This is a preprint of: "On the crossing limit cycles created by discontinuous piecewise differential system formed by three linear Hamiltonian saddles", Maria Elisa Anacleto, Jaume Llibre, Clàudia Valls, Claudio Vidal, *Dyn. Syst.*, vol. 39(4), 799–822, 2024.
DOI: [10.1080/14689367.2024.2391417]

ON THE CROSSING LIMIT CYCLES CREATED BY DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEM FORMED BY THREE LINEAR HAMILTONIAN SADDLES

MARIA ELISA ANACLETO[1], JAUME LLIBRE[2], CLAUDIA VALLS[3] AND CLAUDIO VIDAL[1]

Abstract. We study planar discontinuous piecewise differential systems formed by three linear Hamiltonian saddles separated by the non-regular line $\Sigma = \{(x, y) \in \mathbb{R}^2 : (y = 0) \vee (x = 0 \wedge y \geq 0)\}$. We prove that when the linear Hamiltonian saddles are homogeneous they have no limit cycles, and when they are non homogeneous they can have at most three limit cycles having exactly one point on each branch of Σ, and at most one limit cycle having four intersection points on Σ. Moreover we show that they can have at most one limit cycle having four intersection points on Σ and three limit cycles having three intersection points on Σ, simultaneously. Thus we have solved the extension of the 16th Hilbert problem to this class of piecewise differential systems. Furthermore we show that for the three types of combination of the limit cycles here studied in two of them the upper bound are sharp by providing examples of these systems with the maximum number of possible limit cycles. For the remaining type of limit cycles the upper bound is four and we have examples with three limit cycles. So at this moment is an open problem to know if the sharp upper bound is four or three.

1. Introduction and statement of the main results

Discontinuous piecewise differential systems in the plane $\mathbb{R}^2$ formed by linear differential systems appeared in a natural way studying some mechanical systems, see the pioneering book [1]. Later on these kind of differential systems allowed to study several electrical circuits and many other natural phenomena, see for instance the books [5, 23], the survey [21], and the paper [24]. These differential systems also play a main role in the control theory, see for example the books [2, 9, 12, 13].

The dynamics of the discontinuous piecewise differential systems started to be well defined in the book [6].

One of the main objects in the qualitative theory of planar differential systems are the limit cycles. We recall that a *limit cycle* is an isolated periodic orbit in the set of all orbits of a given differential system.

We remark that, in general, it is not easy to provide an explicit upper bound for the maximum number of limit cycles in a given class of differential systems, and even harder is to prove that this bound is reached. Thus the famous 16th Hilbert

---

2010 *Mathematics Subject Classification.* 34C05, 34C07.
Key words and phrases. Limit cycles, linear Hamiltonian saddles, discontinuous piecewise differential systems, first integrals.

problem, stated in 1900, which asks for an upper bound for the maximum number of limit cycles that the planar polynomial differential systems of a given degree can exhibit remains open, see [10].

It is well known that a linear differential system has no limit cycles, because when a linear differential system has a periodic orbit this is not isolated. But the discontinuous piecewise differential systems formed by linear differential systems can have two kinds of limit cycles, the *crossing limit cycles* which are the limit cycles which only contain isolated points of the line of discontinuity which separate the distinct linear differential systems, and the *sliding limit cycles* which contains arcs of the line of discontinuity, for more details on these two classes of limit cycles see for instance [8].

The crossing limit cycles, due to their importance in the study of the dynamics of the discontinuous piecewise differential systems formed by linear differential systems, have been studied intensively in the present century. Being far to be exhaustive see for instance [3, 4, 7, 11, 14–18].

A linear Hamiltonian system having a saddle will be called simply a *linear Hamiltonian saddle.*

In this paper we study the crossing limit cycles of the planar discontinuous piecewise differential systems formed by three linear Hamiltonian saddles separated by the non-regular line

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : (y = 0) \vee (x = 0 \wedge y \geq 0)\}.$$

The three components of $\mathbb{R}^2 \setminus \Sigma$ are the positive or first quadrant

$$R_1 = \{(x, y) \in \mathbb{R}^2 : x > 0 \wedge y > 0\},$$

the second quadrant

$$R_2 = \{(x, y) \in \mathbb{R}^2 : x < 0 \wedge y > 0\}$$

and the half-plane

$$R_3 = \{(x, y) \in \mathbb{R}^2 : y < 0\}.$$

We denote by class $\mathcal{C}_1$ the class of crossing limit cycles intersecting in a unique point each branch of Σ, and by $\mathcal{C}_2$ the class of crossing limit cycles intersecting Σ in four points. Our main result is the following.

Theorem 1. *The maximum number of crossing limit cycles of the discontinuous piecewise differential systems separated by the line Σ and formed by three linear Hamiltonian saddles is*

(a) *three of type $\mathcal{C}_1$ and this upper bound is reached. Moreover we also provide examples of such systems with* 1*, or* 2 *crossing limit cycles of type $\mathcal{C}_1$.*

(b) *one of type $\mathcal{C}_2$ and this upper bound is reached.*

(c) *three of type $\mathcal{C}_1$ and one of type $\mathcal{C}_2$. We provide an example of these systems with one crossing limit cycle of type $\mathcal{C}_1$ and one or two crossing limit cycles of type $\mathcal{C}_2$, simultaneously. Moreover we give one of the systems satisfying all the conditions for having one limit cycle of type $\mathcal{C}_2$ and three limit cycles of type $\mathcal{C}_1$, but this system has no such four limit cycles.*

The proof of Theorem 1 is given in section 2.

Statement (c) shows that our method for obtaining the upper bound on the number of simultaneous limit cycles of type $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot the improved. That is at most one limit cycle of type $\mathcal{C}_2$ and three limiy cycles of type $\mathcal{C}_1$ is the upper bound, but we also have examples with one limit cycle of type $\mathcal{C}_2$ and two limit cycles of type $\mathcal{C}_1$.

Note that Theorem 1 solves the extension of the 16th Hilbert problem for the class of discontinuous piecewise differential systems separated by the line Σ and formed by three linear Hamiltonian saddles. The 16th Hilbert problem for the class of discontinuous piecewise differential systems separated by the line Σ and formed by three linear centers was solved in [20]. As far as the authors know these two papers are among the very few papers where the extension of the 16th Hilbert problem for discontinuous piecewise differential systems has been proved.

Proposition 2. *Discontinuous piecewise differential systems formed by homogeneous linear Hamiltonian saddles and separated by the line Σ have neither limit cycles of type $\mathcal{C}_1$, nor limit cycles of type $\mathcal{C}_2$.*

The proof of the proposition is given in section 3.

2. Proof of Theorem 1

The following lemma provides a normal form for an arbitrary linear differential system having a Hamiltonian saddle.

Lemma 1. *A linear differential system having a Hamiltonian saddle can be written as*

$$\dot{x} = -Ax - \delta y + B, \quad \dot{y} = \alpha x + Ay + C, \tag{1}$$

with $\alpha \in \{0, 1\}$, $A, B, C \in \mathbb{R}$. If $\alpha = 1$ then $\delta = A^2 - \omega$ with $\omega > 0$, and if $\alpha = 0$ then $A = 1$. Moreover, a first integral of system (1) *is*

$$H = -\frac{\alpha^2}{2}x^2 - Axy - \frac{\delta}{2}y^2 - Cx + By.$$

For a proof of Lemma 1 see Proposition 1 of [19].

We prove each statement of Theorem 1 separately.

