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Three-dimensional limit cycles generated from discontinuous piecewise differential systems separated by two intersecting planes

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Due to their widespread application in modeling natural issues, the study of piecewise linear differential systems has gained relevance in recent years. It is well known that the qualitative theory of piecewise linear differential systems heavily relies on limit cycles. Until now most studies have only considered planar systems by examining the presence and maximum number of limit cycles for piecewise differential systems. However few articles studied this issue in $\mathbb{R}^3$. We remind the problem of the existence and the maximum number of limit cycles for planar discontinuous piecewise differential systems formed by linear differential centers separated by one or two parallel straight lines that have at most one limit cycle, respectively. Although in $\mathbb{R}^3$ the maximal number of limit cycles for the same problem is 0 when the separation surface is a plane and at most four limit cycles if the separation surface is two parallel planes.
In this article we mainly focus on the problem of the existence and the maximum number of limit cycles in $\mathbb{R}^3$, when the separating surface is formed by two intersecting half-planes.
First we prove that when the entire space is divided into two regions, this family can have at most five limit cycles, where one limit cycle intersects the separation surface in two points and the remaining four limit cycles intersect the separation surface in four points. Second when the entire space is divided into three regions, we prove that the maximum number of limit cycles intersecting the separation surface in three points and four points simultaneously is at most eight.

Keywords: Piecewise differential system, $3D$-center, $3D$-limit cycle, separation surface.

1. Introduction and statement of the main results

The 1930s noticed the beginning of the first studies of discontinuous piecewise linear differential systems through the works of Andronov *et al.,* see [Andronov *et al.*, 1996]. After that, due to their extensive use in modeling the mechanisms involved in numerous natural phenomena, piecewise differential systems became more and more used in a variety of applied mathematics fields as well as in mechanics, electronics, economics, neuroscience, and other areas, see for example [Di Bernardo *et al.*, 2008; Makarenkov & Lamb, 2012; Simpson, 2010].

Following the Filippov rules given in [Filippov, 1988], we consider $f : \mathbb{R}^3 \to \mathbb{R}$ a C^k smooth function for $1 \leq k \leq \infty$ having 0 as a regular value and $S = f^{-1}(0)$ the *discontinuity region.* Denoting by $S^+ = \{X = (x, y, z) \in \mathbb{R}^3 : f(X) \geq 0\}$ and $S^- = \{X = (x, y, z) \in \mathbb{R}^3 : f(X) \leq 0\}$, we consider the *discontinuous piecewise differential systems* of the form

$$\dot{X} = \begin{cases} F(X), & \text{if } \ X \in S^+, \\ G(X), & \text{if } \ X \in S^-, \end{cases} \tag{1}$$

where $F; S^+ \to \mathbb{R}^3$ and $G; S^- \to \mathbb{R}^3$ are linear vector fields.

A point, $X \in S$ is a *tangency point* of F (resp. G) if $F(X) = 0$ (resp. $G(X) = 0$). A point X_0 of S is called an *escaping point* if the vector fields $F(X_0)$ and $G(X_0)$ move both either outward, or inward with respect to S, and it is of *sliding* type if $F(X_0)$ and $G(X_0)$ points inward. The point X_0 is of *crossing* type if the vector fields $F(X_0)$ and $G(X_0)$ move in the same direction with respect to S. The Lie derivatives are useful for classifying points on the discontinuity region S, acting as follows. At the point $X_0 \in S$ we know that the Lie derivative has the form

$$F(f(X_0)) = \langle \bigtriangledown f(X_0), F(X_0) \rangle.$$

The transversal points on S with respect to the vector fields F and G are classified by:

The escaping region: $R^e = \{X \in S, \ F(f(X)) > 0 \text{ and } \ G(f(X)) < 0\}$ formed by escaping points.

The sliding region: $R^s = \{X \in S, \ F(f(X)) < 0 \text{ and } \ G(f(X)) > 0\}$ formed by sliding points.

The crossing region: $R^c = \{X \in S, \ F(f(X)) \cdot G(f(X)) > 0\}$ formed by crossing points.

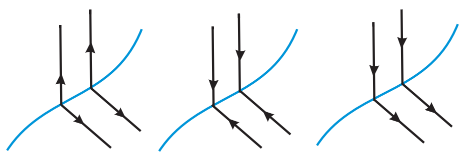

Fig. 1. (a) Escaping, (b) sliding and (c) crossing regions.

We recall that a *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbits of the system. For piecewise vector field in $\mathbb{R}^3$ if the limit cycle contains only crossing points on the discontinuity surface, we say that it is a *crossing limit cycle.* In this paper we are interested in the crossing limit cycle that will be referred to $3D$-limit cycle.

Limit cycles have been considered for the first time by Poincaré [Poincaré, 1891,1897]. Further on, the occurrence of limit cycles has been well observed in the real world, as the generalized Liénard system [Kasbi & Roomi , 2018], the limit cycle of the Van der Pol equation [Van Der Pol, 1920, 1926], the limit cycles of the Belousov Zhavotinskii model [Belousov, 1959], and of the galaxy motion [De Bustos *et al.*, 2016], and many others, which attracted the attention of the mathematical community, and evolved into the primary goal of the second part of the 16th Hilbert's problem. As seen in [Hilbert, 2003; Ilyashenko, 2002; Li , 2003], the second part of the 16th Hilbert's problem asks for an upper bound for the maximum number of limit cycles that planar polynomial differential systems of a given degree can have. The same problem has been extended to piecewise differential systems, and numerous researchers have been working on finding a solution to this problem for some specific classes of piecewise differential systems during these last years.

Over the past two decades there have been many investigations of the limit cycles of piecewise differential systems in the plane, see for example [Braga & Mello, 2014, 2013; Llibre, 2023; Llibre *et al.*, 2014; Llibre & Teixeira, 2018; Zhao *et al.*, 2021] and the references therein. The outcomes in the cited studies have proved that the number of limit cycles that the piecewise differential systems can have is significantly influenced by the form of the discontinuity curve.

It has been shown in [Llibre & Teixeira, 2018] that the simplest class of discontinuous piecewise vector fields in $\mathbb{R}^2$ produced by arbitrary linear centers separated by one straight line has no limit cycles, and it has at most one limit cycle if the discontinuity curve is formed by two parallel straight lines. The same class of discontinuous piecewise differential systems has been considered by Villanueva *et al.* [Villanueva *et al.*, 2022] in $\mathbb{R}^3$ where they separated the entire space by two parallel planes, and such class of differential systems can have at most four limit cycles.

This paper investigates the maximum number of $3D$-limit cycles for two families of discontinuous piecewise differential systems in $\mathbb{R}^3$ separated by two intersecting half-planes instead of two parallel planes and formed by linear differential centers in $\mathbb{R}^3$. More precisely, we consider arbitrary linear differential center in $\mathbb{R}^3$ defined by

$$\dot{X} = M_i X + N_i, \tag{2}$$

with $X = (x, y, z)$ and

$$M_i = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \quad N_i = \begin{pmatrix} a_{i2}a_{i4}c_{i3} - a_{i3}a_{i4}c_{i2} + b_{i2}b_{i4}c_{i3} - b_{i3}b_{i4}c_{i2} \\ -a_{i1}a_{i4}c_{i3} + a_{i3}a_{i4}c_{i1} - b_{i1}b_{i4}c_{i3} + b_{i3}b_{i4}c_{i1} \\ a_{i1}a_{i4}c_{i2} - a_{i2}a_{i4}c_{i1} + b_{i1}b_{i4}c_{i2} - b_{i2}b_{i4}c_{i1} \end{pmatrix}, \tag{3}$$

such that

$$\begin{aligned} &m_{11} = a_{i1}a_{i2}c_{i3} - a_{i1}a_{i3}c_{i2} + b_{i1}b_{i2}c_{i3} - b_{i1}b_{i3}c_{i2}, && m_{12} = c_{i3}\left(a_{i2}^2 + b_{i2}^2\right) - c_{i2}(a_{i2}a_{i3} + b_{i2}b_{i3}), \\ &m_{13} = c_{i3}(a_{i2}a_{i3} + b_{i2}b_{i3}) - c_{i2}\left(a_{i3}^2 + b_{i3}^2\right), && m_{21} = c_{i1}(a_{i1}a_{i3} + b_{i1}b_{i3}) - c_{i3}\left(a_{i1}^2 + b_{i1}^2\right), \\ &m_{22} = -a_{i1}a_{i2}c_{i3} + a_{i2}a_{i3}c_{i1} - b_{i1}b_{i2}c_{i3} + b_{i2}b_{i3}c_{i1}, && m_{23} = c_{i1}\left(a_{i3}^2 + b_{i3}^2\right) - c_{i3}(a_{i1}a_{i3} + b_{i1}b_{i3}), \\ &m_{31} = c_{i2}\left(a_{i1}^2 + b_{i1}^2\right) - c_{i1}(a_{i1}a_{i2} + b_{i1}b_{i2}), && m_{32} = c_{i2}(a_{i1}a_{i2} + b_{i1}b_{i2}) - c_{i1}\left(a_{i2}^2 + b_{i2}^2\right), \\ &m_{33} = a_{i1}a_{i3}c_{i2} - a_{i2}a_{i3}c_{i1} + b_{i1}b_{i3}c_{i2} - b_{i2}b_{i3}c_{i1}, && \end{aligned}$$

with its corresponding two independent first integrals

$$\begin{aligned} H_{i1}(x, y, z) &= c_{i1}x + c_{i2}y + c_{i3}z + c_{i4}, \\ H_{i2}(x, y, z) &= (a_{i1}x + a_{i2}y + a_{i3}z + a_{i4})^2 + (b_{i1}x + b_{i2}y + b_{i3}z + b_{i4})^2. \end{aligned}$$

The arbitrary linear differential system in $\mathbb{R}^3$ defined by the vector field (2), will be referred in this study as a *linear center* in $\mathbb{R}^3$, or simply a $3D$*-center*.

We note that the vector field (2) is obtained after applying an arbitrary affine transformation

$$(x, y, z) \mapsto (a_1x + a_2y + a_3z + a_4, b_1x + b_2y + b_3z + b_4, c_1x + c_2y + c_3z + c_4)$$

to the linear differential system

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0,$$

which has the two independent first integrals $H_1(\mathbf{x}) = z$.and $H_2(\mathbf{x}) = x^2 + y^2$.

First we discuss our findings regarding $3D$-limit cycles for the family of discontinuous piecewise linear differential systems

$$\dot{X} = \begin{cases} M_1X + N_1 & \text{if } X \in R_1, \\ M_2X + N_2 & \text{if } X \in R_2, \end{cases} \tag{4}$$

where M_i and N_i with $i = 1, 2$ are given by (3), and the discontinuity surface $\Gamma = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, \quad x \geq 0\}$ and $\Gamma_2 = \{(x, y, z) \in \mathbb{R}^3 : x = 0, \quad z \geq 0\}$, divides the space into two regions

$$R_1 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, z \geq 0\} \text{ and } R_2 = \{(x, y, z) : x \geq 0, z \leq 0\} \cup \{(x, y, z) : x \leq 0\}.$$

In this case we realized that there are only two different possible types of $3D$-limit cycles for the discontinuous piecewise linear differential system (4) separated by Γ.

