This is a preprint of: "The 16th Hilbert problem for discontinuous piecewise linear Hamiltonian saddles and isochronous centers of degree two separated by a straight line", Jaume Llibre, Clàudia Valls, *Differ. Equ. Dyn. Syst.*, 2024.
DOI: [10.1007/s12591-024-00695-w]

THE 16TH HILBERT PROBLEM FOR DISCONTINUOUS PIECEWISE LINEAR HAMILTONIAN SADDLES AND ISOCHRONOUS CENTERS OF DEGREE TWO SEPARATED BY A STRAIGHT LINE

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

ABSTRACT. In this paper we deal with discontinuous piecewise differential systems formed by two differential systems separated by a straight line when one of these two differential systems is a linear Hamiltonian saddle and the other is a quadratic isochronous center. It is known that there are four families of quadratic isochronous centers.

We provide upper bounds for the maximum number of limit cycles that these four classes of discontinuous piecewise differential systems can exhibit, so we have solved the extension of the 16th Hilbert problem to such piecewise differential systems. Moreover at least in two of the four classes of these discontinuous piecewise differential systems the obtained upper bound for the maximum number of limit cycles is reached.

1. Introduction and main results

Consider discontinuous piecewise differential systems of the form

$$(1)\qquad (\dot{x},\dot{y}) = \mathbf{F}(x,y) = \begin{cases} \mathbf{F}^-(x,y) = (f^-(x,y), g^-(x,y)) \text{ if } x \le 0, \\ \mathbf{F}^+(x,y) = (f^+(x,y), g^+(x,y)) \text{ if } x \ge 0, \end{cases}$$

being bivaluated on the separation line $x = 0$. Following [10] a point $(0, y)$ is a *crossing point* if $f^-(0,y)f^+(0,y) > 0$. A *crossing periodic orbit* is a periodic orbit of the discontinuous differential system (1) that has two crossing points and no more, and a *crossing limit cycle* is an isolated periodic orbit in the set of all crossing periodic orbits of system (1). In all the paper we will say limit cycle instead of crossing limit cycle.

Planar continuous piecewise linear differential systems separated by one straight line appear in several non-linear engineering devices or in mathematical biology, see [6, 7, 33, 35, 34] and the references therein. Their maximal number of limit cycles is one, and there are systems with one limit cycle. So the extension of the 16th Hilbert's problem on the maximum number of limit cycles (see for instance [16, 20, 21] for details) for such continuous piecewise differential systems is solved.

For planar discontinuous piecewise linear differential systems separated by one straight line the extension of the 16th Hilbert's problem is an open problem. This problem has been studied by many authors in the past years and there exists a large bibliography trying to determine how many limit cycles can appear in planar systems separated by one straight line, see for instance [1, 2, 3, 4, 9, 11, 12, 13, 14, 15, 17, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 31] and the references therein. It seems that for this class of discontinuous piecewise linear differential systems the upper bound for their maximal number of limit cycles will be three, but this is an open problem.

An *isochronous center* p is a center (that is, a singularity such that all solutions of the differential system except the singularity are periodic in a neighborhood U of it) such that the period of its periodic orbits is constant for all points in the neighborhood U.

2010 *Mathematics Subject Classification.* Primary 34C05, 34C07, 37G15.

Key words and phrases. discontinuous piecewise linear differential systems, linear Hamiltonian saddles, limit cycles, isochronous center.

In this paper we work with the following five types of systems having a linear Hamiltonian saddle and one of the four families of quadratic polynomial differential systems having an isochronous center. For a proof of the normal form of the linear Hamiltonian saddles see [30] and for a proof of the quadratic systems having an isochronous center, see [32] and page 34 of [5].

(I) Any linear differential system having a linear Hamiltonian saddle can be written as

$$\dot{x} = -Ax - \delta y + B, \quad \dot{y} = \alpha x + Ay + C,$$

with $\alpha \in \{0,1\}$, $A, \delta, B, C \in \mathbb{R}$. Moreover, if $\alpha = 1$ then $\delta = A^2 - \omega$ with $\omega > 0$, and if $\alpha = 0$ then $A = 1$. A first integral of this system is

$$H(x,y) = -\frac{\alpha}{2}x^2 - Axy - \frac{\delta}{2}y^2 - Cx + By.$$

(II) The first whole family of isochronous differential systems of degree two can be obtained after doing the general affine change of variables of the form

$$(x,y) \to (ax + by + c, \alpha x + \beta y + \gamma), \tag{2}$$

with $b\alpha - a\beta \neq 0$, to the differential system

$$\dot{x} = -y + x^2 - y^2, \quad \dot{y} = x(1+2y),$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{x^2+y^2}{1+2y}.$$

