This is a preprint of: "On the dynamics of a hyperjek memristive system", Jaume Llibre, Clàudia Valls, *Applied Physics A*, vol. 130, 903, 2024.
DOI: [10.1007/s00339-024-08073-7]

ON THE DYNAMICS OF A HYPERJEK MEMRISTIVE SYSTEM

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Abstract. We study the existence of zero-Hopf bifurcations in the fourth order ordinary differential equation $\ddddot{x} = -\ddot{x} - a\,\dddot{x} - b\dot{x}^2\,\dddot{x} - (1+x)\dot{x}$ called the hyperjerk memristive system. This system has a line filled with equilibria and it has a polynomial first integral H. Writing this equation as a first order differential system in $\mathbb{R}^4$ we prove that this system has a zero-Hopf equilibrium $(-1, 0, 0, 0)$ and from it, bifurcate two cylinders filled with periodic orbits parameterized by the levels of the first integral. Moreover, the three-dimensional system obtained restricting the differential system in $\mathbb{R}^4$ to the invariant hypersurface $H = h$, exhibits two Hopf bifurcations producing periodic orbits in the center manifold of that restriction.

1. Introduction and statement of the main results

In [12] the authors introduced a cubic two-parametric fourth order differential equation system which generalizes the memristive system introduced in [2], which in its turn generalized the original definition of memristor given in [1]. This equation has a line of equilibria and it has hyperjerk dynamics for some values of the parameters, in the sense that it involves a fourth order differential equation. Moreover, it is chaotic for some values of the parameters and there exist trajectories starting from points in the unstable manifold in a neighborhood of an unstable equilibrium point [7]. Systems exhibiting this chaotic behaviour have attracted the interest of many authors see for instance [9, 10, 11]. From the pioneer work of Chua and Kang in [2] many researches have worked proposing different memristive systems with different applications in different areas depending on their properties and now it is a very active research subject mainly because of its applications, see for instance [3, 4, 6, 13, 14, 15, 17, 18, 19].

Writing this cubic two-parametric fourth order differential equation memristive system introduced in [12] as a first order differential system in $\mathbb{R}^4$ we get

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -y - z - aw - xy - bz^2 w, \tag{1}$$

where a, b are real parameters.

2010 *Mathematics Subject Classification.* 34C05.
Key words and phrases. Hopf bifurcation, zero-Hopf bifurcation.

The first main objective is to study the zero-Hopf equilibria and the zero-Hopf bifurcations of that system. We recall that a *zero-Hopf equilibrium* of an autonomous differential system is an equilibrium of that system whose Jacobian matrix evaluated on it has a pair of purely imaginary eigenvalues and the rest of the eigenvalues are zero. Moreover, a *Hopf equilibrium* of an autonomous differential system is an equilibrium of that system whose Jacobian matrix evaluated on it has a pair of purely imaginary eigenvalues and the rest of the eigenvalues are real and different from zero.

Theorem 1. *The following holds for the differential system* (1)*:*

(i) *It has a line of equilibria* $(x, 0, 0, 0)$ *with* $x \in \mathbb{R}$.
(ii) *It has a polynomial first integral* H.
(iii) *It has no Hopf equilibria.*
(iv) *It has a unique zero-Hopf equilibrium at* $(-1, 0, 0, 0)$ *when* $a = 0$ *and* $b > 0$. *From this zero-Hopf equilibrium bifurcate two cylinders filled with periodic orbits parameterized by the levels* $H = -1/2 + a^2 h$ *of the first integral* H *with* $a > 0$ *sufficiently small. The periodic orbits* $(x_\pm(t, a), y_\pm(t, a), z_\pm(t, a), w_\pm(t, a))$ *bifurcating from this equilibrium point satisfy*

$$(x_\pm(0, a), y_\pm(0, a), z_\pm(0, a), w_\pm(0, a)) = (a \pm \sqrt{2h} - 1, 0, 0, 0) + O(a^2).$$

Moreover, on each level set $H = -1/2 + a^2 h$, *one periodic orbit is unstable and the other is locally asymptotically stable when* $h \in [0, 1/2)$ *and unstable when* $h > 1/2$.

