This is a preprint of: "Phase Portraits of the equation $\ddot{x}+ax\dot{x}+bx^3=0$", Jaume Llibre, Clàudia Valls, *Regul. Chaotic Dyn.*, vol. 29(6), 825–837, 2024.
DOI: [10.1134/S1560354724560053]

PHASE PORTRAITS OF THE EQUATION $\ddot{x}+ax\dot{x}+bx^3=0$

JAUME LLIBRE[1] AND CLAUDIA VALLS[*2]

Abstract. The second-order differential equation $\ddot{x}+ax\dot{x}+bx^3=0$ with $a,b\in\mathbb{R}$ has been studied by several authors mainly due to its applications. Here, for the first time, we classify all its phase portraits in function of its parameters a and b. This classification is done in the Poincaré disc in order to control the orbits which scape or come from infinity. We prove that there are exactly six topologically different phase portraits in the Poincaré disc of the first order differential system associated by the second-order differential equation. Additionally we show that this system is always integrable providing explicitly its first integrals.

1. Introduction and statement of the main results

In the book of Ince [7] or in the work of Painleve [12] appeared the second order ordinary differential equation

$$\ddot{x}+ax\dot{x}+bx^3=0,$$

with $a,b\in\mathbb{R}$. This differential equation is equivalent to the differential system of first order

$$\dot{x}=y,\quad \dot{y}=-axy-bx^3. \tag{1}$$

This differential system arises in many areas of mathematics such as the analysis of fusion of pellets [3], the theory of univalent functions [4], the stability of gaseous spheres [8], in the operator of the Yang-Baxter equations [5, 7, 9], in the description of the motion of a free particle in a space of constant curvature [15],...

This differential system possesses the Painlevé property and it has been studied by many authors due to its simple form, it is a Lienard differential equation and it possesses the algebra $sl(3,\mathbb{R})$ of Lie point symmetries. More concretely it is quite nonlinear and belongs to the class of equations of the form $\ddot{x}=x^3f(\dot{x}/x^2)$ which posses the two Lie point symmetries ∂_t and $t\partial_t-x\partial_x$ for a general function f and also it has the eight Lie point symmetry characteristic of the representative second-order ordinary differential

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05, 37C10.
Key words and phrases. second-order differential equation, Poincaré compactification, global phase portraits.

equation of maximal symmetry. See, for more details on the mentioned symmetries [9]

Due to the rich algebraic structure of the differential system (1) and the fact that many authors have worked with these equations it is somewhat strange that there are no complete studies of their dynamics. This is the main goal of this paper and we shall give the complete description of their phase portraits in the Poincaré disc (i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity) modulo topological equivalence.

Roughly speaking the Poincaré compactification of a polynomial differential system consists in extending this system to an analytic system on the closed disc $\mathbb{D}^2$ of radius one and centered at the origin, whose interior is identified with $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. This closed disc is called the *Poincaré disc*, because the technique for doing such an extension is due to Poincaré [14]. For details on this compactification see [2, chapter 5], or the summary presented in subsection 2.1. In fact we shall present the distinct global phase portraits of system (1) when its parameters a, b varies in $\mathbb{R}$ in the Poincaré disc. In this way we can describe the dynamics of their orbits which come or go to infinity of $\mathbb{R}^2$.

A function $H\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ is a *first integral* of the differential system (1) if

$$\frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0.$$

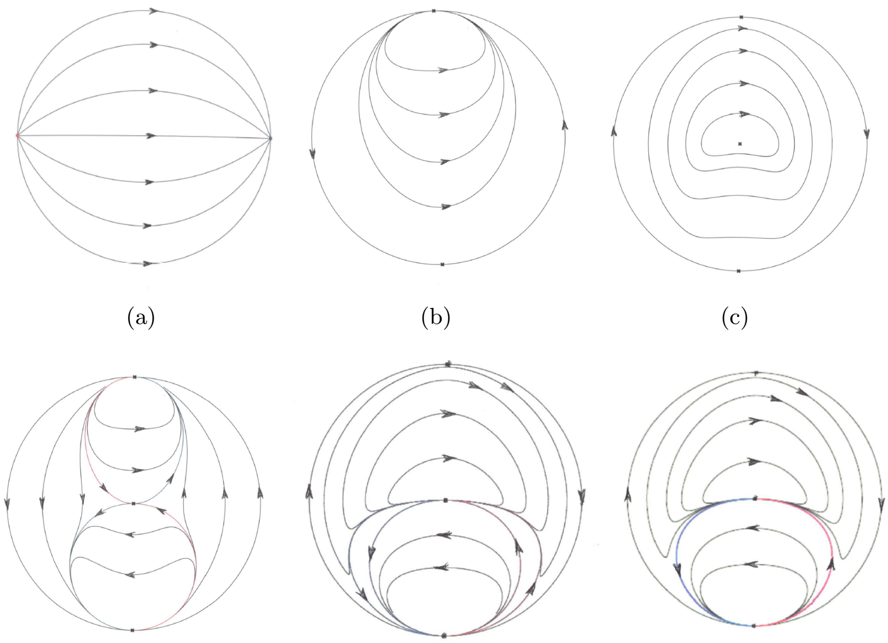
Note that if $b = 0$ then the differential system (1) has a straight line of singular points, and if $b \neq 0$ it has a unique singular point, the origin. We say that the origin is a *center* if there exists a neighborhood $V \subset \mathbb{R}^2$ of the origin such that each orbit of system (1) in $V \setminus \{(0,0)\}$ is periodic. In this case we define the *period annulus* of the center as the maximal open connected set $W \subset \mathbb{R}^2$ such that $W \setminus \{(0,0)\}$ is filled with periodic orbits of system (1). When this period annulus is $\mathbb{R}^2$ we say that the center is *global*.