Thus we obtain the differential system

$$\begin{aligned} \dot{x} = \ & \frac{1}{b\alpha - a\beta}\big(\beta\gamma^2 + 2b\gamma c + bc + \beta\gamma - \beta c^2 + (2ab\gamma + 2\alpha\beta\gamma + ab + \alpha\beta \\ & -2a\beta c + 2\alpha bc)x + (2\gamma+1)(b^2+\beta^2)y + (-a^2\beta + \alpha^2\beta + 2\alpha ab)x^2 \\ & +2\alpha(b^2+\beta^2)xy + \beta(b^2+\beta^2)y^2\big), \\ \dot{y} = \ & \frac{1}{b\alpha - a\beta}\big(-\alpha\gamma^2 - 2a\gamma c - ac - \alpha\gamma + \alpha c^2 - (2\gamma+1)(a^2+\alpha^2)x \\ & +(-2ab\gamma - 2\alpha\beta\gamma - ab - \alpha\beta - 2a\beta c + 2\alpha bc)y - \alpha(a^2+\alpha^2)x^2 \\ & -2\beta(a^2+\alpha^2)xy - (\alpha\beta^2 + 2a\beta b - \alpha b^2)y^2\big), \end{aligned} \tag{3}$$

with the first integral

$$H_2(x,y) = \frac{(c + ax + by)^2 + (x\alpha + y\beta + \gamma)^2}{1 + 2(x\alpha + y\beta + \gamma)}.$$

(III) The second whole family of quadratic isochronous differential systems can be obtained doing the affine change of variables (2) to the differential system

$$\dot{x} = -y + x^2, \quad \dot{y} = x(1+y),$$

whith the first integral

$$\tilde{H}_3(x,y) = \frac{x^2+y^2}{(1+y)^2},$$

obtaining the differential system

$$\begin{aligned} \dot{x} = \ & \frac{1}{\alpha b - a\beta}\big(-b\gamma c - bc - \beta\gamma + \beta c^2 + (-ab\gamma - ab - \alpha\beta + 2a\beta c \\ & -\alpha bc)x - (b^2\gamma + b^2 + \beta^2 + \beta bc)y + a(a\beta - \alpha b)x^2 \\ & -b(\alpha b - a\beta)xy\big), \\ \dot{y} = \ & \frac{1}{\alpha b - a\beta}\big(-a\gamma c - ac - \alpha\gamma + \alpha c^2 - (a^2\gamma + a^2 + \alpha^2 - \alpha ac)x \\ & +(-ab\gamma - ab - \alpha\beta - a\beta c + 2\alpha bc)y - a(a\beta - \alpha b)xy \\ & +b(\alpha b - a\beta)y^2\big), \end{aligned} \tag{4}$$

with the first integral

$$H_3(x,y)=\frac{(ax+by+c)^2+(\gamma+\alpha x+\beta y)^2}{(\gamma+\alpha x+\beta y+1)^2}.$$

(IV) The third whole family of quadratic isochronous differential systems can be obtained doing the general affine transformation (2) to the system

$$\dot x=-y-\frac{4}{3}x^2,\quad \dot y=x\Big(1-\frac{16}{3}y\Big),$$

with the first integral

$$\tilde H_4(x,y)=\frac{(9-24y+32x^2)^2}{3-16y},$$

providing the differential system

$$(5)\qquad \begin{aligned}\dot x=&\ \frac{1}{3(\alpha b-a\beta)}\big(-16b\gamma c+3bc+3\beta\gamma+4\beta c^2+(-16ab\gamma+3ab\\&+3\alpha\beta+8a\beta c-16\alpha bc)x+(-16b^2\gamma+3b^2+3\beta^2-8\beta bc)y\\&+4a(a\beta-4\alpha b)x^2-8b(a\beta+2\alpha b)xy-12b^2\beta y^2\big),\\ \dot y=&\ \frac{1}{3(\alpha b-a\beta)}\big(16a\gamma c-3ac-3\alpha\gamma-4\alpha c^2+(16a^2\gamma-3a^2\\&-3\alpha^2+8\alpha ac)x+(16ab\gamma-3ab-3\alpha\beta+16a\beta c-8\alpha bc)y\\&+12a^2\alpha x^2+8a(2a\beta+\alpha b)xy-4b(\alpha b-4a\beta)y^2\big),\end{aligned}$$

with the first integral

$$\begin{aligned}H_4(x,y)=&\ \frac{1}{16(\gamma+\alpha x+\beta y)-3}\big(-24(ax+by+c)^2(\gamma+\alpha x+\beta y)\\&+9\left((ax+by+c)^2+(\gamma+\alpha x+\beta y)^2\right)+16(ax+by+c)^4\big).\end{aligned}$$

