This is a preprint of: "Limit cycles in a class of planar discontinuous piecewise quadratic differential systems with a non-regular discontinuous boundary (II)", Dongping He, Jaume Llibre, *Mediterranean J. Math.*, vol. 21(174), 2024.
DOI: [10.1007/s00009-024-02714-0]

LIMIT CYCLES IN A CLASS OF PLANAR DISCONTINUOUS PIECEWISE QUADRATIC DIFFERENTIAL SYSTEMS WITH A NON-REGULAR DISCONTINUOUS BOUNDARY (II)

DONGPING HE[1, ✉] AND JAUME LLIBRE[2]

ABSTRACT. In our previous work we have already studied the Poincaré bifurcation for a class of discontinuous piecewise quadratic polynomial differential systems with a non-regular discontinuous boundary, which is formed by two rays starting from the origin and forming an angle $\alpha = \pi/2$. The unperturbed system is the quadratic uniform isochronous center $\dot{x} = -y + xy$, $\dot{y} = x + y^2$ with a family of periodic orbits surrounding the origin. In this paper, we continue to investigate this kind of piecewise differential systems but now the angle between the two rays is $\alpha \in (0, \pi/2) \cup [3\pi/2, 2\pi)$. Using the averaging theory of first order and the Chebyshev theory we prove that the maximum number of hyperbolic limit cycles which can bifurcate from these periodic orbits is exactly 8 for $\alpha \in (0, \pi/2) \cup [3\pi/2, 2\pi)$. Together with our previous work, which concerns on the case of $\alpha = \pi/2$, we can conclude that by using the averaging theory of first order the maximum number of hyperbolic limit cycles is exactly 8, when this quadratic center is perturbed inside the above mentioned classes separated by a non-regular discontinuous boundary with $\alpha \in (0, \pi/2] \cup [3\pi/2, 2\pi)$.

1. Introduction and statement of the main results

The *weak Hilbert's* 16*th problem* proposed by Arnold [2], that concerns about the maximum number of orbits of the centers of polynomial differential systems that persist as limit cycles (i.e., isolated periodic orbits) when they are perturbed inside the class of all polynomial differential systems of degree n. This is one of the classical ways to produce limit cycles, and this mechanism is also called Poincaré bifurcation. Although the weak Hilbert's 16th problem reduces the difficulty of the investigation of the second part of the *Hilbert's* 16*th problem*, which ask for an upper bound of the maximum number of limit cycles in function of the degree of the planar polynomial differential systems, and for the possible distribution of the limit cycles, see [16, 19, 21]. The possible distributions of limit cycles has been solved, see [31], but to find such upper bound remains unsolved.

In this paper we will perturb a uniform isochronous center. Recall that for planar polynomial differential systems, Conti [10] proved that a center is called a *uniform isochronous center* if in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ it can be written in the form $\dot{r} = R(\theta, r), \dot{\theta} = k$, where k is a nonzero real number.

As it is indicated in [27] the quadratic polynomial differential systems with a uniform isochronous center can be written as,

$$\dot{x} = -y + xy, \quad \dot{y} = x + y^2. \tag{1}$$

In fact, this system, under the transformation $x = -Y, y = X$, can be transformed into the system (S_2) of Table 4 in [7], or of Table 1 in [33]. Therefore systems (1) and (S_2) are equivalent, and then we will summarize the existing results for system (1) even if some of them are concerned on system (S_2).

2020 *Mathematics Subject Classification.* Primary: 34C05, 34C07, 34C25, 34C29.

Key words and phrases. Limit cycle, discontinuous piecewise polynomial system, quadratic uniform isochronous center, non-regular discontinuous boundary, averaging theory, Chebyshev theory.

✉ Corresponding author.

There exist some results on the number of limit cycles which bifurcate from the periodic orbits of the uniform isochronous center (1), when it is perturbed inside the class of polynomial differential systems of degree n. For $n = 2$ Chicone and Jacobs [8] proved that at most 2 limit cycles bifurcate from the periodic orbits of the uniform isochronous center (1). If the isochronous center (1) is only perturbed by homogeneous quadratic polynomial perturbations, in 2000 Li, Li, Llibre and Zhang [22] proved that at most 1 limit cycle can bifurcate. However, in 2015 Llibre and Makhlouf [27] obtained at most 2 limit cycles as given in [8] by using the averaging theory of first order. For arbitrary n Li, Li, Llibre and Zhang [23] gave a linear estimate for the number of limit cycles, where 0 for $n = 0, 1$ and n for $n \geq 2$, bifurcating from the periodic orbits of the uniform isochronous center (1), and these bounds are sharp.

When the uniform isochronous center (1) is perturbed inside the class of all discontinuous piecewise polynomial differential systems of degree n with two zones separated by a straight line. If this straight line is $y = 0$ and $n = 2$, Llibre and Mereu [28] in 2014 using the averaging theory of first order proved that at least 5 limit cycles bifurcate from the periodic orbits of the uniform isochronous center (1). Different from the method used in [28], Yang and Zhao [36] using the first order Melnikov method, Picard–Fuchs equation and Chebyshev theory considered the same systems discussed in [28] and proved that the maximum number of limit cycles is exactly 6. If the discontinuous boundary is $x = 0$, Xiong [34] in 2015 showed that, under the piecewise quadratic polynomial perturbations, there exist 6 limit cycles bifurcating from the periodic orbits of the uniform isochronous center (1). In 2019 da Cruz, Novaes and Torregrosa [11] added the nonzero constant terms in the polynomial perturbations of [28] and also using the averaging theory of first order proved that at least 4, 5 and 7 small-amplitude limit cycles can bifurcate from the uniform isochronous center itself (1) when the discontinuity boundary is $x = 0$, $y = 0$ and $y + \sqrt{3}x = 0$ respectively. Recently, Cen, Liu, Yang and Zhang [6] also using the first order Melnikov method considered the piecewise polynomial perturbations of arbitrary degree n with the nonzero constant terms, and showed that $n+1$ for $n = 0, 1$, and $2n+2$ for $n \geq 2$ limit cycles can bifurcate from the periodic orbits of the uniform isochronous center (1), when the discontinuous boundary of [6] is a straight line $y = 0$. It follows from the results of [28] and [6, 34, 36] that the averaging theory of first order and the first order Melnikov function are not equivalent in studying the number of limit cycles for piecewise polynomial differential systems.

Notice that the existing results as mentioned above are concerned on the case where the discontinuous boundary is a straight line. Here we are interested in the case when the discontinuous boundary is non-regular one, i.e., the discontinuous boundary is formed by two rays separating the plane into two angular sectors with angles $\alpha \in (0, \pi)$ and $2\pi - \alpha$. Without loss of generality, we assume that one ray is the positive x-axis, and the other ray starting at O and forming with the positive x-axis an angle $\alpha \in (0, \pi)$. Therefore the non-regular discontinuous boundary can be expressed as follows:

$$\Sigma_\alpha := \begin{cases} \Sigma_0 \cup \left\{(x, y) \in \mathbb{R}^2 : y = \tan(\alpha) \cdot x, y \geq 0\right\}, & \alpha \in (0, \pi/2) \cup (\pi/2, \pi), \\ \Sigma_0 \cup \left\{(x, y) \in \mathbb{R}^2 : x = 0, y \geq 0\right\}, & \alpha = \pi/2, \end{cases} \tag{2}$$

where $\Sigma_0 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 0\}$. Then $\Sigma_\alpha^\pm$ are two angular sectors separated by Σ_α with angles α and $2\pi - \alpha$, respectively. Moreover the discontinuous boundary is a straight line when $\alpha = \pi$, which is called here the regular case.

Following the Filippov convention [12], the *sliding regions* are defined as the set of points in $\Sigma_\alpha/\{O\}$ where the vector fields $\mathcal{X}^\pm$, which are the vector fields of the differential systems in $\Sigma_\alpha^\pm$ respectively, point outward or inward from $\Sigma_\alpha/\{O\}$ simultaneously, and the *crossing regions* are the complement of the sliding regions on $\Sigma_\alpha/\{O\}$ except for the tangency points of $\mathcal{X}^\pm$ with Σ_α. If a periodic orbit intersect Σ_α only in the crossing regions, then this periodic orbit is called a *crossing periodic orbit.* This periodic orbit is a *crossing limit cycle* when it is isolated in the set of all crossing periodic orbits. In what follows the crossing limit cycles will be called simply *limit cycles.*

When the discontinuous boundary is non-regular Σ_α defined in (2), there exist some results on the number of limit cycles for the discontinuous piecewise linear polynomial differential systems, see [1, 26]. In the following we only introduce the results that these limit cycles bifurcate from the periodic orbits of the linear center. For example, by calculating the higher order Melnikov functions Cardin and Torregrosa [4] in 2016 studied the number of limit cycles for a class of piecewise linear differential systems, and proved that the maximum number of the bifurcated limit cycles is 5 when the sixth Melnikov function is not zero identically, and this number can be realized in an explicit example with the non-regular discontinuous boundary $\Sigma_{\pi/2}$ defined in (2). As commented in [35], the method used in [4] can not be applied to the cases which are not suitable for a polar coordinate transformation. Therefore, they overcame this difficulty and derived the explicit formulas of the first and second order Melnikov functions for general piecewise Hamiltonian systems. Then, they used the obtained Melnikov functions to study the number of limit cycles for a class of piecewise linear Hamiltonian systems, whose unperturbed system is piecewise smooth forming by a linear center and a constant differential system. Two years later, Liang, Romanovski and Zhang [25] using the first-order Melnikov method obtained one more limit cycle in a class of planar piecewise linear Hamiltonian systems compared with the case where the discontinuous boundary is a straight line. Note that in the above mentioned works, the considered polynomial perturbations have the same degrees in two zones $\Sigma_\alpha^\pm$ defined below (2). If the considered polynomial perturbations in the two sectors $\Sigma_\alpha^\pm$ have different degrees, Li and Llibre [24] using the averaging theory up to any order provided an upper bound for the maximum number of limit cycles that bifurcate from the periodic orbits of the linear center, where this upper bound can realized for the first two orders. If the linear center is perturbed inside the class of piecewise polynomial Liénard systems, they gave some better upper bounds in comparison with the one obtained in the general piecewise polynomial perturbations.

It follows from [4, 24, 35] that more limit cycles can be obtained in piecewise linear differential systems when the discontinuous boundary becomes a non-regular one. Therefore, it inspires us to consider the following question: *under the discontinuous piecewise quadratic polynomial perturbations, how many limit cycles can bifurcate from the periodic orbits of the uniform isochronous center (1) when the discontinuous boundary becomes a non-regular one with* $\alpha \in (0, \pi)$? More precisely, we consider the following discontinuous piecewise quadratic polynomial differential systems:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + xy + \varepsilon p_1(x,y) \\ x + y^2 + \varepsilon q_1(x,y) \end{pmatrix}, & \text{for } (x,y) \in \Sigma_\alpha^+, \\ \begin{pmatrix} -y + xy + \varepsilon p_2(x,y) \\ x + y^2 + \varepsilon q_2(x,y) \end{pmatrix}, & \text{for } (x,y) \in \Sigma_\alpha^-, \end{cases} \tag{3}$$

where ε is a small parameter, and

$$\begin{aligned} p_1(x,y) &:= \sum_{i+j=0}^{2} a_{ij} x^i y^j, \quad q_1(x,y) := \sum_{i+j=0}^{2} b_{ij} x^i y^j, \\ p_2(x,y) &:= \sum_{i+j=0}^{2} c_{ij} x^i y^j, \quad q_2(x,y) := \sum_{i+j=0}^{2} d_{ij} x^i y^j. \end{aligned}$$

In this work, we consider the case of $\alpha \in (0, \pi/2]$, and the case of $\alpha \in (\pi/2, \pi)$ we leave it for further study. The main result of this paper is the following one.

Theorem 1. *For* $|\varepsilon| \neq 0$ *enough small and* $\alpha \in (0, \pi/2]$, *the maximum number of hyperbolic limit cycles bifurcating from the periodic orbits of the quadratic uniform isochronous center* $(3)|_{\varepsilon=0}$ *is exactly* 8 *using the averaging theory of first order.*

For the result of Theorem 1, it was considered in [17, Theorem 1] when $\alpha = \pi/2$ by using the averaging theory of first order and the Chebyshev theory, therefore this statement generalizes the results of Theorem 1 in [17], and we only discuss the case of $(0, \pi/2)$ in this paper.

Moreover if $\alpha \in [3\pi/2, 2\pi)$, then the non-regular discontinuous boundary Σ_α defined in (2) becomes

$$\widetilde{\Sigma}_\alpha = \begin{cases} \Sigma_0 \cup \left\{(x,y) \in \mathbb{R}^2 : y = \tan(\alpha) \cdot x, y \leq 0\right\}, & \alpha \in (3\pi/2, 2\pi), \\ \Sigma_0 \cup \left\{(x,y) \in \mathbb{R}^2 : x = 0, y \leq 0\right\}, & \alpha = 3\pi/2, \end{cases} \tag{4}$$

where Σ_0 is defined below (2). Under the change of variables $(x, y, t) \mapsto (x, -y, -t)$ and for each pair natural numbers i and j the parametric transformations

$$(a_{i,0}, a_{0,2i}, b_{i,2j+1}, c_{i,0}, c_{0,2i}, d_{i,2j+1}) = -(a_{i,0}, a_{0,2i}, b_{i,2j+1}, c_{i,0}, c_{0,2i}, d_{i,2j+1}),$$

the discontinuous boundary $\widetilde{\Sigma}_\alpha$ defined in (4) is transformed into Σ_α defined in (2), and system (3) is also invariant. Therefore from Theorem 1 and Theorem 1 in [17] we have the following corollary.

