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NEW FAMILIES OF GLOBAL CUBIC CENTERS

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Abstract. An equilibrium point p of a differential system in the plane $\mathbb{R}^2$ is a center if there exists a neighbourhood U of p such that $U \setminus \{p\}$ is filled with periodic orbits. A difficult classical problem in the qualitative theory of differential systems in the plane $\mathbb{R}^2$ is the problem of distinguishing between a focus and a center.

A global center is a center p such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits. Another difficult problem in the qualitative theory of differential systems in $\mathbb{R}^2$ is to distinguish inside a family of centers the ones which are global.

Lloyd, Pearson and Romanovsky characterized when the origin of coordinats is a center for the family of cubic polynomial differential systems

$$\dot{x} = y - Cx^2 + (B + 2D)\,xy + Cy^2 + Px^3 + Gx^2y - (H + 3P)\,xy^2 + Ky^3,$$
$$\dot{y} = -x + Dx^2 + (E + 2C)\,xy - Dy^2 - Kx^3 - (H + 3P)\,x^2y - Gxy^2 + Py^3.$$

Here we characterize when the origin of this family of differential system, is a global center.

1. Introduction and Statement of the Main Results

The notion of center first appears in the work of Huygens in 1656 on the pendulum clock (look at [12, 17]), but only with the works of Poincaré (see [18]) in 1881 and Dulac (see [8]) in 1908 the notion of center was rigorously defined.

A polynomial differential system in the plane $\mathbb{R}^2$ of degree n is a differential system of the form

$$\dot{x} = P\,(x, y)\,, \qquad \dot{y} = Q\,(x, y)\,, \tag{1}$$

being P and Q polynomials in the variables x and y. The n is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the time t.

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The classification of centers has been done just for polynomial differential systems of degree two by the authors Kapteyn [13] and Bautin [4]. For higher degrees only exists partial results.

There are several works studying the global centers of different families of polynomial differential systems, thus in [9] and [14] the authors proved that polynomial differential systems of even degree cannot have global centers because such systems always have orbits coming from and going to infinity. However to classify all polynomial differential systems of odd degree having a global center is a very difficult problem. This last problem was proposed by Conti in 1998, see his Problem 14.1 of [7], and up to now only few partial results exist for certain families of polynomial differential systems of odd degree, for more details see for instance [6, 10, 11, 14].

The objective of this work is to classify the global centers of the following cubic polynomial differential systems

$$\begin{aligned} \dot{x} &= y - Cx^2 + (B+2D)\,xy + Cy^2 + Px^3 + Gx^2y - (H+3P)\,xy^2 + Ky^3, \\ \dot{y} &= -x + Dx^2 + (E+2C)\,xy - Dy^2 - Kx^3 - (H+3P)\,x^2y - Gxy^2 + Py^3. \end{aligned} \tag{2}$$

when they have a center at the origin of coordinates.

It was given in [16] the classification of the cubic systems (2) having a center at the origin of coordinates. This classification is presented in the next theorem.

Theorem 1. *If the origin of system* (2) *is a center, then one of the following sets of conditions holds:*

(a) $H = P = G = K = 0$;
(b) $H = B = E = 0$;
(c) $H = 0$, $(DE - BC)(G - K) - P\left(B^2 + 4BD + 4CE + E^2\right) = 0$, $B^3C + B^3E + 3B^2DE - 3BCE^2 - BE^3 - DE^3 = 0$, $DE - BC \neq 0$;
(d) $H = 0$, $B = -4D$, $E = -4C$, $C^4P + C^3DG - C^3DK - 6C^2D^2P - CD^3G + CD^3K + D^4P = 0$;
(e) $H = P = C = E = 0$;
(f) $H = P = 0$, $B = E$, $C = D$;
(g) $H = P = 0$, $B = -E$, $C = -D$;
(h) $H = P = B = D = 0$;
(h) $H = 0$, $(DE-BC)K = 2P(C^2+D^2)$, $(DE-BC)G = P(B^2-6C^2-6D^2+E^2)$, $BD + 2C^2 + CE + 2D^2 = 0$, $DE - BC \neq 0$;
(j) $H = P = 0$, $B = -2D$, $E = -2C$, $G = -K$;
(k) $H = P = C = D = K = 0$.

In order to determine the global centers of the family of cubic polynomial differential systems (2), the dynamics of these systems near infinity play a main role. So we need the Poincaré compactification for studying these dynamics.

Roughly speaking the Poincaré compactification, first, consists in identifying the plane $\mathbb{R}^2$ with the interior of the unit closed disc $\mathbb{D}^2$ centered at the origin of coordinates, and the boundary of this disc, the circle $\mathbb{S}^1$, with the infinity of $\mathbb{R}^2$. After the polynomial differential system defined in $\mathbb{R}^2$ is extended analytically to the whole closed disc $\mathbb{D}^2$. In this way we can study the dynamics of the polynomial differential systems in a neighbourhood of the infinity. All the details on the Poincaré compactification can be found in [3, Chapter 5].

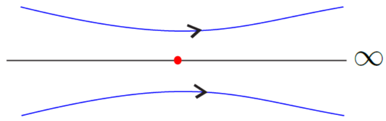

FIGURE 1. An equilibrium point on the infinity line whose local phase portrait is formed by two hyperbolic sectors, whose two separatrices are contained in the infinity line.