(V) The fourth whole family of quadratic isochronous differential systems can be obtained doing the general affine transformation (2) to the differential system

$$\dot x=-y+\frac{16}{3}x^2-\frac{4}{3}y^2,\quad \dot y=x\Big(1+\frac{8}{3}y\Big),$$

with the first integral

$$\tilde H_5(x,y)=\frac{9+96y-256x^2+128y^2}{(3+8y)^4},$$

ingobtain the differential system

$$(6)\qquad \begin{aligned}\dot x=&\ \frac{1}{3(\alpha b-a\beta)}\big(4\beta\gamma^2+8b\gamma c+3bc+3\beta\gamma-16\beta c^2+(8ab\gamma+8\alpha\beta\gamma\\&+3ab+3\alpha\beta-32a\beta c+8\alpha bc)x+(8b^2\gamma+8\beta^2\gamma+3b^2+3\beta^2\\&-24\beta bc)y+4(\alpha^2\beta-4a^2\beta+2\alpha ab)x^2+8(\alpha\beta^2-3a\beta b+\alpha b^2)xy\\&-4\beta y^2(2b^2-\beta^2)y^2\big),\\ \dot y=&\ \frac{1}{3(\alpha b-a\beta)}\big(16\alpha c^2-4\alpha\gamma^2-8a\gamma c-3ac-3\alpha\gamma-(8a^2\gamma+8\alpha^2\gamma\\&+3a^2+3\alpha^2-24\alpha ac)x-(8ab\gamma+8\alpha\beta\gamma+3ab+3\alpha\beta+8a\beta c\\&-32\alpha bc)y+4\alpha(2a^2-\alpha^2)x^2+8(a^2(-\beta)-\alpha^2\beta+3\alpha ab)xy\\&-4(\alpha\beta^2+2a\beta b-4\alpha b^2)y^2\big),\end{aligned}$$

with the first integral

$$\begin{aligned}H_5(x,y)=&\ \frac{1}{(8(\gamma+\alpha x+\beta y)+3)^4}\big(9\left((ax+by+c)^2+(\gamma+\alpha x+\beta y)^2\right)\\&+16(\gamma+\alpha x+\beta y)^4+24(\gamma+\alpha x+\beta y)^3\big).\end{aligned}$$

Our objective is to solve the extension of the 16th Hilbert problem for the four classes of discontinuous piecewise differential systems separated by a straight line and formed by a linear Hamiltonian saddle and an isochronous center of degree 2, i.e. we shall provide for all these four classes an upper bound for the maximum number of limit cycles that each class can exhibit. Moreover as we shall see at least two of these four classes the upper bound that we shall provide is reached.

We recall that linear Hamiltonian differential systems are one of the easiest linear differential systems and that the unique linear differential systems which are Hamiltonian are the linear centers and the linear saddles. So it is natural to start with these systems. However it was proved in [28, 30] that the classes of discontinuous piecewise differential systems separated by a straight line and formed by either two linear centeres, or two linear Hamiltonian saddles, or a linear Hamiltonian saddle and a linear center do not have limit cycles. A forward step is to look for classes of discontinuous piecewise differential systems separated by a straight line and formed by a linear center and an isochronous center of degree 2, or by a linear Hamiltonian saddle and an isochronous center of degree 2. The first of these two classes was studied in [8] and the authors proved the following result.

Theorem 1. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a linear center in $x < 0$, and by a quadratic isochronous center of type either (II), or (III), or (IV), or (V) after an affine change of variables in $x > 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems are*

(a) *at most one for systems of type linear center-(II), and there are systems of this type with exactly one limit cycle;*
(b) *at most one for systems of type linear center-(III), and there are systems of this type with exactly one limit cycle;*
(c) *at most two for systems of type linear center-(IV), and there are systems of this type with exactly one limit cycle;*
(d) *at most two for systems of type linear center-(V), and there are systems of this type with exactly two limit cycles.*

Note that for all systems of type linear center-(k) with $k \in \{II, III, V\}$ the upper bound on the maximum number of limit cycles is reached.

In general it is very difficult (many times for the moment impossible) to provide an upper bound for the maximum number of limit cycles that a class of differential systems in the plane can exhibit, and of course it is even more difficult to provide the exact upper bound, see for instance [16, 20, 21].

