This is a preprint of: “Periodic orbits and non-existence of C^1 first integrals for analytic differential systems exhibiting a zero-Hopf bifurcation in R^4”, Jaume Llibre, Renhao Tian, *Rend. Circ. Mat. Palermo (2)*, vol. 73(2), 2723–2733, 2024.
DOI: [10.1007/s12215-024-01074-8]

PERIODIC ORBITS AND NON-EXISTENCE OF C^1 FIRST INTEGRALS FOR ANALYTIC DIFFERENTIAL SYSTEMS EXHIBITING A ZERO-HOPF BIFURCATION IN $\mathbb{R}^4$

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Abstract. In this paper we investigate the zero-Hopf bifurcation of a four dimensional analytic differential system. We prove that at most five periodic orbits bifurcate from the zero-Hopf equilibrium using the averaging theory of first order and give a specific example to illustrate this conclusion. Moreover we prove the non-existence of C^1 first integrals in a neighbourhood of these periodic orbits.

1. Introduction and statement of the main result

In an n-dimensional analytic autonomous system, there always exist analytic first integrals near a regular point (and there are $n-1$ functionally independent analytic first integrals). However, in general, this is not the case in a neighbourhood of an equilibrium point. The existence of analytic first integrals near them depends on the resonance of the eigenvalues of the linearized system at equilibrium. For details, see the book [11].

Recently Yagasaki [9, 10] studied the analytic non-integrability of three-dimensional analytic differential systems at a zero-Hopf equilibrium. A *zero-Hopf* equilibrium of an n-dimensional autonomous differential system is an equilibrium that has a pair of purely imaginary eigenvalues and the rest are all zero eigenvalues.

Motivated by the works of Yagasaki in [9, 10] Llibre and Zhang [6] studied the dynamics in a neighbourhood of a zero-Hopf equilibrium in $\mathbb{R}^3$. They obtained sufficient conditions for the existence of two periodic orbits bifurcating from the zero-Hopf equilibrium and proved the non-existence of C^1 first integrals in a neighbourhood of these periodic orbits.

In 2021 Llibre and Tian [4] studied a four-dimensional hyperchaotic system depending on six parameters. They characterized the values of the parameters for which their equilibria are zero-Hopf equilibria and obtain the sufficient conditions for the existence of four periodic orbits bifurcating from these zero-Hopf equilibria.

Inspired by the works [4, 6], we consider the following analytic differential system in $\mathbb{R}^4$ which has a zero-Hopf bifurcation at the equilibrium localized at the origin of

2020 *Mathematics Subject Classification.* 34C05, 34C29, 34C07, 37J30,34D08,34D10.
Key words and phrases. Analytic differential systems; zero-Hopf bifurcation; periodic orbits; characteristic multipliers; non-existence of C^1 first integral.

coordinates

$$
\begin{aligned}
\dot{x} =& \varepsilon a_1 x - (\omega + \varepsilon\omega_1) y + \varepsilon a_2 z + \varepsilon a_3 w + a_4 x^2 + a_5 xy + a_6 xz + a_7 wx + a_8 y^2 \\
& + a_9 yz + a_{10} wy + a_{11} z^2 + a_{12} wz + a_{13} w^2 + \text{h.o.t.}, \\
\dot{y} =& (\omega + \varepsilon\omega_1) x + \varepsilon b_1 y + \varepsilon b_2 z + \varepsilon b_3 w + b_4 x^2 + b_5 xy + b_6 xz + b_7 wx + b_8 y^2 \\
& + b_9 yz + b_{10} wy + b_{11} z^2 + b_{12} wz + b_{13} w^2 + \text{h.o.t.}, \\
\dot{z} =& \varepsilon c_0 x + \varepsilon c_1 y + \varepsilon c_2 z + \varepsilon c_3 w + c_4 x^2 + c_5 xy + c_6 xz + c_7 wx + c_8 y^2 \\
& + c_9 yz + c_{10} wy + c_{11} z^2 + c_{12} wz + c_{13} w^2 + \text{h.o.t.}, \\
\dot{w} =& \varepsilon d_0 x + \varepsilon d_1 y + \varepsilon d_2 z + \varepsilon d_3 w + d_4 x^2 + d_5 xy + d_6 xz + d_7 wx + d_8 y^2 \\
& + d_9 yz + d_{10} wy + d_{11} z^2 + d_{12} wz + d_{13} w^2 + \text{h.o.t.},
\end{aligned} \tag{1}
$$

where $\omega > 0$.

For the differential system (1) we study the periodic orbits bifurcating from the zero-Hopf equilibrium when the parameter ε crosses the zero value.

The conclusions are summarized in the following theorem.

Theorem 1. *For the differential system* (1) *the following statements hold.*

(a) *For $|\varepsilon| > 0$ sufficiently small, at most five periodic orbits can bifurcate from the zero-Hopf equilibrium point localized at the origin of coordinates using the first order averaged function. Morever, there are systems* (1) *for which this zero-Hopf bifurcation exhibits the five periodic orbits [see Example 2].*