Corollary 2. *For $|\varepsilon| \neq 0$ enough small and $\alpha \in [3\pi/2, 2\pi)$, the maximum number of hyperbolic limit cycles bifurcating from the periodic orbits of the quadratic uniform isochronous center $(3)|_{\varepsilon=0}$ is exactly 8 using the averaging theory of first order.*

The rest of this paper is organized as follows. In the next section we will introduce the basic averaging theory and the Chebyshev theory, which will be used to prove our main result. Moreover, we also introduce the technique given in [3] for transforming a planar polynomial differential system in the standard form for applying the averaging theory. In section 3, we first derive the recurrent formulas for calculating some definite integrals, which will be used to deduce the explicit expression of the first averaged function of system (3), and then determine the minimum number of the generated functions whose nontrivial linear combination generates that averaged function. Section 4 is devoted to prove Theorem 1 and give two numerical examples to confirm the results of Theorem 1. More precisely, we use the Chebyshev theory to give the exact upper bound for the number of limit cycles, that can be obtained using the averaging theory of first order, of the discontinuous piecewise quadratic differential system (3) when $|\varepsilon| \neq 0$ is sufficiently small. Finally, we give some important and long expressions in Appendix for completeness.

2. Preliminary results

In this section we first introduce the averaging theory of first order (see for instance [18]) which will be used for studying the number of limit cycles of the discontinuous piecewise quadratic polynomial differential system (3) when $|\varepsilon| \neq 0$ is sufficiently small. After we recall the concepts and the known results of the extended complete Chebyshev systems.

2.1. Averaging theory.

In 2015 Llibre, Novaes, Teixeira [30] developed, via Brouwer degree theory, the averaging theory of first order for studying periodic orbits of discontinuous piecewise differential systems in two zones separated by a straight line. Two years later Itikawa, Llibre and Novaes [18] improved the averaging theory at any order for analyzing the periodic solutions of discontinuous piecewise differential systems with two zones, where the discontinuous boundary composed by two half-straight lines starting at the origin and forming an angle α. In the same year, Llibre, Novaes and Rodrigues [29] generalized the results of [18] and considered a class of ε-family of discontinuous differential systems with many zones. Here we introduced the results obtained in [18, 29] that we will use to prove our main results.

Consider a non-autonomous discontinuous piecewise differential system

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon \mathcal{F}_1(\theta, r) + \varepsilon^2 \mathcal{R}_2(\theta, r, \varepsilon), \tag{5}$$

where ε is a real parameter with $0 < |\varepsilon| \ll 1$, $r \in \mathbb{R}$, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$,

$$\mathcal{F}_1(\theta, r) = \sum_{i=1}^{2} \chi_{\mathcal{S}_i}(\theta)\mathcal{F}_i^{\pm}(\theta, r), \qquad \mathcal{R}_2(\theta, r, \varepsilon) = \sum_{i=1}^{2} \chi_{\mathcal{S}_i}(\theta)\mathcal{R}_i^{\pm}(\theta, r, \varepsilon),$$

$\mathcal{F}_1^{\pm} : \mathbb{S}^1 \times \mathcal{D} \to \mathbb{R}$, $\mathcal{R}^{\pm} : \mathcal{D} \times \mathbb{S}^1 \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, are $\mathcal{C}^2$ functions, 2π-periodic in the variable θ, and $\mathcal{D}$ is an open and bounded interval of $(0, \infty)$, and ε_0 is a positive number. Additionally, $\chi_{\mathcal{S}}(\theta)$ is the characteristic function of a set $\mathcal{S} \subset \mathbb{R}^2$, where

$$\chi_{\mathcal{S}}(\theta) = \begin{cases} 1, & \text{for} \quad \theta \in \mathcal{S}, \\ 0, & \text{for} \quad \theta \notin \mathcal{S}. \end{cases}$$

Here the $\mathcal{S}_i$ are the open intervals (θ_i, θ_{i+1}) for $i = 0, 1$ and $0 = \theta_0 < \theta_1 < \theta_2 = 2\pi$. It follows that the set of discontinuity of system (5) is defined by $\Sigma_{\theta_1} = (\{\theta = 0\} \cup \{\theta = \theta_1\}) \cap \mathbb{S}^1 \times \mathcal{D}$.

If $r(\theta, r_0)$ is the solution of the system (5) satisfying $r(0, r_0) = r_0$, then we obtain that

$$r(2\pi, r_0) - r_0 = \varepsilon f(r) + O(\varepsilon),$$

where $f : \mathcal{D} \to \mathbb{R}$ is called *the first averaged function* and of the form

$$f(r) = \int_0^T \mathcal{F}_1(\theta, r)\mathrm{d}\theta. \tag{6}$$

As shown in [18] every positive isolated zero of the averaged function $f(r)$ provides a crossing limit cycle of the discontinuous piecewise differential systems (5) on the cylinder $\mathcal{C} = \{(\theta, r) \in \mathbb{S}^1 \times (0, \infty)\}$. From Theorem 1 of [18] we have the following result on the periodic orbits of system (5).

Lemma 3. *For the non-autonomous discontinuous piecewise differential system* (5)*, if there exists* $r^* \in \mathcal{D}$ *such that* $f(r^*) = 0$ *and* $f'(r^*) \neq 0$*, then for* $|\varepsilon| \neq 0$ *enough small there exists a* 2π*-periodic solution* $r(\theta, \varepsilon)$ *satisfying* $r(0, \varepsilon) \to r^*$ *when* ε *tends to* 0*.*

Notice that under the polar coordinates $(x, y) \to (r\cos\theta, r\sin\theta)$, system (3) can not be written into the standard form (5). In order to obtain the standard form (5) we apply the following result given by Buică and Llibre in [3]. Consider the following perturbed differential systems

$$\begin{aligned} \dot{x} &= P(x, y) + \varepsilon p(x, y), \\ \dot{y} &= Q(x, y) + \varepsilon q(x, y), \end{aligned} \tag{7}$$

where P, Q, p, $q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and ε is a small parameter. Assume that system (7) has a center at $O = (0, 0)$ when $\varepsilon = 0$. It follows that there exist a periodic annulus around this center, which is formed by a continuous family of periodic orbits

$$\{\Gamma_h\} \subset \{(x, y) \in \mathbb{R}^2 : H(x, y) = h, h \in (h_c, h_s)\},$$

where $H(x, y)$ is a first integral of system $(7)|_{\varepsilon=0}$, and h_c and h_s correspond to the values of $H(x, y)$ at the center and at the boundary of the periodic annulus of the center, respectively. Recall that the periodic annulus of a center is the maximal open set U containing the center O and such that $U \setminus \{O\}$ is filled of periodic orbits.

Lemma 4. ([3, Theorem 5.1]) *Consider the unperturbed system* $(7)|_{\varepsilon=0}$*, which has a first integral* $H(x, y)$*. Suppose that* $xQ(x, y) - yP(x, y) \neq 0$ *for all* (x, y) *in the periodic annulus formed by the ovals* $\{\Gamma_h\}$ *and that* $H(x, y)$ *takes non-negative values on the periodic annulus. Let* $\rho : [0, 2\pi) \times (\sqrt{h_c}, \sqrt{h_s}) \to [0, \infty)$ *be a continuous function satisfying*

$$H\big(\rho(\theta, R)\cos\theta, \rho(\theta, R)\sin\theta\big) = R^2$$

for all $R \in (\sqrt{h_c}, \sqrt{h_s})$ *and all* $\theta \in [0, 2\pi)$*. Then the differential equation which indicates the dependence between the square root of the energy* $R = \sqrt{h}$ *and the angle associated to the perturbed system*

(7) *is of the form*

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \varepsilon \frac{\mu(x^2+y^2)(Qp-Pq)}{2R(xQ-yP)} + O(\varepsilon^2),$$

where $\mu = \mu(x,y)$ *is the integrating factor associated to the first integral* H*, and* $x = \rho(\theta, R)\cos\theta$ *and* $y = \rho(\theta, R)\sin\theta$.

2.2. **Chebyshev theory.** Determining the maximum number of isolated zeros of the first averaged function is a difficult task, because in general this function is a nontrivial linear combination of various elementary functions, such as a square root function, an inverse trigonometric function and logarithmic function, as the averaged function given in (18). In order to obtain the exact number of zeros for such averaged function, we introduce the some effective results, which are concerned on the lower bound and the upper bound of the number of zeros for this averaged function.

The following result is effective for the lower bound.

Lemma 5. ([9, Lemma 4.5]) *If the analytic functions* $u_i : \mathbb{I} \to \mathbb{R}$ *for* $i = 0, 1, \cdots, m$ *are linearly independent, where* $\mathbb{I} \subset \mathbb{R}$ *is an interval, and there exists* $j \in \{0, 1, \cdots, m\}$ *such that* $u_j(x)$ *has constant sign, then there exist* $m+1$ *constants* $\alpha_i \in \mathbb{R}$ *for* $i = 0, 1, \cdots, m$ *and* m *constants* $a_j \in \mathbb{I}$ *for* $j = 1, 2, \cdots, m$ *such that for every* $j \in \{1, 2, \cdots, m\}$ *the function* $u(x) = \sum\limits_{i=0}^{m} \alpha_i u_i(x)$ *satisfies* $u(a_i) = 0$ *and* $u'(a_i) \neq 0$.

As it is indicated in [32, Appendix], we say that the functions $u_i : \mathbb{I} \to \mathbb{R}$ for $i = 0, 1, \cdots, m$ are *linearly independent* in the interval $\mathbb{I}$ if and only if for any $a \in \mathbb{I}$ the following condition holds,

$$\sum_{i=0}^{m} \alpha_i u_i(a) = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0,$$

where α_i's are real numbers.

For the upper bound, we introduce the theory of *Extended Complete Chebyshev systems* (abbriv. ECT-system), which is a classical tool to deal with this maximum number of isolated zeros of these averaged functions. Assume that $\mathcal{U} := [u_0, u_1, \cdots, u_m]$ is an ordered set of analytic functions on an open interval $\mathbb{I} := (a, b) \subset \mathbb{R}$. This ordered set forms an ECT-system on $\mathbb{I}$, provided that, for all $k = 0, 1, \cdots, m$, any nontrivial function $\sum\limits_{i=0}^{k} \alpha_i u_m(x)$ in $\mathrm{Span}(\mathcal{U})$ can have at most k isolated zeros on $\mathbb{I}$ counting their multiplicities. Here $\mathrm{Span}(\mathcal{U})$ is the set of functions generated by all nontrivial linear combinations of elements of $\mathcal{U}$, namely, $u(x) = \sum\limits_{i=0}^{m} \alpha_i u_m(x)$, where the coefficients α_i's are real numbers. Moreover, a function $u(x)$ is said to be *nontrivial* if $\sum\limits_{i=0}^{m} \alpha_i^2 > 0$.

Lemma 6. ([20]) *If the ordered set* $\mathcal{U}$ *forms an ECT-system on* $\mathbb{I}$*, then the number of isolated zeros for each element in* $\mathrm{Span}(\mathcal{U})$ *does not exceed* m*. Additionally, for every configuration of* $n \leq m$ *zeros counting their multiplicities, there exists an element in* $\mathrm{Span}(\mathcal{U})$ *having this configuration of zeros.*

Remark 7. ([15, Remark 3.7]) *If the ordered set* $\mathcal{U}$ *forms an ECT-system on* $\mathbb{I}$*, then for every* $k = 0, 1, \cdots, m$ *there exists* $\alpha_i^* \in \mathbb{R}$ $(i = 0, 1, \cdots, k)$ *such that the nontrivial function* $\sum\limits_{i=0}^{k} \alpha_i^* u_k(x)$ *in* $\mathrm{Span}(\mathcal{U})$ *has exactly* k *simple zeros on* $\mathbb{I}$.

In order to apply Lemma 6 we need to prove that the ordered set $\mathcal{U}$ is an ECT-system on $\mathbb{I}$. The following lemma, see for instance [14, 20], gives a method to achieve this.

Lemma 8. *The ordered set $\mathcal{U}$ is an ECT-system on $\mathbb{I}$ if and only if for every $k \in \{0,1,2,\cdots,m\}$ and all $x \in \mathbb{I}$, the Wronskian $W_k(x) \neq 0$, where*

$$W_k(x) := W[u_0, u_1, \cdots, u_k](x) = \begin{vmatrix} u_0(x) & u_1(x) & \cdots & u_k(x) \\ u_0'(x) & u_1'(x) & \cdots & u_k'(x) \\ \vdots & \vdots & \vdots & \vdots \\ u_0^{(k)}(x) & u_1^{(k)}(x) & \cdots & u_k^{(k)}(x) \end{vmatrix}.$$

3. Averaged Function

This section is devoted to deduce the recurrent formulas of some definite integrals that appear in deducing the expression of the first averaged function for the discontinuous piecewise quadratic polynomial differential system (3) with $\alpha \in (0, \pi/2)$, and then to determine the minimum number of functions whose nontrivial linear combination generates that averaged function.