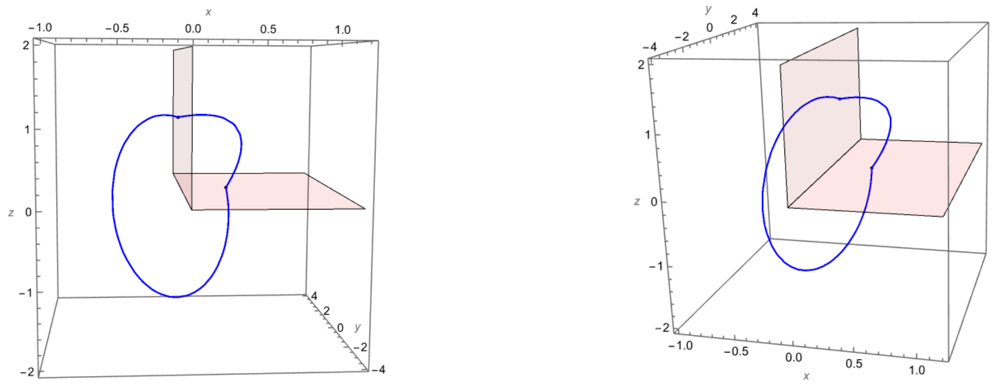

Fig. 2. The unique $3D$-limit cycle of type T_1 for the discontinuous piecewise differential system (7)-(8) in two different views.

- The type T_1 corresponding to the $3D$-limit cycles intersecting the separation surface Γ at two points, one in Γ_1 and the other one in Γ_2, as shown Figure 2.
- The type T_2 corresponding to the limit cycles intersecting Γ at four points, two in Γ_1 and the two others in Γ_2, as shown Figure 3.

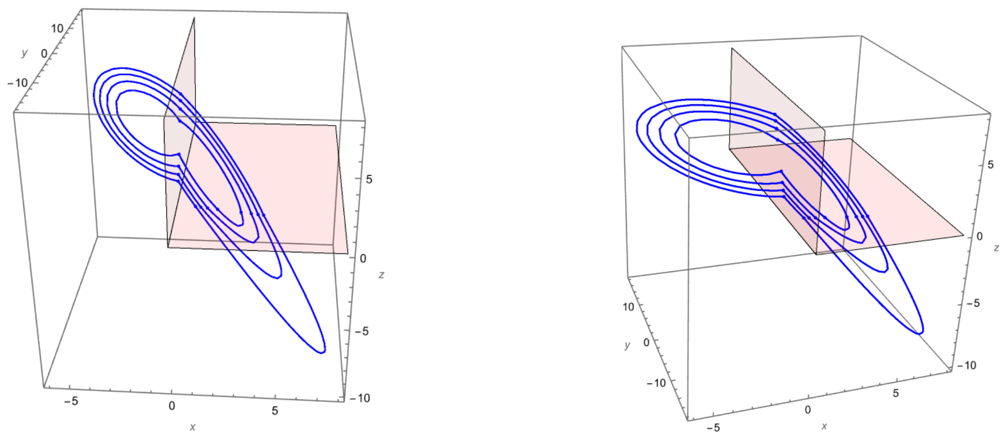

Fig. 3. Two different views of the four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (10)-(11).

Concerning the type of limit cycles intersecting with the separation surface Γ at two points either in Γ_1, or in Γ_2 they do not exist, see for instance [Villanueva *et al.*, 2022].

Our main results are stated in the next theorem.

Theorem 1. *The family of discontinuous piecewise differential system* (4) *separated by* Γ *and formed by* $3D$*-centers has at most*

(a) *one* $3D$*-limit cycle of type* T_1*, see Figure* 2*;*
(b) *four* $3D$*-limit cycles of type* T_2*, see Figure* 3*;*
(c) *five* $3D$*-limit cycles of types* T_1 *and* T_2 *simultaneously.*

There are examples of discontinuous piecewise differential system (4) *having simultaneously one* $3D$*-limit cycle of type* T_1 *and three* $3D$*-limit cycles of type* T_2*, see Figure* 4*.*

We cannot find an example of five $3D$-limit cycles of types T_1 and T_2 simultaneously. But we have provided an example of four $3D$-limit cycles.

Theorem 1 is proved in section 2.

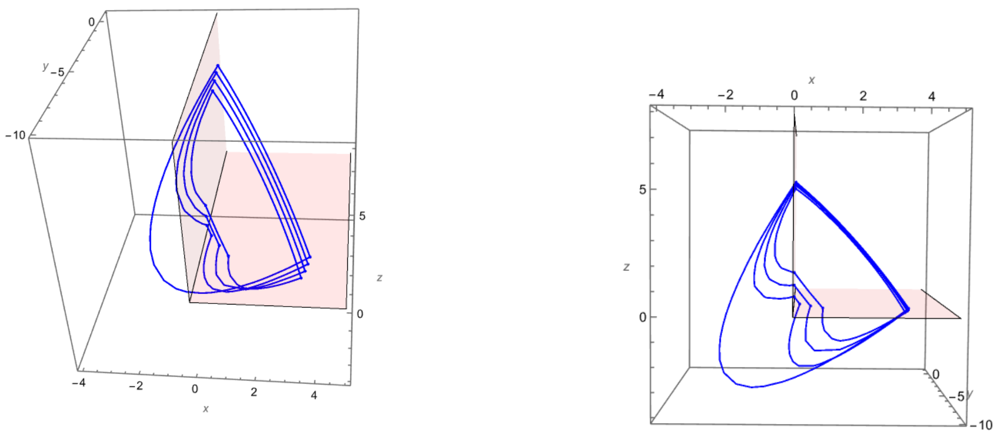

Fig. 4. Two different views of the three $3D$-limit cycles of type T_2 and one limit cycle of type T_1 simultaneously for the discontinuous piecewise differential system (12)-(13).

Second we present our results on the $3D$-limit cycles of the second family of discontinuous piecewise linear differential system

$$\dot{X} = \begin{cases} M_r X + N_r & \text{if } X \in R_r, \\ M_l X + N_l & \text{if } X \in R_l, \\ M_l X + N_l & \text{if } X \in R_d, \end{cases} \tag{5}$$

where M_i and N_i with $i = r, l, d$ are given by (3), and the discontinuity surface $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where

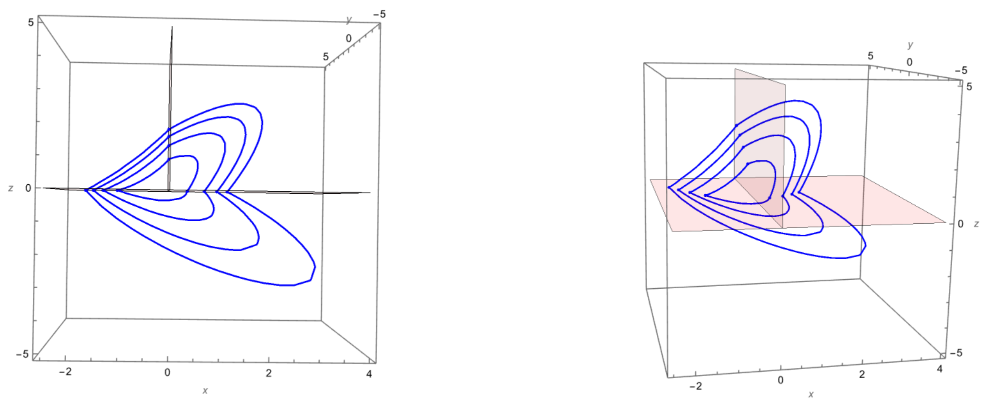

Fig. 5. Two different views of the four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (15)-(16)-(17).

$\Sigma_1 = \{(x,y,z) \in \mathbb{R}^3 : z = 0, \quad x \geq 0\}$, $\Sigma_2 = \{(x,y,z) \in \mathbb{R}^3 : x = 0, \quad z \geq 0\}$ and $\Sigma_3 = \{(x,y,z) \in \mathbb{R}^3 : z = 0, \quad x \leq 0\}$ divided the space into three regions

$$R_r = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, z \geq 0\}, \quad R_l = \{(x,y,z) \in \mathbb{R}^3 : x \leq 0, z \geq 0\},$$

and

$$R_d = \{(x,y,z) \in \mathbb{R}^3 : z \leq 0\}.$$

The discontinuous piecewise linear differential systems (5) have two different types of $3D$-limit cycles separated by Σ.

- The type corresponding to the $3D$-limit cycles intersecting the separation surface Σ at four points, two in Σ_2 and the two others are either in Σ_1 or in Σ_3 is equivalent to the type T_2 of the family (4) but in this case we have three regions instead of two regions. So, from the results of the family (4) it follows that there are at most four limit cycles of this type and there are piecewise differential systems of the second family with four limit cycles of this type.

- The other type named by T_3 corresponding to the $3D$-limit cycles intersecting the separation surface Σ at three points, one point in each of the half-planes Σ_k with $k = 1, 2, 3$, as shown Figure 5.

The second main result of this paper is presented in the following theorem.

Theorem 2. *For the family of discontinuous piecewise differential system* (5) *separated by* Σ *and formed by* $3D$*-centers, there are*

(a) *at most four* $3D$*-limit cycles of type* T_3*, see Figure 5;*
(b) *at most eight* $3D$*-limit cycles of types* T_2 *and* T_3 *simultaneously, see Figure 6.*

Theorem 2 is proved in section 3.

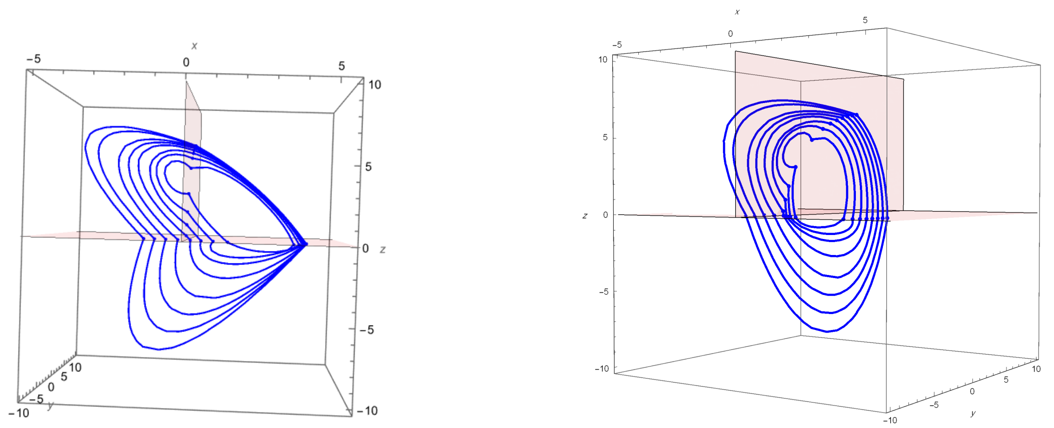

Fig. 6. Two different views of the eight $3D$-limit cycles of types T_2 and T_3 simultaneously for the discontinuous piecewise differential system (18)-(19)-(20).