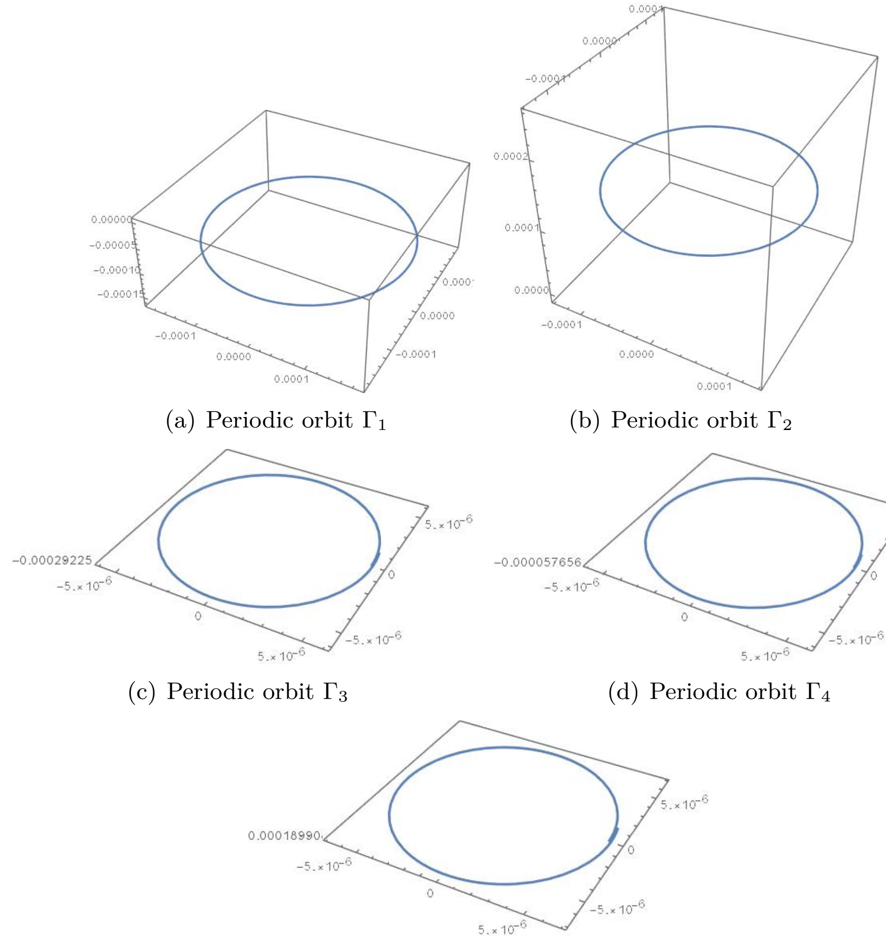
(b) *For $|\varepsilon| > 0$ system* (1) *has two types of periodic orbits bifurcating from the zero-Hopf equilibrium, $\Gamma_{1\varepsilon} : (x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon), w_1(t,\varepsilon))$ and $\Gamma_{2\varepsilon} : (x_2(t,\varepsilon), y_2(t,\varepsilon), z_2(t,\varepsilon), w_2(t,\varepsilon))$. There are $Z_i \in \mathbb{R}$, $W_i \in \mathbb{R}/\{0\}$, $i = 1,2,3$, given in* (14) *and appendix, such that $(x_1(0,\varepsilon), y_1(0,\varepsilon), z_1(0,\varepsilon), w_1(0,\varepsilon)) = (O(\varepsilon^2), O(\varepsilon^3), \varepsilon Z_i + O(\varepsilon^2), \varepsilon W_i + O(\varepsilon^2))$. Morever, if $c_{12} W_i + 2c_{11} Z_i + c_2 + 2d_{13} W_i + d_{12} Z_i + d_3 \neq 0$, system* (1) *has no C^1 first integrals in a neighbourhood of $\Gamma_{1\varepsilon}$. And there are $R_j \in \mathbb{R}/\{0\}$, $Z_j, W_j \in \mathbb{R}$, $j = 4,5$, given in* (16) *and appendix, such that $(x_2(t,\varepsilon), y_2(t,\varepsilon), z_2(t,\varepsilon), w_2(t,\varepsilon)) = \varepsilon(R_j \cos(\omega t), R_j \sin(\omega t), Z_j, W_j) + O(\varepsilon^2)$. Morever, if $\lambda_{1,j}$, $\lambda_{2,j}$ and $\lambda_{3,j}$, the eigenvalues of Jacobian matrix $D_x(\mathbf{g}(R_j, Z_j, W_j))$ defined in* (12)*, are not an integer multiple of $\sqrt{-1}/\varepsilon$, then the system has no C^1 first integrals in a neighbourhood of $\Gamma_{2\varepsilon}$.*

Example 2. *The following particular system* (1)

$$
\begin{aligned}
\dot{x} =& \varepsilon x - y + wx + xz + yz, \\
\dot{y} =& \varepsilon y + x + yz, \\
\dot{z} =& -3\varepsilon w - 5\varepsilon z - x^2 + wz + w^2, \\
\dot{w} =& -5\varepsilon w - 5\varepsilon z + x^2 + z^2 - 5wz - 5w^2,
\end{aligned} \tag{2}
$$

has five periodic orbits bifurcating from the zero-Hopf equilibrium. Morever, the system has no C^1 first integrals in the neighbourhood of these five periodic orbits.

We numerically simulate the five periodic orbits of system (2) (as shown in Figure 1), where we take the parameter $\varepsilon = 1/25000$. Note that we are studying a *four*-dimensional system, but we can only draw *three*-dimensional images. So we consider the projection of these periodic orbits in (x, y, w) space.



(a) Periodic orbit Γ_1 (b) Periodic orbit Γ_2

(c) Periodic orbit Γ_3 (d) Periodic orbit Γ_4

(e) Periodic orbit Γ_5

FIGURE 1. The periodic orbits of Example 2 computed numerically with the initial conditions given in statement (b) of Theorem 1 with $\varepsilon = 1/25000$.

The periodic orbit $\Gamma_1 : (x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon), w_1(t,\varepsilon))$ and $\Gamma_2 : (x_2(t,\varepsilon), y_2(t,\varepsilon), z_2(t,\varepsilon), w_2(t,\varepsilon))$ of system (2) have been obtained with the initial conditions $(x_1(0,\varepsilon), y_1(0,\varepsilon), z_1(0,\varepsilon), w_1(0,\varepsilon)) = (2\sqrt{5}\varepsilon, 0, 0, -2\varepsilon)$ and $(x_2(0,\varepsilon), y_2(0,\varepsilon), z_2(0,\varepsilon), w_2(0,\varepsilon)) = (\frac{16\sqrt{2}}{7}\varepsilon, 0, \frac{18}{7}\varepsilon, \frac{22}{7}\varepsilon)$, respectively.

The periodic orbit $\Gamma_3 : (x_3(t,\varepsilon), y_3(t,\varepsilon), z_3(t,\varepsilon), w_3(t,\varepsilon))$, $\Gamma_4 : (x_4(t,\varepsilon), y_4(t,\varepsilon), z_4(t,\varepsilon), w_4(t,\varepsilon))$ and $\Gamma_5 : (x_5(t,\varepsilon), y_5(t,\varepsilon), z_5(t,\varepsilon), w_5(t,\varepsilon))$ of system (2) have been obtained with the initial conditions $(x_3(0,\varepsilon), y_3(0,\varepsilon), z_3(0,\varepsilon), w_3(0,\varepsilon)) = (3500\varepsilon^2, \varepsilon^3, 6.11887\varepsilon, -7.30628\varepsilon)$, $(x_4(0,\varepsilon), y_4(0,\varepsilon), z_4(0,\varepsilon), w_4(0,\varepsilon)) = (3500\varepsilon^2, \varepsilon^3, 0.993874\varepsilon, -1.44142\varepsilon)$ and $(x_5(0,\varepsilon), y_5(0,\varepsilon), z_5(0,\varepsilon), w_5(0,\varepsilon)) = (4000\varepsilon^2, \varepsilon^3, 32.8873\varepsilon, 4.7477\varepsilon)$, respectively.