A first integral and its associated integrating factor of system (1) are

$$H(x,y) = \frac{x^2+y^2}{(1-x)^2} \quad \text{and} \quad \mu(x,y) = \frac{2}{(x-1)^3},$$

respectively. Moreover the periodic annulus of the uniform isochronous center (1) is formed by a continuous family of periodic orbits $\{\Gamma_h\} \subset \{(x,y) \in \mathbb{R}^2 : H(x,y) = h, h \in (0,1)\}$ because system (1) has the invariant algebraic curve $2x + y^2 - 1 = 0$, namely the minimal distance of the outer boundary of the periodic annulus of the center to the origin is 1. Therefore for system (1) we have $h_c = 0$ and $h_s = 1$, and then the function $\rho(\theta, R)$ satisfying the assumption of Lemma 4 is

$$\rho(\theta, R) = R/(1 + R\cos\theta),$$

for all $R \in (0,1)$.

From Lemma 4 we choose the following transformations

$$x = \frac{R\cos\theta}{1+R\cos\theta}, \quad y = \frac{R\sin\theta}{1+R\cos\theta}, \tag{8}$$

and then using (8) system (3) can be written into the form,

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \begin{cases} \varepsilon\mathcal{F}_1^+(\theta,R) + O(\varepsilon^2), & \text{for } 0 \le \theta \le \alpha, \\ \varepsilon\mathcal{F}_1^-(\theta,R) + O(\varepsilon^2), & \text{for } \alpha \le \theta \le 2\pi, \end{cases} \tag{9}$$

where

$$\mathcal{F}_1^\pm := \frac{1}{1+R\cos\theta}\sum_{i=0}^{3}\left(\mathcal{F}_{1i}^\pm(R)\cos(i\theta) + \mathcal{F}_{2i}^\pm(R)\sin(i\theta)\right),$$

in which $\mathcal{F}_{1i}^-$'s and $\mathcal{F}_{2i}^-$'s are obtained by the substitution $(a_{ij}, b_{ij}) \to (c_{ij}, d_{ij})$ in $\mathcal{F}_{1i}^+$'s and $\mathcal{F}_{2i}^+$'s,

$$\begin{aligned}
\mathcal{F}_{10}^+ &:= -(R/2)\left(4a_{00} + a_{10} + b_{01} + (a_{00} + a_{10} + a_{20} + a_{02})R^2\right), \\
\mathcal{F}_{11}^+ &:= -(1/4)\left(4a_{00} + (11a_{00} + 7a_{10} + 3a_{20} + a_{02} + b_{01} + b_{11})R^2\right), \\
\mathcal{F}_{12}^+ &:= -(R/2)\left(2a_{00} + a_{10} - b_{01} + (a_{00} + a_{10} + a_{20} - a_{02})R^2\right), \\
\mathcal{F}_{13}^+ &:= -(1/4)\left(a_{00} + a_{10} + a_{20} - a_{02} - b_{01} - b_{11}\right)R^2, \\
\mathcal{F}_{20}^+ &:= 0, \quad \mathcal{F}_{21}^+ := -(1/4)\left(4b_{00} + (5a_{01} + a_{11} + b_{00} + b_{10} + b_{20} + 3b_{02})R^2\right), \\
\mathcal{F}_{22}^+ &:= -(R/2)\left(a_{01} + 2b_{00} + b_{10} + (a_{01} + a_{11})R^2\right), \\
\mathcal{F}_{23}^+ &:= -(1/4)\left(a_{01} + a_{11} + b_{00} + b_{10} + b_{20} - b_{02}\right)R^2.
\end{aligned}$$

Note that system (9) is in the form (5) so the averaging theory can be applied to it. From (6) we know that the first averaged function $f:(0,1)\to\mathbb{R}$ associated with system (9) can be expressed as

$$\begin{aligned}f(\alpha,R)=&\int_0^{\alpha}\mathcal{F}_1^{+}(\theta,R)\mathrm{d}\theta+\int_0^{2\pi-\alpha}\mathcal{F}_1^{-}(\theta+\alpha,R)\mathrm{d}\theta,\\ =&\sum_{k=0}^{3}\mathcal{F}_{1k}^{+}(R)\mathcal{I}_k^{\alpha}(\alpha,R)+\sum_{\ell=0}^{3}\mathcal{F}_{2\ell}^{+}(R)\mathcal{J}_\ell^{\alpha}(\alpha,R)\\ &+\sum_{k=0}^{3}\mathcal{F}_{1k}^{-}(R)\mathcal{I}_k^{2\pi-\alpha}(\alpha,R)+\sum_{\ell=0}^{3}\mathcal{F}_{2\ell}^{-}(R)\mathcal{J}_\ell^{2\pi-\alpha}(\alpha,R),\end{aligned}\tag{10}$$

where the functions $\mathcal{F}_{1k}^{\pm}$ and $\mathcal{F}_{2\ell}^{\pm}$ are given below (9),

$$\begin{aligned}\mathcal{I}_k^{\alpha}&:=\int_0^{\alpha}\frac{\cos(k\theta)}{1+R\cos\theta}\mathrm{d}\theta, & \mathcal{J}_\ell^{\alpha}&:=\int_0^{\alpha}\frac{\sin(\ell\theta)}{1+R\cos\theta}\mathrm{d}\theta,\\ \mathcal{I}_k^{2\pi-\alpha}&:=\int_0^{2\pi-\alpha}\frac{\cos\big(k(\theta+\alpha)\big)}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta, & \mathcal{J}_\ell^{2\pi-\alpha}&:=\int_0^{2\pi-\alpha}\frac{\sin\big(\ell(\theta+\alpha)\big)}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta.\end{aligned}\tag{11}$$

In order to simplify the calculations of the integrals in (10), we introduce the following results.

Proposition 9. *For $0<\alpha<\pi/2$ and $0<R<1$ the functions $\mathcal{I}_k^{2\pi-\alpha}$ and $\mathcal{J}_\ell^{2\pi-\alpha}$ defined in (11), can be simplified as*

$$\begin{aligned}\mathcal{I}_k^{2\pi-\alpha}(\alpha,R)=&\sum_{j=0}^{2}\mathcal{I}_k^{j(\pi/2)}(\alpha,R)+\widetilde{\mathcal{I}}_k^{3\pi/2}(\alpha,R),\\ \mathcal{J}_\ell^{2\pi-\alpha}(\alpha,R)=&\sum_{j=0}^{2}\mathcal{J}_\ell^{j(\pi/2)}(\alpha,R)+\widetilde{\mathcal{J}}_\ell^{3\pi/2}(\alpha,R),\end{aligned}\tag{12}$$

where

$$\begin{aligned}\mathcal{I}_k^{j(\pi/2)}&:=\int_0^{\pi/2}\frac{\cos\big(k\left(\theta+\alpha+j(\pi/2)\right)\big)}{1+R\cos\big(\theta+\alpha+j(\pi/2)\big)}\mathrm{d}\theta,\\ \mathcal{J}_\ell^{j(\pi/2)}&:=\int_0^{\pi/2}\frac{\sin\big(\ell\left(\theta+\alpha+j(\pi/2)\right)\big)}{1+R\cos\big(\theta+\alpha+j(\pi/2)\big)}\mathrm{d}\theta,\\ \widetilde{\mathcal{I}}_k^{3\pi/2}&:=\int_0^{\pi/2-\alpha}\frac{\cos\big(k\left(\theta+\alpha+3\pi/2\right)\big)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta,\\ \widetilde{\mathcal{J}}_\ell^{3\pi/2}&:=\int_0^{\pi/2-\alpha}\frac{\sin\big(\ell\left(\theta+\alpha+3\pi/2\right)\big)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta.\end{aligned}\tag{13}$$

Proof. Notice that $\alpha\in(0,\pi/2)$, then $2\pi-\alpha\in(3\pi/2,2\pi)$. When we calculate the integrals $\mathcal{I}_k^{2\pi-\alpha}(\alpha,R)$ and $\mathcal{J}_\ell^{2\pi-\alpha}(\alpha,R)$, we need to apply the change of variables $\cos\theta=(1-\psi^2)/(1+\psi^2)$, $\sin\theta=2\psi/(1+\psi^2)$ and $\psi=\tan(\theta/2)$, under this change the interval $(0,2\pi-\alpha)$ pass to the intervals $(0,+\infty)\cup(-\infty,\tan(\pi-\alpha/2))$. In order to keep the continuity of the integral interval after using the above change of variables, we first convert the interval $(0,2\pi-\alpha)$ into the interval $(0,\pi/2)$.

For $\alpha\in(0,\pi/2)$ the definite integral $\mathcal{I}_k^{2\pi-\alpha}(\alpha,R)$ in (11) can be written as

$$\begin{aligned}\int_0^{2\pi-\alpha}\frac{\cos(k(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta=&\int_0^{\pi/2}\frac{\cos(k(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta+\int_{\pi/2}^{2\pi-\alpha}\frac{\cos(k(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta\\ =&\mathcal{I}_k^{0}(\alpha,R)+\int_{\pi/2}^{3\pi/2}\frac{\cos(k(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta+\int_{3\pi/2}^{2\pi-\alpha}\frac{\cos(k(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta\end{aligned}\tag{14}$$

because $2\pi-\alpha\in(3\pi/2,2\pi)$ in this case. Then we apply the change of variable $\theta=\phi+\pi/2$ twice in the second definite integral of (14), and we obtain

$$\begin{aligned}\int_{\pi/2}^{3\pi/2}\frac{\cos\big(k(\theta+\alpha)\big)}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta=&\int_0^{\pi}\frac{\cos(k(\theta+\alpha+\pi/2))}{1-R\sin(\theta+\alpha)}\mathrm{d}\theta\\ =&\mathcal{I}_k^{\pi/2}(\alpha,R)+\int_{\pi/2}^{\pi}\frac{\cos(k(\theta+\alpha+\pi/2))}{1-R\sin(\theta+\alpha)}\mathrm{d}\theta\\ =&\mathcal{I}_k^{\pi/2}(\alpha,R)+\mathcal{I}_k^{\pi}(\alpha,R),\end{aligned}$$

where we still use θ instead of ϕ for simplicity. Similarly, using the change of variable $\theta=\phi+3\pi/2$ in the third definite integral of (14), we get

$$\int_{3\pi/2}^{2\pi-\alpha}\frac{\cos(i(\theta+\alpha))}{1+R\cos(\theta+\alpha)}\mathrm{d}\theta=\widetilde{\mathcal{I}}_k^{3\pi/2}(\alpha,R).$$

It follows that for $\alpha\in(0,\pi/2)$ we obtain the expression of $\mathcal{I}_k^{2\pi-\alpha}(\alpha,R)$ as given in (12).

The simplification of the integral $\mathcal{J}_\ell^{2\pi-\alpha}(\alpha,R)$ can be deduced similarly. Thus, we complete the proof of this proposition. □

From Proposition 9 we know that it is an important step to calculate the integrals $\mathcal{I}_k^\alpha$ and $\mathcal{J}_k^\alpha$ in (11), $\mathcal{I}_k^{j(\pi/2)}$, $\mathcal{J}_\ell^{j(\pi/2)}$, $\mathcal{I}_k^{3\pi/2}$ and $\mathcal{J}_\ell^{3\pi/2}$ in (13) in order to deduce the expression of the first averaged function for $\alpha\in(0,\pi/2)$. In the following we will give the explicit recurrences for each integral in order to obtain the explicit expressions of theses integrals.

Lemma 10. *For $0<\alpha<\pi/2$ and $0<R<1$ the functions $\mathcal{I}_k^\alpha$ and $\mathcal{J}_\ell^\alpha$ in (11) write as*

$$\mathcal{I}_k^\alpha(\alpha,R)=\begin{cases}\dfrac{2\mathcal{A}(\alpha,R)}{\sqrt{1-R^2}}, & k=0,\\ \dfrac{\alpha-\mathcal{I}_0^\alpha(\alpha,R)}{R}, & k=1,\\ \dfrac{2\sin((k-1)\alpha)}{(k-1)R}-\dfrac{2}{R}\mathcal{I}_{k-1}^\alpha(\alpha,R)-\mathcal{I}_{k-2}^\alpha(\alpha,R), & k\ge2,\end{cases}$$

$$\mathcal{J}_\ell^\alpha(\alpha,R)=\begin{cases}0, & \ell=0,\\ -\dfrac{1}{R}\ln\left(\dfrac{1+R\cos\alpha}{1+R}\right), & \ell=1,\\ \dfrac{2-2\cos((\ell-1)\alpha)}{(\ell-1)R}-\dfrac{2}{R}\mathcal{J}_{\ell-1}^\alpha(\alpha,R)-\mathcal{J}_{\ell-2}^\alpha(\alpha,R), & \ell\ge2,\end{cases}\tag{15}$$

where

$$\mathcal{A}:=\arctan\left(\sqrt{\frac{1-R}{1+R}}\tan\frac{\alpha}{2}\right).\tag{16}$$

Proof. For $k=0,1$ the expressions of the functions $\mathcal{I}_k^\alpha(\alpha,R)$ are given by direct integration. For $k\geq2$ the recurrence for $\mathcal{I}_k^\alpha(\alpha,R)$ can be deduced by the following calculations:

$$\begin{aligned}\int_0^\alpha\cos\left((k-1)\theta\right)\mathrm{d}\theta=&\int_0^\alpha\frac{\cos\left((k-1)\theta\right)(1+R\cos\theta)}{1+R\cos\theta}\mathrm{d}\theta\\ =&\ \mathcal{I}_{k-1}^\alpha(\alpha,R)+\int_0^\alpha\frac{\cos\left((k-1)\theta\right)\cos\theta}{1+R\cos\theta}\mathrm{d}\theta.\\ =&\ \mathcal{I}_{k-1}^\alpha(\alpha,R)+\frac{R}{2}\left(\int_0^\alpha\frac{\cos(k\theta)}{1+R\cos\theta}\mathrm{d}\theta+\int_0^\alpha\frac{\cos\left((k-2)\theta\right)}{1+R\cos\theta}\mathrm{d}\theta\right)\end{aligned}$$

$$= \mathcal{I}_{k-1}^{\alpha}(\alpha, R) + \frac{R}{2}\left(\mathcal{I}_{k}^{\alpha}(\alpha, R) + \mathcal{I}_{k-2}^{\alpha}(\alpha, R)\right).$$

Similarly we can derive the recurrence for $\mathcal{J}_\ell^\alpha(\alpha, R)$ as we wanted to prove, in this case for $\ell = 0, 1$ the expressions of the integral $\mathcal{J}_\ell^\alpha(\alpha, R)$ are given by direct integration, while for $\ell \geq 2$ we should consider the integral $\int_0^\alpha \sin\left((\ell-1)\theta\right)\mathrm{d}\theta$ instead of $\int_0^\alpha \cos\left((k-1)\theta\right)\mathrm{d}\theta$. Hence the proof of Lemma 10 is completed. □

From the expressions of the functions $\mathcal{I}_k^{j(\pi/2)}$, $\mathcal{J}_\ell^{j(\pi/2)}$, $\widetilde{\mathcal{I}}_k^{3\pi/2}$ and $\widetilde{\mathcal{J}}_\ell^{3\pi/2}$ in (13), we know that they can be further simplified. The following two propositions and lemmas are devoted to simplify them in order to reduce the difficulty of the calculations and then to deduce the explicit expressions for these integrals.