2. Proof of Theorem 1

In this section we are going to provide the maximum number of $3D$-limit cycles of type T_1, T_2, and T_1 and T_2 simultaneously, for the discontinuous piecewise differential system (4) separated by Γ and formed by $3D$-centers.

Proof. [Proof of statement (a) of Theorem 1] If there is a limit cycle of type T_1 for the discontinuous piecewise differential system (4) separated by Γ and formed by $3D$-centers, it must intersect the discontinuity surface Γ in the two points $X_r = (x_r, y_r, 0)$ and $X_u = (0, y_u, z_u)$, where $x_r, z_u > 0$. It is clear that these two points must verify the following closing equations with the variables y_u, z_u, x_r and y_r.

$$\begin{aligned} E_1 &= H_{11}(x_r, y_r, 0) - H_{11}(0, y_u, z_u) = 0, \\ E_2 &= H_{12}(x_r, y_r, 0) - H_{12}(0, y_u, z_u) = 0, \\ E_3 &= H_{21}(0, y_u, z_u) - H_{21}(x_r, y_r, 0) = 0, \\ E_4 &= H_{22}(0, y_u, z_u) - H_{22}(x_r, y_r, 0) = 0, \end{aligned} \tag{6}$$

or equivalently

$$\begin{aligned} E_1 &= c_{12}y_u - c_{11}x_r - c_{12}y_r + c_{13}z_u = 0, \\ E_2 &= (a_{14} + a_{12}y_u + a_{13}z_u)^2 - (a_{14} + a_{11}x_r + a_{12}y_r)^2 - (b_{14} + b_{11}x_r + b_{12}y_r)^2 + (b_{14} + b_{12}y_u + b_{13}z_u)^2 = 0, \\ E_3 &= c_{22}y_u - c_{21}x_r - c_{22}y_r + c_{23}z_u = 0, \\ E_4 &= (a_{24} + a_{22}y_u + a_{23}z_u)^2 - (a_{24} + a_{21}x_r + a_{22}y_r)^2 - (b_{24} + b_{21}x_r + b_{22}y_r)^2 + (b_{24} + b_{22}y_u + b_{23}z_u)^2 = 0. \end{aligned}$$

Since the equations $E_1 = 0$ and $E_3 = 0$ are linear, we have to study these two different cases $c_{12}c_{21} - c_{11}c_{22} \neq 0$ or $c_{12}c_{21} - c_{11}c_{22} = 0$.

Case 1. If we assume that $c_{12}c_{21} - c_{11}c_{22} \neq 0$, from the system $E_1 = 0$ and $E_3 = 0$ we get

$$x_r = \frac{c_{13}c_{22}z_u - c_{12}c_{23}z_u}{c_{11}c_{22} - c_{12}c_{21}}, \quad y_r = \frac{c_{11}c_{22}y_u - c_{12}c_{21}y_u + c_{11}c_{23}z_u - c_{13}c_{21}z_u}{c_{11}c_{22} - c_{12}c_{21}}.$$

Replacing these expressions of the two variables x_r and y_r in the remaining equations $E_2 = 0$ and $E_4 = 0$ we find

$$\begin{aligned} E_2 = & -2(c_{12}c_{21} - c_{11}c_{22})(c_{23}(-c_{12}(a_{11}a_{14} + b_{11}b_{14}) + a_{12}a_{14}c_{11} + b_{12}b_{14}c_{11}) + a_{11}a_{14}c_{13}c_{22} \\ & -a_{12}a_{14}c_{13}c_{21} + a_{13}a_{14}(c_{12}c_{21} - c_{11}c_{22}) + b_{11}b_{14}c_{13}c_{22} - b_{12}b_{14}c_{13}c_{21} - b_{13}b_{14}c_{11}c_{22} \\ & +b_{13}b_{14}c_{12}c_{21}) + 2y_u(c_{12}c_{21} - c_{11}c_{22})(a_{12}(a_{11}c_{12}c_{23} - a_{11}c_{13}c_{22} + a_{13}c_{11}c_{22} - a_{13}c_{12} \\ & c_{21}) + a_{12}^2(c_{13}c_{21} - c_{11}c_{23}) + b_{12}(b_{11}c_{12}c_{23} - b_{11}c_{13}c_{22} - b_{12}c_{11}c_{23} + b_{12}c_{13}c_{21} + b_{13}c_{11} \\ & c_{22} - b_{13}c_{12}c_{21})) + z_u(c_{23}^2\left(c_{12}^2\left(a_{11}^2 + b_{11}^2\right) - 2c_{11}c_{12}(a_{11}a_{12} + b_{11}b_{12}) + c_{11}^2\left(a_{12}^2 + b_{12}^2\right)\right) \\ & -2c_{13}c_{23}(c_{12}c_{22}\left(a_{11}^2 + b_{11}^2\right) - a_{11}a_{12}(c_{11}c_{22} + c_{12}c_{21}) + a_{12}^2c_{11}c_{21} - b_{11}b_{12}(c_{11}c_{22} + c_{12} \\ & c_{21}) + b_{12}^2c_{11}c_{21}) + c_{13}^2(c_{22}^2\left(a_{11}^2 + b_{11}^2\right) - 2c_{21}c_{22}(a_{11}a_{12} + b_{11}b_{12}) + c_{21}^2\left(a_{12}^2 + b_{12}^2\right)) \\ & -a_{13}^2(c_{12}c_{21} - c_{11}c_{22})^2 - b_{13}^2(c_{12}c_{21} - c_{11}c_{22})^2) = 0, \end{aligned}$$

$$\begin{aligned} E_4 = & -2(c_{12}c_{21} - c_{11}c_{22})(c_{23}(-c_{12}(a_{21}a_{24} + b_{21}b_{24}) + a_{22}a_{24}c_{11} + b_{22}b_{24}c_{11}) + a_{21}a_{24}c_{13}c_{22} \\ & -a_{22}a_{24}c_{13}c_{21} + a_{23}a_{24}(c_{12}c_{21} - c_{11}c_{22}) + b_{21}b_{24}c_{13}c_{22} - b_{22}b_{24}c_{13}c_{21} - b_{23}b_{24}c_{11}c_{22} \\ & +b_{23}b_{24}c_{12}c_{21}) + 2y_u(c_{12}c_{21} - c_{11}c_{22})(a_{22}(a_{21}c_{12}c_{23} - a_{21}c_{13}c_{22} + a_{23}c_{11}c_{22} - a_{23}c_{12} \\ & c_{21}) + a_{22}^2(c_{13}c_{21} - c_{11}c_{23}) + b_{22}(b_{21}c_{12}c_{23} - b_{21}c_{13}c_{22} - b_{22}c_{11}c_{23} + b_{22}c_{13}c_{21} + b_{23}c_{11} \\ & c_{22} - b_{23}c_{12}c_{21})) + z_u(c_{23}^2(c_{12}^2\left(a_{21}^2 + b_{21}^2\right) - 2c_{11}c_{12}(a_{21}a_{22} + b_{21}b_{22}) + c_{11}^2\left(a_{22}^2 + b_{22}^2\right)) \\ & -2c_{13}c_{23}(c_{12}c_{22}\left(a_{21}^2 + b_{21}^2\right) - a_{21}a_{22}(c_{11}c_{22} + c_{12}c_{21}) + a_{22}^2c_{11}c_{21} - b_{21}b_{22}(c_{11}c_{22} + c_{12} \\ & c_{21}) + b_{22}^2c_{11}c_{21}) + c_{13}^2\left(c_{22}^2\left(a_{21}^2 + b_{21}^2\right) - 2c_{21}c_{22}(a_{21}a_{22} + b_{21}b_{22}) + c_{21}^2\left(a_{22}^2 + b_{22}^2\right)\right) \\ & -a_{23}^2(c_{12}c_{21} - c_{11}c_{22})^2 - b_{23}^2(c_{12}c_{21} - c_{11}c_{22})^2) = 0. \end{aligned}$$

Since $E_2 = E_4 = 0$ is a linear system of equations in the variables y_u and z_u, it has at most one real solution. Then in this case we find that it is possible to have at most one $3D$-limit cycle of type T_1 for the discontinuous piecewise differential system (4) separated by Γ.

Case 2. If we assume that $c_{12}c_{21} - c_{11}c_{22} = 0$, we have to study two subcases.

Subcase 2.1. If $c_{21} \neq 0$ then $c_{12} = (c_{11}c_{22})/c_{21}$ and we get $x_r = (c_{23}z_u - c_{22}y_r + c_{22}y_u)/c_{21}$ from the equation $E_3 = 0$, and by substituting x_r in $E_1 = 0$ we get $z_u(c_{21}c_{13} - c_{11}c_{23}) = 0$ that implies either $z_u = 0$ which contradicts the assumption that $z_u > 0$, or $c_{21}c_{13} - c_{11}c_{23} = 0$, which reduces the number of equations of system (6) to two polynomial equations containing three unknowns y_r, y_u and z_u. Thus this system has infinitely many solutions that produce a continuum of periodic solutions of the discontinuous piecewise differential system (4). Then we cannot have $3D$-limit cycles.

Subcase 2.2. If $c_{21} = 0$ then $c_{11}c_{22} = 0$ and we distinguish two possible subcases.

2.2.1. In case of $c_{21} = c_{11} = 0$ and $c_{22} \neq 0$. If $c_{12} \neq 0$ then from $E_1 = 0$ we get $y_r = (c_{12}y_u + c_{13}z_u)/c_{12}$. We substitute y_r in $E_3 = 0$ to find $E_3 = z_u(c_{12}c_{23} - c_{13}c_{22}) = 0$ which implies either $c_{12}c_{23} - c_{13}c_{22} = 0$ or $z_u = 0$. If $c_{12}c_{23} - c_{13}c_{22} = 0$ we have infinitely many solutions, and if $z_u = 0$ we have a contradiction with the assumptions. Therefore there are no $3D$-limit cycles. But if $c_{21} = c_{11} = c_{12} = 0$ and $c_{22} \neq 0$. The equation becomes $E_1 = c_{13}z_u = 0$, which implies either $z_u = 0$ for $c_{13} \neq 0$ that is a contradiction with the assumption of the theorem, or $c_{13} = 0$ and then we have infinitely many solutions.