To prove the existence of periodic orbits, we utilize the first order averaging theory. For proving the non-existence of the C^1 first integrals near these periodic orbits, we employ the method of Llibre and Valls [5], which has been improved by Llibre and Zhang in [6] computing the characteristic multipliers of the variational equation along a periodic orbit in a relatively easy way.

This paper is organized as follows: We summarize the averaging theory of first order and some basic results on the C^1 non-integrability in section 2 that will be used for proving our main results. In section 3 we prove our Theorem 1 and Example 2.

2. Preliminary results

2.1. **Averaging theory.** In this section we summarize the results on the averaging theory of first order for computing periodic orbits. The following result is widely used, and its proof can be found in [8].

Theorem 3. *Consider the following differential system*

$$\dot{\mathbf{x}} = \varepsilon \mathbf{G}(t, \mathbf{x}) + \varepsilon^2 \mathbf{R}(t, \mathbf{x}, \varepsilon), \quad (t, \mathbf{x}, \varepsilon) \in [0, \infty) \times \Omega \times (-\varepsilon_0, \varepsilon_0), \tag{3}$$

where Ω *is an open subset of* $\mathbb{R}^n$*,* $\mathbf{G}(t, \mathbf{x})$*,* $\mathbf{R}(t, \mathbf{x}, \varepsilon)$ *are* C^2 *functions* T*-periodic in* t*, and* $\varepsilon_0 > 0$ *is a small real constant. The first averaged function is*

$$\mathbf{g}(\mathbf{x}) = \frac{1}{T} \int_0^T \mathbf{G}(t, \mathbf{x}) dt.$$

If $\mathbf{a} \in \Omega$ *is a zero of the averaged function* $\mathbf{g}(\mathbf{x})$ *such that the Jacobian* $\det\big(D_{\mathbf{x}} \mathbf{g}(\mathbf{a})\big) \neq 0$*, then the differential system* (3) *has a* T*-periodic solution* $\mathbf{x}(t, \varepsilon)$ *such that* $\mathbf{x}(0, \varepsilon) \to \mathbf{a}$ *as* $\varepsilon \to 0$*.*

2.2. C^1 **non-integrability.** The following result goes back to Poincaré. Consider the following differential system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{4}$$

where $\mathbf{f} \in C^1$ and Ω is an open subset of $\mathbb{R}^n$. Assume that system (4) has a T-periodic orbit $\mathbf{x}(t)$. Define the *variational differential system* of (4) along the periodic orbit $\mathbf{x}(t)$ as

$$\dot{\mathbf{z}} = D_{\mathbf{x}} \mathbf{f}(\mathbf{x}(t)) \mathbf{z}, \tag{5}$$

where $\mathbf{z}$ is an $n \times n$ matrix. Let $\Phi(t)$ be the fundamental solution matrix of system (5) satisfying $\Phi(0) = id_{n \times n}$. Then the *characteristic multipliers* of the periodic orbit $\mathbf{x}(t)$ are the eigenvalues of the matrix $\Phi(T)$. Note that 1 is always a characteristic multiplier whose the eigenvector is tangent to the periodic orbit. For details, see [5].

Next we give the following theorem derived from Poincaré [7] and its proof can be found in [5].

Theorem 4. *If* 1 *is a characteristic multiplier of a periodic orbit* Γ *of system* (4) *having multiplicity one, then the differential system* (4) *has no* C^1 *first integrals defined in a neighbourhood of* Γ*.*

The following result obtained by Llibre and Zhang [6] provides a relatively easy way to compute the characteristic multipliers.

Theorem 5. *Consider the differential system* (3) *under the assumptions of Theorem 3, such that system* (3) *has a* T*-periodic solution* $\mathbf{x}_\varepsilon(t)$ *such that* $\mathbf{x}_\varepsilon(0) \to \mathbf{a}$ *when* $\varepsilon \to 0$*. Let* $\Phi(t, \varepsilon)$ *be the fundamental solution matrix of the variational system*

$$\dot{\mathbf{z}} = \varepsilon D_x \big(\mathbf{G}(t, \mathbf{x}) + \varepsilon \mathbf{R}(t, \mathbf{x}, \varepsilon)\big)|_{\mathbf{x} = \mathbf{x}_\varepsilon(t)} \mathbf{z}, \tag{6}$$

such that $\Phi(0, \varepsilon) = Id_{n \times n}$*. Then* $\Phi(T, \varepsilon) \to \exp(\varepsilon D_{\mathbf{x}} \mathbf{g}(\mathbf{a}) T)$ *when* $\varepsilon \to 0$*.*

Combining Theorem 4 and 5, we have the following remark.

Remark 6. We emphasize that working with the averaged function we have eliminated the characteristic multiplier 1 of the variational equation associated with the periodic orbit. So in order to prove that there is no C^1 first integral near the periodic orbit, we only need to check that the eigenvalues of the matrix $D_{\mathbf{x}}\mathbf{g}(\mathbf{a})$ in Theorem 3 are not an integer multiple of $2\pi\sqrt{-1}/(T\varepsilon)$.

3. PROOF OF OUR MAIN RESULT

Proof of Theorem 1. For studying the periodic orbits of system (1) bifurcating from the zero-Hopf equilibrium $(0,0,0,0)$, we perform the rescaling $(x,y,z,w)=(\varepsilon X,\varepsilon Y,\varepsilon Z,\varepsilon W)$. Being $\varepsilon>0$ a small parameter, then system (1) becomes

$$
\begin{aligned}
\dot X =& -\omega Y+\varepsilon(-Y\omega_1+a_1X+a_2Z+a_3W+a_4X^2+a_5XY+a_6XZ+a_7WX\\
&+a_8Y^2+a_9YZ+a_{10}WY+a_{11}Z^2+a_{12}WZ+a_{13}W^2),\\
\dot Y =&\omega X+\varepsilon(X\omega_1+b_1Y+b_2Z+b_3W+b_4X^2+b_5XY+b_6XZ+b_7WX\\
&+b_8Y^2+b_9YZ+b_{10}WY+b_{11}Z^2+b_{12}WZ+b_{13}W^2),\\
\dot Z =&\varepsilon(c_0X+c_1Y+c_2Z+c_3W+c_4X^2+c_5XY+c_6XZ+c_7WX\\
&+c_8Y^2+c_9YZ+c_{10}WY+c_{11}Z^2+c_{12}WZ+c_{13}W^2),\\
\dot W =&\varepsilon(d_0X+d_1Y+d_2Z+d_3W+d_4X^2+d_5XY+d_6XZ+d_7WX\\
&+d_8Y^2+d_9YZ+d_{10}WY+d_{11}Z^2+d_{12}WZ+d_{13}W^2).
\end{aligned}
\tag{7}
$$

