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THE DYNAMICS OF THE LADDER SYSTEM

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ABSTRACT. We consider the n-dimensional ladder system, that is the homogeneous differential systems of the form

$$\dot{x}_i = x_i \sum_{j=1}^{n} (i+1-j)x_j, \quad i = 1, \ldots, n$$

introduced by Imai and Hirata for studying the integrability of a new class of Lotka-Volterra systems. Here we describe the dynamics of these Lotka-Volterra systems in arbitrary dimension.

1. Introduction and statement of the main results

Imai and Hirata using Lie point symmetries found in [8] a new integrable family in the class of the Lotka–Volterra systems $\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, \ldots, n$, see also [10]. More precisely, they found the n-dimensional ladder system

$$\dot{x}_i = x_i \sum_{j=1}^{n} (i+1-j)x_j \text{ for } i = 1, \ldots, n. \tag{1}$$

When $n = 1$ system (1) is the well-known Riccati equation. When $n = 2$ the 2-dimensional ladder system is reducible to the scalar second-order Ermakov–Pinney equation (which is well-known in physics). For $n = 3$ the 3-dimensional ladder system is reducible to the scalar third-order equation of maximal symmetry. This is one of the main reasons that this system has been previously studied by many authors. In [1] system (1) was studied using the Painlevé method. In [2] the ladder system is generalised to hyperladder system, and in [11] the superintebraility of the ladder system is analyzed.

In this paper we focus on the dynamical aspects of system (1). The main results of our paper are the following (see again section 2 for the definitions of all the notions that appear in the main theorems).

The first theorem is precisely the statement of the complete integrability of system (1) with very easy rational first integrals. Set $S_n = \sum_{i=1}^{n} x_i$.

Theorem 1. *The ladder system* (1) *with* $n \geq 2$ *has the invariant algebraic hyperplane* $S_n = 0$ *with cofactor* S_n. *Moreover, it is completely integrable with the* $n-1$ *independent rational first integrals*

$$H_k = \frac{x_k S_n}{x_{k+1}} = \frac{x_k \left(\sum_{j=1}^{n} x_j\right)}{x_{k+1}}, \quad k = 1, \ldots, n-1.$$

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The proof of Theorem 1 is given in section 3. Set $R_n = \sum_{i=1}^{n}(n-i+1)x_i$.

Theorem 2. *For $n \geq 2$ all the equilibria of the ladder system* (1) *form the* $(n-2)$*-dimensional hyperplane* $\Gamma_n = \{S_n = 0\} \cap \{R_n = 0\}$.

The proof of Theorem 2 is given in section 4.

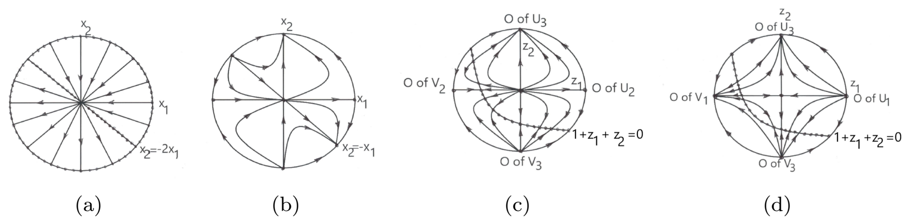

FIGURE 1. (a) The phase portrait in the Poincaré disc of the ladder system (1) restricted to the invariant plane $S_3 = 0$. (b) The phase portrait in the Poincaré disc of the ladder system (1) for $n = 2$. (c) The phase portrait in the Poincaré disc of the ladder system (1) for $n = 3$ restricted to the local chart U_1. (d) The phase portrait in the Poincaré disc of the ladder system (1) for $n = 3$ restricted to the local chart U_2.

Theorem 3. *The phase portrait of the ladder system* (1) *restricted to the invariant hyperplane $S_n = 0$ has the hyperplane $\sum_{i=1}^{n}(n-i)x_i = 0$ filled up with equilibria, and all the points of $S_n = 0$ satisfying $\sum_{i=1}^{n}(n-i)x_i > 0$ have their ω-limit at infinity and their α-limit at the origin, while the points satisfying $\sum_{i=1}^{n}(n-i)x_i < 0$ have their ω-limit at the origin and their α-limit at infinity. See Figure 1(a) for the case* $n = 3$.