The following theorem is our main result.

Theorem 1. *The following statements hold for system* (1).

(i) *It can be transformed via a change of variables and a rescaling of time (if necessary) into one of the following three families of systems*
$\dot{x} = 1$, $\dot{y} = 0$;
$\dot{x} = 1$, $\dot{y} = x$;
$\dot{x} = y$, $\dot{y} = -x^3$;
$\dot{x} = y$, $\dot{y} = x^3$;
$\dot{x} = y$, $\dot{y} = -xy - \delta x^3$ *with* $\delta \in \mathbb{R} \setminus \{0\}$.
called families (I), (II), (III), (IV) *and* (V), *respectively.*

(ii) *Family (I) is integrable with the first integral* $H = y$, *and its global phase portrait in the Poincaré disc is topologically equivalent to the*



(a) (b) (c) (d) (e) (f)

Figure 1. The six different topological phase portraits of system (1).

one presented in Figure 1(*a*)*. All the orbits of system* (1) *are straight lines parallel to the y-axis.*

(iii) *Family (II) is integrable with the first integral $H = y - x^2/\sqrt{2}$, and its global phase portrait in the Poincaré disc is topologically equivalent to the one presented in Figure* 1(*b*)*. In particular all orbits of system* (1) *come from the endpoint of the positive y-half-axis and end also in this endpoint.*

(iv) *Family (III) is integrable with the first integral $H = y^2 + x^4/2$, and its global phase portrait in the Poincaré disc is topologically equivalent to the one presented in Figure* 1(*c*)*. In particular the origin is a global center.*

(v) *Family (IV) is integrable with the first integral $H = y^2 - x^4/2$, and its global phase portrait in the Poincaré disc is topologically equivalent to the one presented in Figure* 1(*d*)*.*

(vi) *Family (V) is integrable with the first integral*

$$H = \begin{cases} (2\delta(1-\Gamma)x^2+8\delta y)^{\Gamma-1}(2\delta(1+\Gamma)x^2+8\delta y)^{\Gamma+1} & \text{if } \delta < 1/8, \\ e^{\frac{4x^2}{x^2+4y}}\left(x^2+4y\right)^4 & \text{if } \delta = 1/8, \\ \left(\delta x^4+x^2y+2y^2\right)^{\sqrt{8\delta-1}}\, e^{-2\arctan\frac{x^2+4y}{\sqrt{8\delta-1}x^2}} & \text{if } \delta > 1/8, \end{cases}$$

where $\Gamma = \sqrt{1-8\delta}$*. Its global phase portrait in the Poincaré disc is topologically equivalent to the one presented in:*
Figure 1(*d*) *if* $\delta < 0$,
Figure 1(*e*) *if* $\delta \in (0, 1/8)$,
Figure 1(*f*) *if* $\delta = 1/8$, *and*
Figure 1(*c*) *if* $\delta > 1/8$.

The proof of Theorem 1 is given in the next section where we prove each of the statements in different subsections. We have included an appendix with some preliminary results.

2. Proof of Theorem 1

2.1. **Proof of Theorem 1(i).** Assume first that $b = 0$, then system (1) becomes

$$\dot{x} = y, \quad \dot{y} = -axy.$$

If $a = 0$ then we have the system $\dot{x} = y$, $\dot{y} = 0$, which correspond to the family (I).

If $a \neq 0$ in the new variables $X = -a^{1/3}x$, $Y = -a^{1/3}y$ and with the rescaling of the time $dt = Y\,ds$ the above system becomes

$$X' = 1, \quad Y' = X,$$

where the prime denotes derivative in the new time s. Note that this is family (II).

Assume now that $b \neq 0$ and $a = 0$. Then system (1) becomes

$$\dot{x} = y, \quad \dot{y} = bx^3.$$

In the new variables $X = \sqrt{|b|}x$, $Y = \sqrt{|b|}y$ system (1) becomes

$$\dot{X} = Y, \quad \dot{Y} = \pm X^3,$$

with the sign $+$ if $b > 0$, and the sign $-$ if $b < 0$. Thus we have families (III) and (IV), respectively.

Assume now that $ba \neq 0$. Setting $X = ax$, $Y = ay$, we get

$$\dot{X} = Y, \quad \dot{Y} = -XY - \frac{b}{a^2}X^3.$$

yielding family (V). This completes the proof of statement (i).

2.2. **Proof of Theorem 1(ii).** We study the global phase portraits of family (I). Since clearly $H = y$ is a first integral of this system the phase portrait is formed by straight lines $y =$ constant. Consequently there is a unique pair of singular points at infinity at the endpoints of the y-axis. Therefore the first portrait is the one given in Figure 1(a).

2.3. **Proof of Theorem 1(iii).** We study the global phase portraits of family (II). It is clear that there are no finite singular points for family (II). Moreover it is easy to check that $H = y - x^2/\sqrt{2}$ is a first integral of this system. Since $y = x^2/\sqrt{2} + h$ are parabolas symmetric with respect to the y-axes, it follows that the phase portrait in the Poincaré disc is topologically equivalent to the one given in Figure 1(b). So at infinity there is only a pair of singular points. Each of this singular points in the Poincaré sphere is formed by one elliptic sector and one hyperbolic sector. This completes the proof of Theorem 1(iii).

2.4. **Proof of Theorem 1(iv).** We study the global phase portraits of family (III). The unique singular point is the origin. It is easy to verify that $H = y^2 + x^4/2$ is a first integral of this system. Since all the curves $y^2 + x^4/2 = h$ are closed simple curves surrounding the origin, it follows that the phase portrait in the Poincaré disc is topologically equivalent to the one given in Figure 1(c). So the phase portrait is a global center.

