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PERIODIC ORBITS OF CONTINUOUS-DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS WITH FOUR PIECES SEPARATED BY THE CURVE $xy = 0$ AND FORMED BY LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. In recent years there has been a significant interest in studying the piecewise differential systems, mainly due to their wide range of applications in modeling natural phenomena. To understand the dynamics of such systems in the plane is a significant challenge, particularly when we want to study their periodic orbits and, more specifically, their limit cycles. Consequently numerous studies have been dedicated to investigate the existence or non-existence of periodic orbits within continuous and discontinuous piecewise differential systems. However to the best of our knowledge, this paper is one of the pioneering works analysing the periodic orbits within a specific class of piecewise differential systems, the ones exhibiting continuity in one part of the separation line while being discontinuous in the other part.

Our study analyzes the periodic orbits of the piecewise differential systems formed by four pieces, having the curve $xy = 0$ as the separation line, and in each piece there is an arbitrary linear Hamiltonian system. Moreover we assume that these piecewise differential systems exhibit continuity along the x-axis while being discontinuous along the y-axis.

1. Introduction and statement of the main results

A dynamical system is any system that changes over time, and ODEs provide a concise and elegant way to capture this behavior. A dynamical system can be defined a function that describes the time dependence of a point in an ambient space, such as a parametric curve. Examples of dynamical systems that can be modeled with ODEs include the oscillation of a clock pendulum, the flow of water through a pipe, the motion of particles in the air, and the population dynamics of a lake's fish species. By modeling these systems with ODEs, we gain insight into their behavior, which can inform important decisions in fields such as physics, engineering, and ecology. Therefore, ODEs play a crucial role in the study of dynamical systems and are essential tools for understanding the behavior of natural phenomena.

Piecewise differential systems in the plane are a particular type of dynamical system that are defined by different sets of differential equations in different regions of the state space. The boundary between regions, called the switching or separation line defines the conditions under which the system switches from one set of equations to another, see for more details [8, 15, 26]. These systems are often used to model complex phenomena that exhibit different behaviors or dynamics under different conditions, such as ecological systems or mechanical systems subject to switching or control inputs.

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The study of dynamical systems and piecewise differential systems involves the analysis of the system's behavior, stability, and bifurcations. The complexity of these systems often limits the effectiveness of the analytical methods. Therefore we use qualitative methods to study and gain a deeper understanding of these systems.

Dynamical systems and in particular piecewise differential systems have numerous applications in science and engineering, see for instance [1, 4, 5, 13, 16, 21, 24]. Understanding the principles of these systems can help researchers and engineers gain insight into the behavior of real-world systems, leading to more effective control and optimization strategies.

While the continuous piecewise differential systems and the discontinuous ones have been studied intersevely, see for instance [**?**, 2, 3, 6, 7, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22, 23, 25], the piecewise differential systems exhibiting continuity in one part of the separation line while being discontinuous in the other part almost have not been studied.

This paper studies the piecewise differential systems

$$(1)\quad (\dot{x},\dot{y}) = X(x,y) = \begin{cases} X_1(x,y) = \left(-\frac{\partial H_1}{\partial y}, \frac{\partial H_1}{\partial x}\right) & \text{if } Q_1 = \{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\geq 0\},\\ X_2(x,y) = \left(-\frac{\partial H_2}{\partial y}, \frac{\partial H_2}{\partial x}\right) & \text{if } Q_2 = \{(x,y)\in\mathbb{R}^2 : x\leq 0,\ y\geq 0\},\\ X_3(x,y) = \left(-\frac{\partial H_3}{\partial y}, \frac{\partial H_3}{\partial x}\right) & \text{if } Q_3 = \{(x,y)\in\mathbb{R}^2 : x\leq 0,\ y\leq 0\},\\ X_4(x,y) = \left(-\frac{\partial H_4}{\partial y}, \frac{\partial H_4}{\partial x}\right) & \text{if } Q_4 = \{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\leq 0\}, \end{cases}$$

characterized by a combination of continuity and discontinuity on the separation line $xy = 0$, where

$$\begin{aligned} H_1(x,y) =& a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2,\\ H_2(x,y) =& b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2,\\ H_3(x,y) =& d_0 + d_1x + d_2y + d_3x^2 + d_4xy + d_5y^2,\\ H_4(x,y) =& c_0 + c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2. \end{aligned}$$

Furthermore we assume that these piecewise differential systems exhibit continuity along the x-axis while being discontinuous along the y-axis.