The proof of Theorem 3 is given in section 5. The dynamics of the ladder system (1) is summarized in the next theorem.

Theorem 4. *Let $\varphi(t, S_0)$ be the solution $(x_1(t), \ldots, x_n(t))$ of the ladder system* (1) *such that* $\sum_{i=1}^{n} x_i(0) = S_0$.

(a) *If $S_0 < 0$ the solution $\varphi(t, S_0)$ comes from the infinity when t decreases tending to $1/S_0$, and goes versus the invariant hyperplane $S_n = 0$ when $t \to \infty$.*
(b) *If $S_0 = 0$ then the solution $\varphi(t, S_0)$ remains on the invariant hyperplane $\sum_{i=1}^{n} x_i = 0$, and its dynamics is described in Theorem 3.*
(c) *If $S_0 > 0$ the solution $\varphi(t, S_0)$ comes from the infinity when $t \to -\infty$, and goes versus the hyperplane $S_n = 0$ when t increases tending to $1/S_0$.*

The proof of Theorem 4 is given in section 6. From Theorem 4 it follows immediately the next corollary.

Corollary 5. *The ladder system* (1) *has no periodic solutions.*

Theorem 6. *The phase portrait of the ladder system* (1) *for $n = 2$ is shown in Figure 1(b).*

Theorem 7. *The ladder system* (1) *for* $n = 3$ *has four invariant planes* $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ *and* $S_3 = 0$. *The phase portrait on the plane* $S_3 = 0$ *is given in Figure 1(a), and the phase portrait on the planes* $x_i = 0$ *is given in Figure 1(b) for* $i = 1, 2, 3$. *On the sphere* $\mathbb{S}^2$ *of the infinity there is a circle filled up with equilibria, this circle is the boundary at infinity of the plane* $S_3 = 0$, *and at each equilibrium point of this circle either arrive or exit two orbits, except in two equilibria where arrives one orbit and exits another orbit. Moreover the origin of the local chart* U_1 *is a hyperbolic unstable node, the origin of the local chart* U_2 *is a hyperbolic saddle, and the origin of the local chart* U_3 *is a hyperbolic stable node. The phase portrait of the local chart* U_1 *is shown in Figure 1(c), while the phase portrait of the local chart* U_2 *is shown in Figure 1(d).*

The proofs of Theorems 6 and 7 are given in sections 7 and 8, respectively.

2. Preliminary notions

2.1. **First integrals.** Let $U \subset \mathbb{R}^n$ be an open subset. We say that the C^1 nonconstant function $H: U \to \mathbb{R}$ is a C^1*-first integral* of system (1) if $H(x_1(t), \ldots, x_n(t))$ is constant for all values of t for which the solution $(x_1(t), \ldots, x_n(t))$ is defined on U. When H is a polynomial we say that H is a *polynomial first integral*, when it is a rational function we say that H is a *rational first integral*. We will say that system (1) is *completely integrable* if it admits $n-1$ functionally independent first integrals. Since each level hypersurface of a first integral is invariant under the flow induced by the system, it is clear that if the system is completely integrable with the first integrals H_i, $i = 1, \ldots, n-1$, then the intersection $\bigcap_{i=1}^{n-1}\{H_i = h_i\}$, for $h_i \in \mathbb{R}$ for $i = 1, \ldots, n-1$, determines invariant curves that are formed by the orbits of the system.

2.2. **Invariant algebraic hypersurfaces.** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial. We say that $f(x_1, \ldots, x_n) = 0$ is *an invariant algebraic hypersurface* of system (1) if it satisfies $\sum_{k=1}^n \frac{\partial f}{\partial x_k} x_k \sum_{j=1}^n (k-j+1)x_j = Kf$, for some polynomial $K \in \mathbb{R}[x_1, \ldots, x_n)]$ called the *cofactor* of the invariant algebraic hypersurface $f = 0$. Note that K has degree at most one and that an invariant algebraic hypersurface with zero cofactor determines a polynomial first integral. For more information on the invariant algebraic hypersurfaces see [7, 12, 13]. The following result is well-known.