On the local chart U_1 family (III) becomes

$$\dot{u} = -1 - u^2v^2, \quad \dot{v} = -uv^3.$$

Note that there are no singular points in the local chart U_1.

On the local chart U_2 family (III) becomes

$$\dot{u} = v^2 + u^4, \quad \dot{v} = u^3v.$$

The origin $(0, 0)$ is a singular point of the local chart U_2. Due to the existence of a global center this singular point on the Poincaré sphere is formed by two hyperbolic sectors. This completes the proof of Theorem 1(iv).

2.5. **Proof of Theorem 1(v).** We study the global phase portraits of family IV). It is clear that the unique finite singular point for family (IV) is the origin. The Jacobian matrix of family (IV) at the origin is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

and so the origin is a nilpotent singular point. It follows from Theorem 3.5 of [2] that the origin is a saddle.

It is easy to check that $H = y^2 - x^4/2$ is a first integral of system (IV).

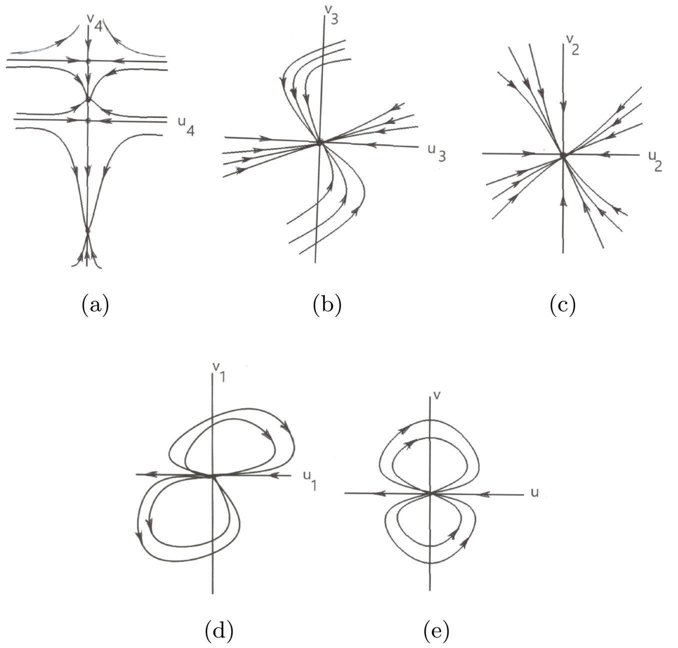


Figure 2. The local phase portraits near the vertical axis: (a) of system (10), (b) of system (8), (c) of system (7), (d) of system (5), (e) of system (3), i.e. at the origin of the local chart U_2 of system (IV).

On the local chart U_1 family (IV) becomes

$$\dot{u} = 1 - u^2 v^2, \quad \dot{v} = -uv^3.$$

Note that there are no singular points in the local chart U_1.

On the local chart U_2 family (IV) becomes

$$\dot{u} = v^2 - u^4, \quad \dot{v} = -u^3 v. \tag{3}$$

The origin $(0,0)$ is a singular point of the local chart U_2. The Jacobian matrix at the origin is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

Therefore the origin is a degenerate singular point. We need to apply the blow-up technique. Since the characteristic directions at the origin of the previous differential system are given by $uv^2 = 0$ we remove the direction $u = 0$ doing the change of variables $u_1 = u + v$, $v_1 = v$. Then system (3) becomes

$$\begin{aligned} \dot{u}_1 &= -v_1^2 - u_1^4 + 3u_1^3 v_1 - 3u_1^2 v_1^2 + u_1 v_1^3, \\ \dot{v}_1 &= -v_1(u_1^3 - 3u_1^2 v_1 + 3u_1 v_1^2 - v_1^3). \end{aligned} \tag{5}$$

Now we do a vertical blow-up. To do so, let $u_2 = u_1$, $v_2 = v_1/u_1$ and we get the system

$$\dot{u}_2 = u_2^2(-u_2^2 + v_2^2 + 3u_2^2 v_2 - 3u_2^2 v_2^2 + u_2^2 v_2^3), \quad \dot{v}_2 = -u_2 v_2^3 \tag{6}$$

Reescaling the independent variable we eliminate the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$, and we obtain the differential system

$$\dot{u}_2 = u_2(-u_2^2 + v_2^2 + 3u_2^2 v_2 - 3u_2^2 v_2^2 + u_2^2 v_2^3), \quad \dot{v}_2 = -v_2^3. \tag{7}$$

Again the unique singular point on the straight line $v_2 = 0$ is the origin. So, we need to do another blow-up. Since the characteristic directions at the origin of the previous system are $v_2 u_2(u_2^2 - 2v_2^2) = 0$ we need to remove the characteristic direction $v_2 = 0$ doing the change of variables $u_3 = u_2 + v_2$ and $v_3 = v_2$, the previous system becomes

$$\begin{aligned} \dot{u}_3 =& -u_3^3 + 3u_2^3 v_3 - 2u_3 v_3^2 - v_3^3 + 3u_3^3 v_3 - 9u_3^2 v_3^2 + 9u_3 v_3^3 - 3v_3^4 \\ & - 3u_3^3 v_3^2 + 9u_3^2 v_3^3 - 9u_3 v_3^4 + 3v_3^5 + u^3 v_3^3 - 3u_3^2 v_3^4 + 3u_3 v_3^5 - v_3^6, \\ \dot{v}_3 =& -v_3^3 \end{aligned} \tag{8}$$