We note that in order that the origin of a polynomial differential system be a global center, that system either does not have equilibrium points at infinity in the Poincaré compactification, or the local phase portraits of all its infinite equilibrium points are formed by two hyperbolic sectors having their two separatrices at infinity, see Figure 1. This implies that the Jacobian matrix at any infinite equilibrium point of the system must be identically zero, otherwise the infinite equilibrium point would be either hyperbolic, or semi-hyperbolic, or nilpotent, and it is known that the local phase portraits of such kind of equilibrium points are not formed by two hyperbolic sectors having their two separatrices contained at infinity, for details see Theorems 2.15, 2.19 and 3.5 of [3].

Our main theorem is the following.

Theorem 2. *If the origin of system* (2) *is a global center and the polynomials* $\dot{x}$ *and* $\dot{y}$ *has no common factors that are non-constant, then one of the following sets of conditions holds:*

(b.1) $H = B = E = C = D = 0$, $G = (2P^2 - K^2)/K$ *and* $K > 0$.

(b.2) $H = B = E = C = D = P = 0$, $G = -K$ *and* $K > 0$.

(d.1) $H = P = C = E = 0$, $B = -4D$, $G = K$ *and* $D^2 - 4K < 0$.

(d.2) $H = P = D = 0$, $B = -4D$, $E = -4C$, $G = K$ *and* $C^2 - 4K < 0$.

(d.3) $H = P = 0$, $B = -4D$, $E = -4C$, $D = \pm C$, $G = K$ *and* $C^2 - 2K < 0$.

(e.1) $H = P = C = E = 0$, $B = -2D$, $K = -G$, $K > 0$ *and* $D^2 + 4G < 0$.

(f.1) $H = P = K = D = C = 0$, $B = E$ *and* $E^2 - 4G < 0$.

(f.2) $H = P = 0$, $G = K$, $B = E$, $C = D$, $E = -4D$ *and* $D^2 - 2K < 0$.

(f.3) $H = P = 0$, $G = -K$, $B = E$, $C = D$, $E = -2D$, *and* $D^2 - K < 0$.

(g.1) $H = P = K = D = C = 0$, $B = -E$ *and* $E^2 - 4G < 0$.

(g.2) $H = P = 0$, $G = K$, $B = -E$, $C = -D$, $E = 4D$ *and* $D^2 - 2K < 0$.

(g.3) $H = P = 0$, $G = -K$, $B = -E$, $C = -D$, $E = 2D$, *and* $D^2 - K < 0$.

(h.1) $H = P = B = D = 0$, $E = -2C$, $G = -K$ *and* $C^2 - 4K < 0$.

(j.1) $H = P = 0$, $B = -2D$, $E = -2C$, $G = -K$, $(C+D)^2 - 4K < 0$, $(C-D)^2 - 4K < 0$ *and* $C^2 + D^2 - 4K \pm \sqrt{(D^2 + 4K - C^2)^2 - 16D^2K} < 0$,.

(k.1) $H = P = C = D = K = 0$, $B^2 - 4G < 0$ *and* $E^2 - 4G < 0$.

In section 2 we present some useful notions and results that we need for proving Theorem 2 in section 3.

2. Preliminary Results

2.1. **Equilibrium Points.** A point (x_0, y_0) is an equilibrium of the differential system (1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. The equilibrium (x_0, y_0) is *hyperbolic* if all eigenvalues of the Jacobian matrix of the differential system evaluated at (x_0, y_0) have real part different from zero, and it is *semi-hyperbolic* if the Jacobian matrix presents only one eigenvalue equal to zero.

The classification of the local phase portraits of the hyperbolic and semi-hyperbolic equilibria can be found in Theorems 2.15 and 2.19 of [3], respectively. In this work when appears concepts related to local phase portraits of hyperbolic or semi-hyperbolic equilibrium points, we will be using implicitly these two theorems. But when the Jacobian matrix evaluated at an equilibrium point is identically zero, the local phase portrait at this equilibrium point must be analysed through special changes of variables called blow up's (for more details on the blow up method, see for instance [1]).

2.2. **Vertical blow up.** Consider a real planar polynomial differential system given by

$$\dot{x} = P(x, y) = P_n(x, y) + \ldots, \quad \dot{y} = Q(x, y) = Q_n(x, y) + \ldots, \tag{3}$$

with P and Q being coprime polynomials, P_n and Q_n being homogeneous polynomials of degree $n \in \mathbb{N}$ and the dots representing higher order terms in x and y. Since $n > 0$ the origin is an equilibrium point of system (3). Then the *characteristic directions* at the origin are given by the straight lines trought the origin defined by the real linear factors of the homogeneous polynomial $P_n(x, y)y - Q_n(x, y)x$. It is known that the orbits that start or end at the origin start or end tangent to the straight lines given by the characteristic directions. For more details on the characteristic directions see for instance [2].

Suppose that we have an equilibrium point at the origin of coordinates, as in the differential system (3) and that this equilibrium is linearly zero. Then for studying its local phase portrait we will do vertical blow up's.

We define the vertical blow up in the y direction as the change of variables $(u, v) = (x, y/x)$. This change transforms the origin of system (3) in the straight line $u = 0$, analyzing the dynamics of the differential system in a neighbourhood of this straight line we are analyzing the local phase portrait of the equilibrium point at the origin of system (3). But before doing a vertical blow up in order that we do not lost information we must avoid that the direction $x = 0$ be a characteristic direction of the origin of system (3). If $x=0$ is a characteristic direction we do a convenient twist $(x,y)=(u,u+\alpha v)$ with $\alpha\neq 0$ in order that new vertica straigh line $u=0$ non be a characteristic direction.

