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Let U be simply connected open subset of $\mathbb{R}^2$, and let $f : U \to \mathbb{R}^2$ be a local diffeomorphism. We study the global injectivity of f using the planar vector fields of type annular, radial or strip. Our main result enables the unification of proofs for classical results on global injectivity, such as the Hadamard global invertibility theorem and the condition related to the connectedness of the levels sets of one of the coordinates of f.

1. Introduction and statement of the main results

We provide a relationship between parallel vector fields and global injectivity of local diffeomorphisms in dimension two.

Let $U \subset \mathbb{R}^2$. We say that a vector field $\mathcal{X} : U \to \mathbb{R}^2$ is *parallel* when it is topologically equivalent to one of the vector fields:

(a) $(-y, x)$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(r) (x, y) in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(s) $(1, 0)$ in $\mathbb{R}^2$.

A parallel vector field is called *annular*, *radial* or *strip* when it is topologically equivalent to the vector field (a), (r) or (s), respectively.

If a vector field $\mathcal{X} : U \to \mathbb{R}^2$ is such that $\mathcal{X}|_{U\setminus\{z\}}$ for some $z \in U$, is annular or radial, we simply say that $\mathcal{X}$ is *annular* or *radial surrounding* z, respectively. This clearly imposes that U is simply connected.

On the other hand, let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be a C^k, $k \geq 1$, local diffeomorphism, i.e., its Jacobian determinant

$$Jf(z) = \det Df(z) = {f_1}_x(z)\, {f_2}_y(z) - {f_1}_y(z)\, {f_2}_x(z), \quad z \in U$$

is nowhere zero. By the inverse function theorem, f is a local diffeomorphism but may fail to be a global one, as for instance, $f(x, y) = (e^x \cos y, e^x \sin y)$ defined in $\mathbb{R}^2$ which is neither injective nor onto. It is an old problem to find additional conditions in order to "globalize" the inverse function theorem, i.e., in order to

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guarantee that f is a global diffeomorphism, or at least globally injective (and so a global diffeomorphism onto its image), see [20; 24].

In the polynomial case the claim that $f : \mathbb{R}^2 \to \mathbb{R}^2$ with constant nonzero Jf is injective is known as the *Jacobian conjecture in* $\mathbb{R}^2$. This is part of the famous *Jacobian conjecture*, claiming that a polynomial map $f : k^n \to k^n$, where k is a field of characteristic zero, such that its Jacobian determinant is an element of k^* is an automorphism, i.e., invertible with polynomial inverse. Both conjectures are completely open, see [13]. If k is algebraically closed (respectively $k = \mathbb{R}$), it is known that a polynomial injective map $f : k^n \to k^n$ is automatically an automorphism [10] (respectively onto [2]). In particular, the injectivity of the map is the essential question in the Jacobian conjecture (at least when k is algebraically closed) and in the Jacobian conjecture in $\mathbb{R}^2$. Back to the case when Jf is nowhere zero but not necessarily constant, the injectivity of f may fail even in the polynomial case in $\mathbb{R}^2$ [21]. Even further assuming $Jf(x) \geq 1$ for all $x \in \mathbb{R}^2$ is not enough to get the global injectivity in the polynomial case [3].

Sabatini [23] proved that a polynomial local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(0) = 0$, is globally injective if and only if a certain vector field in $\mathbb{R}^2$ (constructed from f) is annular surrounding 0, see Theorem 1 below. The same result follows independently when Jf is constant, from the reasons of Gavrilov [14] in the complex context. It turns out that in this result, the map f provides the topological equivalence of this special vector field with the one of definition (a). Our main results generalize the results of Gavrilov [14] and Sabatini [23] to the three types of parallel vector fields.

We begin by recalling the known result and then introducing the new ones in order to compare them.

From now on, if we do not say the converse, we always assume that the open set U will be simply connected. Let $H : U \to \mathbb{R}$ be a C^r function. The C^{r-1} *Hamiltonian vector field associated to* H is the vector field $\nabla H^{\perp} : U \to \mathbb{R}^2$ defined by

$$\nabla H^{\perp}(z) = (-\partial_2 H(z), \partial_1 H(z)), \quad z \in U.$$

We recall that a vector field $\mathcal{X} : U \to \mathbb{R}^2$ is called a *global center* when there exists $z \in U$ such that $\mathcal{X}$ is annular surrounding z.

Theorem 1 [14; 23]. *Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nonvanishing Jf and such that $f(0) = 0$. Then the following properties are equivalent*:

(i) *The origin is a global center for $\nabla H^{\perp}$, where $H = \frac{1}{2}(f_1^2 + f_2^2)$.*

(ii) *f is a global diffeomorphism of the plane onto itself.*

This theorem is also true for more general local diffeomorphisms, see [4].