2.2.2. If $c_{21} = c_{22} = 0$ and $c_{11} \neq 0$ or $c_{21} = c_{22} = c_{11} = 0$, we know that $E_3 = c_{23}z_u$. If $c_{23} = 0$ we have infinitely many solutions, and if $z_u = 0$ we have a contradiction with the assumptions. Therefore there are no $3D$-limit cycles.

Consequently the maximum number of $3D$-limit cycles of type T_1 for the discontinuous piecewise differential system (4) separated by Γ is at most one.

Now to confirm that the upper bound obtained is reached, we consider the differential system

$$\begin{aligned}\dot{x} &\approx 7.34011x - 0.108903y - 11.1167z - 1.90548,\\ \dot{y} &\approx -51.9801x + 4.64989y + 35.5439z - 18.7173,\\ \dot{z} &\approx 23.708x - 2.5y - 11.99z + 11.686,\end{aligned} \tag{7}$$

in the region R_1, where

$$\begin{aligned}H_{11}(x,y,z) &\approx x + 0.8y + 1.4444z + 20.3801,\\ H_{12}(x,y,z) &= (0.5x - 0.3y + 2.5z + 2.3)^2 + (5.1x - 0.5y - 3z + 2.2)^2,\end{aligned}$$

are two independent first integrals. In the second region R_2, we consider the differential center

$$\begin{aligned}\dot{x} &\approx 66.9666x - 75.7593y + 16.8125z - 16.8125,\\ \dot{y} &\approx 986.758x - 9.61355y - 68.3345z + 68.3345,\\ \dot{z} &\approx 807.857x - 2.94128y - 57.353z + 57.353,\end{aligned} \tag{8}$$

having the two independent first integrals

$$\begin{aligned}H_{21}(x,y,z) &\approx -6.37847x + 80y - 97.1874z - 2,\\ H_{22}(x,y,z) &\approx (3.17294x - 0.1y - 0.2z + 0.2)^2 + (0.2x + y - 0.3z + 0.3)^2.\end{aligned}$$

The unique solution of system (6) satisfying $x_{1r}, z_{1u} > 0$ is $(x_{1r}, y_{1r}, y_{1u}, z_{1u}) \approx (0.366549, 0.316725, 1.4891, 0.989095)$, which provides the unique $3D$-limit cycle of type T_1 for the discontinuous piecewise differential system (7)-(8) separated by Γ, see Figure 2. ■

Proof. [Proof of statement (b) of Theorem 1] If there is a limit cycle of type T_2 for the discontinuous piecewise differential system (4) separated by Γ, then it has four intersection points on the discontinuity surface Γ noted by $X_l = (x_l, y_l, 0)$, $X_r = (x_r, y_r, 0)$, $X_u = (0, y_u, z_u)$ and $X_d = (0, y_d, z_d)$, where $x_l, x_r, z_d, z_u > 0$, $X_l \neq X_r$ and $X_d \neq X_u$. Therefore the conditions in the following equations must be achieved.

$$\begin{aligned}E_1 &= H_{11}(x_l, y_l, 0) - H_{11}(0, y_d, z_d) = 0,\\ E_2 &= H_{12}(x_l, y_l, 0) - H_{12}(0, y_d, z_d) = 0,\\ E_3 &= H_{11}(x_r, y_r, 0) - H_{11}(0, y_u, z_u) = 0,\\ E_4 &= H_{12}(x_r, y_r, 0) - H_{12}(0, y_u, z_u) = 0,\\ E_5 &= H_{21}(0, y_u, z_u) - H_{21}(0, y_d, z_d) = 0,\\ E_6 &= H_{22}(0, y_u, z_u) - H_{22}(0, y_d, z_d) = 0,\\ E_7 &= H_{21}(x_l, y_l, 0) - H_{21}(x_r, y_r, 0) = 0,\\ E_8 &= H_{22}(x_l, y_l, 0) - H_{22}(x_r, y_r, 0) = 0,\end{aligned} \tag{9}$$

having x_i, y_i with $i = r, l$ and y_j, z_j with $j = u, d$ as the variables. System (9) is equivalent to

$E_1 = c_{11}x_l - c_{12}y_d + c_{12}y_l - c_{13}z_d = 0,$

$E_2 = (a_{14} + a_{11}x_l + a_{12}y_l)^2 + (b_{14} + b_{11}x_l + b_{12}y_l)^2 - (a_{14} + a_{12}y_d + a_{13}z_d)^2 - (b_{14} + b_{12}y_d + b_{13}z_d)^2 = 0,$

$E_3 = c_{11}x_r + c_{12}y_r - c_{12}y_u - c_{13}z_u = 0,$

$E_4 = (a_{14} + a_{11}x_r + a_{12}y_r)^2 + (b_{14} + b_{11}x_r + b_{12}y_r)^2 - (a_{14} + a_{12}y_u + a_{13}z_u)^2 - (b_{14} + b_{12}y_u + b_{13}z_u)^2 = 0,$

$E_5 = c_{22}y_u - c_{22}y_d - c_{23}z_d + c_{23}z_u = 0,$

$E_6 = (a_{24} + a_{22}y_u + a_{23}z_u)^2 - (a_{24} + a_{22}y_d + a_{23}z_d)^2 - (b_{24} + b_{22}y_d + b_{23}z_d)^2 + (b_{24} + b_{22}y_u + b_{23}z_u)^2 = 0,$

$E_7 = c_{21}x_l - c_{21}x_r + c_{22}y_l - c_{22}y_r = 0,$

$E_8 = (a_{24} + a_{21}x_l + a_{22}y_l)^2 + (b_{24} + b_{21}x_l + b_{22}y_l)^2 - (a_{24} + a_{21}x_r + a_{22}y_r)^2 - (b_{24} + b_{21}x_r + b_{22}y_r)^2 = 0.$

Analysing the linear odd equations $E_j = 0$ with $j = 1, 3, 5, 7$ we distinguish two cases $c_{12}c_{22}(c_{12}c_{23} - c_{13}c_{22}) \neq 0$ and $c_{12}c_{22}(c_{12}c_{23} - c_{13}c_{22}) = 0$.

Case 1. By supposing $c_{12}c_{22}(c_{12}c_{23} - c_{13}c_{22}) \neq 0$ and solving the system of the odd equations $E_j = 0$ with $j = 1, 3, 5, 7$ regarding y_r, y_l, y_d and z_d, we obtain

$$y_r = (c_{13}z_u - c_{11}x_r + c_{12}y_u)/c_{12}, \quad y_l = (c_{13}c_{22}z_u - c_{12}c_{21}x_l - c_{11}c_{22}x_r + c_{12}c_{21}x_r + c_{12}c_{22}y_u)/(c_{12}c_{22}),$$
$$y_d = (c_{12}c_{21}c_{23}x_l - c_{11}c_{22}c_{23}x_l + c_{11}c_{22}c_{23}x_r - c_{12}c_{21}c_{23}x_r - c_{12}c_{22}y_u + c_{13}c_{22}^2y_u)/(c_{22}(c_{13}c_{22} - c_{12}c_{23})),$$
$$z_d = l(c_{11}c_{22}x_l - c_{12}c_{21}x_l - c_{11}c_{22}x_r + c_{12}c_{21}x_r - c_{12}c_{23}z_u + c_{13}c_{22}z_u)/(c_{13}c_{22} - c_{12}c_{23}).$$

Upon replacing y_r, y_l, y_d and z_d in the remaining four equations, we obtain four new equations $E'_j = 0$ for $j = 2, 4, 6, 8$, with big expressions that we omit to write. We notice that the maximum exponent of the equations $E'_2 = 0$ and $E'_4 = 0$ is two and that one is the maximum exponent for each variable in the equations $E'_6 = 0$ and $E'_8 = 0$. Now we solve $E'_6 = 0$ and $E'_8 = 0$ for x_r and z_u, and substituting their expressions in the two equations $E'_2 = 0$ and $E'_4 = 0$ we obtain two quadratic equations having x_r and y_d as the two independent variables. According to Bézout theorem (see [Shafarevich & Lamb, 1974]), we can find in this case at most four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (4) separated by Γ and formed by $3D$-centers.

Case 2. If we consider $c_{12}c_{22}(c_{12}c_{23} - c_{13}c_{22}) = 0$, we distinguish the following subcases: $c_{12} = 0$ and $c_{22}c_{13} \neq 0$, or $c_{12}c_{23} \neq 0$ and $c_{22} = 0$, or $c_{12} = 0$ and $c_{22} = 0$, or $c_{12} = c_{13} = 0$ and $c_{22} \neq 0$, or $c_{12} \neq 0$ and $c_{22}c_{23} = 0$, or $c_{12}c_{22} \neq 0$ and $c_{23} = (c_{13}c_{22})/c_{12}$.

Subcase 2.1. If $c_{12} = 0$ and $c_{22}c_{13} \neq 0$, from the system of linear equations $E_1 = 0$, $E_3 = 0$, $E_5 = 0$ and $E_7 = 0$ we get

$$z_d = (c_{11}x_l)/c_{13}, \quad z_u = (c_{11}x_r)/c_{13}, \quad y_d = (c_{11}c_{23}x_r - c_{11}c_{23}x_l + c_{13}c_{22}y_u)/(c_{13}c_{22}),$$
$$y_l = (c_{21}x_r - c_{21}x_l + c_{22}y_r)/c_{22}.$$

Now by replacing these expressions of the variables y_l, y_d, z_u, and z_d in the remaining equations $E_2 = 0$, $E_4 = 0$, $E_6 = 0$ and $E_8 = 0$, we obtain four new equations $E'_j = 0$ for $j = 2, 4, 6, 8$, with big expressions that we omit to write. Solving the equations $E'_6 = 0$ and $E'_8 = 0$ with respect to y_u and y_r we get

$$y_u = -((-2a_{22}a_{24}c_{13}c_{22}c_{23} + 2a_{23}a_{24}c_{13}c_{22}^2 - 2b_{22}b_{24}c_{13}c_{22}c_{23} + 2b_{23}b_{24}c_{13}c_{22}^2 + a_{22}^2c_{11}c_{23}^2x_l - 2a_{22}a_{23}c_{11}c_{22}c_{23}x_l + a_{23}^2c_{11}c_{22}^2x_l + b_{22}^2c_{11}c_{23}^2x_l - 2b_{22}b_{23}c_{11}c_{22}c_{23}x_l + b_{23}^2c_{11}c_{22}^2x_l - a_{22}^2c_{11}c_{23}^2x_r + a_{23}^2c_{11}c_{22}^2x_r - b_{22}^2c_{11}c_{23}^2x_r + b_{23}^2c_{11}c_{22}^2x_r)/(2c_{13}c_{22}\left(-a_{22}^2c_{23} + a_{22}a_{23}c_{22} - b_{22}^2c_{23} + b_{22}b_{23}c_{22}\right))),$$
$$y_r = -((2a_{21}a_{24}c_{22}^2 - 2a_{22}a_{24}c_{21}c_{22} + 2b_{21}b_{24}c_{22}^2 - 2b_{22}b_{24}c_{21}c_{22} + a_{21}^2c_{22}^2x_l - 2a_{21}a_{22}c_{21}c_{22}x_l + a_{22}^2c_{21}^2x_l + b_{21}^2c_{22}^2x_l - 2b_{21}b_{22}c_{21}c_{22}x_l + b_{22}^2c_{21}^2x_l + a_{21}^2c_{22}^2x_r - a_{22}^2c_{21}^2x_r + b_{21}^2c_{22}^2x_r - b_{22}^2c_{21}^2x_r)/(2c_{22}(a_{21}a_{22}c_{22} - a_{22}^2c_{21} + b_{21}b_{22}c_{22} - b_{22}^2c_{21}))).$$

By substituting these expressions of y_u and y_r in $E'_2 = 0$ and $E'_4 = 0$ we obtain two quadratic equations with the independent variables x_r and x_l. According to Bézout theorem, we know that the discontinuous piecewise differential system (4) has at most four $3D$-limit cycles of type T_2.