2.3. **The Poincaré compactification.** Let $X=(P,Q)$ be a planar polynomial vector field of degree n. The Poincaré compactified system $p\,(X)$ is an analytic vector field on the 2-dimensional sphere $\mathbb{S}^2$.

Initially we identify the plane $\mathbb{R}^2$ with the plane $y_3=1$ of $\mathbb{R}^3$, and define the Poincaré sphere $\mathbb{S}^2=\left\{(y_1,y_2,y_3)\in\mathbb{R}^3; y_1^2+y_2^2+y_3^2=1\right\}$. We consider the northern hemisphere $H_+=\left(y\in\mathbb{S}^2: y_3>0\right\}$, the southern hemisphere $H_-=\left\{y\in\mathbb{S}^2: y_3<0\right\}$ and the equator $\mathbb{S}^1=\left\{y\in\mathbb{S}^2; y_3=0\right\}$ of the sphere $\mathbb{S}^2$.

The projections of the plane $y_3=1$ on the sphere $\mathbb{S}^2$ are defined as $f_\pm:\mathbb{R}^2\longrightarrow H_\pm$, $f_\pm(y_1,y_2,1)=\left(y_1^2+y_2^2+1\right)^{-1/2}(y_1,y_2,1)$. These two diffeomorphisms establish two copies of the vector field X, one X_+ in H_+ and another X_- in H_-. Then we have the vector field $X'=X_+\cup X_-$ defined in $\mathbb{S}^2\backslash\mathbb{S}^1$ and the infinity of the plane $y_2=1$, i.e. of $\mathbb{R}^2$, is identified with the equator $\mathbb{S}^1$.

The *Poincaré compactified vector field* $p\,(X)$ is the analytic extension of X' from $H_+\cup H_-$ to $\mathbb{S}^2$ doing $p\,(X)=y_3^{n-1}X'$, and the *Poincaré disc* $\mathbb{D}$ is obtained by the projection of the closed northern hemisphere in $y_3=0$ under the projection $(y_1,y_2,y_3)\mapsto(y_1,y_2)$. The dynamics presented by the vector field $p\,(X)$ near $\mathbb{S}^1$ is the dynamics near the infinity of the vector field X.

We can define local charts $\phi_i:U_i\longrightarrow\mathbb{R}^2$ and $\psi_i:V_i\longrightarrow\mathbb{R}^2$ by $\phi_i(y_1,y_2,y_3)=\psi_i(y_1,y_2,y_3)=(y_a/y_i,y_b/y_i)=(u,v)$, with $a\neq i$, $b\neq i$ and $a<b$. The expression of the vector field $p\,(X)$ in the chart U_1 is

$$\dot{u}=v^n\left(Q\left(\frac{1}{v},\frac{u}{v}\right)-uP\left(\frac{1}{v},\frac{u}{v}\right)\right),\quad \dot{v}=-v^{n+1}P\left(\frac{1}{v},\frac{u}{v}\right),$$

and in the chart U_2 is

$$\dot{u}=v^n\left(P\left(\frac{u}{v},\frac{1}{v}\right)-uQ\left(\frac{u}{v},\frac{1}{v}\right)\right),\quad \dot{v}=-v^{n+1}Q\left(\frac{u}{v},\frac{1}{v}\right).$$

The expressions of $p\,(X)$ on the charts V_1 and V_2 are the same than in the charts U_1 and U_2 but multiplied by $(-1)^{n-1}$, respectively. In the charts U_i and V_i for $i=1,2$ the points of the infinity are of the form $(u,0)$. The infinity $\mathbb{S}^1$ is invariant under the flow of $p\,(X)$. For more details in Poincaré compactification see [3, Chapter 5].

The equilibrium points of the vector field X are called *finite* equilibrium points of X or of $p(X)$, while the equilibrium points of the vector field $p(X)$ in $\mathbb{S}^1$ are called *infinite* equilibrium points of X or of $p(X)$.

2.4. **Characterization of the global centers.** As we have seen in section 1 we have

Proposition 3. *Assume that a polynomial differential system has a global center and has infinite equilibrium points. Then the local phase portraits of these equilibria must* *be formed by two hyperbolic sectors, and consequently, having their two separatrices contained in the circle of the infinity. Moreover, the Jacobian matrix at these infinite equilibria must be identically zero.*

The next result characterize when a polynomial differential system in $\mathbb{R}^2$ has a global center. For a proof see Theorem 1.2 of [11].

Proposition 4. *A polynomial differential system* (3) *with finitely many infinite equilibria has a global center if and only if it has a unique finite singular point which is a center and all the infinite equilibria have their local phase portraits formed by two hyperbolic sectors.*

3. The proof of Theorem 2

3.1. **Skecth of the proof of Theorem** 2. From statement (a) of Theorem 1 we can not obtain a global center, because with the application of the condition $H = P = G = K = 0$ the vector field obtained has degree two.