Doing the vertical blow-up $u_4 = u_3$, $v_4 = v_3/u_3$, we get that the above system becomes

$$\begin{aligned} \dot{u}_4 =& -u_4^3(1 - 3v_4 - 3u_4 v_4 + 2v_4^2 + 9u_4 v_4^2 + v_4^3 + 3u_4^2 v_4^2 - 9u_4 v_4^3 \\ & - 9u_4^2 v_4^3 + 3u_4 v_4^4 - u_4^3 v_4^3 + 9u_4^2 v_4^4 + 3u_4^3 v_4^4 - 3u_4^2 v_4^5 - 3u_4^3 v_4^5 \\ & + u_4^3 v_4^6), \\ \dot{v}_4 =& u_4^2(v_4 - 1)v_4(-1 + 2v_4 + 3u_4 v_4 + v_4^2 - 6u_4 v_4^2 - 3u_4^2 v_4^2 + 3u_4 v_4^3 \\ & + 6u_4^2 v_4^3 + u_4^3 v_4^3 - 3u_4^2 v_4^4 - 2u_4^3 v_4^4 + u_4^3 v_4^5). \end{aligned} \tag{9}$$

Reescaling the independent variable we eliminate the common factor u_4^2 between $\dot{u}_4$ and $\dot{v}_4$, and we obtain the differential system

$$\begin{aligned} \dot{u}_4 =& -u_4(1 - 3v_4 - 3u_4 v_4 + 2v_4^2 + 9u_4 v_4^2 + v_4^3 + 3u_4^2 v_4^2 - 9u_4 v_4^3 \\ & - 9u_4^2 v_4^3 + 3u_4 v_4^4 - u_4^3 v_4^3 + 9u_4^2 v_4^4 + 3u_4^3 v_4^4 - 3u_4^2 v_4^5 - 3u_4^3 v_4^5 \\ & + u_4^3 v_4^6), \\ \dot{v}_4 =& (v_4 - 1)v_4(-1 + 2v_4 + 3u_4 v_4 + v_4^2 - 6u_4 v_4^2 - 3u_4^2 v_4^2 + 3u_4 v_4^3 \\ & + 6u_4^2 v_4^3 + u_4^3 v_4^3 - 3u_4^2 v_4^4 - 2u_4^3 v_4^4 + u_4^3 v_4^5). \end{aligned} \tag{10}$$

This system has four singular points on the straight line $u_4 = 0$ which are $(0,0)$, $(0,1)$, $(0,-1+\sqrt{2})$ and $(0,-1-\sqrt{2})$. Computing the eigenvalues of the Jacobian matrix at these singular points we get that $(0,0)$ is a saddle (the eigenvalues are $1,-1$), $(0,1)$ is a saddle (the eigenvalues are $-1,2$), $(0,-1+\sqrt{2})$ is a stable node (the eigenvalues are $-4(3+2\sqrt{2}), -3-2\sqrt{2}$) and $(0,-1-\sqrt{2})$ is also a stable node (the eigenvalues are $-4(3-2\sqrt{2}), -3+2\sqrt{2}$).

Now we start with the blow-down process. The local phase portrait of the differential system (10) around the v_4-axis is shown in Figure $2(a)$. Then the local phase portrait of the differential system (9) around the v_4-axis is exactly the same than in Figure 2(a) but the v_4-axis is plenty of singular points.

Going back to system (8) we obtain that its local phase portrait at the origin is given in Figure 2(b). Undoing the change of variables $(u_2, v_2) \to (u_3, v_3)$ we get the local phase portrait at the origin of system (7) given in Figure 2(c). Then the local phase portrait of system (6) is the same as the one given in Figure 2(c) but with the v_2-axis filled up with singular points and the orbits in the half-plane $u_2 < 0$ with the orientation reversed. Undoing the blow-up for going back to the system (5) we get that its local phase portrait at the origin is the one given in Figure 2(d). Finally, going back to system (3) we conclude that the phase portrait at the origin of the local chart U_2 is provided in Figure 2(e), that is, this local phase portrait is formed by two elliptic sectors separated by two parabolic sectors.

From subsection 3.2 in order to obtain the global phase portrait in the Poincaré disc we only need to determine the behaviour of the separatrices of the hyperbolic sectors of the saddle at the origin of coordinates, i.e. where they born and where they die. Since the unique singular points of this system are the origin of coordinates and the endpoints of the y-axis, the mentioned separatrices must end or start at these singular points, because system (IV) has no limit cycles due to the existence of a first integral defined in the whole plane. So the two stable separatrices at the origin start in the infinite pair of singular points, and the two unstable separatrices of the origin end at the infinite pair of singular points. These four separatrices are at the boundary of the two elliptic sectors at infinity. In the two regions outside the closure of the elliptic sectors there are orbits travelling from one point of the infinity to the other. See Figure 1(d). This completes the proof of Theorem 1(v).

2.6. **Proof of Theorem 1(vi).** It is easy to check that the function H provided in the statement (vi) of Theorem 1 is a first integral of system (V). Note that when $\delta \geq 1/8$ is defined in the whole plane, and when $\delta < 1/8$ this first integral together with its inverse cover all the plane. So system (V) cannot have limit cycles for any $\delta \neq 0$.

It is clear that the unique finite singular point is the origin. The Jacobian matrix of family (V) at the origin is the same as in (2) and so the origin is nilpotent. It follows from Theorem 3.5 in [2] that the origin is a saddle when $\delta < 0$, it is formed by the union of an elliptic and a hyperbolic sector separated by parabolic sectors when $\delta \in (0, 1/8)$, it is formed by the union of an elliptic and a hyperbolic sector when $\delta = 1/8$, and it is a center when $\delta > 1/8$ due to the existence of the first integral defined locally at the origin.