Subcase 2.2. If $c_{22}=0$ and $c_{12}c_{23}\neq 0$, then from $E_5=0$ and $E_7=0$ we get $z_u=z_d$ and $x_r=x_l$ by replacing them in $E_1=0$ and $E_3=0$ we get

$$y_r=\frac{c_{12}y_u-c_{11}x_l+c_{13}z_d}{c_{12}},\quad y_l=\frac{c_{12}y_d-c_{11}x_l+c_{13}z_d}{c_{12}}.$$

Replacing y_r and y_l in the remaining equations $E_j=0$ with $j=2,4,6,8$ we get new quadratic equations $E'_j=0$ with $j=2,4,6,8$. We have

$$\begin{aligned}E'_8=&(y_d-y_u)\Big(2c_{12}x_l(a_{21}a_{22}+b_{21}b_{22})-2c_{11}x_l\left(a_{22}^2+b_{22}^2\right)+c_{12}(2a_{22}a_{24}+2b_{22}b_{24}+y_d(a_{22}^2+b_{22}^2)\\&+a_{22}^2y_u+b_{22}^2y_u)+2c_{13}z_d\left(a_{22}^2+b_{22}^2\right)\Big)=0.\end{aligned}$$

Since $y_u\neq y_d$, the unique solution of the equation $E'_8=0$ with respect to y_d is $y_d=\dfrac{c_{11}}{c_{12}}x_l-\dfrac{a_{22}a_{24}+b_{22}b_{24}}{a_{22}^2+b_{22}^2}-\dfrac{a_{21}a_{22}+b_{21}b_{22}}{a_{22}^2+b_{22}^2}x_l-\dfrac{c_{13}}{c_{12}}z_d$. After replacing y_d in $E'_j=0$ with $j=2,4,6$ we get new equations $E''_j=0$ with $j=2,4,6$. Solving $E''_6=0$ with respect to y_u where

$$\begin{aligned}E''_6=&(c_{12}x_l(a_{21}a_{22}+b_{21}b_{22})-c_{11}x_l\left(a_{22}^2+b_{22}^2\right)+c_{12}\left(a_{22}a_{24}+b_{22}b_{24}+y_u\left(a_{22}^2+b_{22}^2\right)\right)+c_{13}z_d\\&\left(a_{22}^2+b_{22}^2\right))(-c_{12}x_l(a_{21}a_{22}+b_{21}b_{22})+c_{11}x_l\left(a_{22}^2+b_{22}^2\right)+c_{12}(a_{22}a_{24}+b_{22}b_{24}+y_u(a_{22}^2\\&+b_{22}^2)+2a_{22}a_{23}z_d+2b_{22}b_{23}z_d)-c_{13}z_d\left(a_{22}^2+b_{22}^2\right))=0.\end{aligned}$$

we get $y_u=\dfrac{c_{11}}{c_{12}}x_l-\dfrac{a_{22}a_{24}+b_{22}b_{24}}{a_{22}^2+b_{22}^2}-\dfrac{a_{21}a_{22}+b_{21}b_{22}}{a_{22}^2+b_{22}^2}x_l-\dfrac{c_{13}}{c_{12}}z_d$ as the unique solution. Then by Bézout Theorem the quadratic system formed by $E''_2=0$ and $E''_4=0$ can have at most four real solutions. Consequently in this case we can find at most four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (4).

Subcase 2.3. If $c_{12}=c_{22}=0$, then from $E_5=E_7=0$ we get $z_u=z_d$ and $x_r=x_l$, by replacing them in $E_1=0$ and $E_3=0$ we get $E=E_1=E_3=c_{11}x_l-c_{13}z_d$. Then a new system of five equations $E=0$, $E_2=0$, $E_4=0$, $E_6=0$ and $E_8=0$ and six unknowns $(x_l,y_l,y_r,y_d,z_d,y_u)$ is obtained. So we cannot have $3D$-limit cycles.

Subcase 2.4. If $c_{22}\neq 0$ and $c_{12}=c_{13}=0$, the equations $E_1=0$ and $E_3=0$ becomes $E_1=c_{11}x_l=0$ and $E_3=c_{11}x_r=0$. If $c_{11}=0$ system (10) becomes a system of six equations with eight unknowns, then we cannot have $3D$-limit cycles. If $c_{11}\neq 0$ and due to the fact that $x_l,x_r>0$, then system (10) has no solutions.

Subcase 2.5. If $c_{12}\neq 0$ and $c_{22}=c_{23}=0$, the equation $E_5\equiv 0$ and system (10) becomes a system of seven equations with eight unknowns that can not provide any $3D$-limit cycles.

Subcase 2.6. If $c_{12}c_{22}\neq 0$ then $c_{23}=\dfrac{c_{13}c_{22}}{c_{12}}$, from $E_1=E_3=0$ we get

$$y_d=(c_{11}x_l+c_{12}y_l-c_{13}z_d)/c_{12},\quad y_u=(c_{11}x_r+c_{12}y_r-c_{13}z_u)/c_{12},$$

by replacing them in $E_5=0$ and $E_7=0$ we find $x_l=x_r$ and $y_l=y_r$ which is a contradiction with the assumption of the theorem. Then no $3D$-limit cycles.

Consequently the maximum number of $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (4) separated by Γ is at most four.

By providing a discontinuous piecewise differential system in $\mathbb{R}^3$ separated by Γ and constructed by $3D$-centers with precisely four limit cycles of type T_2, we can now guarantee that the upper bound obtained is reached. We consider the first $3D$-center

$$\begin{aligned}\dot{x}&\approx -421.382x-124.801y-210.115z+225.766,\\\dot{y}&\approx 775.318x+228.516y+378.809z-406.025,\\\dot{z}&\approx 828.85x+214.044y+192.866z-178.943,\end{aligned}\tag{10}$$

which has

$$H_{11}(x,y,z) \approx 75.139x + 42.4413y - 1.5z + 173.489,$$
$$H_{12}(x,y,z) = (0.7x + 0.4y + 1.7z - 2.)^2 + (-6.x - 1.55y - 1.4z + 1.3)^2,$$

as two independent first integrals defined in R_1, and the $3D$-center

$$\begin{aligned} \dot{x} &\approx -87.1281x - 14.7292y - 33.6255z - 8.03317,\\ \dot{y} &\approx 245.848x + 66.4952y + 92.7775z - 35.8364,\\ \dot{z} &\approx 47.0065x - 21.5974y + 20.633z + 73.6542, \end{aligned} \tag{11}$$

in R_2, having

$$H_{21}(x,y,z) \approx 63.6263x + 19.416y + 16.3863z + 0.8,$$
$$H_{22}(x,y,z) \approx (-2.80146x - 1.43607y - 1.z + 2.)^2 + (-4.96673x - 1.03907y - 1.9z + 0.01)^2,$$

as two independent first integrals in R_2. Then the four solutions of system (9) satisfying $x_{ij}, z_{ik} > 0$, $X_{il} \neq X_{ir}$ and $X_{iu} \neq X_{id}$ with $i = 1,2,3,4$, $j = r,l$ and $k = u,d$ that provide the intersection points with Γ of the four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (10)-(11) are

$$\begin{aligned} (x_{1r}, y_{1r}, x_{1l}, y_{1l}, y_{1d}, z_{1d}, y_{1u}, z_{1u}) &\approx (3.11249, -9.05624, 1.97147, -8.48574, -4.71924, 2.78076,\\ &\quad -2.8636, 4.6364),\\ (x_{2r}, y_{2r}, x_{2l}, y_{2l}, y_{2d}, z_{2d}, y_{2u}, z_{2u}) &\approx (3.60217, -9.30109, 1.48179, -8.24089, -5.38473, 2.11527,\\ &\quad -2.19811, 5.30189),\\ (x_{3r}, y_{3r}, x_{3l}, y_{3l}, y_{3d}, z_{3d}, y_{3u}, z_{3u}) &\approx (3.92854, -9.46427, 1.15542, -8.07771, -5.84482, 1.65518,\\ &\quad -1.73803, 5.76197),\\ (x_{4r}, y_{4r}, x_{4l}, y_{4l}, y_{4d}, z_{4d}, y_{4u}, z_{4u}) &\approx (4.19155, -9.59577, 0.892413, -7.94621, -6.21922, 1.28078,\\ &\quad -1.36362, 6.13638). \end{aligned}$$

This establishes the four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system separated by Γ and produced by the $3D$-centers (10) and (11), see Figure 3. ∎

Proof. [Proof of statement (c) of Theorem 1] From statement (a) of Theorem 1 we know that one is the maximum number of $3D$-limit cycles of the discontinuous piecewise differential system (4) of type T_1, and in the statement (b) of the same theorem we showed that four is the maximum number of $3D$-limit cycles of type T_2 for the same system. Therefore the upper bound number of $3D$-limit cycles for discontinuous piecewise differential system (4) of types T_1 and T_2 simultaneously is at most five.