From statement (b) of Theorem 1 we have six cases to analyze and we find global centers in only two of them. They are given in statements $(b.k)$ for $k = 1, 2$ of Theorem 2. In the first five cases the origin of the local chart U_2 in the Poincaré compactification is not an equilibrium point. In statement (b.2) initially we find $H = B = E = C = D = P = 0$, $G = \pm K$ and $K > 0$ for a global center at the origin, but for $G = K$ we have that the system (2) becomes $\dot{x} = y\left(1 + Kx^2 + Ky^2\right)$, $\dot{y} = -x\left(1 + Kx^2 + Ky^2\right)$. Since $1+Kx^2+Ky^2$ is a common factor of the differential system $\dot{x}$ and $\dot{y}$ rescaling the time this differential system can be reduced to the system $\dot{x} = y$, $\dot{y} = -x$, and we do not consider such systems. When $G = -K$ system (2) becomes $\dot{x} = y\left(1 - Kx^2 + Ky^2\right)$, $\dot{y} = -x\left(1 + Kx^2 - Ky^2\right)$, and again we do not consider it. In the six case the origin of U_2 is an equilibrium point and it will be a particular case from conditions of statement (fg.1), and so presenting a global center.

From statement (c) of Theorem 1 if we try to solve the three equations $(DE - BC)$ $(G - K) - P\left(B^2 + 4BD + 4CE + E^2\right) = 0$, $B^3C + B^3E + 3B^2DE - 3BCE^2 - BE^3 - DE^3 = 0$ and $H = 0$ taking into account that $DE - BC \neq 0$, we do not have solutions for analyze the existence of global centers.

From statement (d) of Theorem 1 when we solve the equation $C^4P + C^3DG - C^3DK - 6C^2D^2P - CD^3G + CD^3K + D^4P = 0$, we get six different cases to analyze. Each case is separated in two subcases, either the origin of U_2 is an equilibrium point, or not. One of the subcases that has global center presents the same parameters conditions of case (b.2) and the subcases that have global center are given in statements $(d.r)$ for $r = 1, 2, 3$ of Theorem 2.

Under the conditions of statement (e) of Theorem 1 we get four cases to study. Each case is separated in two subcases, either the origin of U_2 is an equilibrium point, or not. The subcase that has global center is given in statement $(e.1)$ of Theorem 2. One of the subcases obtained will be a particular case of statement (k.1) and another subcase presents the same parameters conditions of statement (d.1).

From the conditions of statements (f) and (g) we get eight subcases to study, two of them having an equilibrium point at the origin of U_2, and for the other six this origin is not an equilibrium point. The subcases with global center are $(fg.k)$ for $k = 1, 2, 3$. We remark that the conditions for having a global center are the same under conditions (f) and (g).

From the conditions of statement (h) we obtain four cases to analyze and only in one case the origin of U_2 is an equilibrium point. The subcase with global center is given in statement $(h.1)$. We get a subcase that will be a particular case of (k.1) with $B = 0$, and other case is a particular case of (d.2) with the parameter condition $B = 0$.

From the conditions of statement (i) and solving the equations $(DE - BC)\,K = 2P\left(C^2 + D^2\right)$, $(DE - BC)G = P(B^2 - 6C^2 - 6D^2 + E^2)$, $BD + 2C^2 + CE + 2D^2 = 0$, $H = 0$ and taking into account that $DE - BC \neq 0$ we get five cases to analize, but for these cases in order to have a global center we must have $DE - BC = 0$, which is a contradiction. So we cannot have global center for statement (i).

From the conditions of statement (j) we get four cases to study, only in one case the origin of U_2 is an equilibrium point. Only one of these cases allows the existence of a global center, that is given in statement $(j.1)$.

Finally, from statement (g) we have only one subcase to analize and it is a global center, given in statement $(g.1)$.

Many cases not mentioned in Theorem 2 do not have a global center at the origin of system (2) due to one of the following five reasons:

(i) System (2) becomes quadratic for some values of the parameters. For instance, this is the case under the conditions of statement (a) of Theorem 1.
(ii) It is not possible to become zero the Jacobian matrices at the infinite equilibria of the charts U_1 or U_2 in the Poincaré compactification.
(iii) When studying the local phase portrait of some infinite equilibrium point doing blow up's appears some parabolic sector, inside a saddle-node or a node, because they produce in the local phase portrait of the infinite equilibrium point orbits that go to infinity or come from infinity. We show one of these cases in section 3, namely case e.3, for some conditions on the parameters.
(iv) System (2) presents more than one finite equilibria, i.e. some equilibrium point distinct the center localized at the origin of coordinates.
(v) By the existence of some indetermination, thus in one of the cases analyzed the origin of the local chart U_2 is an equilibrium point and the parameter $G = (-C^4P + 6C^2D^2P - D^4P)/(C^3D - CD^3)$ must be zero, but in order that the Jacobian matrix of be zero at the infinite equilibria, we must have $C = P = 0$, but then G becomes indetermined.

The distinct cases described in the statements of Theorem 2 fit in one of following three different patterns:

(I) There are no infinite equilibria, i.e. the infinity is a periodic orbit. Then to prove that system (2) has a global center, it is reduced to prove that the unique finite equilibrium is the origin of coordinates. This sometimes is a difficult computational problem. For instance, this situation occurs in inside the case of statement (c) of Theorem 1 when $D = (-B^3C - B^3E + 3BCE^2 + BE^3)/(E(3B^2 - E^2))$ and $K = (-B^3EG + BE^3G + B^4P - 6B^2E^2P + E^4P)/(BE(-B^2 + E^2))$, is a solution of $(DE - BC)(G - K) - P(B^2 + 4BD + 4CE + E^2) = 0$. Also it occurs in the case of statement (d) of Theorem 1 when $K = (C^3DG - CD^3G + C^4P - GC^2D^2P + D^4P)/(C^3D - CD^3)$, is a solution of $C^4P + C^3DG - C^3DK - 6C^2D^2P - CD^3G + CD^3K + D^4P = 0$.