Theorem 1 is proved in section 2. The main tool for proving it will be the averaging theory of first order, that will be summarized in Theorem 3 of the appendix. Statements (i) and (ii) were proved in [5]. In fact the first integral is

$$H = x + y + az + w + \frac{1}{2}x^2 + \frac{b}{3}z^3.$$

So, in section 2 we will prove only statements (iii) and (iv).

We also consider the reduced system of dimension three by the restriction to the hypersurface $H = h$ taking

$$w = h - x - y - az - \frac{1}{2}x^2 - \frac{b}{3}z^3,$$

i.e the differential system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = h - x - y - az - \frac{1}{2}x^2 - \frac{b}{3}z^3. \tag{2}$$

We will study the dynamics of system (2). The non-existence of a Hopf bifurcation for system (1) does not imply that the restricted system (2) also does not have a Hopf bifurcation. In fact we will show that system (2) has two Hopf bifurcations. The zero-Hopf equilibrium points of system (2) are the restricted zero-Hopf equilibrium points of system (1) due to the fact that the equilibrium points of system (1) are in a straight line filled with equilibria.

Our second main result is the existence of a Hopf bifurcation for system (2).

Theorem 2. *The following statements hold for system* (2) *when* $h > -1/2$.

(i) *For* $a = -\sqrt{1+2h}$, *at the equilibrium point* $p_1 = (-1-\sqrt{1+2h}, 0, 0)$ *system* (2) *has a Hopf bifurcation, that is, system* (2) *has a center manifold around that equilibrium point and on that center manifold a small limit cycle appears which is stable if* $\ell(p_1) < 0$ *and unstable if* $\ell(p_1) > 0$. *The explicit expression of* $\ell(p_1)$ *is given in the proof.*

(ii) *For* $a = \sqrt{1+2h}$, *at the equilibrium point* $p_2 = (-1+\sqrt{1+2h}, 0, 0)$ *system* (2) *has a Hopf bifurcation, that is, system* (2) *has a center manifold around that equilibrium point and on that center manifold a small limit cycle appears which is stable if* $l(p_2) < 0$ *and unstable if* $\ell(p_2) > 0$. *The explicit expression of* $\ell(p_2)$ *is given in the proof.*

The proof of Theorem 2 is given in section 3. The main tool for proving it is stated in Theorem 4 of the appendix.

2. Proof of Theorem 1

The characteristic polynomial of the linear part of system (1) at the equilibrium point $(-1, 0, 0, 0)$ is

$$\Delta(\lambda) = \lambda(1 + x + \lambda + a\lambda^2 + \lambda^3)$$

Since this polynomial $\Delta(\lambda)$ has always $\lambda = 0$ as solution, system (1) has no Hopf equilibrium points, and so statement (iii) is proved. To have a zero-Hopf equilibrium point $\Delta(\lambda) = 0$ must have two zero real eigenvalues and a pair of purely imaginary eigenvalues. This happens if and only if $\Delta(\lambda) = \lambda^2(\lambda^2 + \omega^2)$ with $\omega > 0$. Imposing this condition we get $x = -1$, $a = 0$ and $\omega = 1$ Hence there is a unique zero-Hopf equilibrium point $(-1, 0, 0, 0)$ when $a = 0$. Note that the equlibrium point $(-1, 0, 0, 0)$ when $a = 0$ is a zero-Hopf equilibrium inside a continuum of equilibria.

We shall use the averaging theory of first order described in Theorem 3 of the appendix to study it. The first step is to translate the point $(-1, 0, 0, 0)$ to the origin of coordiantes. We set $x = -1 + X, y = Y, z = Z, w = W$, and system (1) becomes

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \quad \dot{W} = -Z - aW - XY - bWZ^2. \tag{3}$$

The second step is to write the differential system in the normal form for applying the averaging theory of first order. That is, to write the linear part of system (3) when $\varepsilon = 0$ into its real Jordan normal form. Applying the change of variables

$$X = -X_1 + Z_1, \quad Y = W_1 + Y_1, \quad Z = X_1, \quad W = -Y_1 \ ,$$

system (3) in the variables (X_1, Y_1, Z_1, W_1) becomes

$$\begin{aligned}
\dot X_1 &= -Y_1,\\
\dot Y_1 &= X_1 - aY_1 - X_1Y_1 - X_1W_1 + Y_1Z_1 + Z_1W_1 - bX_1^2Y_1,\\
\dot Z_1 &= W_1,\\
\dot W_1 &= aY_1 + X_1Y_1 + X_1W_1 - Y_1Z_1 - Z_1W_1 + bX_1^2Y_1.
\end{aligned} \tag{4}$$