Then we write system (7) in cylindrical coordinates via the change of variables $(X,Y,Z,W)=(R\cos\theta,R\sin\theta,Z,W)$ and system (7) becomes

$$
\begin{aligned}
\dot R =&\varepsilon\Big(a_2\cos\theta Z+a_3\cos\theta W+a_{11}\cos\theta Z^2+a_{12}\cos\theta WZ+a_{13}\cos\theta W^2\\
&+b_2\sin\theta Z+b_3\sin\theta W+b_{13}\sin\theta W^2+b_{12}\sin\theta WZ+b_{11}\sin\theta Z^2\\
&+\big(a_1\cos^2\theta+a_7\cos^2\theta W+a_{10}\sin\theta\cos\theta W+a_6\cos^2\theta Z\\
&+a_9\sin\theta\cos\theta Z+b_1\sin^2\theta+b_{10}\sin^2\theta W+b_7\sin\theta\cos\theta W\\
&+b_9\sin^2\theta Z+b_6\sin\theta\cos\theta Z\big)R+\big(a_4\cos^3\theta+a_5\sin\theta\cos^2\theta\\
&+a_8\sin^2\theta\cos\theta+b_8\sin^3\theta+b_4\sin\theta\cos^2\theta+b_5\sin^2\theta\cos\theta\big)R^2\Big),\\
\dot\theta =&\omega+\frac{1}{R}\varepsilon\Big(-a_{13}\sin\theta W^2-a_3\sin\theta W-a_{12}\sin\theta WZ-a_{11}\sin\theta Z^2\\
&-a_2\sin\theta Z+b_{13}\cos\theta W^2+b_3\cos\theta W+b_{12}\cos\theta WZ+b_{11}\cos\theta Z^2\\
&+b_2\cos\theta Z+\big(-a_1\sin\theta\cos\theta-a_{10}\sin^2\theta W-a_7\sin\theta\cos\theta W\\
&-a_9\sin^2\theta Z-a_6\sin\theta\cos\theta Z+b_1\sin\theta\cos\theta+b_7\cos^2\theta W\\
&+b_{10}\sin\theta\cos\theta W+b_6\cos^2\theta Z+b_9\sin\theta\cos\theta Z+\omega_1\sin^2\theta\\
&+\omega_1\cos^2\theta\big)R+\big(-a_8\sin^3\theta-a_4\sin\theta\cos^2\theta-a_5\sin^2\theta\cos\theta\\
&+b_4\cos^3\theta+b_5\sin\theta\cos^2\theta+b_8\sin^2\theta\cos\theta\big)R^2\Big),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\dot{Z} =& \varepsilon\Big(c_2 Z + c_3 W + c_{13} W^2 + c_{12} WZ + c_{11} Z^2 + \big(c_1 \sin\theta + c_0 \cos\theta \\
& + c_{10} \sin\theta W + c_7 \cos\theta W + c_9 \sin\theta Z + c_6 \cos\theta Z\big) R + \big(c_8 \sin^2\theta \\
& + c_4 \cos^2\theta + c_5 \sin\theta\cos\theta\big) R^2\Big), \\
\dot{W} =& \varepsilon\Big(d_2 Z + d_3 W + c_{13} W^2 + d_{12} WZ + d_{11} Z^2 + \big(d_1 \sin\theta + d_0 \cos\theta \\
& + d_{10} \sin\theta W + d_7 \cos\theta W + d_9 \sin\theta Z + d_6 \cos\theta Z\big) R + \big(d_8 \sin^2\theta \\
& + d_4 \cos^2\theta + d_5 \sin\theta\cos\theta\big) R^2\Big).
\end{aligned}
$$

Taking the variable θ as the new time, system (8) becomes $(dR/d\theta, dZ/d\theta, dW/d\theta)$. Due to $\omega > 0$ this system is well-defined in a neighbourhood of the origin. So we have the differential system

$$
\begin{aligned}
\frac{dR}{d\theta} =& \varepsilon\omega^{-1}\Big(a_2 \cos\theta Z + a_3 \cos\theta W + a_{11} \cos\theta Z^2 + a_{12} \cos\theta WZ \\
& + a_{13} \cos\theta W^2 + b_2 \sin\theta Z + b_3 \sin\theta W + b_{13} \sin\theta W^2 \\
& + b_{12} \sin\theta WZ + b_{11} \sin\theta Z^2 + (a_1 \cos^2\theta + a_7 \cos^2\theta W \\
& + a_{10} \sin\theta\cos\theta W + a_6 \cos^2\theta Z + a_9 \sin\theta\cos\theta Z + b_1 \sin^2\theta \\
& + b_{10} \sin^2\theta W + b_7 \sin\theta\cos\theta W + b_9 \sin^2\theta Z + b_6 \sin\theta\cos\theta Z) R \\
& + \big((a_4 \cos^3\theta + a_5 \sin\theta\cos^2\theta + a_8 \sin^2\theta\cos\theta + b_8 \sin^3\theta \\
& + b_4 \sin\theta\cos^2\theta + b_5 \sin^2\theta\cos\theta\big) R^2\Big) + O(\varepsilon^2) \\
:=& \varepsilon G_1(\theta, R, Z, W) + O(\varepsilon^2), \\
\frac{dZ}{d\theta} =& \varepsilon\omega^{-1}(c_8 \sin^2\theta R^2 + c_4 \cos^2\theta R^2 + c_5 \sin\theta\cos\theta R^2 + c_1 \sin\theta R \\
& + c_0 \cos\theta R + c_{10} \sin\theta RW + c_7 \cos\theta RW + c_9 \sin\theta RZ \\
& + c_6 \cos\theta RZ + c_{13} W^2 + c_{12} WZ + c_3 W + c_{11} Z^2 + c_2 Z) \\
:=& \varepsilon G_2(\theta, R, Z, W) + O(\varepsilon^2), \\
\frac{dW}{d\theta} =& \varepsilon\omega^{-1}(d_8 \sin^2\theta R^2 + d_4 \cos^2\theta R^2 + d_5 \sin\theta\cos\theta R^2 + d_1 \sin\theta R \\
& + d_0 \cos\theta R + d_{10} \sin\theta RW + d_7 \cos\theta RW + d_9 \sin\theta RZ \\
& + d_6 \cos\theta RZ + d_{13} W^2 + d_{12} WZ + d_3 W + d_{11} Z^2 + d_2 Z) \\
:=& \varepsilon G_3(\theta, R, Z, W) + O(\varepsilon^2).
\end{aligned}
\tag{9}
$$