Proof of statement (*a*) *of Theorem* 1. Assume that we have a discontinuous piecewise differential system formed by three linear Hamiltonian saddles separated by the line Σ. By Lemma 1 we can write such a discontinuous piecewise differential system as

$$\begin{aligned} &\dot{x} = -A_1x - \delta_1 y + B_1, \quad \dot{y} = \alpha_1 x + A_1 y + C_1 \quad \text{in } R_1,\\ &\dot{x} = -A_2x - \delta_2 y + B_2, \quad \dot{y} = \alpha_2 x + A_2 y + C_2 \quad \text{in } R_2,\\ &\dot{x} = -A_3x - \delta_3 y + B_3, \quad \dot{y} = \alpha_3 x + A_3 y + C_3 \quad \text{in } R_3. \end{aligned} \tag{2}$$

The linear Hamiltonian saddles in R_1, R_2 and R_3 have the first integrals

$$
\begin{aligned}
H_1 &= -\frac{\alpha_1^2}{2}x^2 - A_1xy - \frac{\delta_1}{2}y^2 - C_1x + B_1y,\\
H_2 &= -\frac{\alpha_2^2}{2}x^2 - A_2xy - \frac{\delta_2}{2}y^2 - C_2x + B_2y,\\
H_3 &= -\frac{\alpha_3^2}{2}x^2 - A_3xy - \frac{\delta_3}{2}y^2 - C_3x + B_3y,
\end{aligned}
\tag{3}
$$

respectively.

Assume that this discontinuous piecewise differential system has some crossing limit cycle of type $\mathcal{C}_1$, with the points intersecting Σ being $(u_1, 0)$ with $u_1 > 0$, $(0, v_1)$ with $v_1 > 0$ and $(u_2, 0)$ with $u_2 < 0$. Then the first integrals given in (3) must satisfy the following system

$$
\begin{aligned}
e_1 &= H_1(u_1, 0) - H_1(0, v_1) = 0,\\
e_2 &= H_2(0, v_1) - H_2(u_2, 0) = 0,\\
e_3 &= H_3(u_2, 0) - H_3(u_1, 0) = 0,
\end{aligned}
\tag{4}
$$

or equivalently

$$
\begin{aligned}
e_1 &= \delta_1 v_1^2 - 2B_1v_1 - \alpha_1^2u_1^2 - 2C_1u_1 = 0,\\
e_2 &= \alpha_2^2u_2^2 + 2C_2u_2 - \delta_2v_1^2 + 2B_2v_1 = 0,\\
e_3 &= (u_1 - u_2)\left(\alpha_3^2u_1 + \alpha_3^2u_2 + 2C_3\right) = 0.
\end{aligned}
\tag{5}
$$

Since $u_1 - u_2 > 0$, by Bezout Theorem (see [22]), system (5) has at most four real solutions. So the discontinuous piecewise differential system (2) can have at most four crossing limit cycles.

If $\alpha_3 = 0$ it follows from Lemma 1 that $A_3 = 1$ and equation $e_3 = 0$ reduces to $C_3 = 0$. Therefore system $e_1 = e_2 = 0$ leads to a continuum of solutions, because we have two equations and three unknowns u_1, u_2, v_1. Consequently system (2) has no limit cycles in this case. Hence, from Lemma 1 we can assume that $\alpha_3 = 1$ and $A_3^2 = \delta_3 + \omega, \omega > 0$.

From the third equation of system (5) we obtain

$$
u_2 = -u_1 - 2C_3. \tag{6}
$$

Substituting u_2 in the first and second equations of (5) we obtain

$$
\begin{aligned}
E_1 &= \delta_1v_1^2 - 2B_1v_1 - \alpha_1^2u_1^2 - 2C_1u_1 = 0,\\
E_2 &= -\delta_2v_1^2 + 2B_2v_1 + \alpha_2^2u_1^2 + \left(4\alpha_2^2C_3 - 2C_2\right)u_1 + 4C_3(\alpha_2^2C_3 - C_2) = 0.
\end{aligned}
\tag{7}
$$

We consider the following cases.

Case 1: $\alpha_1 = \alpha_2 = 0$. Then by Lemma 1 $A_1 = A_2 = 1$, and using the resultant between E_1, E_2 with respect to the variable u_1 we get the polynomial

$$
\frac{1}{2}(C_2\delta_1 + C_1\delta_2)v_1^2 - (B_2C_1 + B_1C_2)v_1 + \frac{2C_1C_2C_3}{\alpha_3^2} = 0.
$$

So in this case the discontinuous piecewise differential system (2) can have at most two crossing limit cycles, and statement (a) is proved in this case.

Case 2: $\alpha_1 = \alpha_2 = 1$. Then by Lemma 1 $A_1^2 = \delta_1 + \omega$, $A_2^2 = \delta_2 + \omega, \omega > 0$, and system (7) becomes

$$\begin{aligned} E_1 &= \delta_1 v_1^2 - 2B_1 v_1 - u_1^2 - 2C_1 u_1 = 0, \\ E_2 &= -\delta_2 v_1^2 + 2B_2 v_1 + u_1^2 + (4C_3 - 2C_2)\, u_1 + 4C_3(C_3 - C_2) = 0. \end{aligned} \tag{8}$$

Note that if $C_3 = 0$, system (8) has the solution $(u_1, v_1) = (0, 0)$ which cannot contribute to a limit cycle, and so in this case, system (5) can have at most three solutions that can provide limit cycles for system (2), proving statement (a) when $C_3 = 0$ in Case 2.

From now on in this case we assume that $C_3 \neq 0$ and we will study system (8) considering the following subcases.

Subcase 2.1: $\delta_1 = \delta_2 = 0$. Then system (8) reduces to

$$\begin{aligned} E_1 &= -2B_1 v_1 - u_1^2 - 2C_1 u_1 = 0, \\ E_2 &= 2B_2 v_1 + u_1^2 + (4C_3 - 2C_2)\, u_1 + 4C_3(C_3 - C_2) = 0. \end{aligned} \tag{9}$$

To compute the common zeroes of $E_1 = E_2 = 0$ we will compute the resultant of E_1 and E_2 respect to the variable u_1, that is, $\mathcal{R}_1 = \mathrm{Res}(E_1, E_2, v_1)$. This resultant is

$$\mathcal{R}_1 = u_1^2(B_2 - B_1) + 2u_1(B_1C_2 - 2B_1C_3 + B_2C_1) + 4B_1C_3(C_2 - C_3) = 0.$$

Then at most there are two positive solutions of $\mathcal{R}_1 = 0$. Therefore there are at most two positive solutions of $E_1 = E_2 = 0$, and so at most two crossing limit cycles for system (2) in this subcase.

Subcase 2.2: $\delta_1 = 0$ and $\delta_2 \neq 0$. System (8) becomes

$$\begin{aligned} E_1 &= -u_1^2 - 2C_1 u_1 - 2B_1 v_1 = 0, \\ E_2 &= u_1^2 + (4C_3 - 2C_2)u_1 + 4C_3(C_3 - C_2) - \delta_2 v_1^2 + 2B_2 v_1 = 0. \end{aligned} \tag{10}$$

By Bezout Theorem there are at most four positive solutions (u_1^i, v_1^i) for $i = 1, 2, 3, 4$ of system (10) with u_1^i and v_1^i positive. We observe that although there can be four possible admissible solutions we will prove that there are at most three. Note that these admissible solutions must satisfy

$$0 < u_1^1 < u_1^2 < u_1^3 \text{ and } 0 < v_1^1 < v_1^2 < v_1^3, \tag{11}$$

otherwise the solutions of system (2) connecting the points $(u^i, 0)$ and $(0, v^i)$ in the region R_1 would intersect, which contradicts the Uniqueness Theorem of solutions of an ordinary differential equation with a given initial value.