For natural numbers k and ℓ we define the following functions:

$$\begin{aligned}
&c_{kj}^{\pi/2} := \int_0^{\pi/2} \frac{\cos\left(k(\theta+\alpha)\right)}{1+(-1)^{\frac{j}{2}} R\cos(\theta+\alpha)}\mathrm{d}\theta, && s_{\ell j}^{\pi/2} := \int_0^{\pi/2} \frac{\sin\left(\ell(\theta+\alpha)\right)}{1+(-1)^{\frac{j}{2}} R\cos(\theta+\alpha)}\mathrm{d}\theta,\\
&\widetilde{c}_{\nu j}^{\pi/2} := \int_0^{\pi/2} \frac{\cos\left(\nu(\theta+\alpha)\right)}{1+(-1)^{\frac{j+1}{2}} R\sin(\theta+\alpha)}\mathrm{d}\theta, && \widetilde{s}_{\nu j}^{\pi/2} := \int_0^{\pi/2} \frac{\sin\left(\nu(\theta+\alpha)\right)}{1+(-1)^{\frac{j+1}{2}} R\sin(\theta+\alpha)}\mathrm{d}\theta,\\
&c_{\nu}^{\pi/2-\alpha} := \int_0^{\pi/2-\alpha} \frac{\cos\left(\nu(\theta+\alpha)\right)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta, && s_{\nu}^{\pi/2-\alpha} := \int_0^{\pi/2-\alpha} \frac{\sin\left(\nu(\theta+\alpha)\right)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta,
\end{aligned} \tag{17}$$

where $\nu \in \{k, \ell\}$.

The following proposition is proved after simple calculations, so we omit its proof.

Proposition 11. *For $0 < \alpha < \pi/2$ and $0 < R < 1$ the functions $\mathcal{I}_k^{j(\pi/2)}$ and $\mathcal{J}_\ell^{j(\pi/2)}$ in (13) are*

$$\mathcal{I}_k^{j(\pi/2)}(\alpha, R) = \begin{cases}
c_{kj}^{\pi/2}(\alpha, R), & j = 4m,\\
(-1)^{\frac{k}{2}}\widetilde{c}_{kj}^{\pi/2}(\alpha, R), & j = 4m+1, \quad k = 4i \quad \text{or} \quad k = 4i+2,\\
(-1)^{\frac{k+1}{2}}\widetilde{s}_{kj}^{\pi/2}(\alpha, R), & j = 4m+1, \quad k = 4i+1 \quad \text{or} \quad k = 4i+3,\\
(-1)^{k} c_{kj}^{\pi/2}(\alpha, R), & j = 4m+2,\\
(-1)^{\frac{k}{2}}\widetilde{c}_{kj}^{\pi/2}(\alpha, R), & j = 4m+3, \quad k = 4i \quad \text{or} \quad k = 4i+2,\\
(-1)^{\frac{k-1}{2}}\widetilde{s}_{kj}^{\pi/2}(\alpha, R), & j = 4m+3, \quad k = 4i+1 \quad \text{or} \quad k = 4i+3,
\end{cases}$$

$$\mathcal{J}_\ell^{j(\pi/2)}(\alpha, R) = \begin{cases}
s_{\ell j}^{\pi/2}(\alpha, R), & j = 4m,\\
(-1)^{\frac{\ell}{2}}\widetilde{s}_{\ell j}^{\pi/2}(\alpha, R), & j = 4m+1, \quad \ell = 4i \quad \text{or} \quad \ell = 4i+2,\\
(-1)^{\frac{\ell-1}{2}}\widetilde{c}_{\ell j}^{\pi/2}(\alpha, R), & j = 4m+1, \quad \ell = 4i+1 \quad \text{or} \quad \ell = 4i+3,\\
(-1)^{\ell} s_{\ell j}^{\pi/2}(\alpha, R), & j = 4m+2,\\
(-1)^{\frac{\ell}{2}}\widetilde{s}_{\ell j}^{\pi/2}(\alpha, R), & j = 4m+3, \quad \ell = 4i \quad \text{or} \quad \ell = 4i+2,\\
(-1)^{\frac{\ell+1}{2}}\widetilde{c}_{\ell j}^{\pi/2}(\alpha, R), & j = 4m+3, \quad \ell = 4i+1 \quad \text{or} \quad \ell = 4i+3,
\end{cases}$$

where i and m are natural numbers.

The following lemma gives the recurrent formulas in terms of k, j, ℓ and ν for the functions $c_{kj}^{\pi/2}$, $s_{\ell j}^{\pi/2}$, $\widetilde{c}_{\nu j}^{\pi/2}$ and $\widetilde{s}_{\nu j}^{\pi/2}$ defined in (17).

Lemma 12. *For $0 < \alpha < \pi/2$ and $0 < R < 1$ the following two statements hold.*

(i) *The functions* $c_{kj}^{\pi/2}$ *and* $s_{\ell j}^{\pi/2}$ *defined in* (17) *are*

$$c_{kj}^{\pi/2}(\alpha,R)=\begin{cases}\dfrac{2}{\sqrt{1-R^2}}\mathcal{A}_0^+(\alpha,R), & k=0,\quad j=4m,\\ \dfrac{2}{\sqrt{1-R^2}}\mathcal{A}_0^-(\alpha,R), & k=0,\quad j=4m+2,\\ \dfrac{2\pi}{R}-\dfrac{1}{R}c_{0j}^{\pi/2}(\alpha,R), & k=1,\quad j=4m,\\ -\dfrac{2\pi}{R}+\dfrac{1}{R}c_{0j}^{\pi/2}(\alpha,R), & k=1,\quad j=4m+2,\\ \mathcal{C}_0(\alpha,R), & k\geq 2,\quad j=4m \quad \text{or} \quad j=4m+2,\end{cases}$$

$$s_{\ell j}^{\pi/2}(\alpha,R)=\begin{cases}0, & \ell=0,\quad j=4m \quad \text{or} \quad j=4m+2,\\ \dfrac{1}{R}\mathcal{L}_0^+(\alpha,R), & \ell=1,\quad j=4m,\\ -\dfrac{1}{R}\mathcal{L}_0^-(\alpha,R), & \ell=1,\quad j=4m+2,\\ \mathcal{S}_0(\alpha,R), & \ell\geq 2,\quad j=4m \quad \text{or} \quad j=4m+2,\end{cases}$$

where

$$\mathcal{C}_0:=\frac{2\sin\big((k-1)(\alpha+\pi/2)\big)-2\cos\big((k-1)\alpha\big)}{(-1)^{\frac{j}{2}}(k-1)R}-\frac{2}{(-1)^{\frac{j}{2}}R}c_{k-1,j}^{\pi/2}(\alpha,R)-c_{k-2,j}^{\pi/2}(\alpha,R),$$

$$\mathcal{S}_0:=\frac{2\cos\big((\ell-1)\alpha\big)-2\cos\big((\ell-1)(\alpha+\pi/2)\big)}{(-1)^{\frac{j}{2}}(\ell-1)R}-\frac{2}{(-1)^{\frac{j}{2}}R}s_{\ell-1,j}^{\pi/2}(\alpha,R)-s_{\ell-2,j}^{\pi/2}(\alpha,R),$$

$$\mathcal{A}_0^{\pm}:=\operatorname{arccot}\left(\frac{1\pm R(\cos\alpha-\sin\alpha)}{\sqrt{1-R^2}}\right),\quad \mathcal{L}_0^{\pm}:=\ln\left(\frac{1\pm R\cos\alpha}{1\mp R\sin\alpha}\right).$$

(ii) *The functions* $\widetilde{c}_{\nu j}^{\pi/2}$ *and* $\widetilde{s}_{\nu j}^{\pi/2}$ *defined in* (17) *for* $\nu\in\{k,\ell\}$ *are*

$$\widetilde{c}_{\nu j}^{\pi/2}(\alpha,R)=\begin{cases}\dfrac{2}{\sqrt{1-R^2}}\mathcal{A}_1^-(\alpha,R), & \nu=0,\quad j=4m+1,\\ -\dfrac{2}{\sqrt{1-R^2}}\mathcal{A}_1^+(\alpha,R), & \nu=0,\quad j=4m+3,\\ -\dfrac{1}{R}\mathcal{L}_1^-(\alpha,R), & \nu=1,\quad j=4m+1,\\ \dfrac{1}{R}\mathcal{L}_1^+(\alpha,R), & \nu=1,\quad j=4m+3,\\ \mathcal{C}_1(\alpha,R), & \nu\geq 2,\quad j=4m+1 \quad \text{or} \quad j=4m+3,\end{cases}$$

$$\widetilde{s}_{\nu j}^{\pi/2}(\alpha,R)=\begin{cases}0, & \nu=0,\quad j=4m+1 \quad \text{or} \quad j=4m+3,\\ -\dfrac{\pi}{2R}+\dfrac{1}{R}\widetilde{c}_{0,j}^{\pi/2}(\alpha,R), & \nu=1,\quad j=4m+1,\\ \dfrac{\pi}{2R}+\dfrac{1}{R}\widetilde{c}_{0,j}^{\pi/2}(\alpha,R), & \nu=1,\quad j=4m+3,\\ \mathcal{S}_1(\alpha,R), & \nu\geq 2,\quad j=4m+1 \quad \text{or} \quad j=4m+3,\end{cases}$$

where

$$\mathcal{C}_1:=\frac{2\cos\big((\nu-1)(\alpha+\pi/2)\big)-2\cos\big((\nu-1)\alpha\big)}{(-1)^{\frac{j+1}{2}}(\nu-1)R}+\frac{2}{(-1)^{\frac{j+1}{2}}R}\widetilde{s}_{\nu-1,j}^{\pi/2}(\alpha,R)+\widetilde{c}_{\nu-2,j}^{\pi/2}(\alpha,R),$$

$$\mathcal{S}_1 := \frac{2\sin\big((\nu-1)(\alpha+\pi/2)\big) - 2\sin\big((\nu-1)\alpha\big)}{(-1)^{\frac{j+1}{2}}(\nu-1)R} - \frac{2}{(-1)^{\frac{j+1}{2}}R}\widetilde{c}^{\,\pi/2}_{\nu-1,j}(\alpha,R) + \widetilde{s}^{\,\pi/2}_{\nu-2,j}(\alpha,R),$$

$$\mathcal{L}_1^{\pm} := \ln\left(\frac{1\pm R\cos\alpha}{1\pm R\sin\alpha}\right), \quad \mathcal{A}_1^{\pm} := \arctan\left(\frac{R\pm\tan\frac{\alpha}{2}}{\sqrt{1-R^2}}\right) - \arctan\left(\frac{R\pm\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{\sqrt{1-R^2}}\right).$$

Proof. The expressions of the integrals $c^{\pi/2}_{kj}(\alpha,R)$ and $\widetilde{c}^{\,\pi/2}_{\nu j}(\alpha,R)$ for $k=0,1$ and $\nu=0,1$ can be obtained by direct integration, and also have been verified by the algebraic manipulators, such as Mathematica or Maple.

