The transition between two ecological limit cycles in one predator-two competitive prey model

*Víctor Castellanos, [a]Eduardo Chan-López, and [b]Jaume Llibre

[a]División Académica de Ciencias Básicas, Universidad Juárez Autónoma de Tabasco
Km.1, Carretera Cunduacán-Jalpa de Méndez, C.P. 86690, Cunduacán, Tabasco, México

[b]Departament de Matemátiques, Universitat Autónoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Abstract

In this paper we analyze an ecological model with a single predator and two competitive prey species. This model incorporates several key elements, including logistic growth dynamics for the prey populations, a Holling type II functional response governing predator-prey interactions, and the inclusion of intraspecific competition among predators.

Our study establishes stringent conditions on the model parameters to ensure the existence of two coexistence equilibrium points (CEPs). Of particular interest is one CEP that undergoes a Hopf bifurcation, resulting in a continuous transition between two limit cycles residing in different dimensions. More precisely, we observe a periodic solution within a three-dimensional phase space, alongside another periodic solution confined to an invariant phase plane. The bialternate sum matrix criterion serves as a vital tool in demonstrating the existence of this Hopf bifurcation.

Furthermore, employing Lyapunov exponents, we provide numerical evidence showcasing the emergence of chaotic dynamics within the model. This comprehensive analysis sheds light on the intricate behavior of the ecological system under consideration, offering valuable insights into the complex interplay of ecological factors and nonlinear phenomena within the predator-prey dynamics that has not been previously detected.

***Keywords*:** Ecological model, limit cycle, biproduct, Hopf bifurcation, stability.

1. Introduction

The dynamics of the trophic relationships have been widely studied during these last years, with predator–prey interactions standing out as recurrent subjects due to their biological significance (see Refs. [1–7]). Another important type of interaction is competition, which can manifest as interspecific or intraspecific. Interspecific mode of this interaction, arises when individuals within a community vie for the same limited resources in a particular habitat, while intraspecific involves competition among members of the same population (see Refs. [8, 9]).

Understanding the local dynamics of these population phenomena hinges on the underlying assumptions employed in their modeling, including factors as mortality rates, environmental conditions, the type of functional response, and numerous other variables. For instance, adopting a specific functional response in our model (see Refs. [10–14]) carries ecological

*Author for correspondence

implications, as it embodies critical traits governing the temporal evolution of niches, thereby influencing population survival across trophic levels (see Ref. [15]). Given the functional responses we introduce density–dependence in the predator–prey systems (see Refs. [16, 17]), they play a pivotal role in forecasting various aspects of the dynamics within a food web, particularly the local stability of the coexistence equilibria (see Ref. [18]). Even amidst intricate interconnections, the qualitative attributes of the resultant dynamical system invariably hinge on the assumptions underlying the model construction.

Prior research has delved into the dynamics of a predator and two competing preys (see Refs. [19–22]), with significant attention given to the seminal work of Gilpin within the Lotka-Volterra framework (see Ref. [23]). Gilpin's investigation garnered considerable interest in demonstrating the emergence of chaotic behavior through numerical solutions under specific parameter configurations (see Ref. [24]). Subsequent advancements in this realm include modifications proposed by Klebanoff and Hastings (see Ref. [21]), who refined Gilpin's model by incorporating the Holling type II functional response, thereby constraining the predator's capture rate. Furthermore, Ali and Chakravarty expanded upon Klebanoff and Hastings' model by integrating a competition term within the predator population (see Ref. [25]). However, certain geometric aspects inherent in the invariant planes of the model analyzed by Ali and Chakravarty have not received the adequate attention, despite their influence on the local dynamics. This gap in understanding serves as motivation for our study, wherein we investigate a model featuring one predator and two preys, accounting for intraspecific predator competition and employing the type II Holling functional response for both preys. Leveraging analytical insights from prior literature (see Ref. [26]), we identify the candidate equilibrium points and establish a criterion for the Hopf bifurcation using the bialternate sum matrix (see Refs. [27–31]). Our research culminates in a discussion of findings derived from numerical simulations conducted with a reduced set of model parameters.