We observe that if $B_1 = 0$, the first equation of (10) has at most one positive solution and therefore, from the second equation of (10) there are at most two solutions.

If $B_1 \neq 0$ the first equation of (10) is a parabola, and we can write it as

$$(u_1 + C_1)^2 = -2B_1 \left(v_1 - \frac{C_1^2}{2B_1} \right). \tag{12}$$

Note that the axis of symmetry of this parabola is parallel to the v_1-axis.

The second equation of (10) for $\delta_2 > 0$ can be written as

$$(u_1 + 2C_3 - C_2)^2 - \left(\sqrt{\delta_2}v_1 - \frac{B_2}{\sqrt{\delta_2}}\right)^2 = C_2^2 - \frac{B_2^2}{\delta_2} = h, \tag{13}$$

which is a hyperbola with its two axes of symmetry parallel to the coordinates axes if $h \neq 0$, and two straight lines intersecting in a point with two axes of symmetry parallel to the coordinates axes if $h = 0$. The second equation of (10) for $\delta_2 < 0$ can be written as

$$(u_1 + 2C_3 - C_2)^2 + \left(\sqrt{-\delta_2}v_1 - \frac{B_2}{\sqrt{-\delta_2}}\right)^2 = C_2^2 - \frac{B_2^2}{\delta_2} = r, \tag{14}$$

which is an ellipse again with its two axes of symmetry parallel to the coordinates axes if $r > 0$, a point if $r = 0$ and the empty set if $r < 0$.

If the hyperbola (13) or the ellipse (14) intersects the piece of the parabola (12) contained in the positive quadrant of the plane (u_1, v_1) in four points, (u_1^i, v_1^i) for $i = 1, 2, 3, 4$, since the axis of symmetry of the parabola is parallel to the v_1-axis, the axes of symmetry of the hyperbola, or of the two straight lines intersecting in a point, or of the ellipse have its two axes of symmetry parallel to the axes of coordinates, two points of the intersection are in the increasing part of the parabola and the other two points are in the decreasing part of the parabola. The coordinates of these four points satisfy $0 < u_1^1 < u_1^2 < u_1^3 < u_1^4$, but they cannot satisfy $0 < v_1^1 < v_1^2 < v_1^3 < v_1^4$. Hence at most the mentioned intersection can have three points, and the statement (a) holds in this case.

Subcase 2.3*:* $\delta_1 \neq 0$ *and* $\delta_2 = 0$. The proof is analogous to the previous case.

Subcase 2.4*:* $\delta_1 \neq 0$ *and* $\delta_2 \neq 0$. Then the first equation of (8) if $\delta_1 > 0$ can be written as

$$\left(\sqrt{\delta_1}v_1 - \frac{B_1}{\sqrt{\delta_1}}\right)^2 - (u_1 + C_1)^2 = \frac{B_1^2}{\delta_1} - C_1^2 = k, \tag{15}$$

which is a hyperbola if $k \neq 0$, and two straight lines intersecting in a point if $k = 0$. The first equation of (8) if $\delta_1 < 0$ can be written as

$$-\left(\sqrt{-\delta_1}v_1 - \frac{B_1}{\sqrt{-\delta_1}}\right)^2 - (u_1 + C_1)^2 = \frac{B_1^2}{\delta_1} - C_1^2 = s, \tag{16}$$

which is an ellipse if $s < 0$, a point if $s = 0$ and the empty-set if $s > 0$.

Note that the hyperbola, or the two straight lines intersecting in a point (15), and the ellipse (16) have their axes of symmetry parallel to the coordinate axes. Consequently when they intersect in four points, (u_1^i, v_1^i) for $i = 1, 2, 3, 4$ with second equation $E_2 = 0$ in (8) their coordinates can never verify the conditions $0 < u_1^1 < u_1^2 < u_1^3 < u_1^4$ and $0 < v_1^1 < v_1^2 < v_1^3 < v_1^4$ simultaneously. So statement (a) is proved in this case.

Case 3: $\alpha_1 = 0$ and $\alpha_2 = 1$. Then system (7) becomes

$$\begin{aligned} E_1 &= \delta_1 v_1^2 - 2B_1 v_1 - 2C_1 u_1 = 0, \\ E_2 &= -\delta_2 v_1^2 + 2B_2 v_1 + \alpha_2^2 u_1^2 + \left(4\alpha_2^2 C_3 - 2C_2\right) u_1 + 4C_3(\alpha_2^2 C_3 - C_2) = 0. \end{aligned} \tag{17}$$

If $\delta_1 = 0$ equation $E_1 = 0$ is an straight line, and by Bezout Theorem the system $E_1 = E_2 = 0$ has at most two points of intersection.

If $\delta_1 \neq 0$ equation $E_1 = 0$ defines a parabola, and the proof of statement (a) in this case follows as in subcase 2.2.

Case 4: $\alpha_1 = 1$ and $\alpha_2 = 0$. The proof is the same than in case 3.

In order to complete the proof of statement (a) we provide now examples of such systems with 1, 2, or 3 crossing limit cycles of type $\mathcal{C}_1$.

First we provide an example with one crossing limit cycle intersecting in a unique point in each branch of Σ. In this case we consider the discontinuous piecewise differential system formed by the following three linear Hamiltonian saddles

$$
\begin{aligned}
&\dot{x} = -x - 7y + \frac{5}{3}, \quad \dot{y} = y + 1 \text{ in } R_1,\\
&\dot{x} = -x - 4y + \frac{1}{3}, \quad \dot{y} = y - 10 \text{ in } R_2,\\
&\dot{x} = -6x - 35y + 30, \quad \dot{y} = x + 6y - 2 \text{ in } R_3,
\end{aligned} \tag{18}
$$

with the first integrals

$$
\begin{aligned}
H_1(x,y) &= -xy - x - \frac{7y^2}{2} + \frac{5y}{3},\\
H_2(x,y) &= -xy + 10x - 2y^2 + \frac{y}{3},\\
H_3(x,y) &= -\frac{x^2}{2} - 6xy + 2x - \frac{35y^2}{2} + 30y,
\end{aligned}
$$

respectively.

Taking into account that we are only interested in the solutions (u_1, v_1, u_2) of system (4) with $u_1 > 0$, $v_1 > 0$ and $u_2 < 0$, the unique good solution of system (4) is $(u_1, v_1, u_2) = (4.33276.., 1.37591.., -0.332762..)$ which provides the crossing limit cycle of Figure 1 which is traveled in counterclockwise sense.

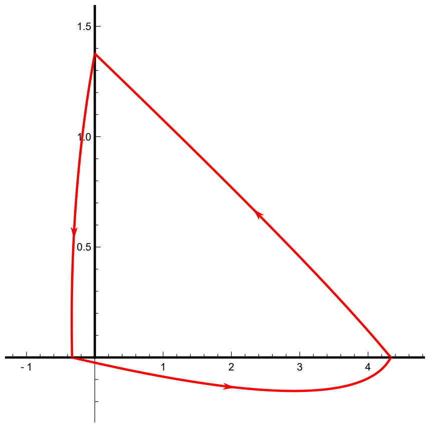


Figure 1: The unique crossing limit cycle of the discontinuous piecewise differential system formed by the linear Hamiltonian saddles (18). This limit cycle is traveled in counterclockwise sense.