The principal outcome of our study can be summarized as follows.

Theorem 1. *The continuous-discontinuous piecewise differential systems* (1) *possess at most one limit cycle.*

2. Proof of Theorem 1

The four linear Hamiltonian systems forming the piecewise differential systems (1) are

$$\begin{array}{lll} \dot{x} = -a_2 - a_4x - 2a_5y, & \dot{y} = a_1 + 2a_3x + a_4y, & \text{in } Q_1,\\ \dot{x} = -b_2 - b_4x - 2b_5y, & \dot{y} = b_1 + 2b_3x + b_4y, & \text{in } Q_2,\\ \dot{x} = -c_2 - c_4x - 2c_5y, & \dot{y} = c_1 + 2c_3x + c_4y, & \text{in } Q_3,\\ \dot{x} = -d_2 - d_4x - 2d_5y, & \dot{y} = d_1 + 2d_3x + d_4y, & \text{in } Q_4. \end{array}$$

Since these piecewise differential systems are continuous on the x-axis they must verify

$$\dot{x}_{Q1} - \dot{x}_{Q4}|_{y=0} = 0,\quad \dot{y}_{Q1} - \dot{y}_{Q4}|_{y=0} = 0,\quad \dot{x}_{Q2} - \dot{x}_{Q3}|_{y=0} = 0,\quad \dot{y}_{Q2} - \dot{y}_{Q3}|_{y=0} = 0.$$

Therefore the following four linear polynomials must vanish

$$\begin{array}{ll} d_2 - a_2 + (d_4 - a_4)x = 0, & a_1 - d_1 + 2(a_3 - d_3)x = 0, \\ c_2 - b_2 + +(c_4 - b_4)x = 0, & b_1 - c_1 + 2(b_3 - c_3)x = 0, \end{array}$$

and consequently we obtain that

$$a_1 = d_1,\ a_2 = d_2,\ a_3 = d_3,\ a_4 = d_4,\ b_1 = c_1,\ b_2 = c_2,\ b_3 = c_4,\ b_4 = c_4.$$

Hence the piecewise differential system (1) becomes

$$\begin{array}{lll} \dot{x} = -a_2 - a_4x - 2a_5y, & \dot{y} = a_1 + 2a_3x + a_4y, & \text{in } Q_1, \\ \dot{x} = -b_2 - b_4x - 2b_5y, & \dot{y} = b_1 + 2b_3x + b_4y, & \text{in } Q_2, \\ \dot{x} = -b_2 - b_4x - 2c_5y, & \dot{y} = b_1 + 2b_3x + b_4y, & \text{in } Q_3, \\ \dot{x} = -a_2 - a_4x - 2d_5y, & \dot{y} = a_1 + 2a_3x + a_4y, & \text{in } Q_4. \end{array}$$

The Hamiltonians of these last Hamiltonian systems are

$$\begin{array}{rl} H_1(x,y) = & a_0 + a_1x + a_3x^2 + a_2y + a_4xy + a_5y^2, \\ H_2(x,y) = & b_0 + b_1x + b_3x^2 + b_2y + b_4xy + b_5y^2, \\ H_3(x,y) = & c_0 + b_1x + b_3x^2 + b_2y + b_4xy + c_5y^2, \\ H_4(x,y) = & d_0 + a_1x + a_3x^2 + a_2y + a_4xy + d_5y^2, \end{array}$$

respectively.