In this manuscript we delve into the complexities of the dynamics of an ecological model featuring a predator population with intraspecific competition and two competing preys. Our exploration is structured as follows, ensuring a comprehensive analysis of the dynamics of the system. Section 1 sets the stage by elucidating the significance of such studies, tracing their historical context, and outlining our motivations to address this problem. To achieve our analysis, Section 2 introduces the model along with its underlying assumptions, with subsections 2.1 and 2.2 dedicated to dissecting the characteristics of trivial and coexistence equilibria, respectively. The core aspects of the present work unfold further in Section 3, where we deal with the Hopf bifurcation using the bialternate sum matrix criterion, corroborating our findings in subsections 3.1 and 3.2. Section 4 serves as the lab for our theoretical propositions, housing a series of simulations meticulously designed to buttress our analytical framework. Finally, in Section 5, we are offering a succinct synthesis of our principal findings to conclude our work.

2. Mathematical model, assumptions and CEPs

We explore the local dynamic of a trophic system consisting of one predator and two competitive preys, considering a logistic growth rate for each prey specie and a Holling type II functional response for predation. We also consider the effects of intraspecific competition among predators and interspecific competition between preys for other resources than food.

Let x, y, and z be the population densities of two competing preys and one predator, respectively, at time $t \geq 0$. The ecological model describing the interaction among these populations consists in the following nonlinear system of coupled differential equations:

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{k_1}\right) - c_1 xy - \frac{a_1 xz}{x + b_1}, \tag{2.1a}$$

$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{k_2}\right) - c_2 xy - \frac{a_2 yz}{y + b_2}, \tag{2.1b}$$

$$\frac{dz}{dt} = \frac{e_1 a_1 xz}{x + b_1} + \frac{e_2 a_2 yz}{y + b_2} - dz - \sigma z^2, \tag{2.1c}$$

where r_1 and r_2 are biotic potentials, k_1 and k_2 are the environmental carrying capacities of two preys, c_1 and c_2 are the coefficients of interspecific competition between the two prey populations, a_1 and a_2 are the predation rates, e_1 and e_2

the efficiency rates that measure the conversion of prey biomass into predator biomass, b_1 and b_2 are the half–saturation constants, d is the natural death rate of predators, and σ is the coefficient associated with the intraspecific competition among predators. The region of interest to analyse the local dynamics of (2.1) is the positive octant of $\mathbb{R}^3$.

2.1. Trivial equilibria with ecological feasibility

The first trivial equilibrium point is the extinction given by $(0, 0, 0)$. The other ones are the equilibrium points $(k_1, 0, 0)$ and $(0, k_2, 0)$ located on the invariant semiaxes $x > 0$ and $y > 0$, and the equilibrium on the invariant plane $z = 0$ given by $(x^*, y^*, 0)$. The other equilibria, not trivial like the previous ones and which are important, belong to two identical Bazykin-type predator-prey systems by considering that one of the prey is extinct, that is, these equilibria corresponds with the invariant planes $x = 0$ and $y = 0$ given by $(0, y^*, z^*)$ and $(x^*, 0, z^*)$, respectively.

For more details about the trivial equilibria of (2.1), Ali and Chakravarty provide an excellent analysis in [25] for his stability, determining the conditions in a suitable rescaled parameter space.

2.2. Coexistence Equilibrium Points (CEPs)

The coexistence equilibrium points of system (2.1) are characterized via the following proposition.