(II) All the infinite equilibria are in the local charts $U_1 \cup V_1$. Since the linear part at these equilibria will be the zero matrix, we need to do blow up's trying to see that the local phase portraits of such equilibria are formed by two hyperbolic sectors.

(III) The origin of the chart U_2 is an equilibrium point. As in the case (II) we must do blow up's.

In what follows we shall illustrate **some cases showing** these patterns.

3.2. **The proofs accoding the previous skecth. Cases (b.1) and (b.2)**. Based on the conditions of statement (b) of Theorem 1, we have $H = B = E = 0$, then system (2) becomes

$$\begin{aligned} \dot{x} &= y - Cx^2 + 2Dxy + Cy^2 + Px^3 + Gx^2y - 3Pxy^2 + Ky^3, \\ \dot{y} &= -x + Dx^2 + 2Cxy - Dy^2 - Kx^3 - 3Px^2y - Gxy^2 + Py^3. \end{aligned} \tag{4}$$

This system in the chart U_2 writes

$$\begin{aligned} \dot{u} = {} & 2Gu^2 + 4Pu\left(u^2 - 1\right) + K\left(1 + u^4\right) + Cv + 3Duv - 3Cu^2v \\ & -Du^3v + v^2 + u^2v^2, \\ \dot{v} = {} & v\left(Gu + Ku^3 + P\left(3u^2 - 1\right) + Dv - 2Cuv - Du^2v + uv^2\right). \end{aligned} \tag{5}$$

First we consider that the origin of system (5) is not an equilibrium, i.e, $K \neq 0$.

In the chart U_1 of the Poincaré compactification system (4) becomes

$$\begin{aligned} \dot{u} = {} & -2Gu^2 + 4Pu\left(u^2 - 1\right) - K\left(1 + u^4\right) + Dv + 3Cuv - 3Du^2v \\ & -Cu^3v - v^2 - u^2v^2, \\ \dot{v} = {} & -v\left(P + Gu - 3Pu^2 + Ku^3 - Cv + 2Duv + Cu^2v + uv^2\right). \end{aligned} \tag{6}$$

We get four infinite equilibria of system (6), namely $p_1 = (u_{-+}, 0)$, $p_2 = (u_{--}, 0)$, $p_3 = (u_{++}, 0)$ and $p_4 = (u_{+-}, 0)$, where

$$\begin{aligned} u_{\pm\pm} = {} & \frac{1}{2K}\Big(2P \pm \sqrt{2}\Big(\sqrt{-K(G+K) + 2P^2} \\ & \pm\sqrt{-GK + K^2 + 4P^2 - 2P\sqrt{-2K(G+K) + 4P^2}}\Big)\Big). \end{aligned}$$

Let J_i be the Jacobian matrix of system (6) evaluated at p_i for $i = 1, 2, 3, 4$. By Proposition 3 these four matrices must be the zero matrix. This provides three conditions, the ones of the following three cases.

Case (b.1). $C = 0$, $D = 0$ and $G = (2P^2 - K^2)/K$. Now we have to do blow up's for studying the local phase portraits at the four infinite equilibria of the chart U_1. We translate the equilibrium p_1 at the origin of coordinates doing the change of variables $(u,v)=(u_1+u_{-+},v_1)$, and we get the system

$$
(7)\qquad
\begin{aligned}
\dot{u_1} = {} & \frac{1}{K^2}\left(-K^3u_1^2\left(4-4\sqrt{1+\frac{P^2}{K^2}}u_1+u_1^2\right)-2P^2v_1^2\right.\\
& +K^2\left(-2+2\sqrt{1+\frac{P^2}{K^2}}u_1-u_1^2\right)v_1^2\\
& \left.-2KP\left(2Pu_1^2+\left(-\sqrt{1+\frac{P^2}{K^2}}+u_1\right)v_1^2\right)\right),\\
\dot{v_1} = {} & -\frac{1}{K}v_1\left(K^2u_1\left(2-3\sqrt{1+\frac{P^2}{K^2}}u_1+u_1^2\right)+K\left(-\sqrt{1+\frac{P^2}{K^2}}+u_1\right)v_1^2\right.\\
& \left.+P\left(2Pu_1+v_1^2\right)\right).
\end{aligned}
$$

Since $u_1=0$ is not a characteristic direction. We do the vertical blow up $(u_1,v_1)=(u_2,u_2v_2)$, and we obtain the system

$$
(8)\qquad
\begin{aligned}
\dot{u_2} = {} & -\frac{1}{K^2}u_2^2\left(4K^3+4KP^2-4K^2\sqrt{(K^2+P^2)}u_2+K^3u_2^2+2Kv_2^2+2P^2v_2^2\right.\\
& \left.-2P\sqrt{K^2+P^2}v_2^2+2KPu_2v_2^2-2K\sqrt{K^2+P^2}u_2v_2^2+K^2u_2^2v_2^2\right),\\
\dot{v_2} = {} & -\frac{1}{K^2}u_2v_2\left(-2K^3-2KP^2+K^2\sqrt{K^2+P^2}u_2-2K^2v_2^2-2P^2v_2^2\right.\\
& \left.+2P\sqrt{K^2+P^2}v_2^2-KPu_2v_2^2+K\sqrt{K^2+P^2}u_2v_2^2\right).
\end{aligned}
$$