On the local chart U_1 family (V) becomes

$$\dot{u} = -\delta - uv - u^2v^2, \quad \dot{v} = -uv^3.$$

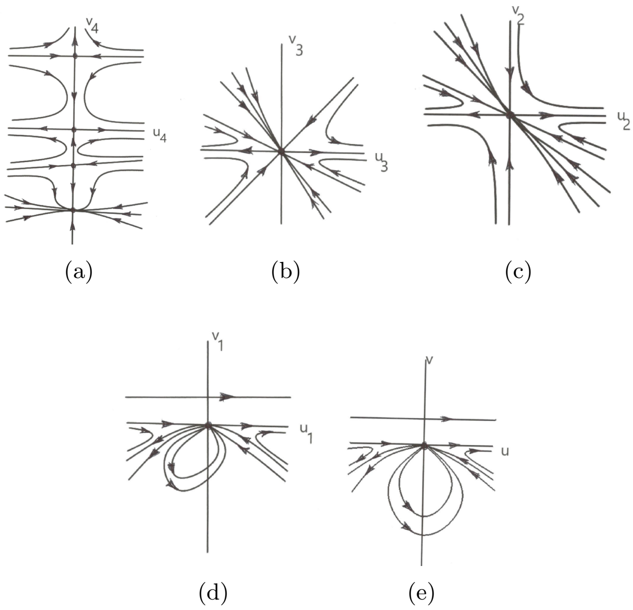


Figure 3. The local phase portraits near the vertical axis: (a) of system (17), (b) of system (15), (c) of system (13), (d) of system (12), (e) of system (11), i.e. at the origin of the local chart U_2 of system (V) for $\delta \in (0, 1/8)$.

Since $\delta \neq 0$ there are no singular points in the local chart U_1.

On the local chart U_2 family (V) becomes

$$\dot{u} = v^2 + u^2 v + \delta u^4, \quad \dot{v} = uv^2 + \delta u^3 v. \tag{11}$$

The origin $(0,0)$ is a singular point of the local chart U_2. The Jacobian matrix at the origin is the same as in (4) and so the origin of U_2 is a degenerate singular point. We need to apply the blow-up technique. Since the characteristic directions at the origin of the previous differential system are given by $uv^2 = 0$ we remove the direction $u = 0$ doing the change of variables $u_1 = u + v$, $v_1 = v$. Then system (11) becomes

$$\begin{aligned} \dot{u}_1 &= v_1^2 + u_1^2 v_1 - u_1 v_1^2 + \delta u_1^4 - 3\delta u_1^3 v_1 + 3\delta u_1^2 v_1^2 - \delta u_1 v_1^3, \\ \dot{v}_1 &= u_1 v_1^2 - v_1^3 + \delta u_1^3 v_1 - 3\delta u_1^2 v_1^2 + 3\delta u_1 v_1^3 - \delta v_1^4. \end{aligned} \tag{12}$$

Now we do a vertical blow-up. To do so, let $u_2 = u_1$, $v_2 = v_1/u_1$ and we get the system

$$\begin{aligned} \dot{u}_2 &= -u_2^2(-\delta u_2^2 - u_2 v_2 - v_2^2 + 3\delta u_2^2 v_2 + u_2 v_2^2 - 3\delta u_2^2 v_2^2 + \delta u_2^2 v_2^3), \\ \dot{v}_2 &= -u_2 v_2^3. \end{aligned} \tag{13}$$

Reescaling the independent variable we eliminate the common factor u_2 between $\dot u_2$ and $\dot v_2$, and we obtain the differential system

$$
\begin{aligned}
\dot u_2 &= -u_2(-\delta u_2^2 - u_2 v_2 - v_2^2 + 3\delta u_2^2 v_2 + u_2 v_2^2 - 3\delta u_2^2 v_2^2 + \delta u_2^2 v_2^3),\\
\dot v_2 &= -v_2^3.
\end{aligned}
\tag{14}
$$

Again the unique singular point on the straight line $v_2 = 0$ is the origin. So, we need to do another blow-up. Since the characteristic directions at the origin of the previous system are $u_2(\delta u_2^2 + u_2 v_2 + 2v_2^3) = 0$ we need to remove the characteristic direction $v_2 = 0$ doing the change of variables $u_3 = u_2 + v_2$ and $v_3 = v_2$, the previous system becomes

$$
\begin{aligned}
\dot u_3 = {}& \delta u_3^3 + (1-3\delta)u_3^2 v_3 + (3\delta-1)u_3 v_3^2 - (1+\delta)v_3^3 + (9\delta-1)u_3^2 v_3^2\\
&+ 3\delta u_3^3 v_3^2 + (2-9\delta)u_3 v_3^3 + (3\delta-1)v_3^4 - 9\delta u_3^2 v_3^3 + 9\delta u_3 v_3^4 - 3\delta v_3^5\\
&- \delta u_3^3 v_3^3 + 3\delta u_3^2 v_3^4 - 3\delta u_3 v_3^5 + \delta v_3^6\\
\dot v_3 = {}& -v_3^3
\end{aligned}
\tag{15}
$$

Doing the vertical blow-up $u_4 = u_3$, $v_4 = v_3/u_3$, we get that the above system becomes

$$
\begin{aligned}
\dot u_4 = {}& u_4^3(\delta + (1-3\delta)v_4 - 3\delta u_4 v_4 + (3\delta-1)v_4^2 + (9\delta-1)u_4 v_4^2\\
&- (1+\delta)v_4^3 + 3\delta u_4^2 v_4^2 + (2-9\delta)u_4 v_4^3 - 9\delta u_4^2 v_4^3 + (3\delta-1)u_4 v_4^4\\
&- \delta u_4^3 v_4^3 + 9\delta u_4^2 v_4^4 + 3\delta u_4^3 v_4^4 - 3\delta u_4^2 v_4^5 - 3\delta u_4^3 v_4^5 + \delta u_4^3 v_4^6)\\
\dot v_4 = {}& u_4^2(1-v_4)v_4(-\delta + (2\delta-1)v_4 + 3\delta u_4 v_4 - (1+\delta)v_4^2\\
&+ (1-6\delta)u_4 v_4^2 - 3\delta u_4^2 v_4^2 + (3\delta-1)u_4 v_4^3 + 6\delta u_4^2 v_4^3 + \delta u_4^3 v_4^3\\
&- 3\delta u_4^2 v_4^4 - 2\delta u_4^3 v_4^4 + \delta u_4^3 v_4^5).
\end{aligned}
\tag{16}
$$