Note that the differential system (9) is written in the normal form (3) for applying the first order averaging theory, where $\mathbf{x} = (R, Z, W)$, $t = \theta$ and $T = 2\pi$. Then we can compute the first averaged function.

$$
\begin{aligned}
\mathbf{g}(R, Z, W) &= \big(g_1(R, Z, W), g_2(R, Z, W), g_3(R, Z, W)\big) \\
&= \int_0^{2\pi} \big(G_1(\theta, R, Z, W), G_2(\theta, R, Z, W), G_3(\theta, R, Z, W)\big) d\theta
\end{aligned}
\tag{10}
$$

and we get

$$\begin{aligned} g_1(R,Z,W) &= \frac{R(a_7W + a_6Z + a_1 + b_{10}W + b_9Z + b_1)}{2\omega}, \\ g_2(R,Z,W) &= \frac{(c_4+c_8)R^2 + 2W(c_{13}W + c_{12}Z + c_3) + 2c_{11}Z^2 + 2c_2Z}{2\omega}, \\ g_3(R,Z,W) &= \frac{(d_4+d_8)R^2 + 2W(d_{13}W + d_{12}Z + d_3) + 2d_{11}Z^2 + 2d_2Z}{2\omega}. \end{aligned} \tag{11}$$

Morever the Jacobian matrix of the averaged function $\mathbf{g}(R,Z,W)$ is

$$D_x\big(\mathbf{g}(R,Z,W)\big) = \begin{pmatrix} \frac{a_7W+a_6Z+a_1+b_{10}W+b_9Z+b_1}{2\omega} & \frac{R(a_6+b_9)}{2\omega} & \frac{R(a_7+b_{10})}{2\omega} \\ \frac{(c_4+c_8)R}{\omega} & \frac{c_{12}W+2c_{11}Z+c_2}{\omega} & \frac{2c_{13}W+c_{12}Z+c_3}{\omega} \\ \frac{(d_4+d_8)R}{\omega} & \frac{d_{12}W+2d_{11}Z+d_2}{\omega} & \frac{2d_{13}W+d_{12}Z+d_3}{\omega} \end{pmatrix}. \tag{12}$$

Next we prove that the averaged function $\mathbf{g}(R,Z,W)$ has at most five non-zero zeros. For the function $g_1(R,Z,W)$ we consider two cases $R=0$ and $R\neq 0$.

First, we assume that $R=0$. Applying the Gröbner basis technique, we can compute a basis Gröbner of the ideal $\mathcal{A}$ of $\mathbb{C}[Z,W]$ generated by the two polynomials $g_2(0,Z,W)$ and $g_3(0,Z,W)$. Using Mathematica we calculated that the ideal has nine generators, two of them are

$$\begin{aligned} \mathbf{a}_1 =& (c_3d_2 - c_2d_3)(c_{11}d_2 - c_2d_{11})W + (c_{11}^2d_3^2 + c_{11}(c_{13}d_2^2 - c_{12}d_3d_2 + 2c_3d_{12}d_2 \\ &- c_2d_{13}d_2 - 2c_3d_3d_{11} - c_2d_3d_{12}) + d_{11}(c_3^2d_{11} - c_3(c_{12}d_2 + c_2d_{12}) \\ &+ c_2(-c_{13}d_2 + 2c_{12}d_3 + c_2d_{13})))W^2 + (2c_{11}^2d_3d_{13} + c_{11}(c_3d_{12}^2 \\ &+ c_{13}(2d_2d_{12} - 2d_3d_{11}) - d_{13}(2c_3d_{11} + c_2d_{12})) + c_{12}^2d_3d_{11} \\ &+ c_{13}d_{11}(2c_3d_{11} - c_2d_{12}) - c_{12}(c_{13}d_2d_{11} + d_{12}(c_{11}d_3 + c_3d_{11}) \\ &+ d_{13}(c_{11}d_2 - 2c_2d_{11})))W^3 + (c_{13}^2d_{11}^2 + d_{13}(c_{11}^2d_{13} - c_{12}c_{11}d_{12} + c_{12}^2d_{11}) \\ &+ c_{13}(c_{11}(d_{12}^2 - 2d_{11}d_{13}) - c_{12}d_{11}d_{12}))W^4 \\ :=& D_0W + C_0W^2 + B_0W^3 + A_0W^4 \end{aligned} \tag{13}$$

and

$$\mathbf{a}_2 = (c_{11}d_{13} - c_{13}d_{11})W^2 + (c_{11}d_3 - c_3d_{11})W + Z\big((c_{11}d_{12} - c_{12}d_{11})W + c_{11}d_2 - c_2d_{11}\big).$$

In fact, we do not consider the case of $W=0$, because if $W=0$, from $g_2(0,0,Z)=0$ and $g_3(0,0,Z)=0$ in equation (11) one gets $Z(c_{11}Z+c_2)=0$ and $Z(d_{11}Z+d_2)=0$, respectively. If $c_2d_{11} - c_{11}d_2 = 0$ then Z can be any non-zero value. Consequently, the determinant of Jacobian matrix $D_x\big(\mathbf{g}(0,Z,0)\big)$ is zero, and the averaging theory does not provide any information about the possible periodic orbits bifurcating from the equilibrium $(0,0,0)$ of the differential system $(dR/\theta, dZ/\theta, dW/\theta)$.

For convenience, we define

$$E_0=\frac{B_0C_0}{6A_0^2}-\frac{B_0^3}{27A_0^3}-\frac{D_0}{2A_0},\quad F_0=\frac{C_0}{3A_0}-\frac{B_0^2}{9A_0^2}.$$

If $E_0^2+F_0^3<0$ and $A_0D_0\neq 0$ we can find three non-zero real zeros from $\mathbf{a}_1=0$, named W_1, W_2 and W_3. Morever, we have

$$\begin{aligned}
W_1&=-\frac{B_0}{3A_0}+\sqrt[3]{E_0+\sqrt{E_0^2+F_0^3}}+\sqrt[3]{E_0-\sqrt{E_0^2+F_0^3}},\\
W_2&=-\frac{B_0}{3A_0}+\frac{-1+\sqrt{-3}}{2}\sqrt[3]{E_0+\sqrt{E_0^2+F_0^3}}+\frac{-1-\sqrt{-3}}{2}\sqrt[3]{E_0-\sqrt{E_0^2+F_0^3}},\\
W_3&=-\frac{B_0}{3A_0}+\frac{-1-\sqrt{-3}}{2}\sqrt[3]{E_0+\sqrt{E_0^2+F_0^3}}+\frac{-1+\sqrt{-3}}{2}\sqrt[3]{E_0-\sqrt{E_0^2+F_0^3}}.
\end{aligned}\tag{14}$$

From $\mathbf{a}_2=0$, the Z_i corresponding to each W_i can be found, $i=1,2,3$. Their expressions are very complex, to avoid reading trouble, we place them in the appendix.