Lemma 8. *The product of* k *invariant algebraic hypersurfaces* $f_k = 0$ *with cofactors* K_k *is an invariant algebraic hypersurface* $f = 0$ *with cofactor* $\sum_k K_k$. *Moreover a rational first integral is a rational function that is either a polynomial first integral, or the quotient* f_1/f_2 *being* $f_i = 0$ *for* $i = 1, 2$ *invariant algebraic hypersurfaces with the same non-zero cofactor.*

2.3. ω **and** α**-limits.** Let $\phi_p(t)$ be the solution of system (1) defined on its maximal interval (α_p, ω_p) and such that $\phi_p(0) = p \in \mathbb{R}^n$. If $\omega_p = \infty$, we define the ω*-limit set* of p as $\omega(p) = \{q \in \mathbb{R}^n : \exists\{t_n\}\text{with } t_n \to \infty \text{ and } \phi_p(t_n) \to q \text{ when } n \to \infty\}$. In a similar way if $\alpha_p = -\infty$, we define the α*-limit set* of p as $\alpha(p) = \{q \in \mathbb{R}^n : \exists\{t_n\}\text{with } t_n \to -\infty \text{ and } \phi_p(t_n) \to q \text{ when } n \to \infty\}$. For more details on the ω– and α–limit sets see for instance section 1.4 of [7].

2.4. **Normally hyperbolicity.** Let Φ_t be a smooth flow on a manifold M and let C be a submanifold of M consisting entirely of equilibrium points of the flow. C is called *normally hyperbolic* if the tangent bundle to M over C splits into three subbundles TC, E^s and E^u invariant under the differential $d\Phi_t$ and satisfying: (i) $d\Phi_t$ contracts E^s exponentially, (ii) $d\Phi_t$ expandsE^u exponentially, and (iii) TC is the tangent bundle of C. For normally hyperbolic submanifolds we have the following theorem, for a proof see [9].

Theorem 9. *Let C be a normally hyperbolic submanifold of equilibrium points for the flow Φ_t. Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$, respectively. Moreover, both C and the stable and unstable manifolds are permanent under small perturbations of the flow.*

2.5. **The Poincaré compactification of $\mathbb{R}^n$.** We follow the description of the Poincaré compactification of $\mathbb{R}^n$ given in [3].

A polynomial vector field X in $\mathbb{R}^n$ can be extended to an analytic vector field on the closed n-dimensional ball of radius one centered at the origin of $\mathbb{R}^{n-1}$, the interior of this ball is diffeomorphic to $\mathbb{R}^n$ and its boundary (a $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$) plays the role of infinity. The technique for making such an extension is called the Poincaré compactification. This technique is as follows: consider in $\mathbb{R}^n$ the polynomial vector field of degree m, $X = (P_1, \ldots, P_n)$, with $P_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $m = \max\{\deg(P_i)\} : i = 1, \ldots, n\}$. Let $y = (y_1, \ldots, y_{n+1})$ and

$$\mathbb{S}^n = \{y : ||y|| = 1\}, \quad \mathbb{S}_+ = \{y \in \mathbb{S}^n : y_{n+1} > 0\}, \quad \mathbb{S}_- = \{y \in \mathbb{S}^n : y_{n+1} < 0\}$$

be the unit sphere in $\mathbb{R}^{n+1}$, the northern hemisphere of $\mathbb{S}^n$ and the southern hemisphere of $\mathbb{S}^n$, respectively. The tangent space of $\mathbb{S}^n$ at the point y will be denoted by $T_y\mathbb{S}^n$ and the tangent plane $T_{(0,\ldots,0,1)}\mathbb{S}^n = \{(x_1, \ldots, x_n, 1) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ is identified with $\mathbb{R}^n$.

Consider the central projections $f_\pm \colon \mathbb{R}^n = T_{(0,\ldots,0,1)}\mathbb{S}^n \to \mathbb{S}_\pm$, where $f_\pm(x) = \pm\frac{(x_1,\ldots,x_n,1)}{\Delta(x)}$ and $\Delta(x) = \left(1 + \sum_{i=1}^n x_i^2\right)^{1/2}$. Using these central projections $\mathbb{R}^n$ is identified with $\mathbb{S}_+$ and $\mathbb{S}_-$. Note that the equator of $\mathbb{S}^n$ is $\mathbb{S}^{n-1} = \{y \in \mathbb{S}^n : y_{n+1} = 0\}$, which is identified with the infinity of $\mathbb{R}^n$.