The main result of this paper is Theorem 2 right below. In order to state it, for a given local diffeomorphism $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and a vector field $\mathcal{X} : V \subset \mathbb{R}^2 \to \mathbb{R}^2$, with $f(U) \subset V$, we define in U the *pullback* of $\mathcal{X}$ by f, denoted by $\mathcal{Y}_{\mathcal{X}}$, by

$$\mathcal{Y}_{\mathcal{X}}(z) = Df(z)^{-1}\mathcal{X}(f(z)), \quad z \in U. \tag{1}$$

Theorem 2. *Let U be a simply connected open subset of $\mathbb{R}^2$. Let $f : U \to \mathbb{R}^2$ be a C^2 local diffeomorphism and $\mathcal{X} : f(U) \to \mathbb{R}^2$ be a C^1 vector field which is annular, radial or strip, surrounding a singular point in the first and second cases. Then the following two conditions are equivalent*:

(i) *The vector field $\mathcal{Y}_{\mathcal{X}}$ is topologically equivalent to the vector field $\mathcal{X}$.*

(ii) *The map f is a global diffeomorphism onto $f(U)$.*

Remark 3. For a given local diffeomorphism $f : U \to \mathbb{R}^2$, let $G : V \to \mathbb{R}$, with $f(U) \subset V$, be a given C^2 function. If $H : U \to \mathbb{R}$ is defined by $H = G \circ f$, then

$$\begin{aligned} \nabla H^{\perp}(z) &= \begin{pmatrix} f_{2y}(z) & -f_{1y}(z) \\ -f_{2x}(z) & f_{1x}(z) \end{pmatrix} \nabla G^{\perp}(f(z)) \\ &= Jf(z)\, Df(z)^{-1} \nabla G^{\perp}(f(z)) = Jf(z)\, \mathcal{Y}_{\nabla G^{\perp}}(z), \end{aligned}$$

That is, $\mathcal{Y}_{\nabla G^{\perp}}$ is a multiple (by a nowhere zero function) of $\nabla H^{\perp}$, and hence these two vectors fields are topologically equivalent.

As a consequence of this remark and Theorem 2, we get:

Corollary 4. *Let U be a simply connected open subset of $\mathbb{R}^2$. Let $f : U \to \mathbb{R}^2$ be a C^2 local diffeomorphism and $G : f(U) \to \mathbb{R}$ be a C^2 function. Assume that the Hamiltonian vector field $\nabla G^{\perp}$ is annular or strip, surrounding a singular point in the first case, and let $H = G \circ f$. Then the following two conditions are equivalent*:

(i) *The Hamiltonian vector fields $\nabla H^{\perp}$ and $\nabla G^{\perp}$ are topologically equivalent.*

(ii) *f is a global diffeomorphism onto $f(U)$.*

We do not have the radial case here as it is clear that if it surrounds a singular point it cannot be a Hamiltonian vector field.

If we want to prove that some local diffeomorphism is globally injective, the following two results are easier to apply, because they do not depend on the knowledge of the set $f(U)$ a priori.

Theorem 5. *Let U be a simply connected open subset of $\mathbb{R}^2$. Let $f : U \to \mathbb{R}^2$ be a local diffeomorphism and $\mathcal{X} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 annular, radial or strip vector field, surrounding a singular point $f(z)$ in the first and second cases. If $\mathcal{Y}_{\mathcal{X}}$ is topologically equivalent to $\mathcal{X}$, then f is globally injective.*

Remark 6. The reciprocal result may be false in general, see Example 17 below, but it is clearly true if one assumes $f(U) = \mathbb{R}^2$ (by Theorem 2). So in the polynomial case $f : \mathbb{R}^2 \to \mathbb{R}^2$, the reciprocal is also true, because an injective polynomial map $\mathbb{R}^n \to \mathbb{R}^n$ is automatically onto, according to [2].

In the specific case of $\mathcal{X}$ being the linear center $(-y, x)$, the reciprocal is also true under the additional assumption that $f(U)$ is a disc centered at the origin, according to [4].

Next corollary is Theorem 5 for Hamiltonian parallel vector fields.

Corollary 7. *Let U be a simply connected open subset of $\mathbb{R}^2$. Let $f : U \to \mathbb{R}^2$ be a C^2 local diffeomorphism and $G : \mathbb{R}^2 \to \mathbb{R}$ be a C^2. Assume that the Hamiltonian vector field $\nabla G^\perp$ is annular or strip, surrounding a singular point $f(z)$ in the first case, and let $H = G \circ f$. If $\nabla H^\perp$ is topologically equivalent to $\nabla G^\perp$, then f is a global diffeomorphism.*

Corollary 7 in the case $G(x, y) = \frac{1}{2}(x^2 + y^2)$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial local diffeomorphism with $f(0) = 0$ recovers Gavrilov and Sabatini's Theorem 1 (for the reciprocal, use Remark 6).

Corollary 7 in case $G(x, y) = x$ recovers the widely known tool used in order to guarantee the global injectivity of a local diffeomorphism

$$f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2,$$

that is, f is injective if the level sets of f_1 are connected. This is so because $\nabla G^\perp = (0, -1)$ is strip, and the Hamiltonian vector field $\nabla H^\perp = \nabla f_1^\perp$ defined in the simply connected set U is strip if and only if the level sets of $H = f_1$ are connected, see Lemma 11 below.

We remark that in the annular and radial cases, the assumption that they surround a singular point in the domain of f cannot be dropped, see, for instance, Example 16.

We now state the known global invertibility Hadamard Theorem 8. Recall that a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *proper* if $f^{-1}(K)$ is compact for all K compact.

Theorem 8. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that Jf is nowhere zero. Then f is a global diffeomorphism if and only if f is proper.*

In Section 3 we provide a proof of Theorem 8 in $\mathbb{R}^2$ by using Theorem 2. We observe that Sabatini [23] used Theorem 8 in order to prove Theorem 1. In [4], an extension of Theorem 1 was proved without the use of Theorem 8. In the present paper, besides not using this theorem in our arguments, it turns out that it is a consequence of them.