Our main result is to provide an upper bound for the the maximum number of limit cycles that can exist for discontinuous piecewise differential systems of the form (1) when in $x < 0$ there is an arbitrary linear Hamiltonian saddle (I), and in $x > 0$ there is one of the four quadratic isochronous differential systems (II), (III), (IV) or (V), after an arbitrary affine change of variables.

Theorem 2. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a linear Hamiltonian saddle (I) in $x > 0$, and by a quadratic isochronous system of type either (II), or (III), or (IV), or (V) after an affine change of variables in $x < 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems are;*

(a) *at most one for systems of type (II)-(I), and there are systems of this type with exactly one limit cycle, see Figure 1(a);*
(b) *at most one for systems of type (III)-(I), and there are systems of this type with exactly one limit cycle, see Figure 1(b);*

(c) *at most two for systems of type (IV)-(I), and there are systems of this type with exactly one limit cycle, see Figure 1(c);*
(d) *at most two for systems of type (V)-(I), and there are systems of this type with exactly two limit cycles, see Figure 1(d).*

Note that at least for all systems of type (k)-(I) with $k \in \{II, III\}$ *the upper bound on the maximum number of limit cycles is reached.*

The proof of Theorem 2 is given in section 2.

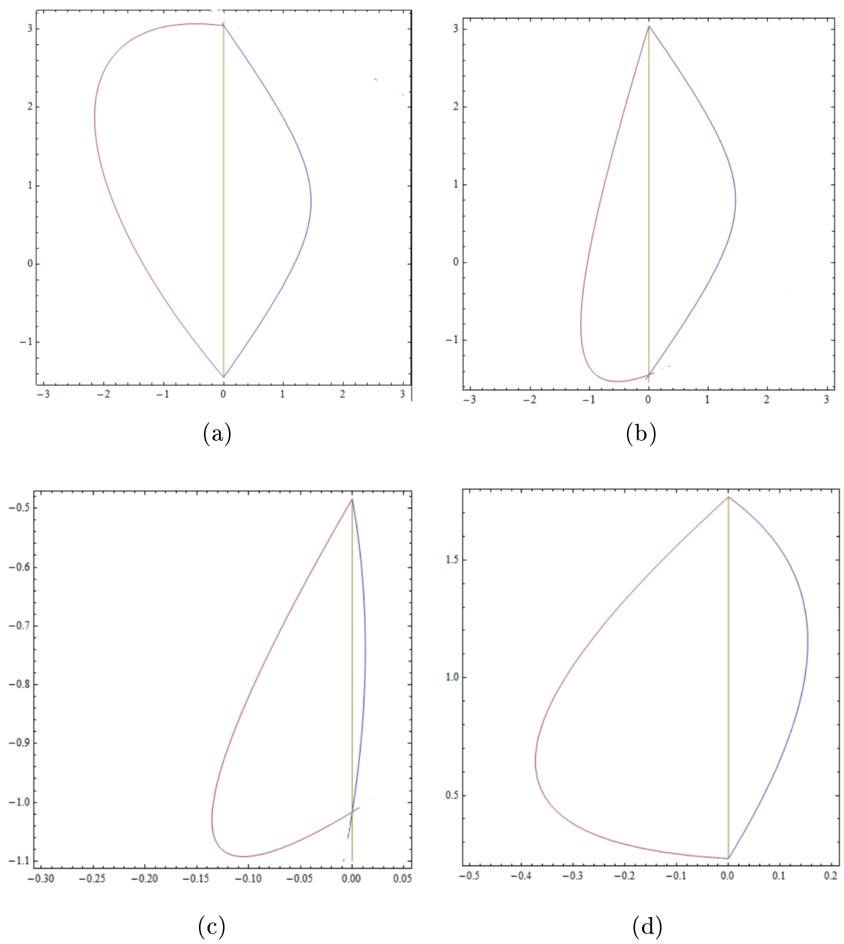

FIGURE 1. (a) The unique limit cycle that exists for system (9)–(8) of class (II)–(I). (b) The unique limit cycle that exists for system (11)–(8) of class (III)–(I). (c) The unique limit cycle that exists for system (14)–(13) of class (IV)–(I). (d) The unique limit cycle that exists for system (17)–(16) of class (V)–(I). These four limit cycles are travelled in counterclocwise sense.