The maps $f_\pm$ define two copies of X on $\mathbb{S}^n$, one $Df_+ \circ X$ in $\mathbb{S}_+$, and the other, $Df_- \circ X$ in $\mathbb{S}_-$. Denote by $\overline{X}$ the vector field on $\mathbb{S}^n \setminus \mathbb{S}^{n-1} = \mathbb{S}_+ \cup \mathbb{S}_-$, that restricted to $\mathbb{S}_+$ coincides with $Df_+ \circ X$, and restricted to $\mathbb{S}_-$ coincides with $Df_- \circ X$. We can extend analytically the vector field $\overline{X}(y)$ to the whole sphere $\mathbb{S}^n$ setting $p(X) = y_{n+1}^{m-1}\overline{X}(y)$. This extended vector field $p(X)$ is called the *Poincaré compactification* of X on $\mathbb{S}^n$.

Using that $\mathbb{S}^n$ is a differentiable manifold, to compute the expression for the vector field $p(X)$, we can consider the $2(n+1)$ local charts (U_i, F_i), (V_i, G_i), where $U_i = \{y \in \mathbb{S}^n : y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^n : y_i < 0\}$, for $i = 1, \ldots, n+1$.

Note that the diffeomorphisms $F_i \colon U_i \to \mathbb{R}^n$ and $G_i \colon V_i \to \mathbb{R}^3$ for $i = 1, \ldots, n+1$ are the inverse of the central projections from the origin to the tangent hyperplane at the points $(\pm1, 0 \ldots, 0), \ldots, (0, \ldots, 0, \pm1)$, respectively.

The analytical vector field $p(X)$ in the local chart $U_1, \ldots, U_{n+1}$ becomes, after a rescaling of the time variable,

$$
\begin{aligned}
& z_n^m(P_2 - z_1P_1, P_3 - z_2P_1, \ldots, P_n - z_{n-1}P_1, -z_nP_1),\ P_i = P_i\Big(\frac{1}{z_n}, \frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\Big) \text{ in } U_1,\\
& z_n^m(P_1 - z_1P_2, P_3 - z_2P_2, \ldots, P_n - z_{n-1}P_2, -z_nP_2),\ P_i = P_i\Big(\frac{z_1}{z_n}, \frac{1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\Big) \text{ in } U_2,\\
& \ldots\\
& z_n^m(P_1 - z_1P_n, P_2 - z_2P_n, \ldots, P_{n-1} - z_{n-1}P_n, -z_nP_n),\ P_i = P_i\Big(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}, \frac{1}{z_n}\Big) \text{ in } U_n,\\
& (P_1, \ldots, P_n),\ P_i = P_i(z_1, \ldots, z_n) \text{ in } U_{n+1}.
\end{aligned}
$$

The expression for $p(X)$ in V_i is the same as in U_i multiplied by $(-1)^{m-1}$, for all $i = 1, \ldots, n+1$.

From now on we will consider only the orthogonal projection of $p(X)$ from $\mathbb{S}_+$ to $y_{n+1} = 0$ and we will denote it again by $p(X)$. Observe that the projection of the closed $\mathbb{S}_+$ is a closed ball of radius one, denoted by B, whose interior is diffeomorphic to $\mathbb{R}^n$. Its boundary $\mathbb{S}^{n-1}$ corresponds to the infinity of $\mathbb{R}^n$. Moreover, $p(X)$ is defined in the whole closed ball B in such way that the flow on the boundary, given by $z_n = 0$ in all the local charts U_i, V_i for $i = 1, \ldots, n$, is invariant. The vector field induced by $p(X)$ on B is called the *Poincaré compactification of* X and B is called the *Poincaré n-ball* and $\mathbb{S}^{n-1}$ the *Poincaré $n-1$-sphere.*

We recall that two polynomial vector fields X and Y on $\mathbb{R}^n$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^n$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$ preserving or reversing the orientation of all the orbits.

3. Proofs of Theorem 1

Note that

$$
\begin{aligned}
\dot S_n &= \sum_{i=1}^n \dot x_i = \sum_{i=1}^n\sum_{j=1}^n (i-j+1)x_ix_j = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \sum_{j=1, j\neq i}^n ((i-j+1)+(j-i+1)x_ix_j\\
&= \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n \sum_{j=1, j\neq i}^n x_ix_j = \Big(\sum_{i=1}^n x_i\Big)^2 = S_n^2.
\end{aligned}
$$

So $S_n = 0$ is an invariant hyperplane with cofactor $K = S_n$.