The next example will have two limit cycles intersecting in one point each branch of Σ. We consider the discontinuous piecewise differential system formed by the following three linear Hamiltonian saddles

$$
\begin{aligned}
&\dot{x} = -3.39116..x - \frac{21y}{2} + \frac{25}{4}, \quad \dot{y} = x + 3.39116..y + 8, \text{ in } R_1,\\
&\dot{x} = -3.51188..x - \frac{34y}{3} - \frac{5}{3}, \quad \dot{y} = x + 3.51188..y - 12, \text{ in } R_2,\\
&\dot{x} = 3y + 30, \quad \dot{y} = x + \frac{1}{2}, \text{ in } R_3,
\end{aligned} \tag{19}
$$

with the first integrals

$$
\begin{aligned}
H_1(x,y) &= -\frac{x^2}{2} - 3.39116..xy - 8x - \frac{21y^2}{4} + \frac{25y}{4},\\
H_2(x,y) &= -\frac{x^2}{2} - 3.51188..xy + 12x - \frac{17y^2}{3} - \frac{5y}{3},\\
H_3(x,y) &- \frac{x^2}{2} - \frac{x}{2} + \frac{3y^2}{2} + 30y.
\end{aligned}
$$

Taking into account that we are only interested in the solutions (u_1, v_1, u_2) with $u_1 > 0$, $v_1 > 0$ and $u_2 < 0$, the unique eligible two solutions of the system (4) are

$$(u_1^1, v_1^1, u_2^1) = (1, 2, -2), \quad (u_1^2, v_1^2, u_2^2) = (3, 3, -4)$$

which provide the two crossing limit cycles of Figure 2 which are traveled in counterclockwise sense.

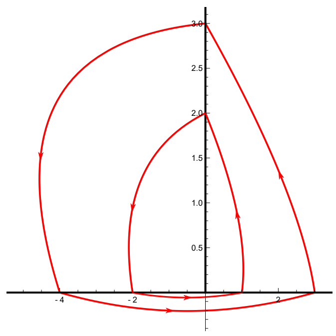


Figure 2: The two crossing limit cycles of the discontinuous piecewise differential system formed by the linear Hamiltonian saddles (19). These limit cycles are traveled in counterclockwise sense.

Now we construct a system having three limit cycles, intersecting in three points on each branch of Σ. We consider the linear differential systems

(20)

$$
\begin{aligned}
&\dot{x} = -3.9988..x - 11.9904..y + 2.86091.., \ \dot{y} = x + 3.9988..y + 0.633453.., \text{ in } R_1,\\
&\dot{x} = 3.58474..x - 11.8504..y + 2.58228.., \ \dot{y} = x - 3.58474..y - 0.483858.., \text{ in } R_2,\\
&\dot{x} = -4x - 2.5y + 3, \ \dot{y} = x + 4y + 0.1, \text{ in } R_3,
\end{aligned}
$$

with first integrales

$$H_1(x,y) = -\frac{x^2}{2} - 3.9988..xy - 0.633453..x - 5.9952..y^2 + 2.86091..y,$$
$$H_2(x,y) = -\frac{x^2}{2} + 3.58474..xy + 0.483858..x - 5.9252..y^2 + 2.58228..y,$$
$$H_3(x,y) = -\frac{x^2}{2} - 4xy - 0.1x - 1.25y^2 + 3y.$$

Taking into account that we are only interested in the solutions (u_1, v_1, u_2) of system (4) with $u_1 > 0$, $v_1 > 0$ and $u_2 < 0$, the unique eligible three solutions of system (4) are

$$(u_1^1, v_1^1, u_2^1) = (0.1, 0.5, -0.3), \quad (u_1^2, v_1^2, u_2^2) = (0.5, 0.6, -0.7),$$

$$(u_1^3, v_1^3, u_2^3) = (0.8, 0.68, -1),$$

which provide the three crossing limit cycles of Figure 3 which are traveled in counterclockwise sense.

This completes the proof of statement (a) of Theorem 1. $\square$

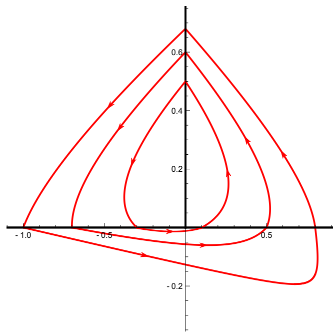


Figure 3: The three crossing limit cycles of the discontinuous piecewise differential system formed by saddles (20). These limit cycles are traveled in counterclockwise.

Proof of statement (*b*) *of Theorem* 1. Assume that we have a discontinuous piecewise differential system formed by three linear Hamiltonian saddles and separated by the line Σ. These linear Hamiltonian saddles in R_1, R_2 and R_3 have the first integrals H_1, H_2 and H_3 described in (3). Assume also that this discontinuous piecewise differential system has a crossing limit cycle of type $\mathcal{C}_2$. The intersection points of this crossing limit cycle with Σ can only be of the following two forms: either $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$ with $0 < y_1 < y_2$, $0 < x_1 < x_2$; or $x_2 < x_1 < 0$. We assume the first case, the proof of the second is similar. In order that these four points correspond to the intersection points of a limit cycle with Σ, they must satisfy the following system

$$
\begin{aligned}
e_1 &= H_1(x_2,0) - H_1(0,y_2) = 0,\\
e_2 &= H_2(0,y_2) - H_2(0,y_1) = 0,\\
e_3 &= H_1(0,y_1) - H_1(x_1,0) = 0,\\
e_4 &= H_3(x_1,0) - H_3(x_2,0) = 0,
\end{aligned} \tag{21}
$$

or equivalently,

$$
\begin{aligned}
e_1 &= \delta_1 y_2^2 - 2B_1 y_2 - \alpha_1^2 x_2^2 - 2C_1 x_2 = 0,\\
e_2 &= (y_1 - y_2)(2B_2 - \delta_2 y_1 - \delta_2 y_2) = 0,\\
e_3 &= \alpha_1^2 x_1^2 + 2C_1 x_1 - \delta_1 y_1^2 + 2B_1 y_1 = 0,\\
e_4 &= (x_1 - x_2)(2C_3 + \alpha_3^2 x_1 + \alpha_3^2 x_2) = 0.
\end{aligned} \tag{22}
$$

Observe first that if $\alpha_3 = 0$ or $\delta_2 = 0$, then system $e_1 = e_3 = 0$ leads to a continuum of solutions, because we have two equations and four unknowns x_1, x_2, y_1, y_2. Consequently, system (2) has no limit cycles. So in what follows we take $\delta_2 \neq 0$ and $\alpha_3 = 1$. From equations $e_2 = 0$ and $e_4 = 0$, we isolate y_2 and x_2, and we obtain

$$
y_2 = -y_1 + \frac{2B_2}{\delta_2}, \quad x_2 = -x_1 - 2C_3. \tag{23}
$$

Substituting (23) into e_1 and summing up the expressions of e_1 and e_3, we get

$$
e_{13} = \left(y_1 - \frac{B_2}{\delta_2}\right)\left(B_1 - \frac{B_2\delta_1}{\delta_2}\right) + (x_1 + C_3)(C_1 - \alpha_1^2 C_3) = 0.
$$

Hence to solve system (22) is equivalent to solve the system

$$
e_{13} = 0 \text{ and } e_3 = 0.
$$

According δ_1 and α_1 we have the following situations:
1) If $\delta_1 = 0$ and $\alpha_1 = 0$ we get

$$
e_{13} = B_1\left(y_1 - \frac{B_2}{\delta_2}\right) + C_1(x_1 + C_3) = 0, \quad e_3 = C_1 x_1 + B_1 y_1 = 0. \tag{24}
$$

Then by Bezout's Theorem the system has a real solution. So, the system (2) has no limit cycle of type $\mathcal{C}_2$.