For $k\geq 2$ we do the following calculations:

$$\begin{aligned}
\int_0^{\pi/2}\cos\left((k-1)(\theta+\alpha)\right)\mathrm{d}\theta &= \int_0^{\pi/2}\frac{\cos\big((k-1)(\theta+\alpha)\big)\big(1+(-1)^{\frac{j}{2}}R\cos(\theta+\alpha)\big)}{1+(-1)^{\frac{j}{2}}R\cos(\theta+\alpha)}\mathrm{d}\theta\\
&= c^{\frac{\pi}{2}}_{k-1,j}(\alpha,R) + (-1)^{\frac{j}{2}}R\int_0^{\pi/2}\frac{\cos\big((k-1)(\theta+\alpha)\big)\cos(\theta+\alpha)}{1+(-1)^{\frac{j}{2}}R\cos(\theta+\alpha)}\mathrm{d}\theta\\
&= c^{\frac{\pi}{2}}_{k-1,j}(\alpha,R) + (-1)^{\frac{j}{2}}\frac{R}{2}\int_0^{\pi/2}\frac{\cos\big(k(\theta+\alpha)\big)}{1+(-1)^{\frac{j}{2}}R\cos(\theta+\alpha)}\mathrm{d}\theta\\
&\quad + (-1)^{\frac{j}{2}}\frac{R}{2}\int_0^{\pi/2}\frac{\cos\big((k-2)(\theta+\alpha)\big)}{1+(-1)^{\frac{j}{2}}R\cos(\theta+\alpha)}\mathrm{d}\theta\\
&= c^{\frac{\pi}{2}}_{k-1,j}(\alpha,R) + (-1)^{\frac{j}{2}}\frac{R}{2}\left(c^{\frac{\pi}{2}}_{k,j}(\alpha,R) + c^{\frac{\pi}{2}}_{k-2,j}(\alpha,R)\right).
\end{aligned}$$

So the recurrence for the function $c^{\frac{\pi}{2}}_{k,j}(\alpha,R)$ is proved.

For $\nu\geq 2$ the recurrence for the function $\widetilde{c}^{\,\pi/2}_{\nu j}(\alpha,R)$ follows doing the next calculations:

$$\begin{aligned}
\int_0^{\pi/2}\sin\big((\nu-1)(\theta+\alpha)\big)\mathrm{d}\theta &= \int_0^{\pi/2}\frac{\sin\big((\nu-1)(\theta+\alpha)\big)\big(1+(-1)^{\frac{j+1}{2}}R\sin(\theta+\alpha)\big)}{1+(-1)^{\frac{j+1}{2}}R\sin(\theta+\alpha)}\mathrm{d}\theta\\
&= \widetilde{s}^{\frac{\pi}{2}}_{\nu-1,j}(\alpha,R) + (-1)^{\frac{j+1}{2}}R\int_0^{\pi/2}\frac{\sin\big((\nu-1)(\theta+\alpha)\big)\sin(\theta+\alpha)}{1+(-1)^{\frac{j+1}{2}}R\sin(\theta+\alpha)}\mathrm{d}\theta\\
&= \widetilde{s}^{\frac{\pi}{2}}_{\nu-1,j}(\alpha,R) + (-1)^{\frac{j+1}{2}}\frac{R}{2}\int_0^{\pi/2}\frac{\cos\big((\nu-2)(\theta+\alpha)\big)}{1+(-1)^{\frac{j+1}{2}}R\sin(\theta+\alpha)}\mathrm{d}\theta\\
&\quad - (-1)^{\frac{j+1}{2}}\frac{R}{2}\int_0^{\pi/2}\frac{\cos\big(\nu(\theta+\alpha)\big)}{1+(-1)^{\frac{j+1}{2}}R\sin(\theta+\alpha)}\mathrm{d}\theta\\
&= \widetilde{s}^{\frac{\pi}{2}}_{\nu-1,j}(\alpha,R) + (-1)^{\frac{j+1}{2}}\frac{R}{2}\left(\widetilde{c}^{\frac{\pi}{2}}_{\nu-2,j}(\alpha,R) - \widetilde{c}^{\frac{\pi}{2}}_{\nu,j}(\alpha,R)\right).
\end{aligned}$$

Similarly we can deduce the recurrences for the functions $s^{\pi/2}_{\ell j}(\alpha,R)$ and $\widetilde{s}^{\,\pi/2}_{\nu j}(\alpha,R)$, in these cases by direct integration we obatin the expressions of the integrals $s^{\pi/2}_{\ell j}(\alpha,R)$ and $\widetilde{s}^{\,\pi/2}_{\nu j}(\alpha,R)$ for $\ell=0,1$ and $\nu=0,1$, while for $\ell\geq 2$ and $\nu\geq 2$ we should consider the integrals $\int_0^{\pi/2}\sin\big((\ell-1)(\theta+\alpha)\big)\mathrm{d}\theta$ and $\int_0^{\pi/2}\cos\big((\nu-1)(\theta+\alpha)\big)\mathrm{d}\theta$ respectively. Hence the proof of this lemma is completed. □

Proposition 13. *For $0<\alpha<\pi/2$ and $0<R<1$ the functions $\widetilde{\mathcal{I}}^{3\pi/2}_k$ and $\widetilde{\mathcal{J}}^{3\pi/2}_\ell$ in* (13) *are*

$$\widetilde{\mathcal{I}}^{3\pi/2}_k(\alpha,R) = \begin{cases} (-1)^{\frac{k}{2}}c^{\pi/2-\alpha}_k(\alpha,R), & k=4m \quad \textit{or} \quad k=4m+2,\\ (-1)^{\frac{k-1}{2}}s^{\pi/2-\alpha}_k(\alpha,R), & k=4m+1 \quad \textit{or} \quad k=4m+3,\end{cases}$$

$$\widetilde{\mathcal{J}}_{\ell}^{3\pi/2}(\alpha,R)=\begin{cases}(-1)^{\frac{\ell}{2}}s_{\ell}^{\pi/2-\alpha}(\alpha,R), & \ell=4m \quad or \quad \ell=4m+2,\\ (-1)^{\frac{\ell+1}{2}}c_{\ell}^{\pi/2-\alpha}(\alpha,R), & \ell=4m+1 \quad or \quad \ell=4m+3,\end{cases}$$

where m is a natural number, $\nu\in\{k,\ell\}$.

The proof of Proposition 13 is omitted here because its results can be deduced directly after simple calculations. In the following lemma we give the recurrent formulas in terms of k and ℓ for the functions $c_{\nu}^{\pi/2-\alpha}$ and $s_{\nu}^{\pi/2-\alpha}$ defined in (17).

Lemma 14. *For $0<\alpha<\pi/2$ and $0<R<1$ the functions $c_{\nu}^{\pi/2-\alpha}$ and $s_{\nu}^{\pi/2-\alpha}$ defined in (17) for $\nu\in\{k,\ell\}$ write as*

$$c_{\nu}^{\pi/2-\alpha}(\alpha,R)=\begin{cases}\dfrac{2}{\sqrt{1-R^2}}\mathcal{A}_3(\alpha,R), & \nu=0,\\ \dfrac{1}{R}\ln\left(\dfrac{1+R}{1+R\cos\alpha}\right), & \nu=1,\\ \dfrac{2\sin\big(\nu(\pi/2)\big)-2\cos\big((\nu-1)\alpha\big)}{(\nu-1)R}+\dfrac{2}{R}s_{\nu-1}^{\pi/2-\alpha}(\alpha,R)+c_{\nu-2}^{\pi/2-\alpha}(\alpha,R), & \nu\geq 2,\end{cases}$$

$$s_{\nu}^{\pi/2-\alpha}(\alpha,R)=\begin{cases}0, & \nu=0,\\ \dfrac{\pi-2\alpha}{2R}-\dfrac{1}{R}c_0^{\pi/2-\alpha}(\alpha,R), & \nu=1,\\ -\dfrac{2\cos\big(\nu(\pi/2)\big)+2\sin\big((\nu-1)\alpha\big)}{(\nu-1)R}-\dfrac{2}{R}c_{\nu-1}^{\pi/2-\alpha}(\alpha,R)+s_{\nu-2}^{\pi/2-\alpha}(\alpha,R), & \nu\geq 2,\end{cases}$$

where $\mathcal{A}_3:=\arctan\left(\sqrt{\frac{1+R}{1-R}}\right)-\arctan\left(\frac{R+\tan\frac{\alpha}{2}}{\sqrt{1-R^2}}\right)$.

Proof. For $\nu=0,1$, direct integrations give the expressions for the function $c_{\nu}^{\pi/2-\alpha}(\alpha,R)$. For $\nu\geq 2$,

$$\begin{aligned}&\int_0^{\pi/2-\alpha}\sin\big((\nu-1)(\theta+\alpha)\big)\mathrm{d}\theta\\ &=\int_0^{\pi/2-\alpha}\frac{\sin\big((\nu-1)(\theta+\alpha)\big)\big(1+R\sin(\theta+\alpha)\big)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta\\ &=s_{\nu-1}^{\pi/2-\alpha}(\alpha,R)+R\int_0^{\pi/2-\alpha}\frac{\sin\big((\nu-1)(\theta+\alpha)\big)\sin\theta}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta\\ &=s_{i-1}^{\pi/2-\alpha}(\alpha,R)+\frac{R}{2}\left(\int_0^{\pi/2-\alpha}\frac{\cos\big((\nu-2)(\theta+\alpha)\big)}{1+R(\sin(\theta+\alpha)}\mathrm{d}\theta-\int_0^{\pi/2-\alpha}\frac{\cos\big(\nu(\theta+\alpha)\big)}{1+R\sin(\theta+\alpha)}\mathrm{d}\theta\right)\\ &=s_{\nu-1}^{\pi/2-\alpha}(\alpha,R)+\frac{R}{2}\left(c_{\nu-2}^{\pi/2-\alpha}(\alpha,R)-c_{\nu}^{\pi/2-\alpha}(\alpha,R)\right).\end{aligned}$$

It follows that the recurrence for the function $c_{\nu}^{\pi/2-\alpha}(\alpha,R)$ is obtained directly.

Similarly we can deduce the recurrence for the function $s_{\nu}^{\pi/2-\alpha}(\alpha,R)$ as in Lemma 14 for $\nu\geq 2$, while in this case we should consider the integral $\int_0^{\pi/2-\alpha}\cos\big((\nu-1)(\theta+\alpha)\big)\mathrm{d}\theta$ instead of $\int_0^{\pi/2-\alpha}\sin\big((\nu-1)(\theta+\alpha)\big)\mathrm{d}\theta$. Moreover, the expressions of the integral $s_{\nu}^{\pi/2-\alpha}(\alpha,R)$ are obtained by direct integration when $\nu=0,1$. Hence the proof of this lemma is completed. □

In the following we will introduce how to use the above recurrences to deduce the expression of the first averaged function $f(\alpha,R)$ defined in (10) for $\alpha\in(0,\pi/2)$, and then determine the minimum number of functions whose nontrivial linear combination generates that averaged function.

Lemma 15. *For $0 < \alpha < \pi/2$ and $0 < R < 1$ the averaged function $f(\alpha, R)$ in (10) can be expressed into the following form*

$$
\begin{aligned}
f(\alpha, R) = {} & k_2 f_2(\alpha, R) + k_5 f_5(\alpha, R) + k_6 f_6(\alpha, R) \\
& + \sum_{i=9}^{12} k_i f_i(\alpha, R) + k_{15} f_{15}(\alpha, R) + k_{16} f_{16}(\alpha, R),
\end{aligned}
\tag{18}
$$

where the functions f_i's are given in the Appendix A,

$$
\begin{aligned}
& k_0 := a_{00}, \qquad k_2 := b_{00} - d_{00} + (a_{00} - c_{00}) n_{00}, \\
& k_5 := b_{10} - b_{02} + d_{02} - d_{10} - (b_{11} - d_{11}) n_{00} + a_{00} n_{01} \\
& \qquad + c_{00} n_{02} + a_{10} n_{03} + c_{10} n_{04} + b_{01} n_{05} + d_{01} n_{06}, \\
& k_6 := (a_{00} + a_{10} + b_{11} - c_{00} - c_{10} - d_{11})\, n_{00} + a_{01} + b_{02} - c_{01} - d_{02}, \\
& k_9 := a_{20} - b_{11}, \qquad k_{10} := c_{20} - d_{11}, \qquad k_{11} := b_{20} - d_{20}, \\
& k_{12} := a_{11} - b_{02} - c_{11} + d_{02}, \quad k_{15} := a_{02}, \qquad k_{16} := c_{02}.
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
& n_{00} := \cot(\alpha/2), && n_{01} := \frac{3\alpha - 2\sin\alpha - \sin\alpha\cos\alpha}{\sin^2\alpha}, \\
& n_{02} := \csc\alpha\,(2 + \cos\alpha + (6\pi - 3\alpha)\csc\alpha), && n_{03} := \frac{\alpha - \sin\alpha}{\sin^2\alpha}, \\
& n_{04} := \frac{2\pi}{\sin^2\alpha} - n_{03}, \quad n_{05} := \frac{\alpha - \sin\alpha\cos\alpha}{\sin^2\alpha}, && n_{06} := \cot\alpha + (2\pi - \alpha)\csc^2\alpha.
\end{aligned}
\tag{20}
$$