On the other hand $x_k = 0$ is also an invariant hyperplane with cofactor $K_k = \sum_{j=1}^n (k-j+1)x_j$ for each $k = 1, \ldots, n$. Therefore by Lemma 8 the cofactor of H_k is given by the sum of the cofactors of the invariant hyperplanes $x_k = 0$, $S_k = 0$ and $x_{k+1} = 0$, i.e. $\sum_{j=1}^n (k-i+1)x_i + \sum_{i=1}^n x_i - \sum_{j=1}^n (k+2-i)x_i = 0$. So H_k is a rational first integral for each $k = 1, \ldots, n-1$. Since H_k for each $k = 1, \ldots, n-1$ are independent this proves that system (1) is completely integrable.

4. Proof of Theorem 2

We proceed by induction over n. For $n = 2$ the ladder system (1) has the form

$$\dot{x}_1 = x_1^2, \quad \dot{x}_2 = x_2(2x_1 + x_2), \tag{2}$$

and $\Gamma_2 = \{x_1 = x_2 = 0\}$. Since the unique equilibrium point of system (2) is $x_1 = x_2 = 0$ (the origin) which is precisely Γ_2 we get that the theorem is proved for $n = 2$.

For $n = 3$ the ladder system (1) has the form

$$\dot{x}_1 = x_1(x_1 - x_3), \quad \dot{x}_2 = x_2(2x_1 + x_2), \quad \dot{x}_3 = x_3(3x_1 + 2x_2 + x_3), \tag{3}$$

and $\Gamma_3 = \{S_3 = 0\} \cap \{R_3 = 0\} = \{x_1 + x_2 + x_3 = 0\} \cap \{3x_1 + 2x_2 + x_3 = 0\} = \{x_3 = x_1,\ x_2 = -2x_1\}$. The equilibrium points of (3) are $x_1 = x_2 = x_3 = 0$ or $\{x_3 = x_1,\ x_2 = -2x_1\}$. Note that they are all in Γ_3. So the theorem is proved for $n = 3$.

Assume it is true until $n-1$ and we will prove it for n. Note that it follows from system (1) that an equilibrium point satisfies

$$x_k \sum_{i=1}^{n} (k - i + 1)x_i = 0, \quad \text{for } k = 1, \ldots, n. \tag{4}$$

Taking $k = n$ we have that the equilibrium point must satisfy that either $x_n = 0$, or $\sum_{i=1}^{n}(n - i + 1)x_i = 0$. We consider two different cases: $x_n = 0$ and $x_n \neq 0$.

Case 1: $x_n = 0$. Note that if $x_n = 0$ then system (1) becomes system (1) but with dimension $n - 1$, and by induction such system has all the equilibrium points on Γ_{n-1}. Note that $\Gamma_{n-1} = \Gamma_n \cap \{x_n = 0\}$. Indeed, $\Gamma_{n-1} = \{S_{n-1} = 0\} \cap \{R_{n-1} = 0\} = \left\{ \sum_{i=1}^{n-1} x_i = 0 \right\} \cap \left\{ \sum_{i=1}^{n-1} (n - i)x_i = 0 \right\}$. Moreover, $\Gamma_n \cap \{x_n = 0\} = \{S_n = 0\} \cap \{R_n = 0\} \cap \{x_n = 0\}$ is equal to

$$\begin{aligned}
&\left\{ \sum_{i=1}^{n} x_i = 0 \right\} \cap \left\{ \sum_{i=1}^{n} (n - i + 1)x_i = 0 \right\} \cap \{x_n = 0\} = \\
&\left\{ \sum_{i=1}^{n-1} x_i = 0 \right\} \cap \left\{ \sum_{i=1}^{n-1} (n - i + 1)x_i = 0 \right\} = \left\{ \sum_{i=1}^{n-1} x_i = 0 \right\} \cap \left\{ \sum_{i=1}^{n-1} (n - i)x_i + \sum_{i=1}^{n-1} x_i = 0 \right\} \\
&= \left\{ \sum_{i=1}^{n-1} x_i = 0 \right\} \cap \left\{ \sum_{i=1}^{n-1} (n - i)x_i = 0 \right\} = \Gamma_{n-1},
\end{aligned}$$

as we wanted to show.