Sabatini [22] proved an extension of Theorem 8 by exploring the dynamics of $\nabla f_1^\perp$, so his proof is related to the strip case of Theorem 2 when $\mathcal{X} = (0, -1)$, as commented above. This is different from our proof, which uses the annular case of Theorem 2.

As a final comment in this section, we mention Cima, Gasull and Mañosas' paper [8], that uses index theory to get conditions ensuring the global injectivity of local polynomial self-diffeomorphisms of $\mathbb{R}^n$. Our approach is quite different from theirs as we rely on qualitative theory of planar differential systems. Anyway, in Example 15, we trivially conclude the injectivity of a suitable locally invertible polynomial map, which cannot be directly obtained from the results of [8].

The paper is organized as follows: in Section 2 we gather some preliminary results. In Section 3 we deliver the proof of our results. In Section 4 we provide examples of applications of our results as well as some counterexamples.

2. Preliminaries

In this section we recall some notions that we shall use in our proofs.

We say that two vector fields $\mathcal{Z} : U \to \mathbb{R}^2$ and $\mathcal{W} : V \to \mathbb{R}^2$ are *topologically equivalent* when there exists a homeomorphism $h : U \to V$ carrying orbits of $\mathcal{Z}$ onto orbits of $\mathcal{W}$ preserving (reversing) the orientation of all the orbits. Let $\phi(t, x)$ and $\psi(t, x)$ be the flows of $\mathcal{Z}$ and $\mathcal{W}$, respectively. We say that $\mathcal{Z}$ and $\mathcal{W}$ are C^r *conjugated*, $r \geq 0$, when there exists a C^r homeomorphism $h : U \to V$ such that

$$h(\phi(t, z)) = \psi(t, h(z))$$

for all (t, z) in the domain of ϕ. That is, a conjugation not only carries orbits to orbits but also preserves the time. Two vector fields $\mathcal{Z}$ and $\mathcal{W}$ are C^r, $r \geq 1$, conjugated if and only if there exists a C^r diffeomorphism $h : U \to V$ such that

$$Dh(z)\,\mathcal{Z}(z) = \mathcal{W}(h(z)) \tag{2}$$

for every $z \in U$, see [11, Lemma 1.11], that is, $\mathcal{Z} = \mathcal{Y}_{\mathcal{W}}$ is the pullback of $\mathcal{W}$ by h.

For a vector field $\mathcal{Z}$ defined in an open set U of $\mathbb{R}^2$ a *separatrix* is a special orbit according to the definitions in the works of Markus [17] and Neumann [19], with a correction provided by Espín Buendía and Jiménez López [12]. The set of all separatrices is closed and each open connected component of its complement in U is called a *canonical region*. Moreover the vector field restricted to a canonical region is parallel. For more details, see [12; 17; 19] .

In the special case when the vector field $\mathcal{Z}$ has no singular points, it defines a foliation in U whose *leaves* are the orbits of $\mathcal{Z}$. Two leaves L_1 and L_2 of $\mathcal{X}$ are called *inseparable* if for any cross sections C_1 and C_2 through L_1 and L_2, respectively, there exists a leave L_3 crossing C_1 and C_2. For a foliation the set of separatrices is precisely the closure of the set of inseparable leaves.

Lemma 9. *Let $U \subset \mathbb{R}^2$ be an open set. Let $f : U \to \mathbb{R}^2$ be a C^2 local diffeomorphism and $\mathcal{X}$ be a C^1 vector field defined in an open set $V \subset \mathbb{R}^2$ such that $f(U) \subset V$. Let $\mathcal{Y}_{\mathcal{X}}$ be the pullback of $\mathcal{X}$ defined in (1).*

(i) *A point $z \in U$ is a singular point of $\mathcal{Y}_{\mathcal{X}}$ if and only if $f(z)$ is a singular point of $\mathcal{X}$.*

(ii) *If γ is an orbit of $\mathcal{Y}_{\mathcal{X}}$ then $(f \circ \gamma)'(t) = \mathcal{X}(f \circ \gamma(t))$ for all t in the maximal interval of the definition of the solution γ, in particular $f \circ \gamma$ is contained in an orbit of $\mathcal{X}$.*

(iii) *For each transversal section S of $\mathcal{Y}_{\mathcal{X}}$ through a point $p \in U$, there exists a neighborhood U_p of p such that $f(S \cap U_p)$ is a transversal section of $\mathcal{X}$ at $f(p)$.*

(iv) *If $G : V \to \mathbb{R}$ is a first integral of $\mathcal{X}$, then $H = G \circ f$ is a first integral of $\mathcal{Y}_{\mathcal{X}}$. A point $z \in U$ is a critical point of H if and only if $f(z)$ is a critical point of G.*

Proof. Statement (i) follows straightforwardly from the definition of $\mathcal{Y}_{\mathcal{X}}$. Further, for an orbit γ of $\mathcal{Y}_{\mathcal{X}}$, it follows that, for each t in the maximal interval of the definition of the solution γ,

$$(f \circ \gamma)'(t) = Df(\gamma(t))\, \gamma'(t) = Df(\gamma(t))\, \mathcal{Y}_{\mathcal{X}}(\gamma(t)) = \mathcal{X}\big(f(\gamma(t))\big),$$

proving statement (ii). Statement (iii) follows because f is a local diffeomorphism.