Proof. For the first and second parts of the first averaged function $f(\alpha, R)$ in (10) can be calculated applying the recurrent formulas of the integrals $\mathcal{I}_k^{\alpha}(\alpha, R)$ and $\mathcal{J}_\ell^{\alpha}(\alpha, R)$ in (15) for $k = 0, 1, 2, 3$ and $\ell = 0, 1, 2, 3$. For the third and fourth parts of the first averaged function $f(\alpha, R)$ in (10), by Proposition 9 they can be reduced to

$$
\begin{aligned}
& \sum_{k=0}^{3} \mathcal{F}_{1k}^{-}(R) \mathcal{I}_k^{2\pi-\alpha}(\alpha, R) + \sum_{\ell=0}^{3} \mathcal{F}_{2\ell}^{-}(R) \mathcal{J}_\ell^{2\pi-\alpha}(\alpha, R) \\
= & \sum_{k=0}^{3} \mathcal{F}_{1k}^{-}(R) \left(\sum_{j=0}^{2} \mathcal{I}_k^{j(\pi/2)}(\alpha, R) + \widetilde{\mathcal{I}}_k^{3\pi/2}(\alpha, R) \right) \\
& + \sum_{\ell=0}^{3} \mathcal{F}_{2\ell}^{-}(R) \left(\sum_{j=0}^{2} \mathcal{J}_\ell^{j(\pi/2)}(\alpha, R) + \widetilde{\mathcal{J}}_\ell^{3\pi/2}(\alpha, R) \right).
\end{aligned}
$$

Then using Propositions 11 and 13, and the recurrent formulas given in Lemmas 12 and 14, we can obtain the explicit expressions for the integrals $\mathcal{I}_k^{j(\pi/2)}(\alpha, R)$ and $\mathcal{J}_\ell^{j(\pi/2)}(\alpha, R)$, and $\widetilde{\mathcal{I}}_k^{3\pi/2}(\alpha, R)$ and $\widetilde{\mathcal{J}}_\ell^{3\pi/2}(\alpha, R)$ when $k, \ell = 0, 1, 2, 3$, and further we can derive the third and fourth parts of the first averaged function $f(\alpha, R)$ in (10). Based on the above calculations we add the obtained results and deduce that the first averaged function $f(\alpha, R)$ in (10) becomes

$$
\begin{aligned}
f(\alpha, R) = {} & a_{00} f_0(\alpha, R) + c_{00} f_1(\alpha, R) + (b_{00} - d_{00}) f_2(\alpha, R) + a_{10} f_3(\alpha, R) \\
& + c_{10} f_4(\alpha, R) + (b_{10} - d_{10}) f_5(\alpha, R) + (a_{01} - c_{01}) f_6(\alpha, R) \\
& + b_{01} f_7(\alpha, R) + d_{01} f_8(\alpha, R) + a_{20} f_9(\alpha, R) + c_{20} f_{10}(\alpha, R) \\
& + (b_{20} - d_{20}) f_{11}(\alpha, R) + (a_{11} - c_{11}) f_{12}(\alpha, R) + b_{11} f_{13}(\alpha, R) \\
& + d_{11} f_{14}(\alpha, R) + a_{02} f_{15}(\alpha, R) + c_{02} f_{16}(\alpha, R) + (b_{02} - d_{02}) f_{17}(\alpha, R),
\end{aligned}
$$

where the functions f_i's are given in the Appendix A. From the expressions of the functions f_i's, we can check that

$$\begin{aligned} f_0 =& n_{00}(f_2+f_6)+n_{01}f_5, & f_1 =& -n_{00}(f_2+f_6)+n_{02}f_5, & f_3 =& n_{03}f_5+n_{00}f_6,\\ f_4 =& n_{04}f_5-n_{00}f_6, & f_7 =& n_{05}f_5, & f_8 =& n_{06}f_5,\\ f_{13} =& n_{00}(f_6-f_5)-f_9, & f_{14} =& (f_5-f_6)n_{00}-f_{10}, & f_{17} =& -f_5+f_6-f_{12}, \end{aligned}$$

where the coefficients n_{0j}'s are given in (20). It follows that the first averaged function $f(\alpha, R)$ in (10) can be written into the form as (18), and the proof of this proposition is completed. □

4. Proof of Theorem 1

Having deduced the expression of the first averaged function $f(\alpha, R)$ in (10) as given in (18), in this section we discuss the number of isolated zeros of this averaged function, which provide the maximum number of limit cycles for the discontinuous piecewise quadratic polynomial differential systems (3) when $|\varepsilon| \neq 0$ is sufficiently small that can be obtained using the averaging theory of first order.

The following proposition is devoted to determine the lower bound of the number of isolated zeros of the function $f(\alpha, R)$ in (18).

Proposition 16. *For $\alpha \in (0, \pi/2)$ the maximum number of simple zeros of the first averaged function $f(\alpha, R)$ given in* (18) *with respect to the variable R is at least* 8 *in the interval* $(0, 1)$.

Proof. In order to determine the maximum number of simple zeros of the first averaged function $f(\alpha, R)$ given in (18) with respect to the variable R when $\alpha \in (0, \pi/2)$, we calculate the Taylor expansions of the functions $f_i, i \in \{2, 5, 6, 9, 10, 11, 12, 15, 16\}$ of (18) in the variable R at $R = 0$, and then we take the coefficients of the monomials R^j for $j = 0, 1, 2, \cdots, 8$ in the obtained nine Taylor expansions forming a 9×9 matrix with nine columns. One can check that the determinant of this matrix is $-(2/4964125)\pi^2 \sin^{20}(\alpha/2) \sin^{12} \alpha \neq 0$, it follows that the functions f_i's are linearly independent.

Moreover, for $i \in \{2, 5, 6, 9, 10, 11, 12, 15, 16\}$ the coefficient k_i given in (19) of the function f_i in (18) are independent because the rank of the Jacobian matrix of these coefficients with respect to the variables a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, c_{00}, c_{10}, c_{01}, c_{20}, c_{11}, c_{02}, d_{00}, d_{10}, d_{01}, d_{20}, d_{11} and d_{02} is 9. Notice that the functions f_5 and f_6 in (18) have constant signs in $(0, 1)$ when $\alpha \in (0, \pi/2)$. It follows from Lemma 5 that there exist coefficients $(a^*, b^*, c^*, d^*) \in \mathbb{R}$ where

$$\begin{aligned} a^* :=& (a^*_{00}, a^*_{10}, a^*_{01}, a^*_{20}, a^*_{11}, a^*_{02}), & b^* :=& (b^*_{00}, b^*_{10}, b^*_{01}, b^*_{20}, b^*_{11}, b^*_{02}),\\ c^* :=& (c^*_{00}, c^*_{10}, c^*_{01}, c^*_{20}, c^*_{11}, c^*_{02}), & d^* :=& (d^*_{00}, d^*_{10}, d^*_{01}, d^*_{20}, d^*_{11}, d^*_{02}) \end{aligned}$$

such that there always exists a nontrivial function generated by the functions f_i for $i \in \{2, 5, 6, 9, 10, 11, 12, 15, 16\}$ with at least 8 simple zeros. From (18) we know that the averaged function $f(\alpha, R)$ is a nontrivial linear combination of the functions f_i's, and then we can conclude that for $\alpha \in (0, \pi/2)$ the maximum number of simple zeros of the first averaged function $f(\alpha, R)$ in (18) with respect to the variable R is at least 8 in the interval $(0, 1)$. Therefore this completes the proof of Lemma 16. □

In the following, we estimate the sharp upper bound of the number of simple zeros for the first averaged function $f(\alpha, R)$ in (18), we will discuss the number of simple zeros of the function $F(\alpha, R) = Rf(\alpha, R)$ with respect to the variable R, which has the same number of zeros as $f(\alpha, R)$ with respect to the variable R in the interval $(0, 1)$.

As it was indicated in Section 2.2 the Chebyshev theory is an effective method to estimate an upper bound for the number of zeros of the function $F(\alpha, R)$, and from Lemma 8 it is enough to prove that the corresponding Wronskians formed by the functions generating the function $F(\alpha, R)$ are non-vanishing. However, the expression of $F(\alpha, R)$ contains some elementary functions, such as $\sqrt{1 - R^2}$, $(1 - R^2)^{3/2}$, $\ln(1 + R\cos\alpha)/(1 + R)$ and some inverse trigonometric functions, therefore it is a challenging work to calculate the Wronskians and to determine whether these Wronskians vanish or not. Hence it is hard to estimate the exact upper bound of the number of isolated zeros of $F(\alpha, R)$.

In order to reduce these difficulties, enlightened by the works in [5, p.321] and [13, p.72] we consider

$$\left(\frac{F(\alpha, R)}{(1-R^2)^{3/2}}\right)' = \frac{G(\alpha, R)}{(1-R^2)^{5/2}(1+R\cos\alpha)}, \tag{21}$$

in which

$$G(\alpha, R) = \sum_{i=0}^{8} m_i g_i(\alpha, R), \tag{22}$$

where the coefficients m_i's are given in the Appendix B, and

$$\begin{aligned} &g_i := R^i, \text{ for } i = 0, 1, \cdots, 4, && g_5 := R(1+R\cos\alpha)\ln\left(\frac{1+R\cos\alpha}{1+R}\right), \\ &g_6 := R^3(1+R\cos\alpha)\ln\left(\frac{1+R\cos\alpha}{1+R}\right), && g_7 := R\sqrt{1-R^2}(1+R\cos\alpha), \\ &g_8 := R\sqrt{1-R^2}(1+R\cos\alpha)\mathcal{A}(\alpha, R), \end{aligned} \tag{23}$$

where $\mathcal{A}(\alpha, R)$ is given in (16). It follows from (21) that

$$F(\alpha, R) = (1-R^2)^{3/2}\int_0^R \frac{G(\alpha, \xi)}{(1-\xi^2)^{5/2}(1+\xi\cos\alpha)}\mathrm{d}\xi, \tag{24}$$

which has at most as much zeros as $G(\alpha, R)$ in R by the Rolle's Theorem.

The following result is concerned on determining the exact number of simple zeros of the function $G(\alpha, R)$ defined in (22) with respect to the variable R when $R \in (0, 1)$ and $\alpha \in (0, \pi/2)$.

Lemma 17. *For $0 < \alpha < \pi/2$ and $0 < R < 1$ the maximum number of simple zeros of the function $G(\alpha, R)$ defined in (22) with respect to the variable R is exactly 8. Then from (24) the maximum number of simple zeros of the function $F(\alpha, R)$ in R is at most 8 in the interval $(0, 1)$.*

Proof. We firstly prove the first part of Lemma 17. Similarly to the proof of Proposition 17 we obtain that for $\alpha \in (0, \pi/2)$ and $R \in (0, 1)$. the maximum number of simple zeros of the function $G(\alpha, R)$ in (22) with respect to the variable R is at least 8.

In the following we prove that this maximum number is at most 8. In fact, it suffices to claim that the ordered set $\mathcal{G} := (g_0, g_1, \cdots, g_8)$, where g_i's are the functions given in (23), forms an ECT-system, which from Lemma 8 is equivalent to check that all the Wronskians $W_k(\alpha, R) \neq 0$ when $\alpha \in (0, \pi/2)$ and $R \in (0, 1)$ for $k = 0, 1, \cdots, 8$. Straightforward calculations show that

$$\begin{aligned} &W_0(\alpha, R) = 1, \quad W_1(\alpha, R) = 1, \quad W_2(\alpha, R) = 2, \quad W_3(\alpha, R) = 12, \\ &W_4(\alpha, R) = 288, \qquad W_5(\alpha, R) = \frac{576(1-\cos\alpha)^2}{(1+R)^5(1+R\cos\alpha)^4}\widetilde{W}_5(\alpha, R), \\ &W_6(\alpha, R) = \frac{6912(1-\cos\alpha)^6}{(1+R)^{10}(1+R\cos\alpha)^8}\widetilde{W}_6(\alpha, R), \\ &W_7(\alpha, R) = \frac{9720(1-\cos\alpha)^6}{(1-R^2)^{13/2}(1+R)^{10}(1+R\cos\alpha)^{10}}\widetilde{W}_7(\alpha, R), \\ &W_8(\alpha, R) = -\frac{9331200(1-\cos\alpha)^8\sin^5\alpha}{(1-R^2)^{15/2}(1+R)^{12}(1+R\cos\alpha)^{15}}\widetilde{W}_8(\alpha, R), \end{aligned} \tag{25}$$

where

$$\widetilde{W}_5 := \sum_{i=0}^{3}\omega_{5i}(\cos\alpha)R^i, \ \widetilde{W}_6 := \sum_{i=0}^{4}\omega_{6i}(\cos\alpha)R^i,$$

$$\widetilde{W}_7 := \sum_{i=0}^{9} \omega_{7i}(\cos\alpha)R^i, \ \widetilde{W}_8 := 8\sum_{i=0}^{6} \omega_{8i}(\cos\alpha)R^i,$$

and ω_{5i}'s, ω_{6i}'s, ω_{7i}'s and ω_{8i}'s are polynomials in $\cos\alpha$ given in the Appendix C.

Since we can easily check that for $k = 0, 1, \cdots, 4$ the Wronskians $W_k(\alpha, R)$ in (25) are non-vanishing in $(0, \pi/2) \times (0, 1)$, we discuss whether the Wronskians $W_j(\alpha, R)$ for $j = 5, 6, 7, 8$ on $(0, \pi/2) \times (0, 1)$ vanish. For the Wronskian $W_5(\alpha, R)$ defined in (25), it suffices to discuss the sign of the function $\widetilde{W}_5(\alpha, R)$ because its factor $576(1 - \cos\alpha)^2/\left((1 + R)^5(1 + R\cos\alpha)^4\right) > 0$ on $(0, \pi/2) \times (0, 1)$. Notice that the coefficients ω_{5i}'s given in the Appendix C of $\widetilde{W}_5(\alpha, R)$ are positive on $(0, \pi/2) \times (0, 1)$. It follows that $W_5(\alpha, R) > 0$ on $(0, \pi/2) \times (0, 1)$.