Now we eliminate the common factor u_2 in the two components of system (8) doing a rescaling of the time, and we get the system

$$
(9)\qquad
\begin{aligned}
\dot{u_2} = {} & -\frac{1}{K^2}u_2\left(4K^3+4KP^2-4K^2\sqrt{(K^2+P^2)}u_2+K^3u_2^2+2Kv_2^2+2P^2v_2^2\right.\\
& \left.-2P\sqrt{K^2+P^2}v_2^2+2KPu_2v_2^2-2K\sqrt{K^2+P^2}u_2v_2^2+K^2u_2^2v_2^2\right),\\
\dot{v_2} = {} & -\frac{1}{K^2}v_2\left(-2K^3-2KP^2+K^2\sqrt{K^2+P^2}u_2-2K^2v_2^2-2P^2v_2^2\right.\\
& \left.+2P\sqrt{K^2+P^2}v_2^2-KPu_2v_2^2+K\sqrt{K^2+P^2}u_2v_2^2\right).
\end{aligned}
$$

Since the equilibria of system (9) on the straight line $u_2=0$ are $p=(0,0)$, $p_\pm=\left(0,\pm\sqrt{K^3+KP^2}/\sqrt{-K^2-P^2+P\sqrt{K^2+P^2}}\right)$, and they are hyperbolic the process of the blow up has ended. Note that $\sqrt{-K^2-P^2+P\sqrt{K^2+P^2}}<0$ because $K\neq 0$, so the equilibria p
pm are not real.

The eigenvalues of the linear part of the system at the equilibrium p are $2K+2P^2/K$ and $-2\left(2K+2P^2/K\right)$, so p is a saddle. If $K>0$ going back through these changes of variables, we obtain that p is formed by two hyperbolic sectors as the ones of Figure 2. If $K<0$ going back through these changes of variables we get the pictures of Figure 3, and the one of the right is in contradiction with the fact that $\dot{u_1}<0$ on $u_1=0$ in a neighborhood of the origin of coordinates.

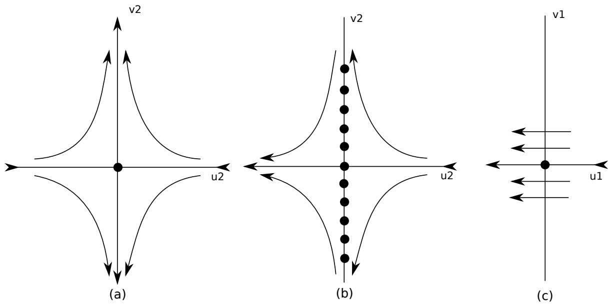

FIGURE 2. Desingularization using blow up's of the origin of coordinates of the chart U_1. (a) The local phase portrait in a neigborhood of the straight line $u_2 = 0$ of system (9). (b) The local phase portrait in a neighborhood of the straigh line $u_2 = 0$ of system (8). (c) The local phase portrait at the origin of the chart U_1.

We do a similar analysis for the equilibrium points p_2, p_3 and p_4, obtaining the same conclusion, i.e. when $K > 0$ these four equilibria are formed by two hyperbolic sectors. So, by Proposition 4 if $K > 0$ then system (4) has a global center if the system has a unique finite equilibrium point.

System (4) after the conditions obtained reduces to

$$\dot{x} = y + Px^3 + \frac{2P^2 - K^2}{K}x^2y - 3Pxy^2 + Ky^3,$$
$$\dot{y} = -x - Kx^3 - 3Px^2y - \frac{2P^2 - K^2}{K}xy^2 + Py^3.$$

Since this system for $K > 0$ has only the origin as an equilibrium point statement (b.1) is proved..

An analogous analysis could be applied to the statements (d.1), (e.3), (h.3), and (j.1) of Theorem 2.

Case (b.2). $C = D = P = 0$ and $G = K$. These conditions transform the equilibrium points p_1 and p_2 into the equilibrium $\left(-\sqrt{-K^2}/K, 0\right)$, and the equilibrium points p_3 and p_4 into the equilibrium $\left(\sqrt{-K^2}/K, 0\right)$. Therefore system (4) has no infinite equilibrium points in the chart U_1, and consequently the infinity is a periodic orbit. Now we study the finite equilbria of system (4), that is reduced to the system

$$\dot{x} = y + Kx^2y + Ky^3, \quad \dot{y} = -x - Kx^3 - Kxy^2.$$

The three equilibria of this system are $\left(x, \pm\sqrt{-1 - Kx^2}/K\right)$ and $(0, 0)$.

If $K > 0$ we have only the origin as a finite equilibrium and then, by Proposition 4, we have a global center at the origin. If $K < 0$ for the values of x such that $K\left(-1 - Kx^2\right) > 0$ we have a continuum of equilibrium points and the system cannot present a global center. This proves the statement (b.2) of Theorem 2.

This kind of proof can be applied to the cases (d.1), (d.2), (d.3), (d.4), (e.2), (fg.2) and (h.2) of Theorem 2.