Case 2: $x_n \neq 0$. In this case we must have $\sum_{i=1}^{n}(n - i + 1)x_i = 0$, that is $R_n = 0$. From there we consider $x_n = -\sum_{i=1}^{n-1}(n - i + 1)x_i$, and we introduce it in (4). In this case (4) becomes

$$\begin{aligned}
x_k \sum_{i=1}^{n} (k - i + 1)x_i &= x_k \left(\sum_{i=1}^{n-1} (k - i + 1)x_i + (k - n + 1)x_n \right) \\
&= x_k \left(\sum_{i=1}^{n-1} (k - i + 1)x_i - (k - n + 1) \sum_{i=1}^{n-1} (n - i + 1)x_i \right) = x_k(n - k) \sum_{i=1}^{n-1} (n - i)x_i,
\end{aligned}$$

for $k = 1, \ldots, n-1$. Therefore the equilibrium points must satisfy that either $x_k = 0$ for $k = 1, \ldots, n-1$, or there is some $k = 1, \ldots, n-1$ with $x_k \neq 0$ and $\sum_{i=1}^{n-1}(n-i)x_i = 0$. In the first case from $x_n = -\sum_{i=1}^{n-1}(n-i+1)x_i$ we have that $x_n = 0$ which is not possible. In the second case, we have $x_n = -\sum_{i=1}^{n-1}(n-i+1)x_i = -\sum_{i=1}^{n-1} x_i$, that yields $\sum_{i=1}^{n} x_i = 0$ which is $S_n = 0$. Therefore the equilibria are on $\{S_n = 0\} \cap \{R_n = 0\} = \Gamma_n$ and the theorem is proved.

5. Proof of Theorem 3

On the invariant hyperplane $S_n = 0$ we have that $x_n = -\sum_{i=1}^{n-1} x_i$, and so system (1) on $S_n = 0$ becomes

$$\begin{aligned} \dot{x}_k &= x_k \sum_{i=1}^{n}(k-i+1)x_i = x_k\left(\sum_{i=1}^{n-1}(k-i+1)x_i + (k-n+1)x_n\right) \\ &= x_k\left(\sum_{i=1}^{n-1}(k-i+1)x_i - (k-n+1)\sum_{i=1}^{n-1} x_i\right) = x_k \sum_{i=1}^{n-1}(n-i)x_i \end{aligned}$$

for all $k = 1, \ldots, n-1$. After a rescaling of the independent variable as $ds = \sum_{i=1}^{n}(n-i)x_i\, dt$, system (1) on $S_n = 0$ becomes

$$x_k' = x_k, \quad k = 1, \ldots, n-1, \tag{5}$$

where the prime denotes derivative with respect to the new variable s. The origin of system (5) is a repeller and all the points different form the origin have their ω-limit at infinity and their α-limit at the origin. Therefore the phase portrait of the ladder system over $S_n = 0$ is given by the hyperplane of equilibria $\sum_{i=1}^{n}(n-i)x_i = 0$, and all the points that are in $\sum_{i=1}^{n}(n-i)x_i > 0$ have their ω-limit at infinity and their α-limit at the origin, while all the points that are in $\sum_{i=1}^{n}(n-i)x_i < 0$ have their ω-limit at the origin and their α-limit at infinity. See Figure 1(a) for the case $n = 3$. This completes the proof of the theorem.

6. Proof of Theorem 4

From the proof of Theorem 1 we have that $\dot{S}_n = S_n^2$. So its solution $S_n(t, S_0)$ satisfying $S_n(t, 0) = S_0$ is $S_n(t, S_0) = S_0/(1 - S_0 t)$, i.e. $\sum_{i=1}^{n} x_i(t) = S_0/(1 - S_0 t)$ with $\sum_{i=1}^{n} x_i(0) = S_0$.

Now we have that if $S_0 < 0$ the solution $(x_1(t), \ldots, x_n(t))$ of the ladder system satisfying $\sum_{i=1}^{n} x_i(0) = S_0$ comes from the infinity when t decreases tending to $1/S_0$, and goes versus the invariant hyperplane $S_n = 0$ when $t \to \infty$. This proves statement (a) of Theorem 4.