We write the differential system (4) in the cylindrical coordinates (r, θ, Z_1, W_1) defined by $X_1 = r\cos\theta$, $Y_1 = r\sin\theta$, and we obtain

$$\begin{aligned}
\dot r &= W_1Z_1\sin\theta - r^2\cos\theta\sin^2\theta - br^3\cos^2\theta\sin^2\theta - rW_1\cos\theta\sin\theta\\
&\quad + r(a - Z_1)\sin^2\theta,\\
\dot\theta &= -1 - r\cos^2\theta\sin\theta - (a - Z_1)\cos\theta\sin\theta - W_1\cos^2\theta - br^2\cos^3\theta\sin\theta\\
&\quad + \frac{1}{r}Z_1W_1\cos\theta,\\
\dot Z_1 &= W_1,\\
\dot W_1 &= r^2\cos\theta\sin\theta + r((a - Z_1)\sin\theta + W_1\cos\theta) - Z_1W_1 + br^3\cos^2\theta\sin\theta.
\end{aligned} \tag{5}$$

Now we rescale the variables (r, θ, Z_1, W_1) as follows $r = a^2R$, $Z_1 = aZ_2$, $W_1 = a^2W_2$ and in this case system (5) becomes

$$\begin{aligned}
\dot R &= a\sin\theta(R(Z_2 - 1)\sin\theta + Z_2W_2) - a^2R\cos\theta\sin\theta(R\sin\theta + W_2)\\
&\quad - a^4bR^3\cos^2\theta\sin^2\theta,\\
\dot\theta &= 1 + \frac{a\cos\theta(R(Z_2 - 1)\sin\theta + Z_2W_2)}{R} - a^2\cos^2\theta(R\sin\theta + W_2)\\
&\quad - a^4bR^2\cos^3\theta\sin\theta,\\
\dot Z_2 &= aW_2,\\
\dot W_2 &= -a(R(Z_2 - 1)\sin\theta + Z_2W_2) + a^2R\cos\theta(R\sin\theta + W_2)\\
&\quad + a^4bR^3\cos^2\theta\sin\theta.
\end{aligned} \tag{6}$$

We take θ as the new independent variable and we obtain

$$\begin{aligned}
R' &= a\sin\theta(R(Z_2 - 1)\sin\theta + Z_2W_2) + O(a^2),\\
Z_2' &= aW_2 + O(a^2),\\
W_2' &= -a(Z_2W_2 + R(Z_2 - 1)\sin\theta) + O(a^2),
\end{aligned} \tag{7}$$

where the prime denotes derivative in the new time θ. We write the energy in the new variables as

$$\tilde{H} = a^3 R\cos\theta + a^2 W_2 + \frac{1}{2}a^4R^2\cos^2\theta - a^3RZ_2\cos\theta + \frac{a^2}{2}Z_2^2 + \frac{1}{3}a^6bR^3\cos^3\theta$$

and we restrict the differential system to the energy level $\tilde{H} = a^2h$ taking

$$W_2 = \frac{1}{6}(6h - 6aR\cos\theta - 3a^2R^2\cos^2\theta + 6aRZ_2\cos\theta - 3Z_2^2 - 2a^4bR^3\cos^3\theta)$$

and we get

$$R' = aF_1(\theta, R, Z_2) + O(a^2), \quad Z_2' = aF_2(\theta, R, Z_2) + O(a^2), \tag{8}$$

where

$$F_1(\theta, R, Z_2) = \frac{1}{2}\sin\theta\big(Z_2(2h - Z_2^2) + 2R(Z_1 - 1)\big),$$
$$F_2(\theta, R, Z_2) = \frac{1}{6}(6h - 3Z_2^2).$$

Using the notation of the appendix we have $t = \theta$, T$= 2\pi$, $x = (R, Z_2)^T$ and

$$F(\theta, R, Z_2) = \begin{pmatrix} F_1(\theta, R, Z_2) \\ F_2(\theta, R, Z_2) \end{pmatrix}, \quad f(\theta, R, Z_2) = \begin{pmatrix} f_1(\theta, R, Z_2) \\ f_2(\theta, R, Z_2) \end{pmatrix}.$$