Here we provide a discontinuous piecewise differential system of the form (4) separated by Γ and formed by $3D$-centers that has exactly one $3D$-limit cycle of type T_1 and three $3D$-limit cycles of type T_2. In region R_1 we consider the differential system

$$\begin{aligned} \dot{x} &\approx -445.784x - 128.3y - 211.z + 229.47,\\ \dot{y} &\approx 291.36x + 81.044y + 124.12z - 138.2,\\ \dot{z} &\approx 1369.36x + 337.878y + 364.74z - 469.3, \end{aligned} \tag{12}$$

with the independent first integrals

$$H_{11}(x,y,z) \approx 20.x + 40.y - 2z + 321.,$$
$$H_{12}(x,y,z) = (0.8x + 0.49y + 1.7z - 1.5)^2 + (-6.2x - 1.5y - 1.5z + 2)^2,$$

and in the region R_2 we consider

$$\begin{aligned} \dot{x} &\approx 17374.6x + 4910.16y - 10140.8z - 6942.8,\\ \dot{y} &\approx -11661.3x - 3188.89y + 6843.63z + 4490.63,\\ \dot{z} &\approx 23791.9x + 5870.45y - 14185.7z - 8153.83, \end{aligned} \tag{13}$$

having

$$H_{21}(x,y,z) \approx 20.x + 40.y + 5z + 321.,$$
$$H_{22}(x,y,z) \approx (-12.9244x - 13.6695y + 4.02816z + 21.0513)^2 + (-23.4506x - 2.8862y + 15z + 3.43746)^2,$$

as two independent first integrals. Now the three solutions of system (9) representing the intersection points of the three $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (12)-(13) with Γ, satisfying $x_{ij}, z_{ik} > 0$, $X_{il} \neq X_{ir}$ and $X_{iu} \neq X_{id}$ with $i = 1,2,3$, $j = r,l$ and $k = u,d$ are

$$(x_{1r}, y_{1r}, x_{1l}, y_{1l}, y_{1d}, z_{1d}, y_{1u}, z_{1u}) \approx (3.49769, -8.24885, 0.94326, -6.97163, -5.12609, 1.37391, -1.27637, 5.22363),$$
$$(x_{2r}, y_{2r}, x_{2l}, y_{2l}, y_{2d}, z_{2d}, y_{2u}, z_{2u}) \approx (3.608, -7.804, 0.551484, -6.27574, -5.19707, 0.802926, -0.60935, 5.39065),$$
$$(x_{3r}, y_{3r}, x_{3l}, y_{3l}, y_{3d}, z_{3d}, y_{3u}, z_{3u}) \approx (3.682, -7.341, 0.196025, -5.59801, -5.21476, 0.285236, 0.00437844, 5.50438),$$

and the unique solution of system (6) representing the two intersection points of the unique $3D$-limit cycle of type T_1 for the discontinuous piecewise differential system (12)-(13) with Γ, satisfying $x_{4r}, z_{4u} > 0$ is

$$(x_{4r}, y_{4r}, y_{4u}, z_{4u}) \approx (3.73181, -6.8659, 0.582336, 5.58234).$$

This proves the existence of four $3D$-limit cycles of types T_1 and T_2 for the discontinuous piecewise differential system separated by Γ and formed by the $3D$-centers (10) and (11), see Figure 3. This example completes the proof of Theorem 1. ■

3. Proof of Theorem 2

In this section we have to provide the maximum number of $3D$-limit cycles of type T_3, or T_2 and T_3 simultaneously, of the discontinuous piecewise differential system (5) separated by Σ and formed by $3D$-centers.

Proof. [Proof of statement (a) of Theorem 2] In order to obtain $3D$-limit cycle of type T_3 which intersects the separation surface Σ in three points $X_r = (x_r, y_r, 0)$, $X_l = (x_l, y_l, 0)$ and $X_u = (0, y_u, z_u)$, where $x_l < 0 < x_r$ and $z_u > 0$, it is necessary that these points satisfy the following system having y_u, z_u, x_i, and y_i with $i = r, l$ as the variables.

$$\begin{aligned} E_1 &= H_{r1}(x_r, y_r, 0) - H_{r1}(0, y_u, z_u) = 0,\\ E_2 &= H_{r2}(x_r, y_r, 0) - H_{r2}(0, y_u, z_u) = 0,\\ E_3 &= H_{l1}(0, y_u, z_u) - H_{l1}(x_l, y_l, 0) = 0,\\ E_4 &= H_{l2}(0, y_u, z_u) - H_{l2}(x_l, y_l, 0) = 0,\\ E_5 &= H_{d1}(x_r, y_r, 0) - H_{d1}(x_l, y_l, 0) = 0,\\ E_6 &= H_{d2}(x_r, y_r, 0) - H_{d2}(x_l, y_l, 0) = 0, \end{aligned} \tag{14}$$

or equivalently

$$E_1 = c_{r2}y_u - c_{r1}x_r - c_{r2}y_r + c_{r3}z_u = 0,$$
$$E_2 = (a_{r4} + a_{r2}y_u + a_{r3}z_u)^2 - (a_{r4} + a_{r1}x_r + a_{r2}y_r)^2 - (b_{r4} + b_{r1}x_r + b_{r2}y_r)^2 + (b_{r4} + b_{r2}y_u + b_{r3}z_u)^2 = 0,$$
$$E_3 = c_{l2}y_u - c_{l1}x_l - c_{l2}y_l + c_{l3}z_u = 0,$$
$$E_4 = (a_{l4} + a_{l2}y_u + a_{l3}z_u)^2 - (a_{l4} + a_{l1}x_l + a_{l2}y_l)^2 - (b_{l4} + b_{l1}x_l + b_{l2}y_l)^2 + (b_{l4} + b_{l2}y_u + b_{l3}z_u)^2 = 0,$$
$$E_5 = c_{d1}x_l - c_{d1}x_r + c_{d2}y_l - c_{d2}y_r = 0,$$
$$E_6 = (a_{d4} + a_{d1}x_l + a_{d2}y_l)^2 + (b_{d4} + b_{d1}x_l + b_{d2}y_l)^2 - (a_{d4} + a_{d1}x_r + a_{d2}y_r)^2 - (b_{d4} + b_{d1}x_r + b_{d2}y_r)^2 = 0.$$

Since the equations $E_1 = 0$, $E_3 = 0$ and $E_5 = 0$ are linear, we have to study these two cases $c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}) \neq 0$ or $c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}) = 0$.

Case 1. By assuming that $c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}) \neq 0$ and solving the system of equations $E_1 = 0$, $E_3 = 0$ and $E_5 = 0$ for the variables y_u, z_u and x_r, we get

$$\begin{aligned}
y_u &= (c_{d1}c_{l3}c_{r1}x_l - c_{d1}c_{l1}x_l - c_{d1}c_{l2}c_{r3}y_l + c_{d2}c_{l3}c_{r1}y_l + c_{d1}c_{l3}c_{r2}y_r - c_{d2}c_{l3}c_{r1}y_r)/(c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}),\\
z_u &= (c_{d1}c_{l1}c_{r2}x_l - c_{d1}c_{l2}x_l + c_{d1}c_{l2}c_{r2}y_l - c_{d2}c_{l2}c_{r1}y_l - c_{d1}c_{l2}c_{r2}y_r + c_{d2}c_{l2}c_{r1}y_r)/(c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}),\\
x_r &= (c_{d1}x_l + c_{d2}y_l - c_{d2}y_r)/c_{d1}.
\end{aligned}$$

After replacing y_u, z_u and x_r in the remaining equations $E_j = 0$ for $j = 2, 4, 6$ we obtain new expressions that we denote by $E_j' = 0$ for $j = 2, 4, 6$. We do not provide the large explicit expressions of the equations $E_j' = 0$ for $j = 2, 4, 6$ that will need several pages to write them. We remark that the two equations $E_2' = 0$ and $E_4' = 0$ are quadratic while $E_6' = 0$ is linear. By solving the equation $E_6' = 0$ for the variable x_l that we substitute in the equations $E_2' = 0$ and $E_4' = 0$, we obtain two quadratic equations namely $E_2'' = 0$ and $E_4'' = 0$ in the variables y_r and y_l. Then based on the Bézout Theorem, there exists at most four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (5) separated by Σ and formed by $3D$-centers.

Case 2. If we consider $c_{d1}(c_{l3}c_{r2} - c_{l2}c_{r3}) = 0$, we find these three subcases $c_{d1} = 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} \neq 0$, or $c_{d1} \neq 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} = 0$, or $c_{d1} = 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} = 0$.

Subcase 2.1. If $c_{d1} = 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} \neq 0$, we get $y_l = y_r$ from the equation $E_5 = 0$. Solving $E_1 = 0$ and $E_3 = 0$ gives

$$y_r = \frac{c_{l1}c_{r3}x_l - c_{l3}c_{r1}x_r - c_{l2}c_{r3}y_u + c_{l3}c_{r2}y_u}{c_{l3}c_{r2} - c_{l2}c_{r3}}, \quad z_u = \frac{c_{l1}c_{r2}x_l - c_{l2}c_{r1}x_r}{c_{l3}c_{r2} - c_{l2}c_{r3}},$$

by replacing y_r and z_u in $E_j = 0$ we get new system of equations $E_j' = 0$ with $j = 2, 4, 6$, where

$$\begin{aligned}
E_6' = (x_l - x_r)\Big(&x_l(-c_{l2}c_{r3}(a_{d1}^2 + b_{d1}^2) + c_{l3}c_{r2}\left(a_{d1}^2 + b_{d1}^2\right) + 2c_{l1}c_{r3}(a_{d1}a_{d2} + b_{d1}b_{d2})) + c_{l3}x_r(c_{r2}\\
&(a_{d1}^2 + b_{d1}^2) - 2c_{r1}(a_{d1}a_{d2} + b_{d1}b_{d2})) - c_{l2}c_{r3}x_r\left(a_{d1}^2 + b_{d1}^2\right) + 2(c_{l3}c_{r2} - c_{l2}c_{r3})(a_{d1}a_{d4} + b_{d1}\\
&b_{d4} + a_{d1}a_{d2}y_u + b_{d1}b_{d2}y_u)\Big) = 0.
\end{aligned}$$

Since $x_r \neq x_l$, the unique solution of the equation $E_6' = 0$ with respect to y_u is

$$y_u = \frac{c_{l1}x_l}{c_{l2}} - \frac{2(a_{d1}a_{d4} + b_{d1}b_{d4}) + x_l\left(a_{d1}^2 + b_{d1}^2\right) + x_r\left(a_{d1}^2 + b_{d1}^2\right)}{2(a_{d1}a_{d2} + b_{d1}b_{d2})} + \frac{c_{l2}c_{l3}c_{r1}x_r - c_{l1}c_{l3}c_{r2}x_l}{c_{l2}(c_{l3}c_{r2} - c_{l2}c_{r3})},$$

when $c_{l2}(a_{d1}a_{d2} + b_{d1}b_{d2}) \neq 0$. Then after replacing y_u in $E_2' = 0$ and $E_4' = 0$ we obtain a new quadratic system of equations $E_2'' = 0$ and $E_4'' = 0$, which implies that system (6) has at most four real solutions. Thus at most four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (5). For the case $c_{l2}(a_{d1}a_{d2} + b_{d1}b_{d2}) = 0$ we notice that the discontinuous piecewise differential system (5) has at most four $3D$-limit cycles of type T_3.

Subcase 2.2. If $c_{d1} \neq 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} = 0$ we have three different subcases.

2.2.1. If $c_{r2} \neq 0$ and $c_{l3} = \dfrac{c_{l2}c_{r3}}{c_{r2}}$, we get

$$x_r = x_l + \frac{c_{d2}}{c_{d1}}(y_l - y_r), \quad y_u = (c_{d1}c_{r1}x_l + c_{d2}c_{r1}y_l + c_{d1}c_{r2}y_r - c_{d2}c_{r1}y_r - c_{d1}c_{r3}z_u)/(c_{d1}c_{r2}),$$

after solving the equations $E_1 = 0$ and $E_5 = 0$. By replacing x_r and y_u in $E_3 = 0$ we distinguish two different cases $c_{l2}c_{r1} - c_{l1}c_{r2} = 0$ or $c_{l2}c_{r1} - c_{l1}c_{r2} \neq 0$.