2) If $\delta_1 \neq 0$ and $\alpha_1 = 0$ we have

$$
\mathcal{L} : \left(y_1 - \frac{B_2}{\delta_2}\right)\left(B_1 - \frac{B_2\delta_1}{\delta_2}\right) + (x_1 + C_3)C_1 = 0, \quad \mathcal{P} : -\delta_1 y_1^2 + 2B_1 y_1 + 2C_1 x_1 = 0. \tag{25}
$$

3) If $\delta_1 > 0$ and $\alpha_1 = 1$ we have

$$
\mathcal{L} : e_{13} = 0, \qquad \mathcal{H} : (x_1 + C_1)^2 - \left(\sqrt{\delta_1} y_1 - \frac{B_1}{\sqrt{\delta_1}}\right)^2 = C_1^2 - \frac{B_1^2}{\delta_1} = r. \tag{26}
$$

4) If $\delta_1 = 0$ and $\alpha_1 = 1$ we obtain

$$
\mathcal{L} : B_1\left(y_1 - \frac{B_2}{\delta_2}\right) + (x_1 + C_3)(C_1 - C_3) = 0, \ \mathcal{P} : (x_1 + C_1)^2 = -2B_1\left(y_1 - \frac{C_1^2}{2B_1}\right) \tag{27}
$$

5) If $\delta_1 < 0$ and $\alpha_1 = 1$ we have

$$\mathcal{L} : e_{13} = 0, \qquad \mathcal{E} : (x_1 + C_1)^2 + \left(\sqrt{-\delta_1} y_1 - \frac{B_1}{\sqrt{-\delta_1}}\right)^2 = C_1^2 - \frac{B_1^2}{\delta_1} = s. \tag{28}$$

By Bezout's Theorem the system (25) or (26) or (27) or (28) has at most two solutions, that is, there are at most one limit cycle of type $\mathcal{C}_2$. This proves the statement (b).

In order to complete the proof of statement (b) we provide an example with one limit cycle of type $\mathcal{C}_2$.

Now we will give an example having a saddle in each of the regions R_1, R_2 and R_3 that has exactly one limit cycle, which intersect the set Σ in four points. We consider the discontinuous piecewise linear differential systems

$$\begin{aligned} &\dot{x} = -0.4x - 0.135135..y + 0.5, \ \dot{y} = x + 0.4y - 1.39189.., \ \text{in } R_1, \\ &\dot{x} = -5x - 0.727273..y + 4, \ \dot{y} = x + 5y - 50, \ \text{in } R_2, \\ &\dot{x} = -0.1x - 0.005y + 4, \ \dot{y} = x + 0.1y - 2, \ \text{in } R_3, \end{aligned} \tag{29}$$

with first integrales

$$\begin{aligned} H_1(x,y) &= -0.5x^2 - 0.4xy + 1.39189..x - 0.0675676..y^2 + 0.5y, \\ H_2(x,y) &= -0.5x^2 - 5xy + 50x - 0.363636..y^2 + 4y, \\ H_3(x,y) &= -0.5x^2 - 0.1xy + 2x - 0.0025..y^2 + 4y. \end{aligned}$$

Taking into account that we are only interested in the solutions of system (21) of the form (x_1, y_1, x_2, y_2) satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$, the unique eligible solution of system (21) is $(x_1, y_1, x_2, y_2) = (1, 3, 3, 8)$ which provides the limit cycle of Figure 4, which is traveled in counterclockwise sense.

This completes the proof of statement (b) of the Theorem 1. $\square$

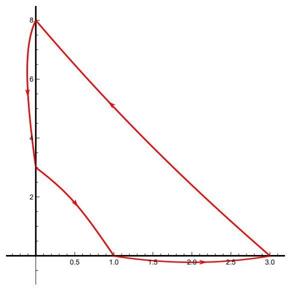


Figure 4: The unique crossing limit cycle of the discontinuous piecewise differential system formed by saddles (29). This limit cycle is traveled in counterclockwise.

Proof of statement (*c*) *of Theorem 1.* The upper bound of the simultaneous limit cycles of three of type $\mathcal{C}_1$ and one of type $\mathcal{C}_2$ follows directly from the statements (a) and (b).

We provide an example of a discontinuous piecewise differential system having one crossing limit cycle of type $\mathcal{C}_2$ and one or two crossing limit cycle of type $\mathcal{C}_1$, simultaneously. Later on, we give one of the systems satisfying all the conditions for having one limit cycle of type $\mathcal{C}_2$ and three limit cycles of type $\mathcal{C}_1$, but this system has no such four limit cycles.

For the first case we consider the discontinuous piecewise differential system formed by the following three linear Hamiltonian saddles

$$\begin{aligned} &\dot{x} = -1.5x - 0.85y + 1.45, \ \dot{y} = x + 1.5y - 2.3, \ \text{in } R_1, \\ &\dot{x} = 2x - 0.5625y + 0.84375, \ \dot{y} = x - 2y - 2, \ \text{in } R_2, \\ &\dot{x} = 6x - 4y + 4, \ \dot{y} = x - 6y - 2.25, \ \text{in } R_3, \end{aligned} \tag{30}$$

with first integrals

$$\begin{aligned} H_1(x,y) &= -0.5x^2 - 1.5xy + 2.3x - 0.425y^2 + 1.45y, \\ H_2(x,y) &= -0.5x^2 + 2xy + 2x - 0.28125y^2 + 0.84375y, \\ H_3(x,y) &= -0.5x^2 + 6xy + 2.25x - 2y^2 + 4y. \end{aligned}$$

Then system (22) is equivalent to system

$$\begin{aligned} e_1 &= -x_2^2 + 4.6x_2 + 0.85y_2^2 - 2.9y_2 = 0, \\ e_2 &= 0.5625y_1 + 0.5625y_2 - 1.6875 = 0, \\ e_3 &= x_1^2 - 4.6x_1 - 0.85y_1^2 + 2.9y_1 = 0, \\ e_4 &= -x_1 - x_2 + 4.5 = 0, \end{aligned}$$

and system (5) becomes

$$\begin{aligned} e_5 &= -u_1^2 + 4.6u_1 + 0.85v_1^2 - 2.9v_1 = 0, \\ e_6 &= u_2^2 - 4u_2 - 0.5625v_1^2 + 1.6875v_1 = 0, \\ e_7 &= u_1 + u_2 - 4.5 = 0. \end{aligned}$$

Taking into account that we are only interested in the solutions (x_1, y_1, x_2, y_2) with $x_2 > x_1 > 0$, $y_2 > y_1 > 0$, the unique solution of system $e_1 = 0, e_2 = 0, e_3 = 0, e_4 = 0$ is $(x_1, y_1, x_2, y_2) = (0.5, 1, 4, 2)$, and the unique solution of system $e_5 = 0, e_6 = 0, e_7 = 0$ satisfying $u_1 > 0$, $v_1 > 0$, $u_2 < 0$ is $(u_1, v_1, u_2) = (5, 4, -0.5)$. So we obtain the two crossing limit cycles of Figure (5), one of type $\mathcal{C}_1$ and the other of type $\mathcal{C}_2$. They are both travelled in counterclockwise sense.

