^2r\sin^2\theta(((((a_2 + e_2)c_2 + 2c_1d_2 + c_{30}f_2)F^2 + (K((a_2 + 2b_2 + c_{30} - 2d_1 + e_2)c_1 + (2g_2 - b_1 + 2b_2 + c_{30} - \\ & e_1)c_2 - c_{30}(f_1 - h_2)) - P(c_1f_2 + c_2^2 + c_2h_2 + 2c_{30}k_2))F - K^2((b_1 + e_1)c_1 + 2c_2g_1 + c_{30}h_1) + KP(c_1^2 + c_1f_1 \\ & + c_2h_1 + 2c_{30}k_1))W - 2Kc_1c_{31}) + F^3\sqrt{-FA_8}A_8^2r^2\sin^3\theta(((g_2 + b_2 + c_{30})c_2^2 + ((a_2 + e_2)c_1 + h_2c_{30})c_2 + c_1c_{30} \\ & f_2 + c_{30}^2k_2 + c_1^2d_2)F - ((-b_2 + d_1)c_1^2 + ((b_1 + e_1)c_2 + c_{30}f_1)c_1 + c_2^2g_1 + c_2c_{30}h_1 + c_{30}^2k_1)K) + F^2\sqrt{-FA_8}rA_8 \\ & \cos\theta\sin\theta((2d_2F^5 + (2K(a_2 + b_2 + c_{30} - d_1 + e_2) - Pf_2)F^4 + (E(c_{30}^2P + F(2c_1d_3 + c_2e_3 + c_{30}f_3) - b_3c_{30}K) \\ & - 2K^2(b_1 - b_2 + e_1 - g_2)F + KP(c_1 - c_2 + f_1 - h_2)F - L((-2b_2 - c_{30} + 2d_1)c_1 + (b_1 + e_1)c_2 + c_{30}f_1)F + \\ & FM((a_2 + e_2)c_2 + 2c_1d_2 + c_{30}f_2) + Sf_2F)F^2 + (E(K(c_1e_3 + 2c_2g_3 + c_{30}(h_3 + b_3))F - P(S + F(c_1f_3 + c_2 \\ & h_3 + c_{30}^2 + 2c_{30}k_3))) - F(2K^3g_1 - K^2Ph_1 + K(L((b_1 + e_1)c_1 + 2c_2g_1 + c_{30}h_1) - M((2b_2 + c_{30} + 2g_2)c_2 + (\\ & a_2 + e_2)c_1 + h_2c_{30})) - P(L(c_1^2 + c_1f_1 + c_2h_1 + 2c_{30}k_1) - M(c_1f_2 + c_2^2 + c_2h_2 + 2c_{30}k_2)) + S(K(c_1 - c_2 + f_1 \\ & - h_2) + 2Pk_2)))F - KSF(Kh_1 - 2Pk_1))W - Fc_{31}(E(Hc_2 + Pc_1) + 2F^2K + 2Lc_1F)) - \sqrt{-FA_8}r^2FA_8 \\ & \sin\theta\cos^2\theta(d_2F^6 + KF(a_2 + b_2 + c_{30} - d_1 + e_2)F^4 + (E(c_{30}^2P + F(2c_1d_3 + c_2e_3 + c_{30}f_3) - b_3c_{30}K) - K^2(b_1 \\ & - b_2 + e_1 - g_2)F - L((-2b_2 - c_{30} + 2d_1)c_1 + (b_1 + e_1)c_2 + c_{30}f_1)F + FM((a_2 + e_2)c_2 + 2c_1d_2 + c_{30}f_2) + S \\ & f_2F)F^3 + (EK(c_1e_3 + 2c_2g_3 + c_{30}(h_3 + b_3)) - K^3g_1 - K(L((b_1 + e_1)c_1 + 2c_2g_1 + c_{30}h_1) - M((2b_2 + c_{30} + \\ & 2g_2)c_2 + (a_2 + e_2)c_1 + h_2c_{30}) + S(c_1 - c_2 + f_1 - h_2)))F^3 + S(E(S + F(c_1f_3 + c_2h_3 + c_{30}^2 + 2c_{30}k_3)) - F(\\ & K^2h_1 + L(c_1^2 + c_1f_1 + c_2h_1 + 2c_{30}k_1) - M(c_1f_2 + c_2^2 + c_2h_2 + 2c_{30}k_2) - Sk_2))F - KS^2k_1F)) = F_1(\theta, r, W), \end{aligned} \tag{36}$$