2.2.1.1. If $c_{l2}c_{r1} - c_{l1}c_{r2} = 0$, i.e. $c_{l1} = \dfrac{c_{l2}c_{r1}}{c_{r2}}$. Since $y_l \neq y_r$, $E_3 = c_{l2}(y_l - y_r)(c_{d2}c_{r1} - c_{d1}c_{r2}) = 0$ has no solution or infinitely many solutions. Then there are no $3D$-limit cycles.

2.2.1.2. If $c_{l2}c_{r1} - c_{l1}c_{r2} \neq 0$, and from $E'_3 = 0$ we get $x_l = \dfrac{c_{l2}(c_{d1}c_{r2} - c_{d2}c_{r1})}{c_{d1}(c_{l2}c_{r1} - c_{l1}c_{r2})}(y_l - y_r)$. By replacing x_l in $E_j = 0$ we get new quadratic equations $E'_j = 0$ with $j = 2, 4, 6$, where

$$
\begin{aligned}
E'_6 = (y_l - y_r)\Big(&2c_{d1}(c_{l2}c_{r1} - c_{l1}c_{r2})(-c_{d2}(a_{d1}a_{d4} + b_{d1}b_{d4}) + a_{d2}a_{d4}c_{d1} + b_{d2}b_{d4}c_{d1}) + y_l(c_{d2}\left(a_{d1}^2 + b_{d1}^2\right) \\
&(-2c_{d1}c_{l2}c_{r2} + c_{d2}c_{l1}c_{r2} + c_{d2}c_{l2}c_{r1}) + 2a_{d1}a_{d2}c_{d1}c_{l2}(c_{d1}c_{r2} - c_{d2}c_{r1}) + a_{d2}^2c_{d1}^2(c_{l2}c_{r1} - c_{l1}c_{r2}) + 2 \\
&b_{d1}b_{d2}c_{d1}c_{l2}(c_{d1}c_{r2} - c_{d2}c_{r1}) + b_{d2}^2c_{d1}^2(c_{l2}c_{r1} - c_{l1}c_{r2})) + y_r(-c_{d2}\left(a_{d1}^2 + b_{d1}^2\right)(-2c_{d1}c_{l2}c_{r2} + c_{d2}c_{l1} \\
&c_{r2} + c_{d2}c_{l2}c_{r1}) + 2a_{d1}a_{d2}c_{d1}c_{r2}(c_{d2}c_{l1} - c_{d1}c_{l2}) + a_{d2}^2c_{d1}^2(c_{l2}c_{r1} - c_{l1}c_{r2}) + 2b_{d1}b_{d2}c_{d1}c_{r2}(c_{d2}c_{l1} \\
&-c_{d1}c_{l2}) + b_{d2}^2c_{d1}^2(c_{l2}c_{r1} - c_{l1}c_{r2}))\Big) = 0.
\end{aligned}
$$

Due to the fact $X_r \neq X_l$ we know that $y_r \neq y_l$, then we can get the linear expression of one of the unknown variables y_r, or y_l from $E'_6 = 0$. By replacing the linear expression of the finding variable in $E'_2 = 0$ and $E'_4 = 0$ we will get a quadratic system of two equations with two variables with at most four real solutions. So there are at most four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (5).

2.2.2. If $c_{r2} = c_{l2} = 0$ and $c_{r3} \neq 0$, from $E_1 = 0$ and $E_5 = 0$ we get

$$
z_u = \frac{c_{r1}}{c_{r3}}x_r, \quad x_l = x_r + \frac{c_{d2}}{c_{d1}}(y_r - y_l),
$$

by replacing z_u and x_r in the remaining equations, and from $E_3 = 0$ we get $y_l = (c_{d1}(c_{l1}c_{r3} - c_{l3}c_{r1})x_r)/(c_{d2}c_{l1}c_{r3}) + y_r$ because if $c_{d2} = 0$ and $c_{l1} \neq 0$, or $c_{l1} = 0$ and $c_{d2} \neq 0$, or $c_{d2} = 0$ and $c_{l1} = 0$ the equations $E_3 = 0$ becomes either $x_r(c_{l3}c_{r1} - c_{l1}c_{r3})$, or $c_{l3}c_{r1}x_r$, or $c_{l3}z_u$, respectively. In all these cases 0 is the unique solution of $E_3 = 0$, which contradicts the assumption of the theorem. Now by replacing y_l in the even equations $E_j = 0$ we find a new quadratic system of equations named by $E'_j = 0$ with $j = 2, 4, 6$, where

$$
\begin{aligned}
E'_6 = x_r(c_{l3}c_{r1} - c_{l1}c_{r3})\Big(&c_{l1}c_{r3}x_r(c_{d2}^2(a_{d1}^2 + b_{d1}^2) - c_{d1}^2(a_{d2}^2 + b_{d2}^2)) + c_{l3}c_{r1}x_r(c_{d2}^2(a_{d1}^2 + b_{d1}^2) - 2c_{d1}c_{d2} \\
&(a_{d1}a_{d2} + b_{d1}b_{d2}) + c_{d1}^2(a_{d2}^2 + b_{d2}^2)) + 2c_{d2}c_{l1}c_{r3}(c_{d2}(a_{d1}a_{d4} + b_{d1}b_{d4} + a_{d1}a_{d2}y_r + b_{d1}b_{d2}y_r) - c_{d1} \\
&(a_{d2}a_{d4} + b_{d2}b_{d4} + y_r(a_{d2}^2 + b_{d2}^2)))\Big) = 0.
\end{aligned}
$$

From the equation $E'_6 = 0$ we can get the linear expression of one of the unknown variables x_r, or y_r. By replacing the linear expression of the finding variable in $E'_2 = 0$ and $E'_4 = 0$ we will obtain a quadratic system of two equations with two variables with at most four real solutions. So there are at most four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (5).

2.2.3. If $c_{r2} = c_{r3} = 0$ and $c_{l2} \neq 0$ or if $c_{r2} = c_{l2} = c_{r3} = 0$, under these conditions the equation $E_1 = 0$ becomes $E_1 = c_{r1}x_r = 0$, that implies either $x_r = 0$ for $c_{r1} \neq 0$ that is a contradiction with the assumption of the theorem, or $c_{r1} = 0$ and then we have infinitely many solutions.

Subcase 2.3. If $c_{d1} = 0$ and $c_{l3}c_{r2} - c_{l2}c_{r3} = 0$, we have two subcases

2.3.1. If $c_{r2} \neq 0$ then $c_{l3} = \dfrac{c_{l2}c_{r3}}{c_{r2}}$, we get $y_l = y_r$ from $E_5 = 0$. Substituting y_l in $E_1 = 0$ and $E_3 = 0$, then if $c_{l1} = 0$ we have $E_1 = x_rc_{r1} = 0$, which implies either $x_r = 0$ for $c_{r1} \neq 0$ which is a contradiction with the assumption of the theorem, or $c_{r1} = 0$, then system (14) has infinitely many solutions.

If $c_{l1} \neq 0$, then solving $E_1 = 0$ and $E_3 = 0$ we get

$$
y_r = \frac{c_{r2}y_u - c_{r1}x_r + c_{r3}z_u}{c_{r2}}, \quad x_l = \frac{c_{l2}c_{r1}x_r}{c_{l1}c_{r2}}.
$$

Replacing y_r and x_l in $E_j = 0$, we get new quadratic equations $E'_j = 0$ with $j = 2, 4, 6$, where we give only the expression $E'_6 = 0$ which is the simplest one

$$
\begin{aligned}
E'_6 = x_r\Big(&c_{r1}x_r(c_{l2}(a_{d1}^2 + b_{d1}^2) - 2c_{l1}(a_{d1}a_{d2} + b_{d1}b_{d2})) + c_{l1}c_{r2}x_r(a_{d1}^2 + b_{d1}^2) + 2c_{l1}(c_{r2}(a_{d1}a_{d4} \\
&+b_{d1}b_{d4} + a_{d1}a_{d2}y_u + b_{d1}b_{d2}y_u) + c_{r3}z_u(a_{d1}a_{d2} + b_{d1}b_{d2}))\Big) = 0.
\end{aligned}
$$

Since $x_r \neq 0$, in this equation we distinguish the two following cases

2.3.1.1. If $c_{l1}c_{r3}(a_{d1}a_{d2}+b_{d1}b_{d2}) \neq 0$ the unique solution of $E_6'=0$ with respect to z_u is

$$z_u = -(c_{r1}x_r\left(c_{l2}\left(a_{d1}^2+b_{d1}^2\right)-2c_{l1}(a_{d1}a_{d2}+b_{d1}b_{d2})\right)+c_{l1}c_{r2}x_r\left(a_{d1}^2+b_{d1}^2\right)+2c_{l1}c_{r2} (a_{d1}a_{d4}+b_{d1}b_{d4}+a_{d1}a_{d2}y_u+b_{d1}b_{d2}y_u))/(2c_{l1}c_{r3}(a_{d1}a_{d2}+b_{d1}b_{d2})),$$

after replacing z_u in $E_2'=0$ and $E_4'=0$ we get a quadratic system of equations $E_2''=0$ and $E_4''=0$ that can have at most four real solutions. Thus the discontinuous piecewise differential system (5) can have at most four $3D$-limit cycles of type T_3.

2.3.1.2. If $c_{l1}c_{r3}(a_{d1}a_{d2}+b_{d1}b_{d2}) = 0$ we have many subcases.

If $c_{l1}=0$ and $c_{r3}(a_{d1}a_{d2}+b_{d1}b_{d2}) \neq 0$ the equation $E_6'=0$ becomes $c_{l2}c_{r1}\left(a_{d1}^2+b_{d1}^2\right)x_r=0$, that implies either $x_r=0$ for $c_{l2}c_{r1}\left(a_{d1}^2+b_{d1}^2\right)\neq 0$ that is a contradiction with the assumption of the theorem, or $c_{l2}c_{r1}\left(a_{d1}^2+b_{d1}^2\right)=0$, and then we have infinitely many solutions.

If $c_{r3}=0$ and $c_{l1}(a_{d1}a_{d2}+b_{d1}b_{d2}) \neq 0$ under these conditions we get

$$y_u = -(2c_{l1}c_{r2}(a_{d1}a_{d4}+b_{d1}b_{d4})+c_{r1}x_r(c_{l2}(a_{d1}^2+b_{d1}^2)-2c_{l1}(a_{d1}a_{d2}+b_{d1}b_{d2}))+c_{l1}c_{r2}x_r (a_{d1}^2+b_{d1}^2))/(2c_{l1}c_{r2}(a_{d1}a_{d2}+b_{d1}b_{d2})),$$

after replacing y_u in $E_2'=0$ and $E_4'=0$ we get a quadratic system of equations $E_2''=0$ and $E_4''=0$ that can have at most four real solutions. Thus the discontinuous piecewise differential system (5) can have at most four $3D$-limit cycles of type T_3.