Proposition 2.1. *Assume that all parameters of system (2.1) are strictly positive. The points $p_1 = (1, 1, 1)$ and $p_2 = (4, 1/2, 1/2)$ are coexistence equilibria of system (2.1) if and only if the following conditions hold:*

$$a_1 = a_{10},\ a_2 = a_{20},\ b_1 = b_{10},\ c_1 = c_{10},\ e_1 = e_{10},\ e_2 = e_{20},\ k_1 = k_{10},\ k_2 = k_{20},\ r_1 = r_{10},\ r_2 = r_{20},\ \sigma = \sigma_0, \tag{2.2}$$

where a_{10}, a_{20}, b_{10}, c_{10}, e_{10}, e_{20}, k_{10}, k_{20}, r_{10}, r_{20} and σ_0 are strictly positive quantities depending on the free parameters b_2, c_2 and d.

Proof. To find coexistence equilibrium points of system (2.1) implies to find solutions in the interior of the positive octant of $\mathbb{R}^3$ for the following system of equations

$$\begin{aligned} r_1 x\left(1 - \frac{x}{k_1}\right) - c_1 xy - \frac{a_1 xz}{x + b_1} &= 0, \\ r_2 y\left(1 - \frac{y}{k_2}\right) - c_2 xy - \frac{a_2 yz}{y + b_2} &= 0, \\ \frac{e_1 a_1 xz}{x + b_1} + \frac{e_2 a_2 yz}{y + b_2} - dz - \sigma z^2 &= 0. \end{aligned}$$

Thus, considering all the conditions (2.2), which are provided in Appendix B because they have large expressions, we can verify that p_1 and p_2 are coexistence equilibria for system (2.1). □

3. Bialternate sum matrix and Hopf bifurcation

In the search for stable limit cycles, many researchers have delved into various analytical tools and methodologies. One of such powerful tools is the bialternate sum matrix, also known as biproduct of matrices, which has become a valuable asset in the analysis of supercritical Hopf bifurcations for the detection of these types of oscillations. This mathematical construction has been refined by Govaerts (see Ref. [32]), offering a systematic approach to the analysis of this local bifurcation, both numerically and analytically. In this way, in the next section we establish our results on the supercritical Hopf bifurcation as a function of the bialternate sum matrix. Thus, in the present section we establish our results on the supercritical Hopf bifurcation following the main results of the Hopf bifurcation using bialternate sum matrix (for more details see Appendix A).

3.1. Hopf bifurcation not takes place at p_1

All the conditions given in Proposition 2.1 guarantee the existence of parameter families for which the equilibrium points p_1 and p_2 live in the positive octant of $\mathbb{R}^3$. Now, to show that the Hopf bifurcation does not take place at p_1, we first determine that its linear approximation is given by

$$A_1 = \begin{pmatrix} j_{1,1} & j_{1,2} & j_{1,3} \\ j_{2,1} & j_{2,2} & j_{2,3} \\ j_{3,1} & j_{3,2} & j_{3,3} \end{pmatrix}, \tag{3.1}$$

whose matrix elements are presented in Appendix B. To search for a possible Hopf bifurcation using Theorem A.3 it is required to have local stability, but we observe that in this case

$$p_0 = -\left|A_1\left(b_2, c_2, d\right)\right| < 0 \tag{3.2}$$

for all combinations of the free parameters b_2, c_2 and d. Therefore, the CEP p_1 is locally unstable and a supercritical Hopf bifurcation does not occur in it.

3.2. Hopf bifurcation takes place at p_2

From Proposition 2.1 we know that the point p_2 is a CEP for the ecological system (2.1). Next, we characterize the emergence of stable limit cycles. To show that a Hopf bicurcation takes place at p_2, we compute the bialternate sum matrix using the symbolic tool given in [33]. Then, we stablish our main results according to the following theorem.

