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Periods of self–maps on $\mathbb{S}^2$ via their homology

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This paper is dedicated to Professor A.N. Sharkovskiĭ

Abstract

As usual we denote by $\mathbb{S}^2$ the 2-dimensional sphere. We study the periods of the periodic orbits of the maps $f : \mathbb{S}^2 \to \mathbb{S}^2$ that are continuous, or C^1 with all their periodic orbits being hyperbolic, or transversal, or holomorphic, or transversal holomorphic.

For the first time we summarize all the known results on the periodic orbits of these distinct kind of self-maps on $\mathbb{S}^2$ together, and we note that every time that a map $f : \mathbb{S}^2 \to \mathbb{S}^2$ increases its structure the number of their periodic orbits provided by its action on the homology increases.

1 Introduction

Usually the periodic orbits play an important role for understanding the dynamics of a map. Two of the best known examples in this direction are the results contained in the seminal papers *Coexistence of the cycles of a continuous mapping of the line into itself* by Sarkovskii [10], and *Period three implies chaos* by Li and Yorke [7] for continuous self–maps on an interval.

Here we summarize all the known results on the periodic orbits of different kind of self–maps on the 2-dimensional sphere $\mathbb{S}^2$, that can be obtained using the action induced by the map on the homological groups of $\mathbb{S}^2$.

[1] *Key words and phrases:* Self-maps of the 2-dimensional sphere, set of periods, periodic points, Lefschetz numbers.
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We shall study the periods of the periodic orbits of the maps $f : \mathbb{S}^2 \to \mathbb{S}^2$ that are continuous, or C^1 having all their periodic orbits hyperbolic, or transversal, or holomorphic, and finally transversal holomorphic. Almost all results here presented have been obtained using the Lefschetz numbers associate to the maps f^n for $n \geq 1$, see [2, 8] for more information about the Lefschetz fixed point theory.

All the results presented in this paper are not new, they have appear published in distinct papers, but for the first time we meet them together here and we observe that every time that a map $f : \mathbb{S}^2 \to \mathbb{S}^2$ increases its structure the number of their periodic orbits provided by its action on the homology increases.

2 Continuous maps

In this section we use the fact that the sphere $\mathbb{S}^2$ is a compact topological space.

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map. A *fixed point* of the map f is a point x of $\mathbb{S}^2$ such that $f(x) = x$. Denote the set of all fixed points by $\mathrm{Fix}(f)$. A point $x \in \mathbb{S}^2$ is *periodic of period* m if $x \in \mathrm{Fix}(f^m)$ but $x \notin \mathrm{Fix}(f^k)$ for $k = 1, \ldots, m-1$. When x is a periodic point of f of period m, then its *periodic orbit* is $\{x, f(x), f^2(x), \ldots, f^{m-1}(x)\}$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f.

It is well known that the rational homological groups of $\mathbb{S}^2$ are $H_j(\mathbb{S}^2; \mathbb{Q}) = \mathbb{Q}$ if $j \in \{0, 2\}$, and $H_1(M; \mathbb{Q}) = 0$. Then a continuous map $f : \mathbb{S}^2 \to \mathbb{S}^2$ induces the homology endomorphism $f_{*j} : H_j(M; \mathbb{Q}) \to H_j(M; \mathbb{Q})$ for $j = 0, 1, 2$. It is known that f_{*0} is the identity and that f_{*2} is determined by the image of 1; i.e. $f_{*2}(1) = d$. The number d is an integer called the *degree* of f. For more information on the homology of $\mathbb{S}^2$ see, for instance, [11].

The next result for the continuous self-maps on $\mathbb{S}^2$ was proved in Theorem 1 of [6].

Theorem 1 *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map of degree d. If $d = -1$, then $\{1, 2\} \subset Per(f)$.*

3 C^1 maps such that all their periodic orbits are hyperbolic

In this section we use the fact that the sphere $\mathbb{S}^2$ is a compact differentiable manifold.

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a C^1 map. If $f(x) = x$ and the Jacobian matrix $df(x)$ of f at x has its two eigenvalues outside the unit circle, then x is a *hyperbolic fixed point*. A periodic point y of the map f of period m is a *hyperbolic periodic point* when y is a hyperbolic fixed point of f^m. When a periodic point y is hyperbolic all the points of its periodic orbit are hyperbolic, and we say that the periodic orbit is *hyperbolic*.