Now we study the zeros with $R\neq 0$ of the averaged function $\mathbf{g}(R,Z,W)$. Then, from $g_1(R,Z,W)=0$ we get $a_7W+a_6Z+a_1+b_{10}W+b_9Z+b_1=0$. In the same way, using the Gröbner basis techniques, we can get a basis Gröbner of the ideal $\mathcal{B}$ of $\mathbb{C}[R,Z,W]$ generated by the three polynomials $a_7W+a_6Z+a_1+b_{10}W+b_9Z+b_1, g_2(R,Z,W)$ and $g_3(R,Z,W)$. Using Mathematica we calculated that the ideal has six generators, three of which are

$$\begin{aligned}
\mathbf{b}_1=&\Big((a_1+b_1)(-(a_1+b_1)(c_{13}(d_4+d_8)-(c_4+c_8)d_{13})+a_7(-c_4d_3-c_8d_3\\
&+c_3(d_4+d_8))+b_{10}(-c_4d_3-c_8d_3+c_3(d_4+d_8)))\Big)+\Big(a_7(a_1c_{12}d_4\\
&+a_1c_{12}d_8+a_6(-c_4d_3-c_8d_3+c_3(d_4+d_8))-a_1c_4d_{12}-a_1c_8d_{12}-b_9c_4d_3\\
&-b_9c_8d_3+b_9c_3d_4+b_1c_{12}d_4+b_9c_3d_8+b_1c_{12}d_8+2b_{10}(c_4d_2+c_8d_2\\
&-c_2(d_4+d_8))-b_1c_4d_{12}-b_1c_8d_{12})+b_{10}((a_1+b_1)(c_{12}(d_4+d_8)\\
&-(c_4+c_8)d_{12})+a_6(-c_4d_3-c_8d_3+c_3(d_4+d_8))+b_9(-c_4d_3-c_8d_3\\
&+c_3(d_4+d_8)))-2(a_1+b_1)(a_6+b_9)(c_{13}(d_4+d_8)-(c_4+c_8)d_{13})\\
&+a_7^2(c_4d_2+c_8d_2-c_2(d_4+d_8))+b_{10}^2(c_4d_2+c_8d_2-c_2(d_4+d_8))\Big)Z\\
&+\Big(a_7((a_6+b_9)(c_{12}(d_4+d_8)-(c_4+c_8)d_{12})-2b_{10}(c_{11}(d_4+d_8)\\
&-(c_4+c_8)d_{11}))+b_{10}(a_6+b_9)(c_{12}(d_4+d_8)-(c_4+c_8)d_{12})\\
&-(a_6+b_9)^2(c_{13}(d_4+d_8)-(c_4+c_8)d_{13})+a_7^2((c_4+c_8)d_{11}\\
&-c_{11}(d_4+d_8))+b_{10}^2((c_4+c_8)d_{11}-c_{11}(d_4+d_8))\Big)Z^2\\
:=&C_1+B_1Z+A_1Z^2,\\
\mathbf{b}_2=&(a_7+b_{10})W+(a_6+b_9)Z+a_1+b_1,\\
\mathbf{b}_3=&\big(d_4+d_8\big)R^2+2d_{13}W^2+2d_{12}WZ+2d_3W+2d_{11}Z^2+2d_2Z.
\end{aligned}\tag{15}$$

If $A_1 \neq 0$ and $B_1^2 - 4A_1C_1 > 0$ we can find two real zeros from $\mathbf{b}_1 = 0$, named Z_4 and Z_5. Morever, we have

$$Z_4 = \frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1}, \quad Z_5 = \frac{-B_1 - \sqrt{B_1^2 - 4A_1C_1}}{2A_1}. \tag{16}$$

Substituting one of their two zeros into $\mathbf{b}_2 = 0$ we obtain a unique solution for W. Then substituting Z and W in $\mathbf{b}_3 = 0$ we obtain at most one positive value for R. We also put the expression of W_4, W_5 and R_4, R_5 in the appendix. This proves half of statement (a) of Theorem 1. To complete the proof we will provide the Example 2 for showing that the five periodic orbit can be reached.

As can see from the proof of statement (a), the averaged function provides two types of zeros, denoted $(0, Z_i, W_i)$ and (R_j, Z_j, W_j). We assume that these zeros are simple zeros, i.e. that the Jacobian $\det(D_{\mathbf{x}}\mathbf{g})$ of these zeros is not zero.

The zero $(0, Z_i, W_i)$ of the averaged function $\mathbf{g}(R, Z, W)$ provide a periodic orbit $(R(\theta, \varepsilon), Z(\theta, \varepsilon), W(\theta, \varepsilon))$ of system (9) satisfying the initial condition:

$$(R(0, \varepsilon), Z(0, \varepsilon), W(0, \varepsilon)) = (0, W_i, Z_i) + O(\varepsilon).$$

This periodic orbit provides the periodic orbit $(R(t, \varepsilon), \theta(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon))$ of system (8) that satisfies the initial condition

$$(R(0, \varepsilon), \theta(0, \varepsilon), Z(0, \varepsilon), W(0, \varepsilon)) = \big(O(\varepsilon), O(\varepsilon), W_i + O(\varepsilon), Z_i + O(\varepsilon)\big).$$

Returning to cartesian coordinates (X, Y, Z, W), we have the periodic orbit $(X(t, \varepsilon),$ $Y(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon))$ such that

$$(X(0, \varepsilon), Y(0, \varepsilon), Z(0, \varepsilon), W(0, \varepsilon)) = \big(O(\varepsilon), O(\varepsilon^2), W_i + O(\varepsilon), Z_i + O(\varepsilon)\big).$$

Finally rescaling back to (x, y, z, w) we obtain the periodic orbit $\Gamma_{1\varepsilon} : (x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon),$ $w(t, \varepsilon))$ such that

$$(x(0, \varepsilon), y(0, \varepsilon), z(0, \varepsilon), w(0, \varepsilon)) = \big(O(\varepsilon^2), O(\varepsilon^3), \varepsilon W_i + O(\varepsilon^2), \varepsilon Z_i + O(\varepsilon^2)\big).$$