Theorem 3.1. *Taking the natural mortality rate of the predator as a bifurcation parameter, the coexistence equilibrium point p_2 is locally stable for $d > d_0$ and exhibits a supercritical Hopf bifurcation for $d < d_0$, where d_0 is the critical Hopf bifurcation value as a function of the free parameters b_2 and c_2 (d_0 is given in Appendix B because it has a big expression).*

Proof. The linear part of the differential system (2.1) p_2 is

$$A_2 = \begin{pmatrix} j^*_{1,1} & j^*_{1,2} & j^*_{1,3} \\ j^*_{2,1} & j^*_{2,2} & j^*_{2,3} \\ j^*_{3,1} & j^*_{3,2} & j^*_{3,3} \end{pmatrix}, \tag{3.3}$$

whose matrix elements are presented in Appendix B. According with Theorem A.3, the conditions for a simple Hopf bifurcation are

$$p_0\left(d_0\right) = \frac{s_3}{s_4} > 0, \quad q_0\left(d_0\right) = -\left|G\left(d_0\right)\right| = 0, -\operatorname{tr} G\left(d_0\right) > 0, \tag{3.4}$$

where s_3 and s_4 are presented in Appendix B, and d_0 is the critical Hopf bifurcation value. Here, BS is the bialternate sum matrix associated to A_2

$$G\left(d\right) = 2A_2\left(d\right) \odot I_3 = \begin{pmatrix} j^*_{1,1} + j^*_{2,2} & j^*_{2,3} & -j^*_{1,3} \\ j^*_{3,2} & j^*_{1,1} + j^*_{3,3} & j^*_{1,2} \\ -j^*_{3,1} & j^*_{2,1} & j^*_{2,2} + j^*_{3,3} \end{pmatrix}. \tag{3.5}$$

Finally, we verify the transversality condition $-D_d\left(\left|G\left(d_0\right)\right|\right) \neq 0$. Therefore a stable limit cycle emerges from p_2 for $d < d_0$. All quantities in this proof are presented in Appendix B. □

4. Numerical simulations

In this section we provide some simulations to support our theoretical findings. In the first place the coexistence of the three populations in a stable limit cycle, and later, the coexistence in a stable limit cycle of a predator-prey system in the invariant plane $x = 0$. Finally, we show a chaotic dynamic when the half-saturation constant b_2 increases.

4.1. Emergence of stable limit cycle from CEP p_2

Taking $b_2 = 4$, $b_5 = 4/3$, $c_2 = 1/5$ and $d = d_0 + 1/10$, from Proposition 2.1 we obtain that the values of a_1, a_2, b_1, c_1, e_1, e_2, k_1, k_2, r_1, r_2 and σ are

$$a_1 = \frac{969777583311724609291}{1385061058477117039917 50}, \ a_2 = \frac{105}{19}, \ b_1 = \frac{17}{5}, \ c_1 = \frac{183471434680596547705 01}{435304904092808212545 50}, \ e_1 = \frac{1399315}{3702553},$$

$$e_2 = \frac{702940201522850283}{1604539988309889400}, \ k_1 = \frac{32}{5}, \ k_2 = 7, \ r_1 = \frac{209681639634967483091 44}{304713432864965748781 85}, \ r_2 = \frac{434}{285}, \ \sigma = \frac{26192132383026647}{98981362915220450}.$$

In the same way if we take $b_2 = 4$, $b_5 = 4/3$, $c_2 = 1/5$ and $d = d_0 - 1/10$, from Proposition 2.1 we obtain that the values of a_1, a_2, b_1, c_1, e_1, e_2, k_1, k_2, r_1, r_2 and σ are

$$a_1 = \frac{276418452212242184274 31}{1385061058477117039917 50}, \ a_2 = \frac{105}{19}, \ b_1 = \frac{17}{5}, \ c_1 = \frac{522953828509647375654 1}{435304904092808212545 50}, \ e_1 = \frac{1399315}{3702553},$$

$$e_2 = \frac{200361037258839603}{1604539988309889400}, \ k_1 = \frac{32}{5}, \ k_2 = 7, \ r_1 = \frac{597661518296739857890 4}{304713432864965748781 85}, \ r_2 = \frac{434}{285}, \ \sigma = \frac{7465617702494527}{98981362915220450}.$$

Thus, considering the above parameter values and the initial condition $X_0 = (41/10, 3/5, 3/5)$, the three populations reach the CEP p_2 (being in the basin of attraction of p_2) for $d > d_0$, and the stable limit cycle emerges from p_2 when $d < d_0$, as shown in Figures 1 and 2 for the time series