Remark. *For the discontinuous piecewise differential systems of type (k)-(I) with* $k \in \{IV, V\}$ *it is possible that the upper bound found for the maximum number of limit cycles cannot be reached. This is due to the fact that the solutions* $(y, \overline{y})$*, which appear in the proof of the upper bound on the maximum number of limit cycles that these systems can exhibit, do not need to correspond necessarily to periodic solutions of the discontinuous piecewise differential systems. In fact there is numerical evidence that probably the upper bound on the maximum number of limit cycles is one instead of two*

2. Proof of Theorem 2

2.1. **Proof of Theorem 2 for system (II)–(I).** We consider the planar linear differential system (I) with first integral $H_1(x, y)$ in the half-plane $x > 0$ and the quadratic isochronous center (3) with first integral $H_2(x, y)$ in the half-plane $x < 0$. If there exists a limit cycle of the discontinuous piecewise differential systems (3)–(I) it must intersect the discontinuity line $x = 0$ in two different points $(0, y)$ and $(0, \overline{y})$. Clearly these two points must satisfy the system

$$\begin{aligned} H_1(0,y) - H_1(0,\overline{y}) = & \quad (\overline{y} - y)P_1(y,\overline{y}), \\ H_2(0,y) - H_2(0,\overline{y}) = & \quad \frac{(\overline{y} - y)Q_2(y,\overline{y})}{[1+2(\beta y+\gamma)][1+2(\beta\overline{y}+\gamma)]} = 0, \end{aligned} \tag{7}$$

where P_1 and Q_2 are polynomials of degrees one and two, respectively. Since the points $(0, y)$ and $(0, \overline{y})$ must be different, from $P_1(y, \overline{y}) = 0$, we get $\overline{y}$ as a function of y, that is, $\overline{y} = f(y)$. Substituting this expression in equation $Q_2(y, \overline{y}) = 0$ we obtain a quadratic equation in the variable y. Then the maximum number of solutions of system (7) is two, namely $(y_1, \overline{y}_1)$ and $(y_2, \overline{y}_2)$, but in fact, these two solutions represent the same limit cycle because $\overline{y}_1 = y_2$ and $\overline{y}_2 = y_1$. So for the discontinuous piecewise differential system (3)–(I), there exists at most one limit cycle.

Now we give an example of a discontinuous piecewise differential system (3)–(I) having one limit cycle. On $x > 0$ we consider the linear differential system

$$\dot{x} = 8 - 10y, \qquad \dot{y} = 2\sqrt{151} - 10x, \tag{8}$$

with the first integral

$$H_1(x, y) = 22 - 2\sqrt{151}x + 8y + 5x^2 - 5y^2,$$

and on $x < 0$ we consider the quadratic isochronous differential system of type (3)

$$\dot{x} = -4 - 5x - 6y - x^2 - 4xy - 2y^2, \qquad \dot{y} = 1 + 3x + y + x^2 + 2xy, \tag{9}$$

with the first integral

$$H_2(x, y) = \frac{(x + y + 1)^2 + (y + 1)^2}{2(x + y + 1) + 1}.$$

The solution $(y, \overline{y})$ of (7) satisfying $y < \overline{y}$, is $(y, \overline{y}) = \left(\frac{1}{5}(4 - 3\sqrt{14}), \frac{1}{5}(4 + 3\sqrt{14})\right)$. This solution provides the limit cycle that exists for the discontinuous differential piecewise system (9)–(8) shown in Figure **??**.

2.2. **Proof of Theorem 2 for system (III)–(I).** We consider the linear differential system (I) with first integral $H_1(x, y)$ on the half-plane $x > 0$, and on the half-plane $x < 0$, we take the quadratic isochronous center (4) with its first integral $H_3(x, y)$. Then if there exists some limit cycle for the discontinuous differential system (4)–(I), it must intersect the discontinuity line $x = 0$ at two different points $(0, y)$ and $(0, \overline{y})$, satisfying the equations

$$\begin{aligned} H_1(0,y) - H_1(0,\overline{y}) = & \quad (\overline{y} - y)P_1(y,\overline{y}) = 0, \\ H_3(0,y) - H_3(0,\overline{y}) = & \quad \frac{(\overline{y} - y)Q_2(y,\overline{y})}{(1+\beta y+\gamma)^2(1+\beta\overline{y}+\gamma)^2} = 0. \end{aligned} \tag{10}$$

In (10) P_1 and Q_2 are polynomials of degrees one and two, respectively. By following the same procedure as for the proof of system (II)–(I), we solve the equation $P_1(y, \overline{y}) = 0$ obtaining the variable $\overline{y}$ as a function of y, that is $\overline{y} = f(y)$. By replacing $\overline{y}$ in the equation $Q_2(y, \overline{y}) = 0$, we obtain again a quadratic polynomial equation in the variable y, so that the equation has at most two different solutions. As in the proof for system (II)–(I), these two solutions represent, if they exist, the same limit cycle. Therefore system (10) has only one solution with $y < \overline{y}$, and then the discontinuous piecewise differential system (4)–(I) has at most one limit cycle.