The zero (R_j, Z_j, W_j) of the averaged function $\mathbf{g}(R, Z, W)$ provide a periodic orbit $(R(\theta, \varepsilon), Z(\theta, \varepsilon), W(\theta, \varepsilon))$ of system (9) satisfying the initial condition:

$$(R(0, \varepsilon), Z(0, \varepsilon), W(0, \varepsilon)) = (R_j, W_j, Z_j) + O(\varepsilon).$$

The periodic orbit $(R(\theta, \varepsilon), Z(\theta, \varepsilon), W(\theta, \varepsilon))$ in the differential system (8) becomes

$$(R(t, \varepsilon), \theta(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon)) = (R_j, \omega t, Z_j, W_j) + O(\varepsilon).$$

And the periodic orbit $(R(t, \varepsilon), \theta(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon))$ in the differential system (7) is

$$(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon)) = (R_j \cos(\omega t), R_j \sin(\omega t), Z_j, W_j) + O(\varepsilon).$$

Finally the periodic orbit $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon), W(t, \varepsilon))$ in the differential system (1) is

$$\Gamma_{2\varepsilon} : (x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon), w(t, \varepsilon)) = \varepsilon\big(R_j \cos(\omega t), R_j \sin(\omega t), Z_j, W_j\big) + O(\varepsilon^2).$$

This proves half of statement (b). Next we study the non-existence of the C^1 first integral in the neighbourhood of $\Gamma_{1\varepsilon}$ and $\Gamma_{2\varepsilon}$.

According to Remark 6, we know that if $\lambda_{1,j}$, $\lambda_{2,j}$ and $\lambda_{3,j}$, the eigenvalues of Jacobian matrix $D_{\mathbf{x}}\mathbf{g}(R_j, Z_j, W_j)$, are distinct from an integer multiple of $\sqrt{-1}/\varepsilon$, the system has no C^1 first integrals in a neighbourhood of $\Gamma_{2\varepsilon}$. Of course, a similar conclusion holds for $\Gamma_{1\varepsilon}$. But we have a simpler way to check the non-existence of the C^1 first integral in the neighbourhood of $\Gamma_{2\varepsilon}$.

Next we prove that if the zeros $(0, Z_i, W_i)$ are simple zeros and satisfy $c_{12}W_i + 2c_{11}Z_i + c_2 + 2d_{13}W_i + d_{12}Z_i + d_3 \neq 0$, then there is no C^1 first integral near their corresponding periodic orbits. In fact, at $(0, Z_i, W_i)$, we have

$$D_x\big(\mathbf{g}(0, Z_i, W_i)\big) =$$
$$\begin{pmatrix} \frac{a_7W_i+a_6Z_i+a_1+b_{10}W_i+b_9Z_i+b_1}{2\omega} & 0 & 0 \\ 0 & \frac{c_{12}W_i+2c_{11}Z_i+c_2}{\omega} & \frac{2c_{13}W_i+c_{12}Z_i+c_3}{\omega} \\ 0 & \frac{d_{12}W_i+2d_{11}Z_i+d_2}{\omega} & \frac{2d_{13}W_i+d_{12}Z_i+d_3}{\omega} \end{pmatrix}$$

and $\det\big(D_x\big(\mathbf{g}(0, Z_i, W_i)\big)\big) \neq 0$. One can easily see that one of the eigenvalues of matrix $D_x\big(\mathbf{g}(0, Z_i, W_i)\big)$ is $(a_7W_i + a_6Z_i + a_1 + b_{10}W_i + b_9Z_i + b_1)/(2\omega)$, and the other two eigenvalues are the zeros of a real quadratic polynomial. Therefore, either the two eigenvalues are real numbers, or they are two conjugate complex numbers $\big($through simple calculations, we can know that their real parts are $(c_{12}W_i + 2c_{11}Z_i + c_2 + 2d_{13}W_i + d_{12}Z_i + d_3)/\omega\big)$. According to Remark 6, if we want to prove that there is no C^1 first integral near the periodic orbit provided by the zero $(0, Z_i, W_i)$ of the averaged function, we must ensure that all eigenvalues of matrix $D_x\big(\mathbf{g}(0, Z_i, W_i)\big)$ are not an integer multiple of $2\pi i/(T\varepsilon) = i/\varepsilon$. Obviously when $c_{12}W_i + 2c_{11}Z_i + c_2 + 2d_{13}W_i + d_{12}Z_i + d_3 \neq 0$, matrix $D_x\big(\mathbf{g}(0, Z_i, W_i)\big)$ has either three real eigenvalues or one real eigenvalue and two conjugate complex eigenvalues. In both cases they cannot be an integer multiple of $\sqrt{-1}/\varepsilon$.

This proves statement (b) of Theorem 1. □

Proof of Example 2. Taking $\omega = a_1 = a_6 = a_7 = a_9 = b_1 = b_9 = c_{12} = c_{13} = d_4 = d_{11} = 1$, $c_3 = -3$, $c_2 = d_2 = d_3 = d_{12} = d_{13} = -5$ and other coefficients being zero, system (1) becomes system (2). Obviously the origin of system (2) is a zero-Hopf equilibrium point when $\varepsilon = 0$.

Next we prove that system (2) indeed has five periodic orbits bifurcating from the zero-Hopf equilibrium point. We can compute the first averaged function (11) and we obtain

$$\mathbf{g}(R, Z, W) = \big(g_1(R, Z, W), g_2(R, Z, W), g_3(R, Z, W)\big) = \big(\frac{1}{2}R(W + 2Z + 2),$$
$$-\frac{R^2}{2} + W^2 + W(Z - 3) - 5Z, \frac{R^2}{2} - 5W^2 - 5W(Z + 1) + X^2 + (Z - 5)Z\big).$$

Solving the equation $g_1(R, Z, W) = g_2(R, Z, W) = g_3(R, Z, W) = 0$ and requiring $R \geq 0$, $|R| + |Z| + |W| \neq 0$, we get five solutions $(R_1, Z_1, W_1) = (2\sqrt{5}, 0, -2)$, $(R_2, Z_2, W_2) = (16\sqrt{2}/7, -18/7, 22/7)$, $(R_3, Z_3, W_3) \approx (0, 6.11887, -7.30628)$, $(R_4, Z_3, W_4) \approx (0, 0.993874, -1.44142)$ and $(R_5, Z_5, W_5) \approx (0, 32.8873, 4.7477)$.