It is easy to see that system (8) satisfies all the assumptions of Theorem 3. Computing the integrals in that theorem we get

$$f_1(R, Z_2) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, R, Z_2)\, d\theta = \frac{1}{2}R(Z_2 - 1),$$
$$f_2(R, Z_2) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta, R, Z_2)\, d\theta = h - \frac{1}{2}Z_2^2.$$

The system $f_1(R, Z_2) = f_2(R, Z_2) = 0$ for $h > 0$ has two solutions $(R, Z_2) = (0, \pm\sqrt{2h})$, which are nonzero and different. We will denote them by $(0, Z_\pm)$. Moreover, the determinant of the Jacobian matrix at $(0, Z_-)$ takes the value

$$\det \frac{\partial(f_1, f_2)}{\partial(R, Z_2)}|_{(R,Z)=(0,Z_-)} = -\sqrt{\frac{h}{2}} - h$$

which is non-zero for $h > 0$. Moreover the eigenvalues of the Jacobian matrix

$$\frac{\partial(f_1, f_2)}{\partial(R, Z_2)}|_{(R,Z)=(0,Z_-)}$$

are $\sqrt{2h}$ and $-(1+\sqrt{2h})/2$. On the other hand, the determinant of the Jacobian matrix at $(0, Z_+)$ takes the value

$$\det \frac{\partial(f_1, f_2)}{\partial(R, Z_2)}|_{(R,Z)=(0,Z_+)} = \sqrt{\frac{h}{2}} - h$$

which is non-zero for $h \in (0, 1/2)$. Moreover the eigenvalues of the Jacobian matrix

$$\frac{\partial(f_1, f_2)}{\partial(R, Z_2)}|_{(R,Z)=(0,Z_+)}$$

are $-\sqrt{2h}$ and $-(1-\sqrt{2h})/2$. The rest of the proof of the theorem follows immediately from Theorem 3. Indeed, applying Theorem 3 we obtain that for a sufficiently small and $h > 0$ the differential system (8) on the energy level $\tilde{H} = a^2 h$ has two periodic orbits $(R_\pm(\theta, a), Z_2^\pm(\theta, a)$ such that

$$(R_\pm(0, a), Z_2^\pm(0, a)) \to (0, \pm\sqrt{2h}) \quad \text{when } a \to 0.$$

The periodic orbit coming from the solution $(0, -\sqrt{2h})$ is unstable, while the periodic orbit coming from the solution $(0, \sqrt{2h})$ is asymptotically stable. The two periodic orbits coming from the differential system (8) provide in the differential system (7) two periodic orbits $(R_\pm(\theta, a), Z_2^\pm(\theta, a), W_2^\pm(\theta, a))$ such that

$$(R_\pm(0, a), Z_2^\pm(0, a), W_2^\pm(0, a)) \to (0, \pm\sqrt{2h}, 0) \quad \text{when } a \to 0.$$

Moreover, the two periodic orbits of the differential system (7) in the differential system (6) become $(R_\pm(t, a), \theta_\pm(t, a), Z_2^\pm(t, a), W_2^\pm(t, a))$ such that

$$(R_\pm(0, a), \theta_\pm(0, a), Z_2^\pm(0, a), W_2^\pm(0, a)) \to (0, 0, \pm\sqrt{2h}, 0) \quad \text{when } a \to 0.$$

The two periodic orbits of the differential system (6) in the differential system (5) provide the periodic orbits $(r_\pm(t, a), \theta_\pm(t, a), Z_1^\pm(t, a), W_1^\pm(t, a))$ such that

$$(r_\pm(0, a), \theta_\pm(0, a), Z_1^\pm(0, a), W_1^\pm(0, a)) = (0, 0, a \pm \sqrt{2h}, 0) + O(a^2).$$

The two periodic orbits of the differential system (5) in the differential system (4) are the periodic orbits $(X_1^\pm(t, a), Y_1^\pm(t, a), Z_1^\pm(t, a), W_1^\pm(t, a))$ such that

$$(X_1^\pm(0, a), Y_1^\pm(0, a), Z_1^\pm(0, a), W_1^\pm(0, a)) = (0, 0, a \pm \sqrt{2h}, 0) + O(a^2),$$

and the two periodic orbits of the differential system (4) in the differential system (3) are the periodic orbits $(X_\pm(t, a), Y_\pm(t, a), Z_\pm(t, a), W_\pm(t, a))$ such that

$$(X_\pm(0, a), Y_\pm(0, a), Z_\pm(0, a), W_\pm(0, a)) = (a \pm \sqrt{2h}, 0, 0, 0) + O(a^2).$$