Next we provide a discontinuous piecewise differential system (4)–(I) having one limit cycle. On the half-plane $x > 0$ we consider the linear differential system (8), and on the half-plane $x < 0$ we consider the quadratic isochronous center of type (4)

$$\dot{x} = -2 - 2y + x^2 - xy, \qquad \dot{y} = -2 + 2x - 3y + xy - y^2, \tag{11}$$

with the first integral

$$H_3(x, y) = \frac{(1 - x + y)^2 + (y + 1)^2}{2(y + 2)^2}.$$

In this case the unique solution for system (10) with $y < \overline{y}$ is

$$(y, \overline{y}) = \left(\frac{1}{5}(4 - 3\sqrt{14}), \frac{1}{5}(4 + 3\sqrt{14})\right),$$

and the corresponding limit cycle of the discontinuous piecewise differential system (11)–(8) associated to this solution is shown in Figure **??**.

2.3. **Proof of Theorem 2 for system (IV)–(I).** We consider again on the half-plane $x > 0$ the linear differential system (I) with its first integral $H_1(x, y)$, and on $x < 0$ we take the quadratic isochronous center (5) with its first integral $H_4(x, y)$. Then if the discontinuous differential system (5)–(I) has a limit cycle, it must intersect the discontinuity line $x = 0$ at two different points $(0, y)$ and $(0, \overline{y})$. These points must satisfy the equations

$$\begin{aligned} H_1(0, y) - H_1(0, \overline{y}) = &\ (\overline{y} - y)P_1(y, \overline{y}) = 0, \\ H_4(0, y) - H_4(0, \overline{y}) = &\ \frac{(\overline{y} - y)Q_4(y, \overline{y})}{(-3 + 16y\beta + 16\gamma)(-3 + 16\overline{y}\beta + 16\gamma)} = 0, \end{aligned} \tag{12}$$

where P_1 and Q_4 are polynomials of degrees one and four, respectively. We solve the equation $P_1(y, \overline{y}) = 0$ obtaining the variable $\overline{y}$ as a function of y, that is $\overline{y} = f(y)$. If we substitute $\overline{y} = f(y)$ in the equation $Q_4(y, \overline{y}) = 0$, we obtain a polynomial equation of degree four in the variable y, and so system (12) has at most four real solutions. Taking into account the symmetry between these solutions, as in the previous proofs there can be only two different solutions $(y, \overline{y})$ of (12) satisfying $y < \overline{y}$.

Now we give an example of a discontinuous piecewise differential system (5)–(I) having one limit cycle. On $x > 0$ we consider the linear differential system

$$\dot{x} = -2 + 2x - \frac{8}{3}y, \qquad \dot{y} = 6 - 2y, \tag{13}$$

with the first integral

$$H_1(x, y) = 6x + 2y - 2xy + \frac{4}{3}y^2,$$

and on $x < 0$ we consider the quadratic isochronous center of type (5)

$$\begin{aligned} \dot{x} &= -19 + 55x - 16x^2 - 34y + 32xy - 12y^2, \\ \dot{y} &= -48 + 129x - 48x^2 - 79y + 80xy - 28y^2, \end{aligned} \tag{14}$$

with the first integral

$$H_2(x, y) = \frac{9 + 32(-1 + 2x - y)^2 - 24(2 - x + y))^2}{3 - 16(2 - x + y)}.$$

The solutions of (12) satisfying $y < \overline{y}$ are the two pairs of solutions

$$(y, \overline{y}) = \frac{1}{8}\Big(-6 - \sqrt{19 - \sqrt{209}}, -6 + \sqrt{19 - \sqrt{209}}\Big),$$
$$(y, \overline{y}) = \frac{1}{8}\Big(-6 + \sqrt{19 - \sqrt{209}}, -6 + \sqrt{19 + \sqrt{209}}\Big),$$

but only the solution $(y,\overline{y}) = \big(-6-\sqrt{19-\sqrt{209}}, -6+\sqrt{19-\sqrt{209}}\big)/8$ provides a limit cycle and it is the limit cycle that exists for the discontinuous differential piecewise system (14)–(13) shown in Figure **??**.