$$
\begin{aligned}
\dot{W} = \ & \frac{\varepsilon}{F^2 A_8\sqrt{-FA_8}}(F\sqrt{-FA_8}r\sin\theta((-Hc_2(Kb_1(a_2c_1+c_{30}h_2)+Pb_2c_1h_2)-GP(c_1^2+c_1f_1+c_2h_1+2c_{30}k_1)F+\\
& H((2b_2^2c_1c_2+(-2b_1c_2^2+c_1^2(a_2+e_2))b_2-b_1c_2^2(c_{30}+2g_2))K+P(-c_1(c_1f_2+c_2^2+2c_{30}k_2)b_2+b_1c_2(c_2h_2+2\\
& c_{30}k_2)))+FG((b_1+e_1)c_1+2c_2g_1+c_{30}h_1)K+(c_{30}^2P+F(2c_1d_3+c_2e_3+c_{30}f_3)-b_3c_{30}K)F^2+(((-2b_2-\\
& c_{30}+2d_1)c_1+(b_1+e_1)c_2+c_{30}f_1)GF-FH((a_2+e_2)c_2+c_{30}f_2+2c_1d_2)+K(c_1e_3+2c_2g_3+c_{30}(h_3+b_3))\\
& F-P(S+F(c_1f_3+c_2h_3+c_{30}^2+2c_{30}k_3)))F-H(K(((-c_{30}-2g_2)c_2-c_{30}h_2)b_2+b_1c_2e_2)c_1-Pb_1c_2(c_1f_2\\
& +c_2^2)))W-c_{31}(F(Hc_2+Pc_1)-2Gc_1F))-\sqrt{-FA_8}r^2\cos\theta\sin\theta((c_{30}^2P+F(2c_1d_3+c_2e_3+c_{30}f_3)-b_3c_{30}K\\
&)F^3+F^3(((-2b_2-c_{30}+2d_1)c_1+(b_1+e_1)c_2+c_{30}f_1)G-((a_2+e_2)c_2+c_{30}f_2+2c_1d_2)H+K(c_1e_3+2c_2g_3+\\
& c_{30}(h_3+b_3)))+(KF(((b_1+e_1)c_1+2c_2g_1+c_{30}h_1)G-((2b_2+c_{30}+2g_2)c_2+(a_2+e_2)c_1+c_{30}h_2)H)+S(S+\\
& F(c_1f_3+c_2h_3+c_{30}^2+2c_{30}k_3)))F+SF((c_1^2+c_1f_1+c_2h_1+2c_{30}k_1)G-H(c_1f_2+c_2^2+c_2h_2+2c_{30}k_2)))-FA_8\\
& r^2\sin^2\theta(F(c_{30}S+(g_3c_2^2+(c_1e_3+c_{30}h_3)c_2+c_{30}^2k_3+c_1^2d_3+c_1c_{30}f_3)F)+((-b_2+d_1)c_1^2+((b_1+e_1)c_2+c_{30}f_1)\\
& c_1+c_2^2g_1+c_2c_{30}h_1+c_{30}^2k_1)FG-((b_2+g_2+c_{30})c_2^2+((a_2+e_2)c_1+c_{30}h_2)c_2+c_1c_{30}f_2+c_{30}^2k_2+c_1^2d_2)FH)-\\
& Fr\cos\theta((2d_3F^5-(P(Ff_3-Kb_3+Pc_{30})+2F(G(b_2+c_{30}-d_1)+d_2H-e_3K))F^3+(F(G(2(b_1+e_1)K-P(c_1\\
& +f_1))-H(2(a_2+e_2)K-Pf_2)+K(2g_3K-(h_3+b_3)P))+S(Ff_3-Kb_3+Pc_{30}))F^2+F^2((2Gg_1-2H(b_2+\\
& g_2))K^2+(S(h_3+b_3)-P(h_1G-(c_2+h_2)H))K+S((c_1+f_1)G-f_2H-2P(k_3+c_{30})))+SF(G(Kh_1-2Pk_1\\
&)-H((c_2+h_2)K-2Pk_2)))W-Fc_{31}(SF-F^2(G-P)))+r^2\cos^2\theta(d_3F^6-F^5(Gb_2+Gc_{30}-Gd_1+Hd_2-K\\
& e_3)+(KF((b_1+e_1)G-H(a_2+e_2)+g_3K)+S(Ff_3-Kb_3+Pc_{30}))F^3+F^3((Gg_1-H(b_2+g_2))K^2+S((c_1+\\
& f_1)G-f_2H+K(h_3+b_3)))+SF^2((h_1G-(c_2+h_2)H)K+S(k_3+c_{30}))+S^2F(Gk_1-Hk_2))+(d_3F^6+(-F(G\\
& (b_2+c_{30}-d_1)+d_2H-e_3K)-P(Ff_3-Kb_3+Pc_{30}))F^4+F^4(G((b_1+e_1)K-P(c_1+f_1))-H((a_2+e_2)K-\\
& Pf_2)+K^2g_3-KP(h_3+b_3)+P^2(k_3+c_{30}))+F^3(G(K^2g_1-KPh_1+P^2k_1)-H((b_2+g_2)K^2-P(c_2+h_2)K\\
& +P^2k_2)))W^2+F^4GWc_{31})=F_2(\theta,r,W).
\end{aligned}
$$