If $S_0 = 0$ then the solution $(x_1(t), \ldots, x_n(t))$ of the ladder system satisfying $\sum_{i=1}^{n} x_i(0) = 0$ remains on the invariant hyperplane $\sum_{i=1}^{n} x_i = 0$, and its dynamics is described in Theorem 3. So statement (b) of Theorem 4 follows.

Finally if if $S_0 > 0$ the solution $(x_1(t), \ldots, x_n(t))$ of the ladder system satisfying $\sum_{i=1}^{n} x_i(0) = S_0$ comes from the infinity when $t \to -\infty$ and goes versus the hyperplane $S_n = 0$ when t increases tending to $1/S_0$. This completes the proof of statement (c) of Theorem 4.

7. Proof of Theorem 6

System (1) with $n = 2$ becomes system (2). We already know that it has the rational first integral $H_1 = \frac{x_1(x_1+x_2)}{x_2}$, being $x_1 = 0$, $x_2 = 0$ and $S_2 = x_1 + x_2 = 0$ three invariant straight lines passing through the origin of coordinates.

From subsection 2.5 the expression of the Poincaré compactification $p(X)$ of system (2) in the local chart U_1 is given by $\dot{u} = (1+u)u$, $v' = v$, and so there are two infinite equilibrium points which are $(-1, 0)$ and $(0, 0)$. Computing the eigenvalues of the Jacobian matrix associated to these infinite equilibrium points we obtain that $(-1, 0)$ is a hyperbolic stable node and $(0, 0)$ is a hyperbolic saddle, for more details see [7, Theorem 2.15].

On the local chart U_2 system (2) becomes $\dot{u} = -u(1+u)$, $\dot{v} = (1+2u)v$, and so the origin is an infinite equilibrium point. Computing the eigenvalues of the Jacobian matrix associated to the origin of the local chart U_2 we get that it is an hyperbolic stable node.

We thus have that the three invariant straight lines reach the infinity in the 6 infinite equilibrium points which are all of them hyperbolic. Taking into account the three straight lines and the behaviour of the equilibria at infinity it follows the phase portrait of the 2-dimensional ladder system (1) is the one described in Theorem 6 (i)–(v) and given in Figure 1(b). This proves Theorem 6.

8. Proof of Theorem 7

System (1) with $n = 3$ becomes system (3). We already know that it has the two rational first integrals $H_1 = \frac{x_1(x_1+x_2+x_3)}{x_2}$ and $H_2 = \frac{x_2(x_1+x_2+x_3)}{x_3}$, being $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $S_3 = x_1 + x_2 + x_3 = 0$ four invariant planes passing through the origin of coordinates.

On the plane $S_3 = 0$ the dynamics has been studied in Theorem 3. On the planes $x_1 = 0$ and $x_3 = 0$ the dynamics is the same as the one of system (1) for $n = 2$, because it is clear that system (3) restricted to $x_3 = 0$ becomes system (2), and system (3) restricted to $x_1 = 0$ after denoting (x_2, x_3) as (x_1, x_2) we get the same system (2).

System (3) restricted to $x_2 = 0$ becomes

$$\dot{x}_1 = x_1(x_1 - x_2), \quad \dot{x}_2 = x_2(3x_1 + x_2), \tag{6}$$

where we have renamed x_3 as x_2. We recall that all the quadratic homogeneous polynomial differential systems $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, in the plane $\mathbb{R}^2$ have the same phase portrait in the Poincaré disc when its homogeneous infinite polynomial $P(x, y)y - Q(x, y)x$ has three real linear factors, see for more details [4, 5, 6]. Since the homogeneous polynomial differential system (2) has its infinite polynomial equal to $-x_1x_2(x_1 + x_2)$, and the homogeneous polynomial differential system (6) has its infinite polynomial equal to $-2x_1x_2(x_1 + x_2)$, it follows that both systems have topological equivalent phase portraits.

Now we analyse the dynamics of system (3) at infinity using the Poincaré compactification of the system in $\mathbb{R}^3$. From subsection 2.5 the expresion of the Poincaré compactification of system (3) in the local chart U_1 is

$$\dot{z}_1 = z_1(1 + z_1 + z_2), \quad \dot{z}_2 = 2z_2(1 + z_1 + z_2), \quad \dot{z}_3 = (-1 + z_2)z_3. \tag{7}$$

For $z_3 = 0$, i.e. for the infinite points in this local chart, system (7) reduces to

$$\dot{z}_1 = z_1(1 + z_1 + z_2), \quad \dot{z}_2 = 2z_2(1 + z_1 + z_2). \tag{8}$$

This system has the straight line of equilibria $1+z_1+z_2 = 0$. After removing it with the rescaling of time $ds = (1+z_1+z_2)\,dt$ the system becomes $\dot{z}_1 = z_1$, $\dot{z}_2 = 2z_2$, and so the origin of U_1 is the unique equilibrium point which is an unstable hyperbolic star node, i.e. all the orbits live on invariant straight lines through the origin of coordinates.