Reescaling the independent variable we eliminate the common factor u_4^2 between $\dot u_4$ and $\dot v_4$, and we obtain the differential system

$$
\begin{aligned}
\dot u_4 = {}& u_4(\delta + (1-3\delta)v_4 - 3\delta u_4 v_4 + (3\delta-1)v_4^2 + (9\delta-1)u_4 v_4^2\\
&- (1+\delta)v_4^3 + 3\delta u_4^2 v_4^2 + (2-9\delta)u_4 v_4^3 - 9\delta u_4^2 v_4^3 + (3\delta-1)u_4 v_4^4\\
&- \delta u_4^3 v_4^3 + 9\delta u_4^2 v_4^4 + 3\delta u_4^3 v_4^4 - 3\delta u_4^2 v_4^5 - 3\delta u_4^3 v_4^5 + \delta u_4^3 v_4^6)\\
\dot v_4 = {}& (1-v_4)v_4(-\delta + (2\delta-1)v_4 + 3\delta u_4 v_4 - (1+\delta)v_4^2\\
&+ (1-6\delta)u_4 v_4^2 - 3\delta u_4^2 v_4^2 + (3\delta-1)u_4 v_4^3 + 6\delta u_4^2 v_4^3 + \delta u_4^3 v_4^3\\
&- 3\delta u_4^2 v_4^4 - 2\delta u_4^3 v_4^4 + \delta u_4^3 v_4^5).
\end{aligned}
\tag{17}
$$

This system has the singular points on the straight line $u_4 = 0$

$$
p_1 = (0,0), \quad p_2 = (0,1), \quad p_\pm = \left(0, \frac{2\delta - 1 \pm \sqrt{1-8\delta}}{2(1+\delta)}\right).
$$

Note that system (17) has four singular points if $\delta < 1/8$ and different from zero. For $\delta = 1/8$ this system has three singular points p_1, p_2 and $p_+ = p_-$; and if $\delta > 1/8$ this system has only two singular points p_1, p_2.

Computing the eigenvalues of the Jacobian matrix at these singular points we get that p_1 is a saddle (the eigenvalues are $\delta, -\delta$) and p_2 is a saddle (the eigenvalues are $-1, 2$). Moreover the Jacobian matrix at the singular point p_+ has the eigenvalues

$$\frac{-2\delta\left(\delta+\sqrt{1-8\delta}-3\right)+\sqrt{1-8\delta}-1}{2(\delta+1)^2},$$
$$\frac{\left(\sqrt{1-8\delta}-3\right)\sqrt{1-8\delta}\left(2\delta+\sqrt{1-8\delta}-1\right)}{4(\delta+1)^2}.$$

It is easy to check that both eigenvalues are negative if $\delta < 0$ so p_+ is a stable node when $\delta < 0$. If $\delta \in (0, 1/8)$ it is a saddle and if $\delta = 1/8$ it is a nilpotent singular point formed by one elliptic sector and one hyperbolic sector. Finally the Jacobian matrix at the singular point p_- has the eigenvalues

$$-\frac{2\delta\left(\delta-\sqrt{1-8\delta}-3\right)+\sqrt{1-8\delta}+1}{2(\delta+1)^2},$$
$$-\frac{\left(\sqrt{1-8\delta}+3\right)\sqrt{1-8\delta}\left(-2d+\sqrt{1-8\delta}+1\right)}{4(\delta+1)^2}.$$

When $\delta < 1/8$ and different from zero, this eigenvalues are always negative. So in this case p_- is a stable node.

We start with the blow-down process. If $\delta < 0$ the blow-down process is the same to the blow-down done in system (IV).

Now we study the case $\delta \in (0, 1/8)$. The local phase portrait of the differential system (17) around the v_4-axis is shown in Figure 3(a). Then the local phase portrait of the differential system (16) around the v_4-axis is exactly the same than in Figure 3(a) but the v_4-axis is plenty of singular points.

Going back to system (15) we obtain that its local phase portrait at the origin is given in Figure 3(b). Undoing the change of variables $(u_2, v_2) \to (u_3, v_3)$ we get the local phase portrait at the origin of system (14) given in Figure 3(c). Then the local phase portrait of system (13) is the same as the one given in Figure 3(c) but with the v_2-axis filled up with singular points and the orbits in the half-plane $u_2 < 0$ with the orientation reversed. Undoing the blow-up for going back to the system (12) we get that its local phase portrait at the origin is the one given in Figure 3(d). Finally, going back to system (11) we conclude that the phase portrait at the origin of the local chart U_2 is provided in Figure 3(e), that is, this local phase portrait is formed by three hyperbolic sectors and one elliptic sector separated from the hyperbolic ones by two parabolic ones.