Computing the integrals (5) we obtain

$$
f_1(r,W)=\frac{Fr(T_8\ W+N_8)}{2AE\sqrt{FA}},\ \ f_2(r,W)=-\frac{\left(D_8W^2+R_8r^2+C_8W\right)}{2FA\sqrt{FA}}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*)=\left(\frac{1}{T_8}\sqrt{\frac{C_8N_8T_8-D_8N_8^2}{R_8}},-\frac{N_8}{T_8}\right),
$$

if $T_8>0$, $R_8(D_8N_8^2-C_8N_8T_8)>0$, and the Jacobian (6) at (r^*,W^*) is $N_8(C_8T_8-N_8D_8)/(2T_8A^3FE)\neq 0$, where

$$
\begin{aligned}
T_8 = \ & (2d_3E+2L(b_2+c_{30}-d_1)+2Md_2)F^3+(E(2Ke_3-Pf_3)-2K((b_1+e_1)L-M(a_2+e_2))+P((c_1+f_1)L\\
& -Mf_2)-((a_2+e_2)c_2+2d_2c_1+c_{30}f_2)A_8)F^2+((2Eg_3-2g_1L+2M(b_2+g_2))K^2-(P((c_2+h_2)M+Eh_3\\
& -Lh_1)+A_8((b_2+c_{30}-2d_1+e_2)c_1+(2g_2+b_2-e_1)c_2+(-f_1+h_2)c_{30}-F+K))K+PA_8(c_1f_2+c_2^2+\\
& c_2h_2+2c_{30}k_2)-E(P^2c_{30}-Sf_3)-S((c_1+f_1)L-Mf_2))F+A_8((b_1+e_1)c_1+2g_1c_2+c_{30}h_1)K^2+(((c_2+\\
& h_2)M+Eh_3-Lh_1)S-A_8(c_1^2+c_1f_1+c_2h_1+2c_{30}k_1)P)K-((c_{30}+2k_3)E-2Lk_1+2Mk_2)SP,\\
D_8 = \ & 2d_3F^5-(2(b_2+c_{30}-d_1)G+2d_2H-2e_3K+2f_3P)F^4+(2G((b_1+e_1)K-P(c_1+f_1))-2(K(a_2+e_2)-\\
& f_2P)H+2g_3K^2-2h_3PK+2k_3P^2)F^3+(2G(K^2g_1-KPh_1+P^2k_1)-2H((b_2+g_2)K^2-P(c_2+h_2)K\\
& +k_2P^2))F^2,\\
R_8 = \ & d_3F^5+(e_3K-(b_2+c_{30}-d_1)G-d_2H)F^4+(K((b_1+e_1)G-H(a_2+e_2))+g_3K^2+Sf_3)F^3+(((-b_2-\\
& g_2)H+Gg_1)K^2-A_8(k_3c_{30}^2+(c_1f_3+c_2h_3)c_{30}+c_1^2d_3+c_1c_2e_3+g_3c_2^2)+((c_1+f_1)G+Kh_3-Hf_2+Pc_{30}\\
&)S)F^2+(GKSh_1-HKSc_2-HKSh_2+c_{30}S^2+S^2k_3-A_8(G((d_1-b_2)c_1^2+((b_1+e_1)c_2+c_{30}f_1)c_1+g_1\\
& c_2^2+c_2c_{30}h_1+k_1c_{30}^2)-((b_2+c_{30}+g_2)c_2^2+(h_2c_{30}+c_1(a_2+e_2))c_2+c_1^2d_2+c_1f_2c_{30}+k_2c_{30}^2)H+Sc_{30}))F\\
& +S^2(Gk_1-Hk_2),\\
N_8 = \ & -c_{31}(EF(G+P)+2F^2L-2KA_8c_1+ES),\\
C_8 = \ & 2F^3Gc_{31}.
\end{aligned}
$$

then, from Theorem 3, for $\varepsilon>0$ sufficiently small and going back through the changes of variables system (35) has a periodic solution $(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon))$ which tends to the equilibrium $(0,0,0)$ when $\varepsilon\to 0$. Therefore there is a periodic solution starting at the zero-Hopf equilibrium $(1,1,1)$ when $\varepsilon=0$. This completes the proof of Theorem 2 under the condition (viii). □