Case (b.3) Now we assume that $K = 0$, so the origin of the chart U_2 is an infinite equilibrium point. From system (5) the Jacobian matrix at the origin of U_2 is

$$\begin{pmatrix} -4P & C \\ 0 & -P \end{pmatrix}.$$

From Proposition 3 this matrix must be the zero matrix if we want to have a global center, so $P = C = 0$.

From system (6), now we get that in the local chart U_1 the system (4) writes

$$\dot{u} = -2Gu^2 - v\left(-D + 3Du^2 + v + u^2v\right), \quad \dot{v} = -uv\left(G + v\left(2D + v\right)\right). \tag{10}$$

The unique infinite equilibria of this system is $p = (0, 0)$. The Jacobian matrix of the system at p is

$$\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}.$$

Due to Proposition 3 if we want to obtain a global center for system (4), this matrix must be the zero matrix, hence $D = 0$. Therefore system (4) under the conditions $K = P = C = D = 0$ reduces to the system

$$\dot{x} = y + Gx^2y, \quad \dot{y} = -x - Gxy^2. \tag{11}$$

The five finite equilibria of these system are $\left(\pm 1/\sqrt{-G}, \pm 1/\sqrt{-G}\right)$ and $(0, 0)$. So if $G > 0$ the origin is the unique finite equilibria.

Now we study the local phase portrait at the origin of the chart U_1, and we need to do blow up's because this equilibrium point is linearly zero. Since $u = 0$ is not a characteristic direction we do the change of variables $(u.v) = (u_1, u_1v_1)$ and the system (10) becomes

$$\dot{u_1} = -u_1^2\left(2G + \left(1 + u_1^2\right)v_1^2\right), \quad \dot{v_1} = u_1v_1\left(G + v_1^2\right). \tag{12}$$

Doing a rescaling of the time we eliminate the common factor u_1 from system (12) and we obtain the system

$$\dot{u_1} = -u_1\left(2G + \left(1 + u_1^2\right)v_1^2\right), \ \dot{v_1} = v_1\left(G + v_1^2\right). \tag{13}$$

The equilibrium points of this system on $u_1 = 0$ are $(0, 0)$ and $\left(0, \pm\sqrt{-G}\right)$, but with the restriction $G > 0$ we only have the equilibria $(0, 0)$. The Jacobian matrix at the origin of the system has eigenvalues $-2G$ and G, so the $(0, 0)$ is a hyperbolic saddle. Going back through the change of variables, we get that the origin of U_1 is formed by two hyperbolic sectors, see Figure 2.

The origin of the chart U_2 from system (5) with $K = P = C = D = 0$ is a linearly zero equilibrium point. Then for studying its local phase portrait we must do blow up's. Since $u = 0$ is not a characteristic direction we do the change of variables $(u, v) = (u_1, u_1v_1)$ getting the system

$$\dot{u_1} = u_1^2\left(2G + \left(1 + u_1^2\right)v_1^2\right), \quad \dot{v_1} = -u_1v_1\left(G + v_1^2\right). \tag{14}$$

Rescaling the time we eliminate the common factor u_1, and we obtain the system

$$\dot{u_1} = u_1 \left(2G + \left(1 + u_1^2\right) v_1^2\right), \quad \dot{v_1} = -v_1 \left(G + v_1^2\right). \tag{15}$$

The equilibrium points of this system on $u_1 = 0$ are $(0,0)$ and $\left(0, \pm\sqrt{-G}\right)$. Again with the restriction $G > 0$ the unique equilibrium is the $(0,0)$. The Jacobian matrix at the origin has the eigenvalues $2G$ and $-G$. So the origin is a saddle. Going back through the change of variables we get that the origin of U_2 is formed by two hyperbolic sectors as shown in Figure 3. By Proposition 4 under all these conditions system (4) has a global center at the origin, and statement (b.3) is proved.

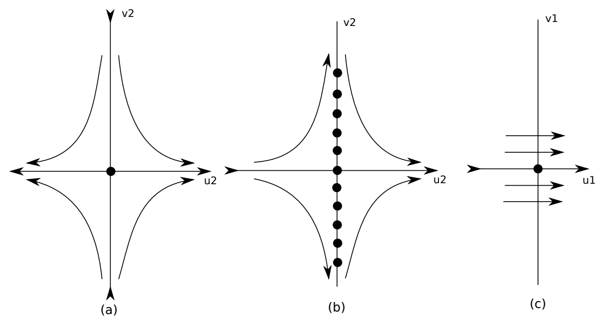

FIGURE 3. Desingularization using blow up's of the origin of coordinates of the chart U_1. (a) The local phase portrait in a neigborhood of the straigh line$u_1 = 0$ of system (15). (b) The local phase portrait in a neighborhood of the straight line $u_1 = 0$ of system (14). (c) The local phase portrait at the origin of the chart U_1.

The cases (e.1), (fg.1), (h.1) and (k.1) are proved in an analogous way.