On the equilibria of the straight line $1+z_1+z_2 = 0$ the eigenvalues of the Jacobian matrix of system (7) are $0, -(2 + z_1), -(2 + z_1)$, so the eigenvalues restricted to infinity are $0, -(2 + z_1)$. By Theorem 9 at the equilibrium point $(z_1, -2 - z_1)$ it arrives two orbits if $-(2 + z_1) < 0$, it exits two orbits if $-(2 + z_1) > 0$, and if $-(2 + z_1) = 0$ arrives one orbit and exits another one. See Figure 1(c).

The expression of the Poincaré compactification of system (3) in the local chart U_2 is

$$\dot{z}_1 = -z_1(1 + z_1 + z_2), \quad \dot{z}_2 = z_2(1 + z_1 + z_2), \quad \dot{z}_3 = -(1 + 2z_1)z_3. \tag{9}$$

System (9) restricted to $z_3 = 0$ reduces to

$$\dot{z}_1 = -z_1(1 + z_1 + z_2), \quad \dot{z}_2 = z_2(1 + z_1 + z_2). \tag{10}$$

We are only interested in studying the infinite equilibrium points of the local chart U_2 which have not been studied in the local chart U_1, i.e. the infinite equilibrium points with $z_3 = 0$ and $z_1 = 0$. Then system (10) becomes $\dot{z}_2 = z_2(1+z_2)$. Therefore in the straight line of the infinity $z_3 = z_1 = 0$ there are only two equilibrium points $(0, 0, 0)$ and $(0, -1, 0)$. The eigenvalues of the Jacobian matrix of system (9) at the origin $(0, 0, 0)$ of U_2 are $-1, -1, 1$, and the eigenvalues restricted to infinity are $-1, 1$, so this equilibrium point on the local chart U_2 is a hyperbolic saddle. More precisely, after removing the straight line $1 + z_1 + z_2 = 0$ with the rescaling of time $ds = (1 + z_1 + z_2)\,dt$ system (10) becomes $\dot{z}_1 = -z_1$, $\dot{z}_2 = z_2$, and so the origin is the unique equilibrium point which is the mencioned hyperbolic saddle.

On the other hand the eigenvalues of the Jacobian matrix of system (9) at the equilibrium point $(0, -1, 0)$ are $-1, -1, 0$, and the eigenvalues restricted to infinity are $-1, 0$. This last equilibrium point belongs to the straight line $1 + z_1 + z_2 = 0$ filled up with equilibria. Consequently the phase portrait of the local chart U_2 is shown in Figure 1(d).

The two straight lines filled up with equilibria of the local charts U_1 and U_2 form at infinity a circle filled up with equilibria. We note that this circle of equilibrium points detected on the local charts U_1 and U_2 is the intersection of the invariant plane $S_3 = 0$ with the sphere $\mathbb{S}^2$ of the infinity.

In the local chart U_3 we only need to study if its origin is an equilibrium point because all the other infinite equilibria have been studied in the local charts U_1 and U_2. Then system (1) in the local chart U_3 becomes

$$\dot{z}_1 = -2z_1(1 + z_1 + z_2), \quad \dot{z}_2 = -z_2(1 + z_1 + z_2), \quad \dot{z}_3 = -(1 + 3z_1 + 2z_2)z_3. \tag{11}$$

The eigenvalues of the Jacobian matrix of system (11) at the equilibrium point $(0,0,0)$ of U_3 are $-2,-1,-1$, so the local phase portrait at this equilibrium point on the local chart U_3 is a hyperbolic stable node with eigenvalues $-2,-1$.

We recall that in Figures 1(c) and 1(d) it already appears the stable node at the origin of U_3. In fact the origins of all the six charts U_i, V_i for $i = 1, \dots, 3$ appear already in Figures 1(c) and 1(d).

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