Finally, the function H of statement (iv) is not constant in any open set because f is open. It is constant along orbits of $\mathcal{Y}_X$ from statement (ii). Then the chain rule finishes the proof of statement (iv). □

It is clear that if f is a diffeomorphism onto V, then $f \circ \gamma$ is an orbit of $\mathcal{X}$, because $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{X}}$ are conjugated. This may be not the case if f is only a local diffeomorphism, as illustrated in the following example.

Example 10. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonglobally injective local diffeomorphism, and consider the strip vector field $\mathcal{X} = (0, 1)$ defined in $\mathbb{R}^2$: its orbits are the vertical lines. The orbits of $\mathcal{Y}_{\mathcal{X}}$ are the connected components of the level sets of f_1.

Since f is not injective, there must exist nonconnected level sets of f_1, by using the mentioned tool in order to guarantee global injectivity recalled at the introduction section. So there will exist at least two inseparable leaves γ_1 and γ_2 of $\mathcal{Y}_{\mathcal{X}}$, see [18, Theorem 1] or [6, Proposition A]. Since they are in the same level set of f_1, it follows that $f(\gamma_1)$ and $f(\gamma_2)$ are contained in the same vertical line.

Now, the function f_2 is monotone along each orbit of $\mathcal{Y}_{\mathcal{X}}$, because Jf is nowhere zero. So, from the inseparability of γ_1 and γ_2, it follows that $f_2(\gamma_1) \cap f_2(\gamma_2)$ must be empty. In particular, neither $f(\gamma_1)$ nor $f(\gamma_2)$ can be an entire vertical line.

Observe that this happens independently whether f is onto $\mathbb{R}^2$ or not.

We recall that any parallel vector field $\mathcal{X}$ of class C^k, $k = 1, \ldots, \infty$, has a first integral of class C^k, see [16], having no critical points. Precisely, since the vector fields given in (a), (r) and (s) have the first integrals $x^2 + y^2$, $\arctan y/x$

and y, defined in $\mathbb{R}^2 \setminus \{0\}$, $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^2$, respectively, we can define a *canonical* first integral G by composing the previous functions with T^{-1}, where T is the C^k diffeomorphism, according to [16], transforming (a), (r) and (s) onto $\mathcal{X}$, respectively.

Lemma 11. *Let $\mathcal{X} : U \to \mathbb{R}^2$ be a C^1 vector field and $F : U \to \mathbb{R}$ be a C^2 first integral of $\mathcal{X}$ having no critical points. If $\mathcal{X}$ is strip* (*respectively annular*), *then the level sets of F are connected. Reciprocally, assuming that $\mathcal{X}$ has no singular points and U is simply connected, if the level sets of F are connected then $\mathcal{X}$ is strip.*

Proof. Let $T_a : \mathbb{R}^2 \setminus \{(0,0)\} \to U \setminus \{z\}$ and $T_s : \mathbb{R}^2 \to U$ be the C^1 [16] topological equivalences between $\mathcal{X}$ and the vector fields defined in (a) and (s), respectively. The connected components of $(F \circ T_a)^{-1}(c)$ and $(F \circ T_s)^{-1}(c)$ with $c \in F(U)$ are circles centered at $(0,0)$ and straight horizontal lines, respectively. Since $F \circ T_a$ and $F \circ T_s$ have no critical points, the first one is monotone along any radius, while the second one is so for any vertical segment. In particular, the sets $(F \circ T_a)^{-1}(c)$ and $(F \circ T_a)^{-1}(c)$, $c \in F(U)$, are connected. Then the levels sets of F in each case must be connected.

On the other hand, assume that $\mathcal{X}$ has no singular points and F has no critical points. It is clear that $\mathcal{X}$ and $\nabla F^{\perp}$ are topologically equivalent. Assume that the level sets of F are connected. If $\mathcal{X}$ is not strip, and so $\nabla F^{\perp}$ is not strip, then a pair of inseparable leaves of the foliation of U given by F must exist, see [18, Theorem 1] or [6, Proposition A]. But clearly two inseparable leaves are in the same level set of F, a contradiction. □

Even if U is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, the assumption that the level sets of F are connected is not enough to conclude that $\mathcal{X}$ is annular. For instance, take $\mathcal{X} = \nabla F^{\perp}$ with $F(x, y) = x^2 + y^2$ defined in $U = \mathbb{R}^2 \setminus \{(t, 0) \mid t = 0 \text{ or } t \geq 1\}$, which is diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$ from the Riemann open mapping theorem [9].

3. Proof of the main theorem

The proof that (ii) implies (i) in Theorem 2 is immediate.

Proof that (i) *implies* (ii) *in Theorem 2, proof of Theorem 5.* Given γ_1 and γ_2 two distinct nonsingular orbits of $\mathcal{Y}_{\mathcal{X}}$, we claim that the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are contained in different orbits of $\mathcal{X}$. In particular, taking into account (i) of Lemma 9 for the annular and radial cases, it will be enough to prove the injectivity of f over each nonsingular orbit of $\mathcal{Y}_{\mathcal{X}}$.