2.4. **Proof of Theorem 2 for system (V)–(I).** We take again the linear differential system (I) with its first integral $H_1(x,y)$ on the half-plane $x>0$, and on $x<0$ we consider the quadratic isochronous center (6) with its first integral $H_5(x,y)$. Thus if the discontinuous differential system (6)–(I) has a limit cycle, it must intersect the discontinuity line $x=0$ at two different points $(0,y)$ and $(0,\overline{y})$. These points must satisfy the equations

$$
\begin{aligned}
H_1(0,y)-H_1(0,\overline{y}) = &\quad (\overline{y}-y)P_1(y,\overline{y}) = 0,\\
H_5(0,y)-H_5(0,\overline{y}) = &\quad \frac{(\overline{y}-y)Q_5(y,\overline{y})}{(3+8y\beta+8\gamma)^4(3+8\overline{y}\beta+8\gamma)^4} = 0,
\end{aligned} \tag{15}
$$

where P_1 and Q_5 are polynomials of degrees one and five, respectively. We solve again the equation $P_1(y,\overline{y})=0$ obtaining the variable $\overline{y}$ as a function of y, that is $\overline{y}=f(y)$. If we substitute $\overline{y}=f(y)$ in the equation $Q_5(y,\overline{y})=0$, we obtain a polynomial equation of degree 4 in the variable y, and so system (12) has at most four real solutions. Taking into account the symmetry between these solutions, as in the previous proofs, there can be only two different solutions $(y,\overline{y})$ of (15) satisfying $y<\overline{y}$.

Now we give an example of a discontinuous piecewise differential system (6)–(I) having one limit cycle. On $x>0$ we consider the linear differential system

$$
\dot{x} = 2+2x-2y, \qquad \dot{y} = 6-2y, \tag{16}
$$

whose first integral is

$$
H_1(x,y) = 6x-2y-2xy+y^2,
$$

and on $x<0$ we consider the quadratic isochronous differential system of type (5)

$$
\begin{aligned}
\dot{x} &= -9x+4x^2+15y+8xy-28y^2,\\
\dot{y} &= -6x-4x^2+9y+32xy-44y^2,
\end{aligned} \tag{17}
$$

whose first integral is

$$
H_2(x,y) = \frac{9+96(-x+y)+128(y-x)^2-256(-x+2y)^2}{(3+8(y-x))^4}.
$$

The solutions of (15) satisfying $y<\overline{y}$ are the two pairs of solutions

$$
\begin{aligned}
(y,\overline{y}) &= \frac{1}{1616}\Big(1616-\sqrt{2222(1013-9\sqrt{1257})}, 1616+\sqrt{2222(1013-9\sqrt{1257})}\Big),\\
(y,\overline{y}) &= \frac{1}{1616}\Big(1616-\sqrt{2222(1013+9\sqrt{1257})}, 1616-\sqrt{2222(1013+9\sqrt{1257})}\Big),
\end{aligned} \tag{18}
$$

but only the first solution in (18) provides a limit cycle and it is the limit cycle that that exists for the discontinuous differential piecewise system (17)–(16) shown in Figure **??**.

Acknowledgements

The first author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant PID2019-104658GB-I00, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2019.

References

[1] J.C. Artés, J. Llibre, J.C. Medrado and M.A. Teixeira , *Piecewise linear differential systems with two real saddles,* Math. Comput. Simul. **95** (2013), 13–22.

[2] D.C. Braga and L.F. Mello, *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane,* Nonlinear Dynam. **73** (2013), 1283–1288.

[3] C. Buzzi, C. Pessoa and J. Torregrosa, *Piecewise linear perturbations of a linear center,* Discrete Contin. Dyn. Syst. **33** (2013), 3915–3936.

[4] J. Castillo, J. Llibre and F. Verduzco, *The pseudo-Hopf bifurcation for planar discontinuous piecewise linear differential systems,* Nonlinear Dynam. **90** (2017), 1829–1840.

[5] J. Chavarriga and M. Sabatini, *A survey on isochronous centers,* Qual. Th. Dyn. Syst. **1** (1999), 1–70.

[6] S. Coombes, *Neuronal networks with gap junctions: a study of piecewise linear planar neuron models,* SIAM J. Appl. Dyn. Syst. **7** (2008), 1101–1129.

[7] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-smooth dynamical systems: theory and applications*, Applied Mathematical Sciences, Springer, 2008.

[8] M. Esteban, J. Llibre and C. Valls, *The 16th Hilbert problem for discontinuous isochronous centers of degree one or two separated by a straight line,* Chaos **31** (2021), 043112, 19 pp.

[9] R.D. Euzébio and J. Llibre, *On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line,* J. Math. Anal. Appl. **424** (2015), 475–486.