If $c_{r3}c_{l1}\neq 0$ and $a_{d1}a_{d2}+b_{d1}b_{d2}=0$ then $a_{d2}=-(b_{d1}b_{d2})/a_{d1}$ if $a_{d1}\neq 0$ (If $a_{d1}=0$ then $b_{d1}b_{d2}=0$ in both cases $b_{d1}=0$ or $b_{d2}=0$ the system (14) has infinitely many solutions). For $c_{l1}c_{r2}+c_{l2}c_{r1}\neq 0$ the equation $E_6'=-2c_{l2}c_{r1}(a_{d1}a_{d4}+b_{d1}b_{d4})=0$ which means there are infinitely many solutions of system (14). If $c_{l1}c_{r2}+c_{l2}c_{r1}\neq 0$ we get $x_r=-2c_{l1}c_{r2}(a_{d1}a_{d4}+b_{d1}b_{d4})/(\left(a_{d1}^2+b_{d1}^2\right)(c_{l1}c_{r2}+c_{l2}c_{r1}))$ by solving the equation $E_6'=0$. Then after replacing x_r in $E_2'=0$ and $E_4'=0$ we get a quadratic system of equations $E_2''=0$ and $E_4''=0$ that can have at most four real solutions. Consequently the discontinuous piecewise differential system (5) can have at most four $3D$-limit cycles of type T_3.

2.3.2. If $c_{r2}=0$ then $c_{l2}c_{r3}=0$, which implies $c_{l2}=0$ or $c_{r3}=0$ in both cases system (5) has infinitely many solutions. Thus the discontinuous piecewise differential system (5) has no limit cycles.

Consequently the maximum number of $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (5) separated by Σ is at most four.

Now we verify that this upper bound is reached. In the region R_r, we consider the $3D$-center

$$\begin{aligned}\dot{x} &\approx 271.405x-3.34397y-249.703z+33.4397,\\ \dot{y} &\approx -524.593x+6.54954y+481.651z-65.4954,\\ \dot{z} &\approx 503.536x-22.2316y-277.955z+222.316,\end{aligned}\tag{15}$$

and we consider the following $3D$-center

$$\begin{aligned}\dot{x} &\approx 17.0601x+40.0748y-48.9849z+1.57584,\\ \dot{y} &\approx 35.5255x+37.0185y-34.6504z+5.66785,\\ \dot{z} &\approx 53.7525x+57.1489y-54.0786z+8.51736,\end{aligned}\tag{16}$$

in the region R_l. In the region R_d we consider

$$\begin{aligned}\dot{x} &\approx -9.41978x-11.0591y-19.3411z+13.386,\\ \dot{y} &\approx -3.83949x-4.50483y-7.8807z+5.45179,\\ \dot{z} &\approx 7.76689x+6.99384y+13.9246z-7.81658.\end{aligned}\tag{17}$$

The differential system (15) has two independent first integrals defined in R_r given by

$$\begin{aligned}H_{r1}(x,y,z) &\approx 35.8085x+18.622y+0.1z+0.214711,\\ H_{r2}(x,y,z) &= (0.2x-0.2y+2.1z+2)^2+(5x-0.2y-3z+2)^2,\end{aligned}$$

and the differential system (16) has the following two independent first integrals in R_l

$$H_{l1}(x,y,z) \approx x + 29.1772y - 19.6009z - 5,$$
$$H_{l2}(x,y,z) \approx (1.0669x + 0.532235y - 0.2z + 0.2)^2 + (-0.880266x - 1.70412y + 2z - 0.1)^2.$$

Finally in R_d two independent first integrals of the differential system (17) are

$$H_{d1}(x,y,z) \approx -60.5234x + 148.892y + 0.2z + 5,$$
$$H_{d2}(x,y,z) \approx (-0.00997777x - 0.200281y - 0.2z + 0.3)^2 + (-0.204347x - 0.114304y - 0.3z + 0.1)^2.$$

The four solutions of system (14) that provide the intersection points of the four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (15)-(16)-(17) with Σ, satisfying $x_{ir}, z_{iu} > 0$ and $x_{il} < 0$ with $i = 1, 2, 3, 4$ are

$$(x_{1r}, y_{1r}, y_{1u}, z_{1u}, x_{1l}, y_{1l}) \approx (0.433032, 0.283484, 1.69163, 1.19163, -1.21924, 1.10962)\,,$$
$$(x_{2r}, y_{2r}, y_{2u}, z_{2u}, x_{2l}, y_{2l}) \approx (0.836428, 0.581786, 2.75671, 1.75671, -1.58126, 1.79063)\,,$$
$$(x_{3r}, y_{3r}, y_{3u}, z_{3u}, x_{3l}, y_{3l}) \approx (1.14474, 0.927628, 3.66268, 2.16268, -1.84819, 2.4241)\,,$$
$$(x_{4r}, y_{4r}, y_{4u}, z_{4u}, x_{4l}, y_{4l}) \approx (1.40588, 1.29706, 4.49566, 2.49566, -2.06795, 3.03397)\,.$$

This proves the existence of the four $3D$-limit cycles of type T_3 for the discontinuous differential system separated by Σ and formed by the $3D$-centers (15), (16) and (17), see Figure 5. ■

Proof. [Proof of statement (b) of Theorem 2] In the statement (a) of Theorem 2 we proved that four is the maximum number of $3D$-limit cycles for the discontinuous piecewise differential system (5) in each of type T_3 and from the proof of Theorem 1 also there are at most four $3D$-limit cycles of type T_2. Then we know that the upper bound for the number of $3D$-limit cycles for the discontinuous piecewise differential system (5) of type T_2 and T_3 simultaneously is at most eight.

Now to complete the proof of statement (b) Theorem 2. We must give an example with exactly eight $3D$-limit cycles, four from each type. In the region R_r we consider the differential system

$$\begin{aligned} \dot{x} &\approx -1189.8x - 362.1y - 370.164z + 587.326, \\ \dot{y} &\approx 3316.4x + 900.249y + 372.358z - 1124.77, \\ \dot{z} &\approx 2349.12x + 641.944y + 289.547z - 816.754. \end{aligned} \tag{18}$$

In the left region R_l we consider the differential system

$$\begin{aligned} \dot{x} &\approx -485.793x - 336.105y - 386.782z + 571.602, \\ \dot{y} &\approx 3052.22x - 1689.63y + 2176.27z - 12924.7, \\ \dot{z} &\approx 3051.17x - 1690.36y + 2175.43z - 12923.4, \end{aligned} \tag{19}$$

and in the region R_d we consider the system

$$\begin{aligned} \dot{x} &\approx -84.1147x - 125.494y - 69.4585z - 777.228, \\ \dot{y} &\approx 90.3947x + 75.8781y + 36.5198z + 372.176, \\ \dot{z} &\approx 222.352x + 60.4018y + 8.2366z - 75.6165. \end{aligned} \tag{20}$$

The differential system (18) has

$$H_{r1}(x,y,z) \approx -10x + 61.3844y - 91.7249z + 627.189,$$
$$H_{r2}(x,y,z) = (-6x - 1.6y - 0.5z + 1.9)^2 + (0.8x + 0.5y + 1.8z - 1.6)^2,$$

as two independent first integrals defined on R_r. For the differential system (19)

$$H_{l1}(x,y,z) \approx 0.1x + 46y - 46z + 369,$$
$$H_{l2}(x,y,z) \approx 64(x - 0.634489y + 0.707606z - 4.43318)^2 + (1.5x + 2.62081y + 1.3z + 2.12173)^2,$$

are two independent first integrals in R_l. In R_d two independent first integrals of the differential system (20) are

$$H_{d1}(x,y,z) \approx 23x + 46y - 10z + 369,$$
$$H_{d2}(x,y,z) \approx (2.5x + y + 0.3z + 1.66891)^2 + (0.394982x + 1.63185y + z + 11.8345)^2.$$

The four solutions of system (9) satisfying $x_{ij}, z_{ik} > 0$, $X_{il} \neq X_{ir}$ and $X_{iu} \neq X_{id}$ with $i = 1,2,3,4$, $j = r,l$ and $k = u,d$ are

$$(x_{1r}, y_{1r}, x_{1l}, y_{1l}, y_{1d}, z_{1d}, y_{1u}, z_{1u}) \approx (3.69845, -8.84922, 1.48945, -7.74472, -3.86958, 3.13042, -2.0397, 4.9603),$$
$$(x_{2r}, y_{2r}, x_{2l}, y_{2l}, y_{2d}, z_{2d}, y_{2u}, z_{2u}) \approx (3.88748, -8.44374, 0.986067, -6.99303, -4.61913, , 1.88087, -0.755096, 5.7449),$$
$$(x_{3r}, y_{3r}, x_{3l}, y_{3l}, y_{3d}, z_{3d}, y_{3u}, z_{3u}) \approx (4.00795, -8.00398, 0.551249, -6.27562, -4.98698, 1.01302, 0.147807, 6.14781),$$
$$(x_{4r}, y_{4r}, x_{4l}, y_{4l}, y_{4d}, z_{4d}, y_{4u}, z_{4u}) \approx (4.08912, -7.54456, 0.155737, -5.57787, -5.22022, 0.279781, 0.916095, 6.4161).$$

These four solutions provide the four $3D$-limit cycles of type T_3 for the discontinuous piecewise differential system (18)-(19)-(20) separated by Σ. Now The four solutions of system (14) satisfying $x_{ir}, z_{iu} > 0$ and $x_{il} < 0$ with $i = 5,6,7,8$ are

$$(x_{5r}, y_{5r}, y_{5u}, z_{5u}, x_{5l}, y_{5l}) \approx (4.14392, -7.07196, 1.60816, 6.60816, -0.290098, -5.44015),$$
$$(x_{6r}, y_{6r}, y_{6u}, z_{6u}, x_{6l}, y_{6l}) \approx (4.17943, -6.58971, 2.24918, 6.74918, -0.712536, -5.45239),$$
$$(x_{7r}, y_{7r}, y_{7u}, z_{7u}, x_{7l}, y_{7l}) \approx (4.20003, -6.10002, 2.85281, 6.85281, -1.10956, -5.47514),$$
$$(x_{8r}, y_{8r}, y_{8u}, z_{8u}, x_{8l}, y_{8l}) \approx (4.20865, -5.60433, 3.4274, 6.9274, -1.49172, -5.5),$$

where these four solutions provide the four $3D$-limit cycles of type T_2 for the discontinuous piecewise differential system (18)-(19)-(20) separated by Σ, see Figure 6. This example completes the proof of Theorem 2. ■

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