For the strip (respectively annular) case, we let G be a canonical first integral of $\mathcal{X}$ (respectively of $\mathcal{X}$ restricted to $f(U) \setminus \{f(z)\}$ in the proof of Theorem 2 or $\mathbb{R}^2 \setminus \{f(z)\}$ for the proof of Theorem 5) and define $H = G \circ f$ in U (respectively in $U \setminus \{z\}$). From (iv) of Lemma 9 it follows that H is a first integral of $\mathcal{Y}_{\mathcal{X}}$ (respectively of $\mathcal{Y}_{\mathcal{X}}$ restricted to $U \setminus \{z\}$) without critical points. Then if $f(\gamma_1)$ and $f(\gamma_2)$ are contained in the same orbit of $\mathcal{X}$, it follows in particular that $G \circ f(\gamma_1) = G \circ f(\gamma_2)$. But then γ_1 and γ_2 will be in the same level set of H, which is connected from Lemma 11, so $\gamma_1 = \gamma_2$.

For the radial case we let U_z be a neighborhood of z such that $f|_{U_z} : U_z \to f(U_z)$ is a diffeomorphism. Since all the orbits in a radial vector surrounding a singular point z has in their boundary the point z, it follows that $f(\gamma_1 \cap U_z)$ and $f(\gamma_2 \cap U_z)$ are in distinct orbits of $\mathcal{X}$. Then the claim follows from (ii) of Lemma 9.

Now we prove the injectivity of f over each nonsingular orbit of $\mathcal{Y}_{\mathcal{X}}$. Since the nonsingular orbits of a strip or radial vector field are homeomorphic to $\mathbb{R}$, it follows from (ii) of Lemma 9 that f is injective over each orbit of $\mathcal{Y}_{\mathcal{X}}$ for the strip and radial cases.

In the annular case the proof is not so direct. This is so because if γ is an orbit of $\mathcal{Y}_{\mathcal{X}}$, it may happen that $f \circ \gamma$ rounds more than once over the closed curve $f(\gamma)$. As we shall see right below, this is not possible because, by continuity, we can prove that this would necessarily happen so close to the singular point z as we want, a contradiction with the fact that in a neighborhood U_z of z the map $f|_{U_z} : U_z \to f(U_z)$ is a diffeomorphism.

Indeed, we first parametrize the nonsingular orbits of the annular vector field $\mathcal{Y}_{\mathcal{X}}|_{U\setminus\{z\}}$ by the levels of a canonical first integral H defined in $U \setminus \{z\}$ such that for each $h \in H(U)$, $h > 0$, the level set $H^{-1}(h)$ is a periodic orbit of $\mathcal{Y}_{\mathcal{X}}$, denoted by γ_h, according to Lemma 11 and the discussion right before it. Moreover, we have that $0 < h_1 < h_2$ if and only if the curve γ_{h_1} is contained in the bounded region limited by γ_{h_2}. This comes from the monotonicity of the function $x^2 + y^2$ along rays.

We consider the set

$$T = \{h \in H(U),\ h > 0 \mid f \text{ is not injective in } \gamma_h\}.$$

It is enough to prove that T is empty.

Suppose on the contrary that T is not empty an let h_α be the infimum of T. Since f is locally injective in z, it follows that $h_\alpha > 0$.

We *claim that $h_\alpha \notin T$*. Indeed, if on the contrary there exist $p, q \in \gamma_{h_\alpha}$ with $p \neq q$ and $f(p) = f(q)$, we consider neighborhoods U_p, U_q and V of p, q and $f(p)$, respectively, with $U_p \cap U_q = \varnothing$, such that the maps $f|_{U_p} : U_p \to V$ and $f|_{U_q} : U_q \to V$ are diffeomorphisms. Let C be the intersection of a transversal section to the flow of $\mathcal{X}$ connecting $f(z)$ to $f(p)$ with the open set V, and we define the curves $C_p = f|_{U_p}^{-1}(C)$ and $C_q = f|_{U_q}^{-1}(C)$. The curves C_p and C_q are contained in the compact region bounded by the curve γ_{h_α}. In particular, for $h < h_\alpha$ close enough to h_α, the orbit γ_h will cut C_p and C_q. But then $f(C_p \cap \gamma_h) = f(C_q \cap \gamma_h)$, and hence f is not injective in γ_h. So $h \in T$, a contradiction with the fact that $h_\alpha = \inf T$. This proves the claim.

Now from the definition of h_α, there exist sequences $\{h_n\}$, $h_n > h_\alpha$ and $h_n \to h_\alpha$, $\{p_n\}$ and $\{q_n\}$ in γ_{h_n}, such that $p_n \neq q_n$, $f(p_n) = f(q_n)$ and $p_n \to p$, $q_n \to q$, $p, q \in \gamma_{h_\alpha}$. Clearly $f(p) = f(q)$, following from the above claim that $p = q$. But as f is locally injective in p, we obtain a contradiction with $p_n \neq q_n$, $f(p_n) = f(q_n)$, and $p_n \to p$ and $q_n \to q$. This contradiction proves that T is empty finishing the proof. □

Proof of Hadamard's result in $\mathbb{R}^2$. We assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is proper. Since f is a local diffeomorphism it follows that its image is an open set, that must be the whole $\mathbb{R}^2$ because f is proper, otherwise we can take a convergent sequence $f(x_n)$ to a boundary point of $f(U)$, and this will give a contradiction. In particular, there exists a point $z_0 \in \mathbb{R}^2$ such that $f(z_0) = 0$.

Now we consider the annular vector field $\mathcal{X}(x, y) = (-y, x)$ defined in $\mathbb{R}^2$, surrounding the singular point 0.