Now we study the case $\delta = 1/8$. It follows the same process as the case $\delta \in (0, 1/8)$ but at the end we get Figure 3(e) without the parabolic sectors,

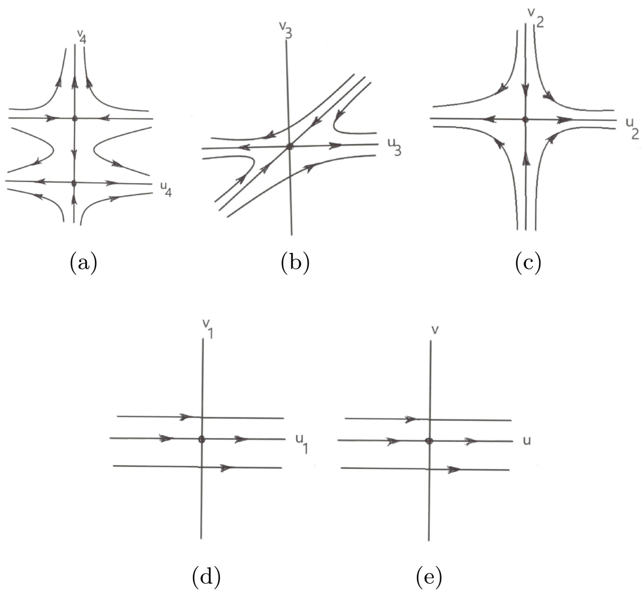


Figure 4. The local phase portraits near the vertical axis: (a) of system (17), (b) of system (15), (c) of system (13), (d) of system (12), (e) of system (11), i.e. at the origin of the local chart U_2 of system (V) for $\delta = 1/8$.

that is, the local phase portrait at the origin of the local chart U_2 is formed by one elliptic sector and three hyperbolic sectors.

Finally, we consider the case $\delta > 1/8$. The local phase portrait of the differential system (17) around the v_4-axis is shown in Figure 4(a). Then the local phase portrait of the differential system (16) around the v_4-axis is exactly the same than in Figure 4(a) but the v_4-axis is plenty of singular points.

Going back to system (15) we obtain that its local phase portrait at the origin is given in Figure 4(b). Undoing the change of variables $(u_2, v_2) \to (u_3, v_3)$ we get the local phase portrait at the origin of system (14) given in Figure 4(c). Then the local phase portrait of system (13) is the same as the one given in Figure 4(c) but with the v_2-axis filled up with singular points and the orbits in the half-plane $u_2 < 0$ with the orientation reversed. Undoing the blow-up for going back to the system (12) we get that its local phase portrait at the origin is the one given in Figure 4(d). Finally, going back to system (11) we conclude that the phase portrait at the origin of the local chart U_2 is provided in Figure 4(e), that is, the local phase portrait at the origin of the local chart U_2 is formed by two hyperbolic sectors.

The global phase portrait in the Poincaré disc when $\delta < 0$ is obtained as in system (IV).

When $\delta \in (0, 1/8)$ the origin O of the local chart V_2 has four separatrices, two at infinity and two in the finite plane. Additionally system (V) has two separatrices corresponding to the hyperbolic sector at the origin. The two finite separatrices of O are one stable and one unstable. The stable one can only start at the origin of coordinates and the unstable one can only end at the origin of coordinates. On the other hand, the stable separatrix of the origin of coordinates can only start at O and the unstable separatrix of the origin of coordinates can only end at O. This is due to the fact that on the positive x-axis the flow of system (V) goes down and on the negative x-axis the flow goes up. Due to the existence of the two parabolic sectors at the origin of coordinates separating the elliptic sector from the hyperbolic one, the two separatrices of the point O cannot connect with the two separatrices of the origin of coordinates. Then for this value of δ the phase portrait of the Poincaré disc is topologically equivalent to Figure 1(e).

When $\delta = 1/8$ the unique difference with the case $\delta \in (0, 1/8)$ is that the parabolic sectors at the origin of coordinates disappear and consequently the two finite separatrices of the point O connect with the two finite separatrices of the origin of coordinates, providing the topological phase portrait given in Figure 1(f).

Finally when $\delta > 1/8$ in the Poincaré disc there are only three singular points: the origin of coordinates which is a center and the two endpoins of the y-axis having each one only one hyperbolic sector, whose separatrices are at the line at infinity. The periodic orbits of the center fill up the whole plane, otherwise if there exist a last periodic orbit since the Poincaré map in a neighborhood of this last periodic orbit is an analytic function of one variable, and it is the identity in the bounded region of this last orbit, it must also be the identity on the unbounded region of this last orbit, consequently such last periodic orbit cannot exist. In short, the origin of system (V) is a global center whose global phase portrait is topologically equivalent to Figure 1(c). This completes the proof of Theorem 1(vi).

3. Appendix

3.1. **Poincaré compactification.** In order to classify the global dynamics of a polynomial differential system the first crucial step is to characterize their finite and infinite singular points in the Poincaré compactification. The second main step for determining the global dynamics in the Poincaré disc of a polynomial differential system is the characterization of their separatrices. For the polynomial differential systems in the Poincaré disc it is known that the *separatrices* are the infinite orbits, the finite singular points, the separatrices of the hyperbolic sectors of the finite and infinite singular points, and the limit cycles.

If Σ denotes the set of all separatrices in the Poincaré disc $\mathbb{D}^2$, Σ is a closed set and the components of $\mathbb{D}^2 \setminus \Sigma$ are called the canonical regions.

We consider the set of all polynomial vector fields in $\mathbb{R}^2$ of the form

$$(\dot{x}, \dot{y}) = X(x, y) = (P(x, y), Q(x, y)), \tag{18}$$

where P and Q are real polynomials in the variables x and y of degrees d_1 and d_2, respectively. Take $d = \max\{d_1, d_2\}$.

Denote by $T_p\mathbb{S}^2$ be the tangent space to the 2-dimensional sphere

$$\mathbb{S}^2 = \{\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$$

at the point p. Assume that X is defined in the tangent plane to $\mathbb{S}^2$ at the point $(0, 0, 1)$ denoted by $T_{(0,0,1)}\mathbb{S}^2 = \mathbb{R}^2$. Consider the central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of X, one in the open northern hemisphere and the other in the open southern hemisphere. Denote by X' the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{s \in \mathbb{S}^2 : s_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. If X is a planar polynomial vector field of degree d, then $p(X)$ is the only analytic extension of $s_3^{d-1}X'$ to $\mathbb{S}^2$. The vector field $p(X)$ is called the *Poincaré compactification* of the vector field X, for more details see [2, chapter 5].