The following resuls comes from Teorem 1 of [4].

Theorem 2 *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a C^1 map of degree d having all its periodic orbits hyperbolic.*

(a) If $d = 0, 1$, then $1 \in Per(f)$.

(b) If $d = -1$, then $\{1, 2\} \subset Per(f)$.

(c) If $d \notin \{-1, 0, 1\}$, then f has infinitely many periodic orbits.

4 Transversal maps

In this section we continue using the fact that the sphere $\mathbb{S}^2$ is a compact differentiable manifold.

As usual we denote by $\mathbb{N}$ the set of all positive integers. A C^1 map $f : \mathbb{S}^2 \to \mathbb{S}^2$ is called *transversal* if for all $m \in \mathbb{N}$ at each point $x \in \mathrm{Fix}(f^m)$ we have $\det(I - df^m(x)) \neq 0$, where I is the 2×2 identity matrix and $df^m(x)$ is the Jacobian matrix of the function f^m evaluated at the point x, in other words 1 is not an eigenvalue of the matrix $df^m(x)$. Geometrically the fact that f is transversal means that for all $m \in \mathbb{N}$ the graph of f^m intersects transversally the diagonal $\{(y, y) : y \in \mathbb{S}^2\}$ at each point (x, x) such that $x \in \mathrm{Fix}(f^m)$.

From Theorem 3 of [9] and Theorem 2 with $n = 1$ of [5] it follows the next result.

Theorem 3 *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a transversal map of degree d.*

(a) If $d = 0, 1$ then $1 \in Per(f)$.

(b) If $d = -1$, then $\{1, 2\} \subset Per(f)$.

(c) If $d \notin \{-1, 0, 1\}$, then $\{1, 3, 5, 7, \dots\} \subset Per(f)$, and $Per(f)$ contains infinitely many even periods. More precisely, for m even if $m \notin Per(f)$, then $\{m/2, 2m\} \subset Per(f)$.

5 Holomorphic maps

In this section we use the fact that the sphere $\mathbb{S}^2$ admits the structure of a complex manifold.

The sphere $\mathbb{S}^2$ can be identified with the Riemann sphere, which is the compactication of the complex plane $\mathbb{C}$ obtained by adding the point at infinity. Therefore the sphere $\mathbb{S}^2$ admits the structure of a complex manifold. In fact $\mathbb{S}^2 = \mathbb{C}P^1$. Let f be a holomorphic map. Since a holomorphic map $f : \mathbb{S}^2 \to \mathbb{S}^2$ preserves the orientation at all points, the degree d is the number of preimages of any regular point, for more details see [3]. So for holomorphic maps $d \geq 0$.

The following result is valid for all holomorphic maps $f : \mathbb{C}P^n \to \mathbb{C}P^n$, and of course in particular for $\mathbb{S}^2 = \mathbb{C}P^1$. For a proof see Theorem 4.2 of [3].

Theorem 4 *$f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a holomorphic map of degree $d \geq 2$.*

(a) The map f has infinitely many periodic orbits.

(b) There exists $M > 0$ such that $p \in Per(f)$ for all $p > M$ prime.

Baker [1] in 1964 has a better version of Theorem 4 for $\mathbb{S}^2 = \mathbb{C}P^1$. Thus Baker in the next result described all possible set of periods of the holomorphic maps on the Riemann sphere with degree $d \geq 2$, that is of the rational maps on $\mathbb{S}^2$.

Theorem 5 *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a holomorphic map (i.e. a rational map), with degree $d \geq 2$.*

(a) The map f has periodic orbits of all periods except, perhaps of periods 2, 3 or 4.

(b) If f has no periodic points of period n, then (d, n) is one of the pair $(2, 2), (2, 3), (3, 2), (4, 2)$.

A shorter proof of Theorem 5 can be found in [3].

6 Transversal holomorphic maps

In this section we continue using the fact that the sphere $\mathbb{S}^2$ admits the structure of a complex manifold.

The richest self-maps on $\mathbb{S}^2$ with more periodic orbits are the transveral holomorphic maps, as the next result shows.

Theorem 6 *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a transversal holomorphic map of degree d.*

(a) If $d \in \{0, 1\}$, then $Per(f) = \{1\}$

(b) If $d > 1$, then $Per(f) = \mathbb{N}$.

Theorem 6 follows from Theorem 2 of [9] and Theorem B of [3].

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