Now we study the possible limit cycles of these continuous-discontinuous piecewise differential systems that intersect the positive and negative x and y axes in points of the form $(\alpha, 0)$, $(0, \beta)$, $(\delta, 0)$ and $(0, \gamma)$ with $\alpha > 0$, $\beta > 0$, $\delta < 0$ and $\gamma < 0$. If such limit cycles exist these four points must satisfy the following equations

$$\begin{array}{ll} H_1(\alpha, 0) - H_1(0, \beta) = 0, & H_2(0, \beta) - H_2(\delta, 0) = 0, \\ H_3(\delta, 0) - H_3(0, \gamma) = 0, & H_4(0, \gamma) - H_4(\alpha, 0) = 0, \end{array}$$

i.e. we have

$$\begin{array}{ll} \alpha^2a_3 + \alpha a_1 - \beta a_2 - \beta^2a_5 = 0, & \beta^2b_5 + \beta b_2 - \delta b_1 - \delta^2b_3 = 0, \\ \delta b_1 - \gamma b_2 + \delta^2b_3 - \gamma^2c_5 = 0, & -\alpha^2a_3 - \alpha a_1 + \gamma a_2 + \gamma^2d_5 = 0. \end{array} \tag{2}$$

In order to simplify the computations we do the the change of variables $\alpha = a^2$, $\beta = b^2$, $\delta = -c^2$, and $\gamma = -b^2$. Then the previous system is transformed as follows

$$\begin{array}{ll} a_3a^4 + a_1a^2 - a_2b^2 - a_5b^4 = 0, & b_5b^4 + b_2b^2 + b_1c^2 - b_3c^4 = 0, \\ -b_1c^2 + b_2d^2 + b_3c^4 - c_5d^4 = 0, & -a_3a^4 - a_1a^2 - a_2d^2 + d_5d^4 = 0. \end{array}$$

From these equations we obtain

$$a_5 = \frac{a^4a_3 + a^2a_1}{b^4} - \frac{a_2}{b^2},\ b_5 = \frac{b_3c^4 - b_1c^2}{b^4} - \frac{b_2}{b^2},\ c_5 = \frac{b_3c^4 - b_1c^2}{d^4} + \frac{b_2}{d^2},\ d_5 = \frac{a^4a_3 + a^2a_1}{d^4} + \frac{a_2}{d^2}.$$

Substituting these expressions into system (2), we obtain the system

$$\begin{array}{l} \alpha^2a_3 + \alpha a_1 - \beta a_2 - \beta^2(\dfrac{a^4a_3 + a^2a_1}{b^4} - \dfrac{a_2}{b^2}) = 0, \\ \beta^2(\dfrac{b_3c^4 - b_1c^2}{b^4} - \dfrac{b_2}{b^2}) + \beta b_2 - \delta b_1 - \delta^2b_3 = 0, \\ \delta b_1 - \gamma b_2 + \delta^2b_3 - \gamma^2(\dfrac{b_3c^4 - b_1c^2}{d^4} + \dfrac{b_2}{d^2}) = 0, \\ -\alpha^2a_3 - \alpha a_1 + \gamma a_2 + \gamma^2(\dfrac{a^4a_3 + a^2a_1}{d^4} + \dfrac{a_2}{d^2}) = 0. \end{array}$$

Doing tedious computations we have solved this algebraic system obtaining the following sets of solutions, that they have been also verified using algebraic manipulators as maple and mathematica:

$$
\begin{aligned}
S_1 =& \{\alpha = 0,\ \beta = 0,\ \delta = 0,\ \gamma = 0\}, \\
S_2 =& \{\alpha = 0,\ \beta = 0,\ \delta = -b_1/b_3,\ \gamma = 0\}, \\
S_3 =& \{\alpha = -a_1/a_3,\ \beta = 0,\ \delta = 0,\ \gamma = 0\}, \\
S_4 =& \{\alpha = -a_1/a_3,\ \beta = 0,\ \delta = -b_1/b_3,\ \gamma = 0\}, \\
S_5 =& \left\{\alpha = a^2,\ \beta = b^2,\ \delta = -c^2,\ \gamma = -d^2\right\}, \\
S_6 =& \left\{\alpha = a^2,\ \beta = b^2,\ \delta = c^2 - b_1/b_3,\ \gamma = -d^2\right\}, \\
S_7 =& \left\{\alpha = -(a^2 a_3 + a_1)/a_3,\ \beta = b^2,\ \delta = -c^2,\ \gamma = -d^2\right\}, \\
S_8 =& \left\{\alpha = -(a^2 a_3 + a_1)/a_3,\ \beta = b^2,\ \delta = c^2 - b_1/b_3,\ \gamma = -d^2\right\}, \\
S_9 =& \left\{\alpha = \alpha_1,\ \beta = 2b^4/(b^2+d^2) - b^2,\ \delta = \delta_1,\ \gamma = d^2(b-d)(b+d)/(b^2+d^2)\right\},
\end{aligned}
$$