In order to finish the proof, we will show that $\mathcal{Y}_X$ is also annular surrounding the singular point z_0, according to Lemma 9, because then $\mathcal{X}$ and $\mathcal{Y}_\mathcal{X}$ will be topologically equivalent and the result will follow from Theorem 2.

The singular points of $\mathcal{Y}_\mathcal{X}$ are the preimages of 0 by f, by Lemma 9. Since f is a local diffeomorphism, it follows that they are all centers. On the other hand, each orbit of $\mathcal{Y}_\mathcal{X}$, which is not a singular point, is contained in the preimage of a circle $x^2 + y^2 = r$, which is a compact set, by the properness of f. Then by the Poincaré–Bendixson theorem, it follows that such an orbit must be periodic. That is, all the orbits different from singular points are periodic and the singular points are centers.

We claim that $\mathcal{Y}_\mathcal{X}$ has a unique center. Indeed, if there are more than one center, there must be a separatrix γ, which is forced to be a periodic orbit, in the boundary of the period annulus of one center. But this is a contradiction, because by using continuous dependence on the initial conditions, γ will be inside the period annulus of the center. □

4. Examples

Let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be a C^2 map such that Jf is nowhere zero and such that there is $z \in \mathbb{R}^2$ such that $f(z) = 0$. Below we provide the explicit expressions of $\mathcal{Y}_\mathcal{X}$ for the canonical annular, radial and strip vector fields, surrounding the 0 in the first and second cases.

Let the canonical annular vector field $\mathcal{X}(x, y) = (-y, x)$, which surrounds the 0. From the definition of $\mathcal{Y}_\mathcal{X}$ we get

$$Jf\,\mathcal{Y}_\mathcal{X} = (-f_1 f_{1y} - f_2 f_{2y}, f_1 f_{1x} + f_2 f_{2x}) = \nabla H^\perp, \tag{3}$$

where $H(x, y) = \frac{1}{2}\big(f_1(x, y)^2 + f_2(x, y)^2\big)$.

Let the canonical radial vector field $\mathcal{X}(x,y)=(x,y)$, which surrounds the 0. From the definition of $\mathcal{Y}_{\mathcal{X}}$ we get

$$Jf\,\mathcal{Y}_{\mathcal{X}}=(f_1\,f_{2y}-f_2\,f_{1y},-f_1\,f_{2x}+f_2\,f_{1x}). \tag{4}$$

Let the canonical strip vector field $\mathcal{X}(x,y)=(1,0)$. From the definition of $\mathcal{Y}_{\mathcal{X}}$ we get

$$Jf\,\mathcal{Y}_{\mathcal{X}}=(f_{2y},-f_{2x})=-\nabla f_2^{\perp}(x,y). \tag{5}$$

In what follows we will freely use the *Poincaré compactification* with the usual notations, as for instance, [11, Chapter 5].

Example 12. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$f=(f_1,f_2)=(x+p_1x^3+p_2x^2y-p_3xy^2,\,y+p_1x^2y+p_2xy^2-p_3y^3).$$

Observe that $f(0)=0$. We have

$$Jf(x,y)=(1+3p_1x^2+3p_2xy-3p_3y^2)(1+p_1x^2+p_2xy-p_3y^2),$$

which does not vanish if and only if either $p_3<0$ and $p_2^2+4p_1p_3<0$, or $p_2=p_3=0$ and $p_1\geq 0$. So under each of these conditions, f is a local diffeomorphism.

We now compare the use of the three parallel canonical systems for proving that f is a global diffeomorphism.

We begin by the canonical radial vector field $\mathcal{X}(x,y)=(x,y)$, that surrounds the 0. From (4) we get

$$Jf(x,y)\,\mathcal{Y}_{\mathcal{X}}(x,y)=(1+p_1x^2+p_2xy-p_3y^2)^2(x,y),$$

which is radial surrounding the 0 because $1+p_1x^2+p_2xy-p_3y^2>0$. Thus, it follows by Theorem 5 that f is a global diffeomorphism (it is globally injective and then, since it is polynomial, it is onto).

Now by considering $\mathcal{X}$ the canonical annular system surrounding the 0, we have from (3) that $Jf(x,y)\,\mathcal{Y}_{\mathcal{X}}(x,y)=\nabla H^{\perp}(x,y)$, where

$$H(x,y)=\tfrac{1}{2}(x^2+y^2)(1+p_1x^2+p_2xy-p_3y^2)^2.$$

Although we know that this is an annular system surrounding the 0 (by using the global injectivity of f), it is not trivial to conclude this by the expression of $\mathcal{Y}_{\mathcal{X}}$ or H: In case $p_3<0$ and $p_2^2+4p_1p_3<0$, it follows that the higher homogeneous part of $H(x,y)$, namely $(x^2+y^2)(p_1x^2+p_2xy-p_3y^2)$ annihilates only at $(0,0)$, so that $\nabla H^{\perp}$ has no infinite singular points. Since $\nabla H^{\perp}$ has a center at $(0,0)$ it follows (by the Poincaré–Bendixson theorem, see, for instance, [11, Chapter 1]) that this center is global. Now in case $p_2=p_3=0$ and $p_1>0$, we get that for all $c > 0$ the equation $H(x, y) = c$ is $y^2(1 + p_1 x^2)^2 = c - x^2(1 + p_1 x^2)$, which is clearly a closed curve, and so we get a global center.