Application 8. Consider the Kolmogorov system

$$
\begin{aligned}
\dot{x} = \ & x(x-2+y+(x-1)^2+(x-1)(y-1)+(z-1)^2),\\
\dot{y} = \ & y(-2x+2-y+z+(x-1)^2+(x-1)(z-1)+(y-1)^2+(z-1)^2),\\
\dot{z} = \ & z(x-1-y+z+(x-1)^2+(y-1)^2+(y-1)(z-1)+(z-1)^2).
\end{aligned}
\tag{37}
$$

This system in the variables (X, Y, Z) writes

$$\begin{aligned}
\dot{X} &= (X+1)(X^2 + XY - 2X\varepsilon + Z^2 + Y),\\
\dot{Y} &= (Y+1)(X^2 + XZ + Y^2 + Z^2 - 2X - Y + Z),\\
\dot{Z} &= (Z+1)((-2X+Z)\varepsilon - 2X - Y + Z + X^2 + Y^2 + YZ + Z^2).
\end{aligned}$$

The two components of the corresponding system associated to system (36) are

$$\begin{aligned}
F_1(\theta, r, W) = {} & -\frac{1}{8}(\sqrt{2}(-2\cos(\theta)W(12\ W+4) - 4\cos^3(\theta)r^2 + 2\cos^2(\theta)r(10\ W+2) + 14\sin(\theta)\sqrt{2}W^2 + 20\sin^2(\theta)\\
& rW + 8\sqrt{2}\sin^3(\theta)r^2 - 2\cos(\theta)\sin(\theta)\sqrt{2}r(15\ W+1) + 12\sin(\theta)\sqrt{2}\cos^2(\theta)r^2)),\\
F_2(\theta, r, W) = {} & -\frac{\sqrt{2}}{4}(\sin(\theta)\sqrt{2}r(W-1) + \cos(\theta)\sin(\theta)\sqrt{2}r^2 + 2\sin^2(\theta)r^2 - \cos(\theta)r(-6\ W+2) - 2W^2).
\end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r, W) = -\frac{r(20\ W+2)\sqrt{2}}{8} = 0, \quad f_2(r, W) = -\frac{(-4W^2 + 2r^2)\sqrt{2}}{8} = 0.$$

This system has the solutions

$$(0,0), \quad (\sqrt{2}/10, -1/10), \quad (-\sqrt{2}/10, -1/10)$$

Here we have only one good solution, the $(\sqrt{2}/10, -1/10)$. The determinant (6) at this solution is $-1/20$. Hence system (37) has one limit cycle bifurcating from the equilibrium point $(1,1,1)$ using the averaging theory of first order. We plot this bifurcated limit cycle for $\varepsilon = 10^{-5}$ in Fig. 8.

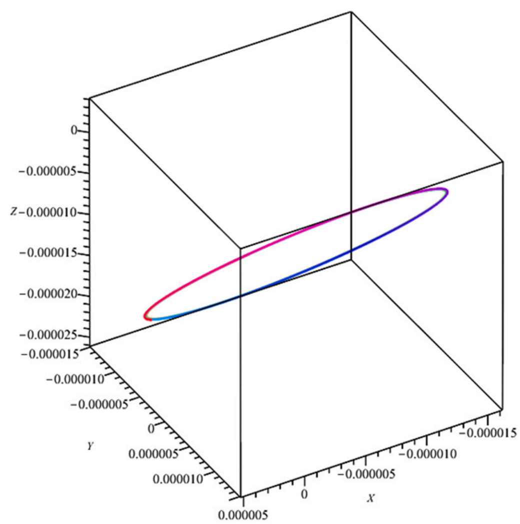

Fig. 8. $X(0) = \epsilon(1+\sqrt{2})/10,\ Y(0) = \epsilon,\ Z(0) = -\epsilon/10.$

The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(\sqrt{2}/10, -1/10)$ are $(-\sqrt{2} \pm \sqrt{22})/20$. So this limit cycle is unstable. This completes Application 8.

4. Conclusions

We have classified all the zero-Hopf equilibria of the Kolmogorov systems of degree 3 in $\mathbb{R}^3$, see Proposition 1, obtaining eight families of such Kolmogorov systems exhibiting zero-Hopf equilibria. Moreover using the averaging theory of first order in Theorem 2 we have characterize the limit cycles that bifurcate from such zero-Hopf equilibria. Finally we provide explicit examples of the limit cycles bifurcating from the eight families of zero-Hopf equilibria.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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