Finally, the two periodic orbits of the differential system (3) become in the differential system (1) the periodic orbits $(x_\pm(t, a), y_\pm(t, a), z_\pm(t, a), w_\pm(t, a))$ such that

$$(x_\pm(0, a), y_\pm(0, a), z_\pm(0, a), w_\pm(0, a)) = (a \pm \sqrt{2h} - 1, 0, 0, 0) + O(a^2).$$

In short, we have two cylinders of periodic orbits in a neighborhood of the equilibrium point $(-1, 0, 0, 0)$ bifurcating from this point. This completes the proof of the theorem.

3. Proof of Theorem 2

First note that system (2) has the two equilibrium points

$$p_1 = (-1 - \sqrt{1+2h}, 0, 0) \quad \text{and} \quad p_2 = (-1 + \sqrt{1+2h}, 0, 0),$$

if $h > -1/2$. The Jacobian matrix evaluated at p_1 has the characteristic polynomial

$$\Delta_1(h,a) = \sqrt{1+2h} - \lambda - a\lambda^2 - \lambda^3 = 0,$$

and the Jacobian matrix evaluated at p_2 has the characteristic polynomial

$$\Delta_2(h,a) = -\sqrt{1+2h} - \lambda - a\lambda^2 - \lambda^3 = 0.$$

We study both equilibrium points separately.

3.1. **The point p_1.** To have a Hopf equilibrium point we must have that $\Delta_1 = 0$ has one real eigenvalue different from zero and a pair of purely imaginary eigenvalues. This happens if and only if $\Delta(\lambda) = (\lambda - r)(\lambda^2 + \omega^2)$ with $r \neq 0$ and $\omega > 0$. Imposing this condition we get $a = -\sqrt{1+2h}$, $r = \sqrt{1+2h}$ and $\omega = 1$ Hence the equilibrium point p_1 is a Hopf equilibrium when $a = -\sqrt{1+2h}$. We translate p_1 to the origin by setting $x = X - 1 - \sqrt{1+2h}, y = Y, z = Z$ and system (2) becomes

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = \frac{1}{6}(6\sqrt{1+2h}X - 6Y - 6aZ - 3X^2 - 2bZ^3). \tag{9}$$

From system (9) and using the notation of Theorem 4 of the appendix we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{1+2h} & -1 & -a \end{pmatrix}, \; B(x,y) = \begin{pmatrix} 0 \\ 0 \\ -x_1 y_1 \end{pmatrix}, \; C(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ -2b x_1 y_1 z_1 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda = \sqrt{1+2h}$ and $\pm i$. The eigenvector q satisfying $Aq = iq$ and normalized so that $\overline{q} \cdot q = 1$ is

$$q = \frac{1+2h}{3+6h+4h^2} \begin{pmatrix} 1 \\ \sqrt{1+2h} \\ 1+2h \end{pmatrix}.$$

Note that $3 + 6h + 4h^2 \neq 0$. The eigenvector p such that $A^T p = -ip$ and so that $\overline{p} \cdot q = 1$ is

$$p = \frac{3+6h+4h^2}{2(1+h)(1+2h)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

So we compute

$$C(q,q,\overline{q}) = -\frac{2b(1+2h)^6}{(3+6h+4h^2)^3}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$B(q,\overline{q}) = B(q,q) = -\frac{(1+2h)^2}{(3+6h+4h^2)^2}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$B(q,A^{-1}B(q,\overline{q})) = \frac{(1+2h)^{5/2}}{(3+6h+4h^2)^3}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$B(\overline{q},(2i\mathrm{Id}-A)^{-1}B(q,q)) = \frac{(1+2h)^3}{(3+6h+4h^2)^3(3+2h-6i)}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then, from (12) of the appendix, the first Liapunov coefficient is

$$\ell(p_1) = -\frac{1}{12(1+h)(5+2h)(3+6h+4h^2)^2}\big(30b+312bh+1320bh^2+2880bh^3 + 3360bh^4+1920bh^5+384bh^6+(30+12h)(1+2h)^{3/2}-(1+2h)^{5/2}\big).$$