We verify that they are simple, i.e. the determinant of the Jacobian matrix of the function $\mathbf{g}(R,Z,W)$ at these zeros is non-zero. Indeed,

$$\begin{aligned}
&\det\big(D(\mathbf{g}(R_1,Z_1,W_1))\big)=180,\quad \det\big(D(\mathbf{g}(R_2,Z_2,W_2))\big)=-4608/49,\\
&\det\big(D(\mathbf{g}(R_3,Z_3,W_3))\big)=145.463,\quad \det\big(D(\mathbf{g}(R_4,Z_4,W_4))\big)=-10.3414,\\
&\det\big(D(\mathbf{g}(R_5,Z_5,W_5))\big)=-50905.1.
\end{aligned}$$

This shows that these five zeros are all simple and that they provide five periodic orbits of system (2).

Next we prove that there is no C^1 first integral near these five periodic orbits. We denote λ_{ij} as the j-th eigenvalue of matrix $D(\mathbf{g}(R_i,Z_i,W_i))$, then through some calculations, we can get

$$\begin{array}{lll}
\lambda_{11}\approx 13.454, & \lambda_{12}\approx -2.72702+2.43767i, & \lambda_{13}\approx -2.72702-2.43767i;\\
\lambda_{21}\approx -22.6309, & \lambda_{22}\approx -1.39884+1.48279i, & \lambda_{23}\approx -1.39884-1.48279i;\\
\lambda_{31}\approx 23.3659, & \lambda_{32}\approx 3.46573, & \lambda_{33}\approx 1.79629;\\
\lambda_{41}\approx -4.01809, & \lambda_{42}\approx 2.02151, & \lambda_{43}\approx 1.27316;\\
\lambda_{51}\approx -223.448, & \lambda_{52}\approx 36.2611, & \lambda_{53}\approx 6.28266;
\end{array}$$

According to Remark 6 there is no C^1 first integral near the five periodic orbits. This ends the proof. □

4. APPENDIX

In the last section, we summarize the expressions for $Z_1, Z_2, Z_3, W_4, W_5, R_4$ and R_5.

$$\begin{aligned}
Z_1 =&\Big(3A_0\big(\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}+\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)-B_0\big)\\
&\big(3A_0\big(d_{11}\big(c_{13}\big(\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}+\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)+c_3\big)\\
&-c_{11}\big(d_{13}\big(\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}+\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)+d_3\big)\big)\\
&+B_0\big(c_{11}d_{13}-c_{13}d_{11}\big)\Big)\Big/\Big(3A_0\big(3A_0\big(d_{11}\big(c_{12}\big(\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&+\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)+c_2\big)-c_{11}\big(d_{12}\big(\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&+\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)+d_2\big)\big)+B_0\big(c_{11}d_{12}-c_{12}d_{11}\big)\big)\Big),\\
Z_2 =&\Big(-\frac{2B_0}{A_0}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\\
&\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big(d_{11}\big(c_3+\frac{1}{6}c_{13}\big(-\frac{2B_0}{A_0}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big),
\end{aligned}$$

$$
\begin{aligned}
&\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)\\
&-c_{11}\big(d_3+\frac{1}{6}d_{13}\big(-\frac{2B_0}{A_0}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)\Big)\Big/\Big(6\big(c_{11}\big(d_2\\
&+\frac{1}{6}d_{12}\big(-\frac{2B_0}{A_0}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)+d_{11}\big(-c_2\\
&-\frac{1}{6}c_{12}\big(-\frac{2B_0}{A_0}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
Z_3=&\Big(-\frac{2B_0}{A_0}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\\
&\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big(d_{11}\big(c_3+\frac{1}{6}c_{13}\big(-\frac{2B_0}{A_0}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\\
&\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)-\\
&c_{11}\big(d_3+\frac{1}{6}d_{13}\big(-\frac{2B_0}{A_0}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)\Big)\Big/\Big(6\big(c_{11}\big(d_2\\
&+\frac{1}{6}d_{12}\big(-\frac{2B_0}{A_0}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)+d_{11}\big(-c_2\\
&-\frac{1}{6}c_{12}\big(-\frac{2B_0}{A_0}+3\sqrt{-1}\big(\sqrt{3}+\sqrt{-1}\big)\sqrt[3]{E_0-\sqrt{F_0^3+E_0^2}}\\
&-3\sqrt{-1}\big(\sqrt{3}-\sqrt{-1}\big)\sqrt[3]{\sqrt{F_0^3+E_0^2}+E_0}\big)\big)\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
W_4&=-\frac{(a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)+2a_1A_1+2A_1b_1}{2A_1(a_7+b_{10})},\\
W_5&=\frac{(a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}+B_1\big)-2a_1A_1-2A_1b_1}{2A_1(a_7+b_{10})},
\end{aligned}
$$

$$
\begin{aligned}
R_4 = {} & \frac{1}{\sqrt{2d_4+2d_8}}\Big(A_1^{-2}\Big(2A_1d_2\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)+d_{11}\big(B_1-\sqrt{B_1^2-4A_1C_1}\big)^2 \\
& -\frac{2A_1d_3\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)+2a_1A_1+2A_1b_1\big)}{a_7+b_{10}} \\
& -\frac{d_{12}\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)+2a_1A_1+2A_1b_1\big)}{a_7+b_{10}} \\
& +\frac{d_{13}\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)+2a_1A_1+2A_1b_1\big)^2}{(a_7+b_{10})^2}\Big)\Big)^{\frac{1}{2}}, \\
R_5 = {} & \frac{1}{\sqrt{2d_4+2d_8}}\Big(-A_1^{-2}\Big(2A_1d_2\big(\sqrt{B_1^2-4A_1C_1}+B_1\big)+d_{11}\big(B_1+\sqrt{B_1^2-4A_1C_1}\big)^2 \\
& -\frac{2A_1d_3\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}+B_1\big)+2a_1A_1+2A_1b_1\big)}{a_7+b_{10}} \\
& -\frac{d_{12}\big(\sqrt{B_1^2-4A_1C_1}+B_1\big)\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}-B_1\big)-2a_1A_1-2A_1b_1\big)}{a_7+b_{10}} \\
& -\frac{-d_{13}\big((a_6+b_9)\big(\sqrt{B_1^2-4A_1C_1}+B_1\big)+2a_1A_1+2A_1b_1\big)^2}{\big(a_7+b_{10}\big)^2}\Big)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Acknowledgements

The first author is partially supported by the Agencia Estatal de Investigación grant PID2022-136613NB-100, the H2020 European Research Council grant MSCARISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Acadèmia de Ciències i Arts de Barcelona.

The second author is supported by International Program for Candidates, Sun Yat-Sen University.

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