Finally, let $\mathcal{X}(x, y) = (1, 0)$ be the canonical strip vector field. From (5) we get $\mathcal{Y}_{\mathcal{X}}(x, y) = -\nabla f_2^{\perp}(x, y)$. Again, this vector field is strip, but it is not trivial to conclude this by using the expression of $\mathcal{Y}_{\mathcal{X}}$. Indeed, we have to prove that the level sets of f_2 are connected. In case $p_3 < 0$ and $p_2^2 + 4 p_1 p_3 < 0$, the higher homogeneous part of f_2 is $y(p_1 x^2 + p_2 xy - p_3 y^2)$. So the only infinite singular point of $\nabla f_2^{\perp}$ is in the direction $y = 0$. Since the factor y appears with exponent 1 in the higher homogeneous part of f_2, it follows that $\nabla f_2^{\perp}$ has no infinite nondegenerate hyperbolic sectors, according to [7, Theorem 2]. Then $\nabla f_2^{\perp}$ has no inseparable leaves (according to [15, Section 2], for instance) and then all the level sets of f_2 are connected (as in Example 10). In case $p_2 = p_3 = 0$ and $p_1 > 0$, it trivially follows that f_2 has connected level sets.

Example 13. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a local polynomial diffeomorphism and $z \in \mathbb{R}^2$ such that $f(z) = 0$.

By considering the canonical annular vector field $\mathcal{X} = (-y, x)$ it is possible to find dynamical conditions on the infinite singular points of $Jf\,\mathcal{Y}_{\mathcal{X}}$ in order to guarantee that it is topologically equivalent to $\mathcal{X}$ and so f is a global diffeomorphism. This was done in [5], for instance, by using the results of Gavrilov and Sabatini, now generalized in the present paper. The condition is the following: the vector field $Jf\,\mathcal{Y}_{\mathcal{X}}$ will have a global center at z, and so it will be topological equivalent to $\mathcal{X}$, if and only if each infinite singular point is formed by two *degenerate* hyperbolic sectors, that is, the separatrices of the hyperbolic sectors are the same and contained in the infinite line. The case when $Jf\,\mathcal{Y}_{\mathcal{X}}$ has no infinite singular points is of course included here. In [5] the higher homogeneous parts of suitable polynomials in f_1, f_2, f_{1x}, f_{1y}, f_{2x} and f_{2y} were considered to deliver a sufficient condition for this.

A similar reason can be done by using the canonical radial vector field surrounding the 0, $\mathcal{X} = (x, y)$. We claim that: *if each infinite singular point of $Jf\,\mathcal{Y}_{\mathcal{X}}$ is formed by two degenerate hyperbolic sectors then $Jf\,\mathcal{Y}_{\mathcal{X}}$ must be radial, and so topologically equivalent to $\mathcal{X}$*. Indeed, in this case none of the orbits noncontained at infinity can have α or ω limit being a point at infinity. Moreover, any point w such that $f(w) = 0$ is a node or a focus of $\mathcal{Y}_{\mathcal{X}}$, and has index 1, because f is a local diffeomorphism. In particular, the only such point is z, because the sum of the indices in the Poincaré sphere must be 2. Then, from the Poincaré–Bendixson theorem, it follows that the α and ω limit sets of a given finite orbit different from z is z or the infinite (which is a graphic). Hence $Jf\,\mathcal{Y}_{\mathcal{X}}$ is radial.

But now this is not a necessary condition for $Jf\,\mathcal{Y}_{\mathcal{X}}$ to be radial: for instance, when f is the identity we have $Jf\,\mathcal{Y}_{\mathcal{X}} = \mathcal{X}$, whose infinite singular points are nodes. See also Examples 12 and 14.

Moreover, here we cannot impose that $Jf\,\mathcal{Y}_{\mathcal{X}}$ has no infinite singular points, because it always has. Indeed, the infinite singular points are given by the linear factors of the homogeneous polynomial

$$y(f_1 f_{2y} - f_2 f_{1y})_k - x(-f_1 f_{2x} + f_2 f_{1x})_k,$$

where $k = \max\{\deg(f_1 f_{2y} - f_2 f_{1y}),\ \deg(-f_1 f_{2x} + f_2 f_{1x})\}$ and, for a polynomial $h : \mathbb{R}^2 \to \mathbb{R}$, we are denoting by h_k its homogeneous part of degree k. But from the Euler theorem for homogeneous polynomial the above expression is $(m-n)\, f_{1n} f_{2m}$, where $n = \deg f_1$ and $m = \deg f_2$ (so if $m \neq n$, then $k = m+n-1$). Since the vector field $\nabla f_1^{\perp}$ has no finite singular points, it will necessarily have infinite ones by the Poincaré–Hopf theorem. These points are the linear factors of f_{1n}. In particular, if $m \neq n$, $(m-n)\, f_{1n} f_{2m}$ has linear factors, and so the vector field $Jf\,\mathcal{Y}_{\mathcal{X}}$ always has infinite singular points. If $m = n$, then all the points at infinity are singular points.

Example 14. Let $f = (f_1, f_2) = (x + x^3 + y^3, y)$.

Here the global injectivity of f follows trivially by using the strip canonical vector field $\mathcal{X}$. The vector field $\mathcal{Y}_{\mathcal{X}}$ is strip because the level sets of y are connected.