Similarly we can obtain that $W_6(\alpha, R) > 0$, $W_7(\alpha, R) > 0$ and $W_8(\alpha, R) < 0$ on $(0, \pi/2) \times (0, 1)$. Therefore we have that for $k = 0, 1, \cdots, 8$ the Wronskians $W_k(\alpha, R)$ are non-vanishing on $(0, \pi/2) \times (0, 1)$, and then the claim above (25) is proved.

It follows from the claim above (25) that the ordered set $\mathcal{G}$ forms an ECT-system when $R \in (0, 1)$ and $\alpha \in (0, \pi/2)$. Then from Lemma 6 and Remark 7 the number of simple zeros for each nontrivial function generated by the functions g_i, $i \in \{0, 1, \cdots, 8\}$ is at most 8, and this number can be realized. Notice that for $\alpha \in (0, \pi/2)$ the function $G(\alpha, R)$ in (22) is a nontrivial linear combination of the functions g_i's. Therefore for $\alpha \in (0, \pi/2)$ the number of simple zeros of the function $G(\alpha, R)$ in (22) with respect to the variable R is at most 8 in $(0, 1)$, and this number can be reached. Hence the first part of Lemma 17 holds.

For the second part of Lemma 17 we obtain the results directly by using the relation (24). In short the proof of this lemma is completed. □

Next we turn to give the proof of Theorem 1.

Proof of Theorem 1. From Proposition 16 we know that for $\alpha \in (0, \pi/2)$ the maximum number of simple zeros of the first averaged function $f(\alpha, R)$ in (18) with respect to the variable R is at least 8 in the interval $(0, 1)$. On the other hand, it follows from $f(\alpha, R) = F(\alpha, R)/R$ and together with Lemma 17 that the first averaged function $f(\alpha, R)$ given by (18) with respect to the variable R has at most 8 simple zeros in the interval $(0, 1)$. Thus we can conclude that for $\alpha \in (0, \pi/2)$ the maximum number of simple zeros of the first averaged function $f(\alpha, R)$ in (18) with respect to the variable R is exactly 8 in the interval $(0, 1)$. Moreover by Lemma 3 we obtain that the discontinuous piecewise quadratic polynomial differential systems (3) can exactly produce 8 hyperbolic limit cycles when $|\varepsilon| \neq 0$ is sufficiently small. □

Finally we give two numerical examples to confirm the results of our Theorem 1. Consider system (3) with $\alpha = \pi/4$ and

$$\begin{aligned}
&a_{00} = -1.00006, \quad a_{01} = 13.5715, \quad a_{11} = 99911.1, \quad a_{02} = -167942,\\
&b_{20} = -17735.7, \quad b_{11} = 136164, \quad b_{02} = -180404, \quad c_{02} = 0.001,\\
&d_{11} = 1.00019, \qquad a_{10} = a_{20} = b_{00} = b_{10} = b_{01} = 1, \quad c_{00} = -1,\\
&c_{10} = c_{01} = c_{20} = c_{11} = d_{00} = d_{10} = d_{01} = d_{20} = d_{02} = 1.
\end{aligned} \tag{26}$$

It follows that the discontinuous boundary Σ_α defined in (2) becomes

$$\Sigma_{\pi/4} = \Sigma_0 \cup \left\{(x, y) \in \mathbb{R}^2 : y = x, y \geq 0\right\},$$

and the polynomial perturbations p_i's and q_i's defined below (3) reduce to the form

$$\begin{aligned}
p_1(x, y) &= -1.00006 + x + 13.5715y + x^2 + 99911.1xy - 167942y^2,\\
q_1(x, y) &= 1 + x + y - 17735.7x^2 + 136164xy - 180404y^2,\\
p_2(x, y) &= -1 + x + y + x^2 + xy + 0.001y^2,
\end{aligned}$$

$$q_2(x,y) = 1 + x + y + x^2 + 1.00019xy + y^2.$$

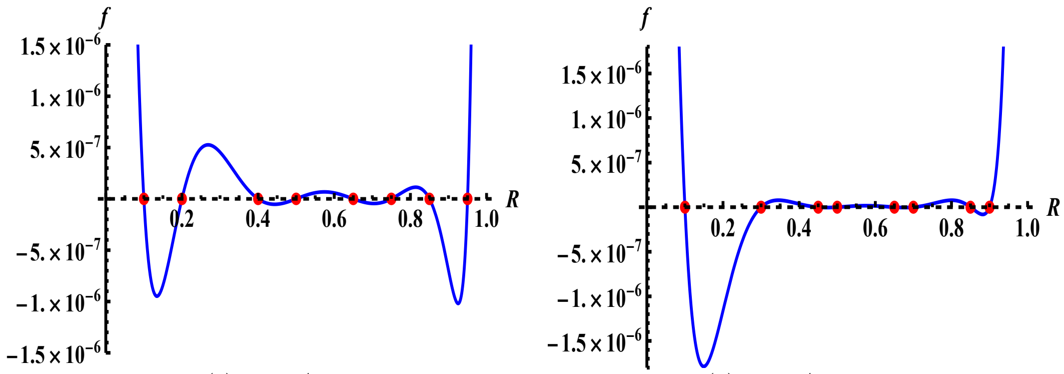
Then computing the averaged function $f(R)$ in (18) for this system we have

$$\begin{aligned} f(R) = {} & 198434\frac{1}{R}\big(-0.125784 + 0.500817R + 0.356753R^2 - 0.745939R^3 \\ & -3.16638\times 10^{-8}\sqrt{1-R^2}(1.19164 - R^2) \\ & +1.69268\sqrt{1-R^2}(0.18923 - R^2)\arctan\left(\sqrt{\frac{1-R}{1+R}}\tan\frac{\pi}{8}\right) \\ & +1.41264(0.936726 - R^2)\ln\left(\frac{\sqrt{2}+R}{\sqrt{2}(1+R)}\right)\bigg). \end{aligned} \tag{27}$$

Straightforward calculations show that the zeros of $f(R)$ in (27) are

$$R_1 = \frac{1}{10},\quad R_2 = \frac{1}{5},\quad R_3 = \frac{2}{5},\quad R_4 = \frac{1}{2},\quad R_5 = \frac{13}{20},\quad R_6 = \frac{3}{4},\quad R_7 = \frac{17}{20},\quad R_8 = \frac{19}{20}$$

and $f'(R_i) \neq 0$ for $i = 1, \cdots, 7$, see Fig.1(a).



(a) $\alpha = \pi/4$ (b) $\alpha = \pi/3$.

FIGURE 1. The graphics of the averaged functions $f(R)$ in (27) and (28).

When $\alpha = \pi/3$ and

$$\begin{aligned} & a_{00} = -1.00005,\quad a_{01} = 9.38065,\quad a_{11} = 2980,\quad a_{02} = -3312.7, \\ & b_{20} = -1172.92,\quad b_{11} = 6267.14,\quad b_{02} = -5615.58,\quad c_{02} = 0.001,\quad d_{11} = 1.00022, \end{aligned}$$

and the other parameters are the same as (26), it follows that the discontinuous boundary Σ_α defined in (2) becomes

$$\Sigma_{\pi/3} = \Sigma_0 \cup \left\{(x,y) \in \mathbb{R}^2 : y = \sqrt{3}x, y \geq 0\right\},$$

and the polynomial perturbations p_i's and q_i's defined below (3) reduce to the form

$$\begin{aligned} p_1(x,y) &= -1.00005 + x + 9.38065y + x^2 + 2980xy - 3312.7y^2, \\ q_1(x,y) &= 1 + x + y - 1172.92x^2 + 6267.14xy - 5615.58y^2, \\ p_2(x,y) &= -1 + x + y + x^2 + xy + 0.001y^2, \\ q_2(x,y) &= 1 + x + y + x^2 + 1.00022xy + y^2. \end{aligned}$$

Therefore the averaged function $f(\alpha,R)$ in (18) is written into the form

$$
\begin{aligned}
f(R) = 1476.72\frac{1}{R}\big(&2.0944 + 0.780839R - 0.149695R^2 - 2.95424R^3\\
&-4.25483\times 10^{-6}\sqrt{1-R^2}(1.21941 - R^2)\\
&-4.48657\sqrt{1-R^2}(0.891549 + R^2)\arctan\left(\sqrt{\frac{1-R}{3(1+R)}}\right)\\
&+5.82073(0.863427 - R^2)\ln\left(\frac{1+R}{2(1+R)}\right)\Big),
\end{aligned}
\tag{28}
$$

where its zeros are

$$R_1=\frac{1}{10},\quad R_2=\frac{3}{10},\quad R_3=\frac{9}{20},\quad R_4=\frac{1}{2},\quad R_5=\frac{13}{20},\quad R_6=\frac{7}{10},\quad R_7=\frac{17}{20},\quad R_8=\frac{9}{10}$$

and $f'(R_i)\neq 0$ for $i=1,2,\cdots,7$, see Fig.1(b).

By Lemma 3 it follows that for $\varepsilon\neq 0$ small enough the above two discontinuous piecewise differential systems can have 8 isolated periodic solutions, which confirm that under the assumptions of Theorem 1 there are systems (3) with exactly 8 hyperbolic limit cycles bifurcating from the periodic orbits of the quadratic isochronous center (1) when $\varepsilon\neq 0$ is sufficiently small.

Acknowledgements

The first author is partially supported by the China Scholarship Council (No. 202206240081). The second author is supported by the Agencia Estatal de Investigación grant PID2022-136613NB-100, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Acadèmia de Ciències i Arts de Barcelona.

Appendices

In this section we provide the explicit expressions of the generated functions f_i's of the averaged function $f(\alpha,R)$ in (18), the coefficients that appear in the function $G(\alpha,R)$ in (22), and the coefficients that appear in Wronskians W_k's in (25), which will give an important information in determining the sign of these Wronskians.

A. **The generated functions of $f(\alpha,R)$ in (18).**

$$
\begin{aligned}
&f_0 := -\sin\alpha - \frac{R}{2}(3\alpha + \sin\alpha\cos\alpha) - R^2\sin\alpha,\\
&f_1 := \sin\alpha - \frac{1}{2}(6\pi - 3\alpha - \sin\alpha\cos\alpha)R + R^2\sin\alpha,\\
&f_2 := -1 + \cos\alpha - \frac{R}{2}\sin^2\alpha, \qquad f_3 := -\frac{1}{2}R(\alpha + \cos\alpha\sin\alpha) - R^2\sin\alpha,\\
&f_4 := -\frac{R}{2}(2\pi - \alpha - \cos\alpha\sin\alpha) + R^2\sin\alpha, \qquad f_5 := -\frac{1}{2}R\sin^2\alpha,\\
&f_6 := -\frac{1}{2}R\sin^2\alpha - (1-\cos\alpha)R^2, \qquad f_7 := -\frac{R}{2}(\alpha - \cos\alpha\sin\alpha),\\
&f_8 := -\frac{1}{2}R(2\pi - \alpha + \cos\alpha\sin\alpha),\\
&f_9 := -\frac{\alpha}{R} + \sin\alpha + \frac{1}{4}R(2\alpha - \sin(2\alpha)) - R^2\sin\alpha + \frac{2\sqrt{1-R^2}}{R}\mathcal{A}(\alpha,R),\\
&f_{10} := -\frac{2\pi-\alpha}{R} - \sin\alpha + \frac{1}{4}R(4\pi - 2\alpha + \sin(2\alpha)) + R^2\sin\alpha + \frac{2\sqrt{1-R^2}}{R}(\pi - \mathcal{A}(\alpha,R)),
\end{aligned}
$$

$$
\begin{aligned}
f_{11} &:= 1-\cos\alpha-\frac{1}{2}R\sin^2\alpha+\frac{1}{R}\ln\left(\frac{1+R\cos\alpha}{1+R}\right),\\
f_{12} &:= 1-\cos\alpha-\frac{1}{2}R\sin^2\alpha-(1-\cos\alpha)R^2+\left(\frac{1}{R}-R\right)\ln\left(\frac{1+R\cos\alpha}{1+R}\right),\\
f_{13} &:= \frac{\alpha}{R}-\sin\alpha-\frac{1}{2}R\left(\alpha-\cos\alpha\sin\alpha\right)+\frac{2\sqrt{1-R^2}}{R}\mathcal{A}(\alpha,R),\\
f_{14} &:= \frac{2\pi-\alpha}{R}+\sin\alpha-\frac{1}{2}R\left(2\pi-\alpha+\cos\alpha\sin\alpha\right)-\frac{2\sqrt{1-R^2}}{R}\left(\pi-\mathcal{A}(\alpha,R)\right),\\
f_{15} &:= \frac{\alpha}{R}-\sin\alpha-\frac{1}{2}R\left(3\alpha-\cos\alpha\sin\alpha\right)+R^2\sin\alpha-\frac{2(1-R^2)^{3/2}}{R}\mathcal{A}(\alpha,R),\\
f_{16} &:= \frac{2\pi-\alpha}{R}+\sin\alpha-\frac{1}{2}R\left(6\pi-3\alpha+\cos\alpha\sin\alpha\right)\\
&\quad -R^2\sin\alpha-\frac{2(1-R^2)^{3/2}}{R}\left(\pi-\mathcal{A}(\alpha,R)\right),\\
f_{17} &:= -1+\cos\alpha+\frac{1}{2}R\sin^2\alpha+\left(\frac{1}{R}-R\right)\ln\left(\frac{1+R\cos\alpha}{1+R}\right),
\end{aligned}
$$

where $\mathcal{A}(\alpha,R)$ is given in (16).