Case (f.3) This case happens when we analyze the case (f) or (g) of Theorem 1 assuming that the origin of the chart U_2 is not an equilibrium. Since $H = P = 0$, $B = E$ and $C = D$ system (2) becomes

$$\begin{aligned} \dot{x} &= y - Dx^2 + (2D + E)\, xy + Gx^2y + Dy^2 + Ky^3, \\ \dot{y} &= -x + Dx^2\, (2D + E)\, xy - Dy^2 - Kx^3 - Gxy^2. \end{aligned} \tag{16}$$

This system in the chart U_2 is

$$\begin{aligned} \dot{u} &= 2Gu^2 + K\left(1 + u^4\right) + v\left(D(1 + 3u - u^3) + Eu - (3D + E)u^2 + v + u^2v\right), \\ \dot{v} &= v\left(Gu + Ku^3 + v\left(D - 2Du - Eu - Du^2 + uv\right)\right). \end{aligned} \tag{17}$$

We have that the origin is not a equilibria when $K \neq 0$.

In the chart U_1 system (16) becomes

$$\begin{aligned} \dot{u} &= -2Gu^2 - K\left(1 + u^4\right) - (u - 1)\, uv - D\left(1 + 3u - 3u^2 - u^3\right) + \left(1 + u^2\right) v, \\ \dot{v} &= -v\left(Gu + Ku^3 + v\left(D\left(-1 + 2u + u^2\right) + u\,(E + v)\right)\right). \end{aligned} \tag{18}$$

The infinite equilibrium points of this system are the four equilibria

$$p_\pm = \left(\pm\sqrt{-(G + \sqrt{G^2 - K^2})/K}, 0\right) \text{ and } q_\pm = \left(\pm\sqrt{-(G - \sqrt{G^2 - K^2})/K}, 0\right).$$

From Proposition 3 we do zero the four Jacobian matrices of system (18) at these four equilibria and we obtain the solution $E = -2D$ and $G = -K$, which origins $p_\pm = q_\pm = (\pm 1, 0)$.

Moving the equilibria $(-1,0)$ to the origin of coordinates system (18) becomes

$$\begin{aligned} \dot{u_1} &= -K\left(-2+u_1\right)^2 u_1^2 - v_1\left(D\left(-2+u_1\right)u_1^2 + \left(2-2u_1+u_1^2\right)v_1\right), \\ \dot{v_1} &= -v_1\left(Ku_1(2-3u_1+u_1^2) + v_1\left(D_1\left(-2+u_1\right)u_1 + \left(-1+u_1\right)v_1\right)\right). \end{aligned} \tag{19}$$

Then we apply the blow up $u_1 = u_2$, $v_1 = u_2 v_2$ and system (19) is given by

$$\begin{aligned} \dot{u_2} &= -u_2^2\left(4K - 4Ku_2 + Ku_2^2 - 2Du_2v_2 + Du_2^2v_2 + 2v_2^2 - 2u_2v_2^2 + u_2^2v_2^2\right), \\ \dot{v_2} &= \left(2-u_2\right)u_2v_2\left(K+v_2^2\right). \end{aligned} \tag{20}$$

Doing a rescaling of the time we eliminate the common factor u_2 from system (20), and we obtain

$$\begin{aligned} \dot{u_2} &= -u_2\left(4K - 4Ku_2 + Ku_2^2 - 2Du_2v_2 + Du_2^2v_2 + 2v_2^2 - 2u_2v_2^2 + u_2^2v_2^2\right), \\ \dot{v_2} &= \left(2-u_2\right)v_2\left(K+v_2^2\right). \end{aligned} \tag{21}$$

The equilibrium points of system (21) at $u_2 = 0$ are $(0,0)$, $\left(0, -\sqrt{-K}\right)$, $\left(0, \sqrt{-K}\right)$. The Jacobian matrix calculated at origin has eigenvalues $-4K$ and $2K$, so the origin is a hyperbolic saddle. If $K > 0$ only the origin is the equilibria of system (21) on $u_2 = 0$. Going back through the change of variables, we obtain that the origin of the system (20) is formed by two hyperbolic sectors when $K > 0$, as shown in Figure 2. If $K < 0$ we have the three equilibrium points of system (21) but the eigenvalues of $\left(0, \pm\sqrt{-K}\right)$ are $-2K$ and $-4K$, so these equilibrium points are nodles and the system can not have a global center.

Doing the same analysis made for the equilibrium $(-1,0)$ for the other equilibrium $(1,0)$ of system (18), i.e. first we translate the equilibrium $(1,0)$ to the origin of coordinates, after we do the blow up process and the rescaling of the time by the common factor u_2, and we obtain the following three equilibrium points $(0,0)$, $\left(0, -D \pm \sqrt{D^2-K}\right)$ on $u_2 = 0$. The origin is a saddle and if the other two equilibria are not real, going back through the change of variables we get again that the origin of system (20) is formed by two hyperbolic sectors.

Before studying when the equilibrium points $\left(0, -D \pm \sqrt{D^2-K}\right)$ exist or not, we study the finite equilibrium points. The system (16) with the conditions $E = -2D$ and $G = -K$ becomes

$$\dot{x} = y - Dx^2 + Dy^2 - Kx^2y + Ky^3, \quad \dot{y} = -x + Dx^2 - Dy^2 - Kx^3 + Kxy^2. \tag{22}$$

Then the unique finite equilibrium point of system (22) is the origin if $D^2 - K < 0$. So the equilibrium points $\left(0, -D \pm \sqrt{D^2-K}\right)$ do not exist and consequently the system has a global center under the conditions $H = P = 0$, $B = E$, $C = D$, $K > 0$, $E = -2D$, $G = -K$ and either $-\sqrt{K} < D < 0$, or $0 < D < \sqrt{K}$.

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