By using the radial vector field $\mathcal{X} = (x, y)$, we get from (4) that

$$Jf\,\mathcal{Y}_{\mathcal{X}} = (x + x^3 - 2y^3,\ y + 3x^2 y).$$

It is simple to conclude the following: the unique finite singular point of $\mathcal{Y}_{\mathcal{X}}$ is the origin, which is a hyperbolic unstable node. Also, the singular points at infinite are, in the local chart U_1, the origin, that is a hyperbolic saddle, and the point $(-1, 0)$, which is a hyperbolic stable node. Finally, $\mathcal{Y}_{\mathcal{X}}$ does not have a periodic orbit otherwise, from (ii) of Lemma 9, $\mathcal{X}$ itself would have a periodic nonsingular orbit. These facts forces that all the orbits of $\mathcal{Y}_{\mathcal{X}}$ have ω-limit at infinity and α-limit being the origin. So $\mathcal{Y}_{\mathcal{X}}$ is radial surrounding 0.

Finally, by taking the canonical annular vector field $\mathcal{X} = (-y, x)$, we get from (3)

$$Jf\,\mathcal{Y}_{\mathcal{X}} = \big(-y(1 + 3xy + 3x^3 y + 3y^4),\ (1 + 3x^2)(x + x^3 + y^3)\big).$$

Here the unique finite singular point is the origin, which is a center (because f is a local diffeomorphism). The infinity has only one pair of equilibrium, which in the chart U_1 has coordinates $(-1, 0)$. It turns out that this equilibrium is degenerate and we have to apply blow-ups in order to study its local phase portrait. We follow the notation of [1]. After translating this equilibrium to the origin of U_1, we see that both horizontal and vertical directions are characteristic ones. We apply an odd x-directional blow up, obtaining a saddle at the origin, with separatrices being the axes, the stable being the vertical one, and no other equilibria in $x = 0$. Now after an odd y-directional blow up we get a degenerate equilibrium at the origin and no other equilibria in $y = 0$. The only characteristic direction for this point is the horizontal axis, and after an odd x-directional blow up we get a saddle at the origin just as the one found right above, and no other equilibria in $x = 0$. By blowing down this process, it is simple to conclude that the only equilibrium in the chart U_1 is formed by two degenerate hyperbolic sectors. Then, as mentioned in Example 13, $\mathcal{Y}_{\mathcal{X}}$ is annular in $\mathbb{R}^2$ surrounding 0, equivalently the center at the origin is global.

In this example, the sufficient condition for global injectivity given in [5], which comes from the results of [7], is not valid.

Example 15. Let $f(x, y) = (y + y^3, x + xy^2)$. This map is from [8]. It is an example that does not satisfy the sufficient condition given by [8, Theorem A], a result that ensures global injectivity of locally invertible polynomial maps.

We show here that by using either the canonical strip or the canonical radial vector field, it is immediate to obtain the injectivity of f throughout our main result. We have $f(0) = 0$ and $Jf(x, y) = -(1+4y^2+3y^4)$, which is everywhere negative.

Since the level sets of f_1 are connected, it follows that f is injective. As commented in the introduction section, this comes from our main result by considering the strip canonical vector field $\mathcal{X} = (0, 1)$, so that $Jf\,\mathcal{Y}_{\mathcal{X}} = -(1+3y^2)(0, 1)$, which is clearly strip.

By using the radial canonical vector field $\mathcal{X} = (x, y)$, it follows from (4) that $Jf\,\mathcal{Y}_{\mathcal{X}} = -(1 + y^2)^2(x, y)$, which is radial.

Example 16. Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ be the local diffeomorphism defined in polar coordinates as $(r, \theta) \mapsto (r, 2\theta)$, which is a double recovering of $\mathbb{R}^2 \setminus \{0\}$. In cartesian coordinates f has the expression

$$f(x, y) = \sqrt{x^2 + y^2}\left(\cos\left(2 \arctan \frac{y}{x}\right), \sin\left(2 \arctan \frac{y}{x}\right)\right).$$

This map is analytic in $\mathbb{R}^2\setminus\{0\}$ because we can use $\arccos(x/\sqrt{x^2 + y^2})$ (respectively $\arccos(x/\sqrt{x^2 + y^2} + \pi)$ for the angle for x close to zero and $y > 0$ (respectively $y < 0$). Here $Jf = 2$.

It is straightforward to see that if $\mathcal{X} = 2(-y, x)$, then $\mathcal{Y}_{\mathcal{X}} = (-y, x)$. Also if $\mathcal{X} = (x, y)$, then $\mathcal{Y}_{\mathcal{X}} = (x, y)$. Therefore, in both cases, $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{X}}$ are topologically equivalent, but f is not injective.

So this example shows the necessity of U being simply connected in Theorem 2.

Example 17. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the injective local diffeomorphism given by $f(x, y) = (\arctan x, y)$. The pullback $\mathcal{Y}_{\mathcal{X}}$ of the linear center $\mathcal{X}(x, y) = (-y, x)$ is not a center, because the orbits passing through points (x, y) with $|y| \geq \frac{\pi}{2}$ are not bounded. Indeed, such an orbit is the preimage by f of the periodic curve $(r \cos t, r \sin t)$, $t \in \mathbb{R}$, with $r \geq \frac{\pi}{2}$. It is so one of the curves $(\tan(r \cos t), r \sin t)$, with $|\cos t| < 1$.

This example shows that the reciprocal of Theorem 5 is false in general.

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