B. **Coefficients of the function $G(\alpha,R)$ in (22).**

$$
\begin{aligned}
m_0 &:= -k_2(1-\cos\alpha),\\
m_1 &:= -k_2(\cos\alpha-\cos(2\alpha))-(k_5+k_6)\sin^2\alpha-k_{10}(4\pi-2\alpha)-2k_9\alpha,\\
m_2 &:= -(1/4)k_2(8-7\cos\alpha-\cos(3\alpha))-(1/4)k_6(12-11\cos\alpha-\cos(3\alpha))\\
&\quad -(1/2)k_9\cos\alpha\big(4\alpha+\sin(2\alpha)\big)-(1/2)k_{10}\cos\alpha\big(8\pi-4\alpha-\sin(2\alpha)\big)\\
&\quad +k_{11}(2+\cos\alpha)(1-\cos\alpha)^2-(1/2)(k_5+k_{12})\sin\alpha\sin(2\alpha)\\
&\quad -(k_{15}-k_{16})\sin^3\alpha,\\
m_3 &:= -(1/2)k_2(1+5\cos\alpha)(1-\cos\alpha)-(1/2)k_6(1+7\cos\alpha)(1-\cos\alpha)\\
&\quad +(1/2)k_9(\alpha-3\cos\alpha\sin\alpha)+(1/4)k_{10}\big(4\pi-2\alpha+3\sin(2\alpha)\big)\\
&\quad -(1/2)k_{11}(1-3\cos\alpha)(1-\cos\alpha)-(1/2)(k_5+3k_{12})\sin^2\alpha\\
&\quad -(3/4)k_{15}\big(2\alpha-\sin(2\alpha)\big)-(3/4)k_{16}\big(4\pi-2\alpha+\sin(2\alpha)\big),\\
m_4 &:= -(1/4)(k_2+k_5+k_6+k_{11}+k_{12})\sin\alpha\sin(2\alpha)\\
&\quad +(1/4)k_9\cos\alpha\big(2\alpha-\sin(2\alpha)\big)+(1/4)k_{10}\cos\alpha\big(4\pi-2\alpha+\sin(2\alpha)\big)\\
&\quad -(1/8)k_{15}\big(12\alpha\cos\alpha-9\sin\alpha-\sin(3\alpha)\big)\\
&\quad -(1/8)k_{16}\big(12(2\pi-\alpha)\cos\alpha+9\sin\alpha+\sin(3\alpha)\big),\\
m_5 &:= 3k_{11}+k_{12},\quad m_6 := -k_{12},\quad m_7 := 4k_{10}\pi,\quad m_8 := 4(k_9-k_{10}).
\end{aligned}
$$

C. **Coefficients of the Wronskians $W_j(\alpha,R)$ for $j=5,6,7,8$ in (25).**

$$
\begin{aligned}
&\omega_{50} := 5(3+2\cos\alpha+\cos^2\alpha), \qquad \omega_{51} := 3+56\cos\alpha+29\cos^2\alpha+2\cos^3\alpha,\\
&\omega_{52} := 10\cos\alpha(1+7\cos\alpha+\cos^2\alpha), \qquad \omega_{53} := 10\cos^2\alpha(1+2\cos\alpha),\\
&\omega_{60} := 20(3+2\cos\alpha), \qquad \omega_{61} := 25(3+\cos\alpha)(1+3\cos\alpha),\\
&\omega_{62} := 15(2+3\cos\alpha)(1+6\cos\alpha+\cos^2\alpha),\\
&\omega_{63} := 3+68\cos\alpha+183\cos^2\alpha+138\cos^3\alpha+8\cos^4\alpha,\\
&\omega_{64} := 5\cos\alpha(1+\cos\alpha)^2(1+4\cos\alpha), \quad \omega_{70} := 140(5+14\cos\alpha+12\cos^2\alpha+4\cos^3\alpha),\\
&\omega_{71} := 8(491+1721\cos\alpha+2286\cos^2\alpha+1216\cos^3\alpha+236\cos^4\alpha),\\
&\omega_{72} := 4(2260+10516\cos\alpha+17901\cos^2\alpha+14216\cos^3\alpha+4816\cos^4\alpha+516\cos^5\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\omega_{73} := {} & 11555 + 68950\cos\alpha + 151419\cos^2\alpha + 156624\cos^3\alpha \\
& +79184\cos^4\alpha + 16464\cos^5\alpha + 904\cos^6\alpha, \\
\omega_{74} := {} & 2(4375 + 33785\cos\alpha + 94005\cos^2\alpha + 124479\cos^3\alpha + 83354\cos^4\alpha \\
& +26784\cos^5\alpha + 3104\cos^6\alpha + 64\cos^7\alpha), \\
\omega_{75} := {} & 5(769 + 8114\cos\alpha + 28599\cos^2\alpha + 47724\cos^3\alpha + 41246\cos^4\alpha \\
& +17976\cos^5\alpha + 3392\cos^6\alpha + 160\cos^7\alpha), \\
\omega_{76} := {} & 10(90 + 1441\cos\alpha + 6606\cos^2\alpha + 13908\cos^3\alpha + 15248\cos^4\alpha \\
& +8715\cos^5\alpha + 2310\cos^6\alpha + 192\cos^7\alpha), \\
\omega_{77} := {} & 5(1+\cos\alpha)^2(18 + 512\cos\alpha + 2432\cos^2\alpha + 4068\cos^3\alpha + 2583\cos^4\alpha + 432\cos^5\alpha), \\
\omega_{78} := {} & 10\cos\alpha(1+\cos\alpha)^2(22 + 178\cos\alpha + 447\cos^2\alpha + 423\cos^3\alpha + 120\cos^4\alpha), \\
\omega_{79} := {} & 5\cos^2\alpha(1+\cos\alpha)^2(3+4\cos\alpha)(6+16\cos\alpha+13\cos^2\alpha), \\
\omega_{80} := {} & 224, \qquad \omega_{81} := 483(1+2\cos\alpha), \qquad \omega_{82} := 420(1+2\cos\alpha)^2, \\
\omega_{83} := {} & 2(89 + 572\cos\alpha + 1144\cos^2\alpha + 750\cos^3\alpha), \\
\omega_{84} := {} & 36(1+2\cos\alpha)(1+8\cos\alpha+16\cos^2\alpha+10\cos^3\alpha), \\
\omega_{85} := {} & 3(1+16\cos\alpha+80\cos^2\alpha+168\cos^3\alpha+160\cos^4\alpha+58\cos^5\alpha), \\
\omega_{86} := {} & 2\cos\alpha(1+\cos\alpha)^3(1+5\cos\alpha+8\cos^2\alpha),
\end{aligned}
$$

References

[1] A.M. Alves, R.D.Euzébio, Periodic trajectories in planar discontinuous piecewise linear systems with only centers and with a nonregular switching line, *Nonlinearity*, **36**, 2023, 6747–6776.

[2] V.I. Arnold, Ten problems, *Adv. Soviet Math.*, **1**, 1990, 1–8.

[3] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.*, **128**, 2004, 7–22.

[4] P.T. Cardin, J. Torregrosa, Limit cycles in planar piecewise linear differential systems with nonregular separation line, *Physica D*, **337**, 2016, 67–82.

[5] X. Cen, S. Li, Y. Zhao, On the number of limit cycles for a class of discontinuous quadratic differential systems, *J. Math. Anal. Appl.*, **449**, 2017, 314–342.

[6] X. Cen, C. Liu, L. Yang, M. Zhang, Limit cycles by perturbing quadratic isochronous centers inside piecewise polynomial differential systems, *J. Differential Equations*, **265**, 2018, 6083–6126.

[7] J. Chavarriga, M. Sabatini, A survey of isochronous centers, *Qual. Theory Dyn. Syst.*, **1**, 1999, 1–70.

[8] C. Chicone, M. Jacobs, Bifurcation of limit cycles from quadratic isochrones, *J. Differential Equations*, **91**, 1991, 268–326.

[9] B. Coll, A. Gasull, R. Prohens, Bifurcation of limit cycles from two families of centers, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **12**, 2005, 275–287.

[10] R. Conti, Uniformly isochronous centers of polynomial systems in $\mathbb{R}^2$, *Lecture Notes in Pure and Appl. Math.*, **152**, 1994, 21–31.

[11] L.P.C. da Cruz, D.D. Novaes, J. Torregrosa, New lower bound for the Hilbert number in piecewise quadratic differential systems, *J. Differential Equations*, **266**, 2019, 4170–4203.

[12] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.

[13] L. Gavrilov, I.D. Iliev, Quadratic perturbations of quadratic codimension-four centers, *J. Math. Anal. Appl.*, **357**, 2009, 69–76.

[14] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, *Trans. Am. Math. Soc.*, **363**, 2011, 109–129.

[15] M. Grau, J. Villadelprat, Bifurcation of critical periods from Pleshkan's isochrones, *J. London Math. Soc.*, **81**, 2010, 142–160.

[16] D. Hilbert, Mathematische Probleme, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. Göttingen Math. Phys. KL.*, 1900, 253–297; English transl., *Bull. Amer. Math. Soc.*, **8**, 1902, 437–479; *Bull. (New Series) Amer. Math. Soc.*, **37**, 2000, 407–436.

[17] D. He, J. Llibre, Limit cycles in a class of planar discontinuous piecewise quadratic differential systems with a non-regular discontinuous boundary (I), submitted.

[18] J. Itikawa, J. Llibre, D.D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.*, **33**, 2017, 1247–1265.

[19] Y. Ilyashenko, Centennial history of Hilbert's 16th problem, *Bull. (New Series) Amer. Math. Soc.*, **39**, 2002, 301–354.

[20] S. Karlin, W.J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Pure and Applied Mathematics, vol. XV, Interscience Publishers John Wiley and Sons, New York-London-Sydney, 1966.

[21] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, *Int. J. Bifurc. Chaos Appl. Sci. Eng.*, **13**, 2003, 47–106,

[22] C. Li, W. Li, J. Llibre, Z.-F. Zhang, On the limit cycles of polynomial differential systems with homogeneous nonlinearities, *Proc. Edinburgh Math. Soc.*, **43**, 2000, 529–543.

[23] C. Li, W. Li, J. Llibre, Z.-F. Zhang, Linear estimate for the number of zeros of Abelian integrals for quadratic isochronous centres, *Nonlinearity*, **13**, 2000, 1775–1800.

[24] T. Li, J. Llibre, Limit cycles in piecewise polynomial systems allowing a non-regular switching boundary, *Physica D*, **419**, 2021, 132855, 13 pp.

[25] F. Liang, V.G. Romanovski, D.X. Zhang, Limit cycles in small perturbations of a planar piecewise linear Hamiltonian system with a non-regular separation line, *Chaos Solitons Fractals*, **111**, 2018, 18–34.

[26] H. Liu, Z. Wei, I. Moroz, Limit cycles and bifurcations in a class of planar piecewise linear systems with a nonregular separation line, *J. Math. Anal. Appl.* **526**, 2023, 127318, 25 pp.

[27] J. Llibre, A. Makhlouf, Bifurcation of limit cycles from some uniform isochronous centers, *Dyn. Contin. Discrete Impuls. Syst. Ser A: Math Anal.*, **22**, 2015, 381–394.

[28] J. Llibre, A.C. Mereu, Limit cycles for discontinuous quadratic differential systems, *J. Math. Anal. Appl.*, **413**, 2014, 763–775.

[29] J. Llibre, D.D. Novaes, C.A.B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, *Physica D*, **353–354**, 2017, 1–10.

[30] J. Llibre, D.D. Novaes, M.A. Teixeira, On the birth of limit cycles for non-smooth dynamical systems, *Bull. Sci. Math.*, **139**, 2015, 229–244.

[31] J. Llibre, G. Rodríguez, Configurations of limit cycles and planar polynomial vector fields, *J. Differential Equations*, **198**, 2004, 374–380.

[32] J. Llibre, M.A. Teixeira, Limit cycles for m-piecewise discontinuous polynomial Liénard differential equations, *Z. Angew. Math. Phys.*, **66**, 2015, 51–66.

[33] P. Mardesic, C. Rousseau, B. Toni, Linearization of isochronous centers, *J. Differential Equations*, **121**, 1995, 67–108.

[34] Y. Xiong, Limit cycle bifurcations by perturbing piecewise smooth Hamiltonian systems with multiple parameters, *J. Math. Anal. Appl.*, **421**, 2015, 260–275.

[35] P. Yang, Y. Yang, J. Yu, Up to second order Melnikov functions for general piecewise Hamiltonian systems with nonregular separation line, *J. Differential Equations*, **285**, 2021, 583–606.

[36] J. Yang, L. Zhao, Bounding the number of limit cycles of discontinuous differential systems by using Picard-Fuchs equations, *J. Differential Equations*, **264**, 2018, 5734–5757.

[1](Dongping He) School of Mathematics, Sichuan University, Chengdu 610065, China

E-mail address: hedongping1994@163.com

[2](Jaume Llibre) Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jaume.llibre@uab.cat