Hence if $\ell(p_1) > 0$ there is a supercritical Hopf bifurcation at the equilibrium p_1 for $a = -\sqrt{1+2h}$. More precisely, on the center manifold of the equilibrium point p_1 there is a stable strong focus without any period orbit around p_1 for $a > -\sqrt{1+2h}$, and a unstable strong focus for $a < -\sqrt{1+2h}$ with a small unstable limit cycle surrounding p_1. For $a = -\sqrt{1+2h}$ it is a stable weak focus of order 1.

On the other hand if $\ell(p_1) < 0$ there is a subcritical Hopf bifurcation at the equilibrium p_1 for $a = -\sqrt{1+2h}$. More precisely, on the center manifold the equilibrium point p_1 is an unstable strong focus without any period orbit around p_1 for $a < -\sqrt{1+2h}$, and a stable strong focus for $a > -\sqrt{1+2h}$ with a small stable limit cycle surrounding p_1. For $a = -\sqrt{1+2h}$ it is a unstable weak focus of order 1.

3.2. **The point p_2.** To have a Hopf equilibrium point we must have that $\Delta = 0$ has one real eigenvalue different from zero and a pair of purely imaginary eigenvalues. This happens if and only if $\Delta_2(\lambda) = (\lambda - r)(\lambda^2 + \omega^2)$ with $r \neq 0$ and $\omega > 0$. Imposing this condition we get $a = \sqrt{1+2h}$, $r = -\sqrt{1+2h}$ and $\omega = 1$ Hence, the equilibrium point p_2 is a Hopf equilibrium when $a = \sqrt{1+2h}$. We translate p_2 to the origin by setting $x = X - 1 + \sqrt{1+2h}, y = Y, z = Z$ and system (2) becomes

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = -\frac{1}{6}(6\sqrt{1+2h}X + 6Y + 6aZ + 3X^2 + 2bZ^3) \tag{10}$$

From system (10) and using the notation of Theorem 4 in the appendix we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sqrt{1+2h} & -1 & -a \end{pmatrix}, \; B(x,y) = \begin{pmatrix} 0 \\ 0 \\ -x_1y_1 \end{pmatrix}, \; C(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ -2bx_1y_1z_1 \end{pmatrix}$$

The eigenvalues of the matrix A are $\lambda = -\sqrt{1+2h}$ and $\pm i$. The eigenvector q satisfying $Aq = iq$ and normalized so that $\overline{q} \cdot q = 1$ is

$$q = \frac{1+2h}{3+6h+4h^2} \begin{pmatrix} 1 \\ -\sqrt{1+2h} \end{pmatrix}.$$

Note that $3+6h+4h^2 \neq 0$. The eigenvector p such that $A^T p = -ip$ and so that $\overline{p} \cdot q = 1$ is

$$p = \frac{3+6h+4h^2}{2(1+h)(1+2h)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

So we have the following equations for (12)

$$C(q,q,\overline{q}) = -\frac{2b(1+2h)^6}{(3+6h+4h^2)^3} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$B(q,\overline{q}) = B(q,q) = -\frac{(1+2h)^2}{(3+6h+4h^2)^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$B(q, A^{-1}B(q,\overline{q})) = -\frac{(1+2h)^{5/2}}{(3+6h+4h^2)^3} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$B(\overline{q}, (2i\mathrm{Id} - A)^{-1}B(q,q)) = -\frac{(1+2h)^3}{(3+6h+4h^2)^3(3\sqrt{1+2h}+6i)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then the first liapunov coefficient is (see (12)

$$\ell(p_2) = -\frac{1}{12(1+h)(5+2h)(3+6h+4h^2)^2}\big(30b + 312bh + 1320bh^2 + 2880bh^3 + 3360bh^4 + 1920bh^5 + 384bh^6 - (30+12h)(1+2h)^{3/2} + (1+2h)^{5/2}\big).$$

Hence if $\ell(p_2) > 0$ there is a supercritical Hopf bifurcation at the equilibrium p_2 for $a = \sqrt{1+2h}$ at the equilibrium point p_2. More precisely, on the center manifold the equilibrium point p_2 is a stable strong focus without any period orbit around p_2 for $a > \sqrt{1+2h}$, and a unstable strong focus for $a < \sqrt{1+2h}$ with a small unstable limit cycle surrounding p_2. For $a = \sqrt{1+2h}$ it is a stable weak focus of order 1.