On the Poincaré sphere $\mathbb{S}^2$ we use the following six local charts, which are given by $U_i = \{\mathbf{s} \in \mathbb{S}^2 : \; s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{S}^2 : \; s_i < 0\}$, for $i = 1, 2, 3$, with the corresponding diffeomorphisms

$$\varphi_i : \; U_i \to \mathbb{R}^2, \qquad \psi_i : \; V_i \to \mathbb{R}^2,$$

defined by $\varphi_i(\mathbf{s}) = -\psi_i(\mathbf{s}) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$. Thus (u, v) will play different roles in the distinct local charts. The expressions of the vector field $p(X)$ are

$$(\dot{u}, \dot{v}) = \left(v^d \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right), -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right)\right) \qquad \text{in } U_1,$$

$$(\dot{u}, \dot{v}) = \left(v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right), -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right)\right) \qquad \text{in } U_2,$$

$$(\dot{u}, \dot{v}) = (P(u, v), Q(u, v)) \qquad \text{in } U_3.$$

We note that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is equal to the expression in the local chart (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$.

The orthogonal projection under $\pi(y_1, y_2, y_3) = (y_1, y_2)$ of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $s_3 = 0$ is a closed disc $\mathbb{D}^2$ of radius one centered at the origin of coordinates called the *Poincaré disc.* Since a copy of the vector field X on the plane $\mathbb{R}^2$ is in the open northern hemisphere of $\mathbb{S}^2$, the interior of the Poincaré disc $\mathbb{D}^2$ is identified with $\mathbb{R}^2$ and the boundary of $\mathbb{D}^2$, the equator $\mathbb{S}^1$ of $\mathbb{S}^2$, is identified with the infinity of $\mathbb{R}^2$. Consequently the phase portrait of the vector field X extended to the infinity corresponds to the projection of the phase portrait of the vector field $p(X)$ on the Poincaré disc $\mathbb{D}^2$.

The singular points of $p(X)$ in the Poincaré disc lying on $\mathbb{S}^1$ are the *infinite singular points* of the corresponding vector field X. The singular points of $p(X)$ in the interior of the Poincaré disc, i.e. on $\mathbb{S}^2 \setminus \mathbb{S}^1$, are the *finite singular points*. We note that in the local charts U_1, U_2, V_1 and V_2 the infinite singular points have their coordinate $v = 0$.

For a polynomial vector field (18) if $s \in \mathbb{S}^1$ is an infinite singular point, then $-s \in \mathbb{S}^1$ is another infinite singular point. Thus the number of infinite singular points is even and the local phase portrait of one is that of the other multiplied by $(-1)^{d+1}$.

3.2. **Separatrix skeleton.** Given a flow $(\mathbb{D}^2, \phi)$ by the *separatrix skeleton* we mean the union of all the separatries of the flow together with one orbit from each one of the canonical regions. Let C_1 and C_2 be the separatrix skeletons of the flows $(\mathbb{D}^2, \phi_1)$ and $(\mathbb{D}^2, \phi_2)$, respectively. We say that C_1 and C_2 are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ which sends orbits to orbits preserving or reversing the direction of all orbits. From Markus [10], Neumann [11] and Peixoto [13] it follows the next theorem which shows that is enough to describe the separatrix skeleton in order to determine the topological equivalence class of a differential system in the Poincaré disc $\mathbb{D}^2$.

Theorem 2 (Markus–Neumann–Peixoto Theorem)**.** *Assume that* $(\mathbb{D}^2, \phi_1)$ *and* $(\mathbb{D}^2, \phi_2)$ *are two continuous flows with only isolated singular points. Then these flows are topologically equivalent if and only if their separatrix skeletons are equivalent.*

Acknowledgements

The first author is supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00 (FEDER), and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2019.

References

[1] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, J. Math. Anal. Appl., **147** (1990), 420–448.
[2] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer, 2006.
[3] V. J. Ervin, W. F. Ames, and E. Adams, *Nonlinear Waves in Pellet Fusion in Wave Phenomena*, Modern Theory and Applications, edited by Rogers, C. and Moodie, T. B., North Holland, Amsterdam, 1984.
[4] V.V. Golubev, *Lectures on Analytical Theory of Differential Equations*, Gostekhizdat, Moscow, Leningrad, 1950.

[5] I. Z. Golubchik, and V. V. Sokolov, *On some generalisations of the factorisation method*, Theor. Math. Phys. **110** (1997), 267–278.
[6] I.Z. Golubchik and V. V. Sokolov, *Operator Yang-Baxter equations, integrable ODEs and nonassociative algebras*, J. Nonlinear Math. Phys. **7** (2000), 184–197.
[7] E.L. Ince, *Ordinary Differential Equations*, Longmans, Green & Co, London, 1927.
[8] P. G. L. Leach, *First integrals for the modified Emden equation* $\ddot{q} + a(t)\dot{q} + q^n = 0$, J. Math. Phys. **26** (1985), 2510–2514.
[9] P. G. Leach, S. Cotsakis, and G. P. Flessas, *Symmetry, singularity and integrability in complex dynamics: I The reduction problem*, J. Nonlinear Math. Phys. **7** (2000), 445–479.
[10] L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc. **76** (1954), 127–148.
[11] D.A. Neumann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.
[12] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math. **25** (1902), 1–86.
[13] M. Peixoto, *Dynamical Systems*, Proceedings of a Symposium held at the University of Bahia, Acad. Press, New York, 1973, pp.389–420.
[14] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, J. Math. **37** (1881), 375–422; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp 3–84.
[15] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York, 1972.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.tecnico.ulisboa.pt