where

$$
\alpha_1 = \tfrac{-1}{2a_3}\left(a_1 \pm \sqrt{\tfrac{4a_3(b-d)(b+d)(a^4 a_3(b-d)(b+d)+2a_2 b^2 d^2)}{(b^2+d^2)^2} + \tfrac{4a^2 a_1 a_3 (b^2-d^2)^2}{(b^2+d^2)^2} + a_1^2}\right),
$$

$$
\delta_1 = \tfrac{-1}{2b_3}\left(b_1 \pm \sqrt{\tfrac{b^4\left((b_1-2b_3c^2)^2+8b_2b_3d^2\right)+2b^2d^2\left(b_1^2+4b_1b_3c^2-4b_3(b_2d^2+b_3c^4)\right)+d^4(b_1-2b_3c^2)^2}{(b^2+d^2)^2}}\right).
$$

Among the previous sets of solutions, the only ones that satisfy the conditions $\alpha > 0$, $\beta > 0$, $\delta < 0$, and $\gamma < 0$ are S_5, S_6, S_7 and S_8. However, it should be noted that these four solutions can produce at most one limit cycle because they all share the same values for β and γ. This completes the proof of Theorem 1.

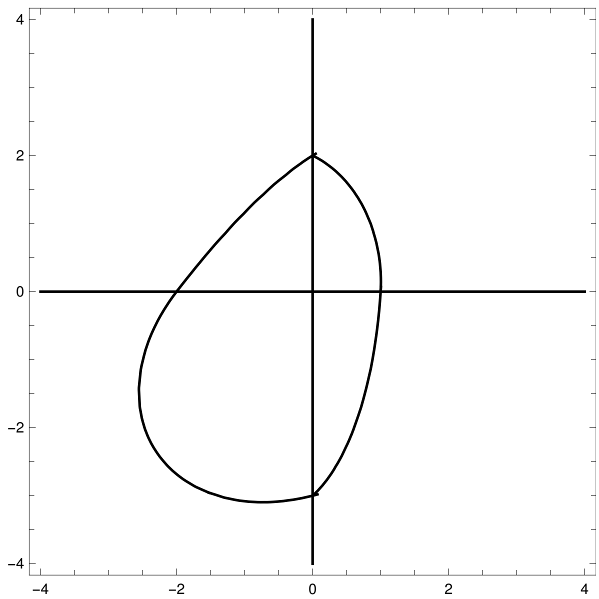

FIGURE 1. The limit cycle of the continuous-discontinuous piecewise differential system (3).

Now we provide a continuous-discontinuous piecewise differential system, continuous on the x-axis and discontinuous on the y-axis formed by the following four linear Hamiltonian systems

$$
\begin{array}{ll}
\dot{x} = 2 - 27y/2, & \dot{y} = 22x + 12, \\
\dot{x} = 3x - y - 11, & \dot{y} = 10x - 3y - 2, \\
\dot{x} = 3x - 38y/3 - 11, & \dot{y} = 10x - 3y - 2, \\
\dot{x} = 2 - 34y/9, & \dot{y} = 22x + 12,
\end{array}
\tag{3}
$$

having the Hamiltonians

$$
\begin{aligned}
H_1 =& 11x^2 + 12x + 27y^2/4 - 2y, \\
H_2 =& 5x^2 - 3xy - 2x + y^2/2 + 11y, \\
H_3 =& 5x^2 - 3xy - 2x + 19y^2/3 + 11y, \\
H_4 =& 11x^2 + 12x + 17y^2/9 - 2y,
\end{aligned}
$$

respectively. This piecewise differential system possess a unique limit cycle that intersects the axes at the points $(1, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -3)$. See figure 1.

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