On the other hand if $\ell(p_2) < 0$ there is a subcritical Hopf bifurcation at the equilibrium p_2 for $a = \sqrt{1+2h}$. More precisely, on the center manifold the equilibrium point p_2 is a unstable strong focus without any period orbit around p_2 for $a < \sqrt{1+2h}$, and a stable strong focus for $a > \sqrt{1+2h}$ with a small stable limit cycle surrounding p_2. For $a = \sqrt{1+2h}$ it is a unstable weak focus of order 1.

APPENDIX

The next result is proved in Theorems 11.5 and 11.6 of [16].

Theorem 3. *Consider the initial value problem*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 G(t,x,\varepsilon) \quad \text{with} \quad x(0) = x_0, \tag{11}$$

where ε is a small parameter, $F\colon \mathbb{R}_0^+ \times D \to \mathbb{R}^n$ being $D \subset \mathbb{R}^n$ and $G\colon \mathbb{R}_0^+ \times D \times (0,\varepsilon_0] \to \mathbb{R}^n$. Moreover we assume that $F, D_xF, D_{xx}F, G, D_xG$ are continuous and bounded by a constant independent of ε in $\mathbb{R}_0^+ \times D$ for $\varepsilon \in (0,\varepsilon_0]$ and are periodic in the variable t of period T (independent of ε). Consider now the initial value problem, called the averaged value problem,

$$\dot{y} = \varepsilon f(y) \quad \text{with} \quad y(0) = x_0, \quad \text{where} \quad f(y) = \frac{1}{T}\int_0^T F(t,y)\,dt.$$

The following statements hold.

(1) *For $t \in [0, 1/\varepsilon]$ we have $x(t) = y(t) + O(\varepsilon)$*
(2) *If p is an equilibrium point of the averaged problem such that $\det(D_y f(p)) \neq 0$, then there exists a periodic solution $x(t,\varepsilon)$ of period T of system* (11) *such that $x(0,\varepsilon) = p + O(\varepsilon)$*
(3) *The periodic solution $x(t,\varepsilon)$ is locally asymptotically stable if all the eigenvalues of $D_y f(p)$ have negative real part and it is unstable if there exists at least one eigenvalue of $D_y f(p)$ with positive real part.*

The following theorem is given in [8, pp 177-180].

Theorem 4. *Let $\dot{x} = F(x)$ be a differential system having an equilibrium point x_0. Write*

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{3!}C(x,x,x) + O(|x|^4)$$

Assume that A has a pair of purely imaginary eigenvalues $\pm\lambda i$. Let q the eigenvector of A corresponding to the eigenvalue λi normalized so that $\overline{q}\cdot q = 1$, where $\overline{q}$ is the conjugate vector of q and the dot is the usual inner product in $\mathbb{R}^n$. Let

p be the vector satisfying $A^T p = -\lambda i p$ satisfying $\overline{p} \cdot q = 1$. Then the Liapunov coefficient of that system at x_0 is given by

$$\frac{1}{2\lambda}\mathrm{Re}\big(\overline{p}\cdot C(q,q,\overline{q}) - 2\overline{p}\cdot B(q, A^{-1}B(q,\overline{q})) + \overline{p}\cdot B(\overline{q}, (2\lambda i\mathrm{Id} - A)^{-1}B(q,q))\big), \tag{12}$$

where $\mathrm{Re}(z)$ *denotes the real part of a complex number z, and* Id *is the* 6×6 *identity matrix.*

1. *If the Liapunov coefficient* (12) *is negative (supercritical Hopf bifurcation), the point x_0 is a weak focus of the system restricted to the center manifold of x_0 and a small stable limit cycle emerges from x_0.*
2. *If the Liapunov coefficient* (12) *is positive (subcritical Hopf bifurcation), the point x_0 is a weak focus of the system restricted to the center manifold of x_0 and a small unstable limit cycle emerges from x_0.*

Funding

The first author is partially supported by the Agencia Estatal de Investigación grant PID2022-136613NB-100, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR grant 2021SGR00113, and by the Reial Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

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