[10] A.F. Filippov, *Differential equations with discontinuous right-hand sides*, translated from Russian. Mathematics and its Applications (Soviet Series) vol. 18, Kluwer Academic Publishers Group, Dordrecht, 1988.

[11] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones,* Int. J. Bifurcation and Chaos. **8** (1998), 2073–2097.

[12] E. Freire, E. Ponce and F. Torres, *Canonical discontinuous planar piecewise linear systems,* SIAM J. Applied Dynamical Systems. **11** (2012), 181–211.

[13] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones,* Nonlinear Dynamics **78** (2014), 251–263.

[14] F. Giannakopoulos and K. Pliete, *Planar systems of piecewise linear differential equations with a line of discontinuity,* Nonlinearity **14** (2001), 1611–1632.

[15] M.R.A. Gouveia, J. Llibre and D D. Novaes, *On limit cycles bifurcating from the infinity in discontinuous piecewise linear differential systems,* Appl. Math. Comput. **271** (2015), 365–374.

[16] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. G"ottingen Math. Phys. KL. (1900), 253–297; English transl. Bull. Amer. Math. Soc. **8** (1902), 437–479; Bull. (New Series) Amer. Math. Soc. **37** (2000), 407–436.

[17] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise systems,* Discrete Cont. Dyn. Syst. Series A **32** (2012), 2147–2164.

[18] S.M. Huan and X.S. Yang, *Existence of limit cycles in general planar piecewise linear systems of saddle-saddle dynamics,* Nonlinear Anal. **92** (2013), 82–95.

[19] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems of node-node types,* J. Math. Anal. Appl. **411** (2014), 340–353.

[20] Yu. Ilyashenko, *Centennial history of Hilbert's* 16*th problem*, Bull. (New Series) Amer. Math. Soc. **39** (2002), 301–354.

[21] J. Li, *Hilbert's* 16*th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.

[22] L. Li, *Three crossing limit cycles in planar piecewise linear systems with saddle-focus type,* Electron. J. Qual. Theory Differ. Equ. **70** (2014), 14 pp.

[23] J. Llibre, D.D. Novaes and M.A. Teixeira, *Limit cycles bifurcating from the periodic orbits of a discontinuous piecewise linear differential center with two zones,* Int. J. Bifurcation and Chaos **25** (2015), 1556144, 11 pp.

[24] J. Llibre, D.D. Novaes and M.A. Teixeira, *Maximum number of limit cycles for certain piecewise linear dynamical systems,* Nonlinear Dyn. **82** (2015), 1159–1175.

[25] J. Llibre, D.D. Novaes and M.A. Teixeira, *On the birth of limit cycles for non-smooth dynamical systems,* Bull. Sci. Math. **139** (2015), 229–244.

[26] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in planar piecewise linear systems without symmetry,* Nonlinear Anal. Series B: Real World Appl. **14** (2013), 2002–2012.

[27] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones,* Dyn. Cont. Disc. Impul. Syst., Series B **19** (2012), 325–335.

[28] J. Llibre and M.A. Teixeira, *Piecewise linear differential systems without equilibria produce limit cycles?,* Nonlinear Dyn. **88** (2017), 157–164.

[29] J. Llibre, M.A. Teixeira and J. Torregrosa, *Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation,* Int. J. Bifurcation and Chaos **23** (2013), 1350066, 10 pp.

[30] J. Llibre and C. Valls, *Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles,* preprint (2021).

[31] J. Llibre and X. Zhang, *Limit cycles for discontinuous planar piecewise linear differential systems separated by one straight line and having a center,* J. Math. Anal. Appl. **467** (2018), 537–549.

[32] W.S. Loud, *Behavior of the period of solutions of certain plane autonomous systems near centers,* Contributions to Differential Equations **3** (1964), 21–36.

[33] R. Thul and S. Coombes, *Understanding cardiac alternans: a piecewise linear modeling framework,* Chaos **20** (2010), 045102, 13 pp.

[34] A. Tonnelier, *McKean caricature of the FitzHugh-Nagumo model: traveling pulses in a discrete diffusive medium,* Physical Review E. Statistical, Nonlinear, and Soft Matter Physics **67** (2003), 036105, 9 pp.

[35] A. Tonnelier and W. Gerstner, *Piecewise linear differential equations and integrate-and-fire neurons: insights from two-dimensional membrane models,* Physical Review E. Statistical, Nonlinear, and Soft Matter Physics **67** (2003), 021908, 16 pp.

[36] A. Visintin, *Differential models of hysteresis*, Applied Mathematical Sciences vol. 111, Springer-Verlag, Berlin 1994.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: `cvalls@math.ist.utl.pt`