For the second case we consider the discontinuous piecewise differential system formed by the following three linear Hamiltonian saddles

$$\begin{aligned} &\dot{x} = -x - 0.764706y + 1.73529, \ \dot{y} = x + y - 2.44118, \ \text{in } R_1, \\ &\dot{x} = 4x - 0.559089y + 1.39772, \ \dot{y} = x - 4y - 1.17727, \ \text{in } R_2, \\ &\dot{x} = 3x - 4y + 10, \ \dot{y} = x - 3y - 2.5, \ \text{in } R_3, \end{aligned} \tag{31}$$

with first integrales

$$\begin{aligned} H_1(x,y) &= -0.5x^2 - xy + 2.44118x - 0.382353y^2 + 1.73529y, \\ H_2(x,y) &= -0.5x^2 + 4xy + 1.17727x - 0.279544y^2 + 1.39772y, \\ H_3(x,y) &= -0.5x^2 + 3xy + 2.5x - 2y^2 + 10y. \end{aligned}$$

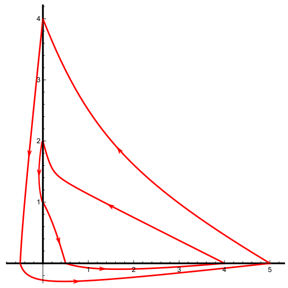


Figure 5: Two crossing limit cycles: one of type $\mathcal{C}_1$ and one of type $\mathcal{C}_2$ of the discontinuous piecewise differential system formed by saddles (30). These limit cycles are traveled in counterclockwise.

Then system (22) is equivalent to system

$$\begin{aligned}
e_1 &= -x_2^2 + 4.88235x_2 + 0.764706y_2^2 - 3.47059y_2 = 0,\\
e_2 &= 0.559089y_1 + 0.559089y_2 - 2.79544 = 0,\\
e_3 &= x_1^2 - 4.88235x_1 - 0.764706y_1^2 + 3.47059y_1 = 0,\\
e_4 &= -x_1 - x_2 + 5 = 0,
\end{aligned}$$

and system (5) becomes

$$\begin{aligned}
e_5 &= -u_1^2 + 4.88235u_1 + 0.764706v_1^2 - 3.47059v_1 = 0,\\
e_6 &= u_2^2 - 2.35453u_2 - 0.559089v_1^2 + 2.79544v_1 = 0,\\
e_7 &= u_1 + u_2 - 5 = 0.
\end{aligned}$$

Taking into account that we are only interested in the solutions (x_1, y_1, x_2, y_2) with $x_2 > x_1 > 0$, $y_2 > y_1 > 0$, the unique solution of system $e_1 = 0, e_2 = 0, e_3 = 0, e_4 = 0$ is $(x_1, y_1, x_2, y_2) = (1, 2, 4, 3)$, and the solutions of system $e_5 = 0, e_6 = 0, e_7 = 0$ satisfying $u_1 > 0$, $v_1 > 0$, $u_2 < 0$ are $(u_1^1, v_1^1, u_2^1) = (6, 6, -1)$ and $(u_1^2, v_1^2, u_2^2) = (8, 8.41454, -3)$. So we obtain the three crossing limit cycles of Figure 6, one of type $\mathcal{C}_2$ and two of type $\mathcal{C}_1$. They are both travelled in counterclockwise sense.

Now we use the notations given in the proofs of statements (a) and (b). First we assume that system (22) has a limit cycle intersecting Σ in four points, and without loss of generality, we can further assume that two of these four points are located on the positive x-axis. The solution of (22) providing the limit cycle intersecting Σ in four points is $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$ with $0 < x_1 < x_2$ and $0 < y_1 < y_2$. Next, we are going to study the system (22) from a geometrical point of view according the parameters.

We consider the case $\alpha_1 = 1$ and $\delta_1 > 0$ we observe that system (26) has a limit cycle intersecting Σ in the four points if $\mathcal{L}$ intersects $\mathcal{H}$ in the two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Note that the second equation of (26) has the two

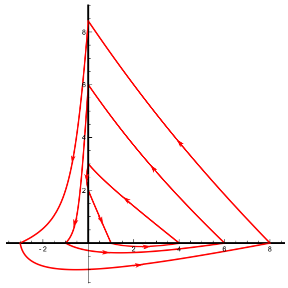


Figure 6: Three crossing limit cycles: one of type $\mathcal{C}_2$ and two of type $\mathcal{C}_1$ of the discontinuous piecewise differential system formed by saddles (31). These limit cycles are traveled in counterclockwise.

asymptotes,

$$
\begin{aligned}
&L^1 : x_1 + \sqrt{\delta_1} y_1 + C_1 - \frac{B_1}{\sqrt{\delta_1}} = 0, \\
&L^2 : x_1 - \sqrt{\delta_1} y_1 + C_1 + \frac{B_1}{\sqrt{\delta_1}} = 0,
\end{aligned} \tag{32}
$$

which intersect in $I_{\mathcal{H}} = \left(-C_1, \frac{B_1}{\delta_1}\right)$. The restrictions $0 < x_1 < x_2$ and $0 < y_1 < y_2$ together with (23) forces that

$$
0 < x_1 < -C_3 < x_2, \quad 0 < y_1 < \frac{B_2}{\delta_2} < y_2. \tag{33}
$$

Note that $I_{\mathcal{L}} = \left(-C_3, \frac{B_2}{\delta_2}\right)$ is a point of the line $\mathcal{L}$ in (26). It follows from (33) that the straight line $\mathcal{L}$ must have positive slope, i.e.,

$$
\frac{\delta_2(C_3 - C_1)}{\delta_2 B_1 - \delta_1 B_2} > 0. \tag{34}
$$

We assume without loss of generality that the hyperbola $\mathcal{H}$ is of left-right type and that the P_1 and P_2 are located on the left branch of $\mathcal{H}^l$ with the slope of $\mathcal{L}$ larger than L^2.

Assume that there exists a discontinuous piecewise differential system separated by the line Σ and formed by three linear Hamiltonian saddles and having one limit cycle of type $\mathcal{C}_2$ and three limit cycles of type $\mathcal{C}_1$ satisfying equations (22) and (8), respectively. Using the previous notation for the coordinates of the intersection points of these limit cycles with the line Σ, these coordinates must satisfy

$$
0 < x_1 < x_2 < u_1^1 < u_1^2 < u_1^3 \quad \text{and} \quad 0 < y_1 < y_2 < v_1^1 < v_1^2 < v_1^3, \tag{35}
$$

where $Q_i = (u_1^i, v_1^i)$ for $i = 1, 2, 3$ are the coordinates associated to the intersection points of the three limit cycles of type $\mathcal{C}_1$.

Next, we prove that systems (8) and (22) have a solution satisfying the conditions (35), but that the closed orbits of these solutions only define one limit cycle of type $\mathcal{C}_1$.

We assume the existence of one limit cycle of type $\mathcal{C}_1$ intersecting Σ in three points $(u_1, 0)$, $(0, v_1)$ and $(u_2, 0)$ with $u_1 > 0$, $v_1 > 0$ and $u_2 < 0$. We point out that under the condition (6) we must have the restriction

$$u_1 > -2C_3. \tag{36}$$

In this case we observe that the first equation of (8) is the same than the third equation of (22), whose form has already been analyzed. Now we consider the second equation of (8) which is a hyperbola in the variables u_1 and v_1, denoted by $\mathcal{H}_1$ having the asymptotes

$$\begin{aligned} L^1_{\mathcal{H}_1} &: u_1 + \sqrt{\delta_2} v_1 + 2C_3 - C_2 - \frac{B_2}{\sqrt{\delta_2}} = 0, \\ L^2_{\mathcal{H}_1} &: u_1 - \sqrt{\delta_2} v_1 + 2C_3 - C_2 + \frac{B_2}{\sqrt{\delta_2}} = 0, \end{aligned} \tag{37}$$

which intersect at the point $I_{\mathcal{H}_1} = \left(C_2 - 2C_3, \frac{B_2}{\delta_2}\right)$. Observe that if there is a limit cycle of type $\mathcal{C}_1$ it cuts the negative x-axis. Therefore the region bounded by this limit cycle must contain the limit cycle of type $\mathcal{C}_2$ in its interior.

To study the maximum number of limit cycles of system (8) of type $\mathcal{C}_1$ provided the existence of a limit cycle of type $\mathcal{C}_2$, is equivalent to find the maximum number of intersection points of the hyperbolas $\mathcal{H}$ and $\mathcal{H}_1$, whose coordinates must satisfy (35).

Knowing that $\mathcal{L}$ intersects $\mathcal{H}$ in the two points P_1 and P_2 (of the limit cycle of type $\mathcal{C}_2$). Note that since $I_{\mathcal{H}_1}$ has the same horizontal coordinate as $I_{\mathcal{L}}$, in order that there exist at most three points Q_i that satisfies (35), we must have that $I_{\mathcal{L}} \prec I_{\mathcal{H}_1}$, i.e. the second coordinate of the point $I_{\mathcal{L}}$ is smaller than the second coordinate of the point $I_{\mathcal{H}_1}$ (when $C_2 > C_3$), or $I_{\mathcal{H}_1} \prec I_{\mathcal{L}}$ (when $C_2 < C_3$), and the points Q_i are located above the horizontal line $y = B_2/\delta_2$. Moreover these points must satisfy

$$P_1 \prec P_2 \prec Q_1 \prec Q_2 \prec Q_3. \tag{38}$$

From the existence of P_1 and P_2 we have that they can be located both on the left branch $\mathcal{H}^l$. In this case we can have that the hyperbola $\mathcal{H}_1$ is of left-right type or of upper-down type and that there is at most three intersection points Q_1, Q_2 and Q_3 of $\mathcal{H}_1$ with $\mathcal{H}$ which satisfy (35), see Figure (7).

Now we provide an example of a discontinuous differential system formed by the following three linear Hamiltonian saddles

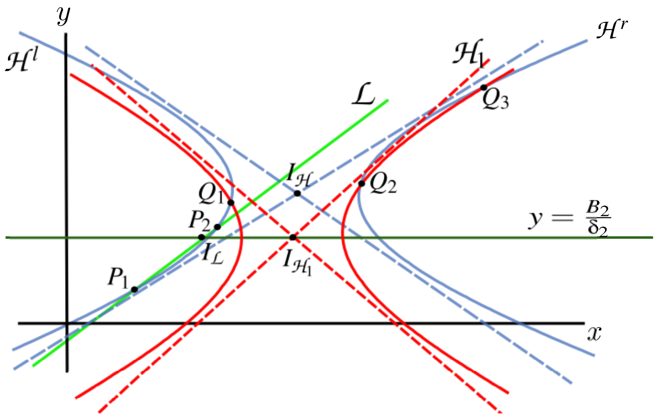
$$\begin{aligned} &\dot{x} = -1.64317x - 1.7y + 7.5,\ \dot{y} = x + 1.64317y - 6.4,\ \text{in } R_1, \\ &\dot{x} = 2.54951x - 5.5y + 1.9,\ \dot{y} = x - 2.54951y - 4.9,\ \text{in } R_2, \\ &\dot{x} = -3x - 4y + 40,\ \dot{y} = x + 3y - 0.4,\ \text{in } R_3. \end{aligned} \tag{39}$$

Then $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{H}_1$ defined in (26) and (8) are the following

$$\mathcal{L} : 30.25(-6.4(x_1 - 0.4) + 0.4x_1 + 7.5y_1 - 0.16) - 10.45(1.7y_1 + 7.5) + 6.137 = 0,$$

$$\mathcal{H} : x_1^2 - 12.8x_1 - 1.7y_1^2 + 15.y_1 = 0,$$

$$\mathcal{H}_1 : (u_1 - 0.8)^2 + 9.8(u_1 - 0.8) - 5.5v_1^2 + 3.8v_1 = 0.$$



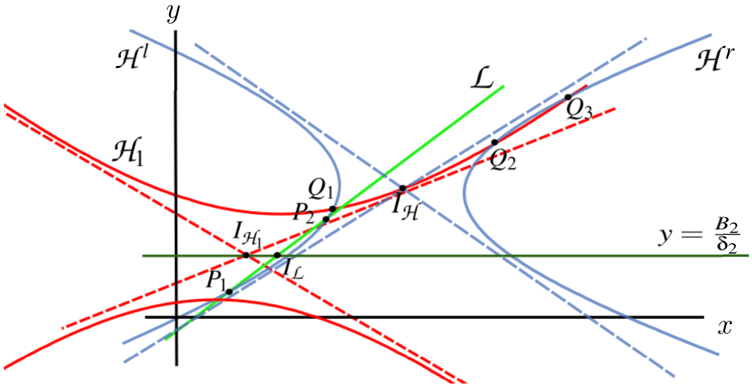


Figure 7: Relative positions of the hyperbola $\mathcal{H}$ of (26) with the hyperbola $\mathcal{H}_1$ of (8) and the three possible intersection points Q_1, Q_2 and Q_3.

It follows from the previous proof that the intersection points of $\mathcal{L}$ and $\mathcal{H}$ are $P_1 = (0.140234.., 0.119987..)$ and $P_2 = (0.659766.., 0.570922..)$, while that $\mathcal{H}$ and $\mathcal{H}_1$ intersect in three points $Q_1 = (0.802264.., 0.696702..)$, $Q_2 = (2.33881.., 2.15977..)$ and $Q_3 = (9.82011.., 5.91185..)$. These coordinates satisfy the condition (35), and the solutions of the system (26) and (8) are the following

$$
\begin{aligned}
(x_1, y_1, x_2, y_2) &= (0.140234.., 0.119987.., 0.659766.., 0.570922..),\\
(u_1^1, v_1^1, u_2^1) &= (0.802264.., 0.696702.., -0.0022644..),\\
(u_1^2, v_1^2, u_2^2) &= (2.33881.., 2.15977.., -1.53881..),\\
(u_1^3, v_1^3, u_2^3) &= (9.82011.., 5.91185.., -9.02011..).
\end{aligned}
$$

But only the last one defines a limit cycle travelled counterclockwise sense.

This completes the proof of statement (c) of Theorem 1. $\square$

3. Proof of Proposition 2

Assume that we have a discontinuous piecewise differential system homogeneous formed by three linear Hamiltonian saddles and separated by the set Σ. By Lemma 1 with $B = C = 0$, we can write such a discontinuous piecewise linear differential system as

$$\begin{aligned} \dot{x} &= -A_1x - \delta_1 y, \quad \dot{y} = \alpha_1 x + A_1 y \quad \text{in } R_1,\\ \dot{x} &= -A_2x - \delta_2 y, \quad \dot{y} = \alpha_2 x + A_2 y \quad \text{in } R_2,\\ \dot{x} &= -A_3x - \delta_3 y, \quad \dot{y} = \alpha_3 x + A_3 y \quad \text{in } R_3. \end{aligned} \tag{40}$$

The linear Hamiltonian saddles in R_1, R_2 and R_3 have the first integrals

$$\begin{aligned} H_1 &= -\frac{\alpha_1^2}{2}x^2 - A_1xy - \frac{\delta_1}{2}y^2,\\ H_2 &= -\frac{\alpha_2^2}{2}x^2 - A_2xy - \frac{\delta_2}{2}y^2,\\ H_3 &= -\frac{\alpha_3^2}{2}x^2 - A_3xy - \frac{\delta_3}{2}y^2, \end{aligned} \tag{41}$$

respectively. Assume that this discontinuous piecewise differential system has a crossing limit cycle of type $\mathcal{C}_1$ and let $(x_1, 0)$ with $x_1 > 0$, $(0, y_1)$ with $y_1 > 0$ and $(x_2, 0)$ with $x_2 < 0$ be the three points intersecting Σ (one in each branch). Then the following system must hold

$$\begin{aligned} e_1 &= H_1(x_1, 0) - H_1(0, y_1) = 0,\\ e_2 &= H_2(0, y_1) - H_2(x_2, 0) = 0,\\ e_3 &= H_3(x_2, 0) - H_3(x_1, 0) = 0, \end{aligned} \tag{42}$$

or equivalently

$$\begin{aligned} e_1 &= \delta_1 y_1^2 - \alpha_1^2 x_1^2 = 0,\\ e_2 &= \alpha_2^2 x_2^2 - \delta_2 y_1^2 = 0,\\ e_3 &= \alpha_3^2 x_1 + \alpha_3^2 x_2 = 0. \end{aligned} \tag{43}$$

If system (43) has not a continuum of solutions, then it has only the solution $(x_1, y_1, x_2) = (0, 0, 0)$, and so system (42) can not have limit cycles of class $\mathcal{C}_1$.

On the other hand, we can assume that the discontinuous piecewise differential system (40) has a crossing limit cycle of type $\mathcal{C}_2$ and let $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$ with $x_2 > x_1 > 0$ and $y_2 > y_1 > 0$, they must satisfy the following system

$$\begin{aligned} e_1 &= H_1(x_2, 0) - H_1(0, y_2) = 0,\\ e_2 &= H_2(0, y_2) - H_2(0, y_1) = 0,\\ e_3 &= H_1(0, y_1) - H_1(x_1, 0) = 0,\\ e_4 &= H_3(x_1, 0) - H_3(x_2, 0) = 0, \end{aligned} \tag{44}$$

or equivalently

$$\begin{aligned} e_1 &= -\alpha_1^2 x_2^2 + \delta_1 y_2^2 = 0,\\ e_2 &= \delta_2(y_1 + y_2) = 0,\\ e_3 &= \alpha_1^2 x_1^2 - \delta_1 y_1^2 = 0,\\ e_4 &= \alpha_3^2(x_1 + x_2) = 0. \end{aligned} \tag{45}$$

Note that if $\delta_2 = 0$ and $\alpha_3 = 0$, then system $e_1 = e_3 = 0$ has a continuum of solutions and system (40) has no limit cycles. So from $e_2 = 0$ and $e_4 = 0$ we obtain that

$$x_2 = -x_1, \quad y_2 = -y_1.$$

But observe that in this case system (45) has no solution such that $x_2 > x_1 > 0$ and $y_2 > y_1 > 0$. So system (40) can not have limit cycles of type $\mathcal{C}_2$.

This completes the proof of the proposition.

Acknowledgements

The second author is supported by the Agencia Estatal de Investigación grants PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2019. The present paper is part of the thesis of the first author.

References

[1] Andronov, A., Vitt, A., Khaikin, S.: Theory of oscillations. Pergamon Press, Oxford, 1966.
[2] Atherton, D.P.: Nonlinear Control Engineering. Van Nostrand Reinhold Co., Ltd., New York, 1982.
[3] Braga, D.C., Mello, L.F.: Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane. Nonlinear Dyn. 73, 1283–1288, (2013).
[4] Buzzi, C., Pessoa, C., Torregrosa, J.: Piecewise linear perturbations of a linear center. Discrete Contin. Dyn. Syst. 9, 3915–3936, (2013).
[5] Di Bernardo, M., Budd, C. J., Champneys, A. R., Kowalczyk, P.: Piecewise-Smooth Dynamical Systems: Theory and Applications. Applied mathematical sciences series 163. Springer, London, 2008.
[6] Filippov, A.F.: *Differential equations with discontinuous right–hand sides*, translated from Russian. Mathematics and its Applications (Soviet Series), vol. 18, Kluwer Academic Publishers Group, Dordrecht, 1988.
[7] Freire, E., Ponce, E., Torres, F.: A general mechanism to generate three limit cycles in planar Filippov systems with two zones. Nonlinear Dyn. 78, 251-263, (2014).
[8] Giannakopoulos, Pliete, K.: Planar systems of piecewise linear differential equations with a line of discontinuity. Nonlinearity 14, 1611–1632, (2001).
[9] Henson, M.A., Seborg, D.E.: Nonlinear Process Control. Prentice-Hall, New Jersey, 1997.
[10] Hilbert, D.; Mathematische Probleme, Lecture, Second Internat.Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. G"ottingen Math. Phys. KL., 253–297, (1900); English transl., Bull. Amer. Math. Soc. 8, 437–479, (1902); Bull. (New Series) Amer. Math. Soc. 37 407–436, (2000).
[11] Huan, S.M., Yang, X.S.: On the number of limit cycles in general planar piecewise linear systems. Discrete Contin. Dyn. Syst. Ser. A 32, 2147–2164, (2012).
[12] Isidori, A.: Nonlinear Control Systems. Springer, London, 1996.
[13] Katsuhiko, O.: Modern Control Engineering, 2nd edn. Prentice-Hall, Upper Saddle River, 1990.
[14] Li, L.: Three crossing limit cycles in planar piecewise linear systems with saddle-focus type. Electron. J. Qual. Theory Differ. Equ. 70, 1–14, (2014).
[15] Llibre, J., Novaes, D., Teixeira, M.A.: Maximum number of limit cycles for certain piecewise linear dynamical systems. Nonlinear Dynamics 82, 1159–1175, (2015).
[16] Llibre, J., Ordóñez, M., Ponce, E.: On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry. Nonlinear Anal. Ser. B RealWorld Appl. 14, 2002–2012, (2013).
[17] Llibre, J., Ponce, E.: Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Dyn. Contin. Discrete Impul. Syst. Ser. B 19, 325–335, (2012).

[18] Llibre, J., Teixeira, M.A.: Piecewise linear differential systems with only centers can create limit cycles?. Nonlinear Dynam. 91, 249–255, (2018).
[19] Llibre, J., Valls, C.: Limit cycles of piecewise differential systems with only linear Hamiltonian saddles. Symmetry 13, 1128, 10 pp., (2021).
[20] Llibre, J., Zhang, X.: Limit cycles created by piecewise linear centers. Chaos 29, 053116, 14 pp., (2019).
[21] Makarenkov, O., Lamb, J.S.W.: Dynamics and bifurcations of nonsmooth systems: a survey. Phys. D 241, 1826–1844, (2012).
[22] I.R. Shafarevich, I.R.: Basic Algebraic Geometry, Springer, 1974.
[23] Simpson, D.J.W.: Bifurcations in Piecewise-Smooth Continuous Systems, World Scientific Series on Nonlinear Science A, vol. 69. World Scientific, Singapore, 2010.
[24] Teixeira, M.A.: Perturbation theory for non-smooth systems. In: Robert, A. M., (ed.) Mathematics of Complexity and Dynamical Systems, vol. 1-3, pp. 1325–1336. Springer, New York, 2012.

[1] Departamento de Matemática, Universidad del Bío-Bío, Concepción, Avda. Collao 1202, Chile

Email address: `manacleto@ubiobio.cl, clvidal@ubiobio.cl`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal

Email address: `cvalls@math.ist.utl.pt`