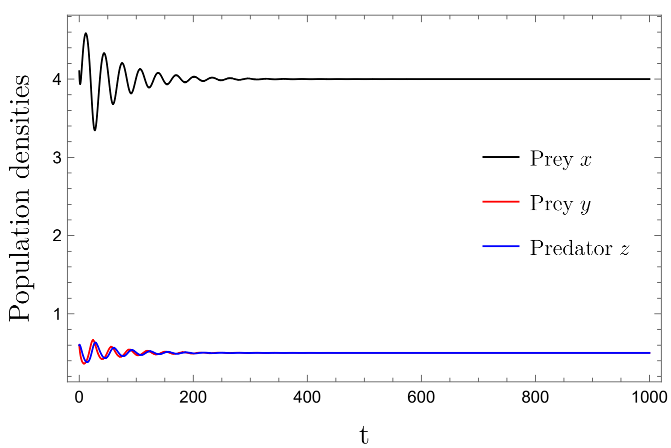

Figure 1: Local behaviour $d > d_0$.

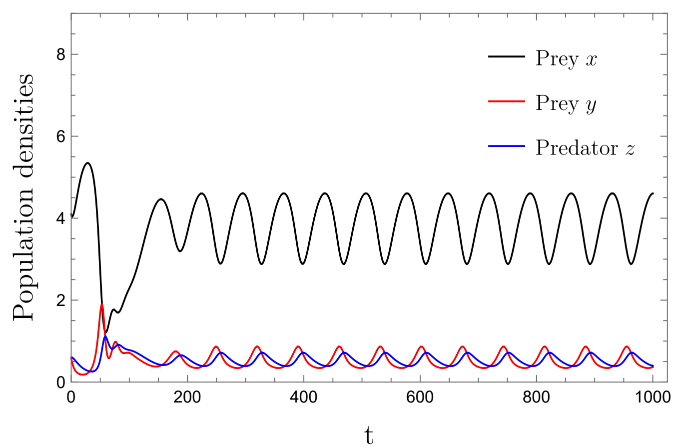

Figure 2: Local behaviour $d < d_0$.

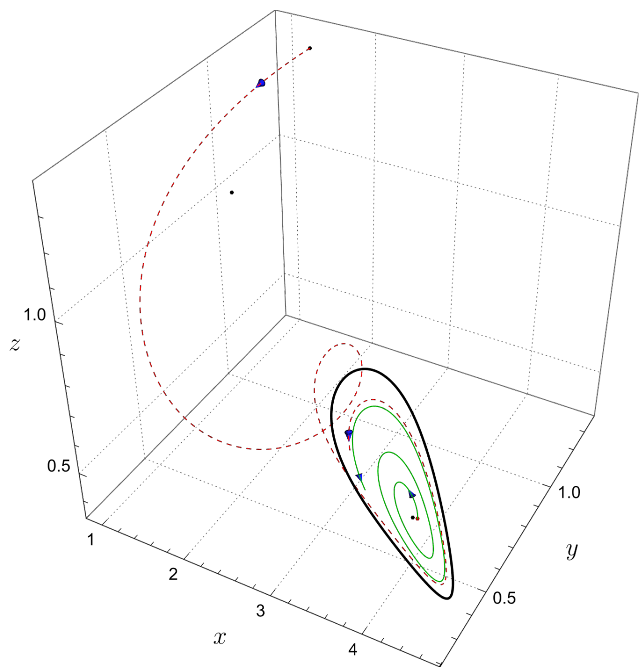

Figure 3: Two orbits tending to the stable limit cycle for $d < d_0$.

We can nicely visualize in Figure 3 the stable limit cycle using the initial conditions

$$X_{01} = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right), \; X_{02} = \left(\frac{201}{50}, \frac{13}{25}, \frac{29}{60}\right)$$

4.2. A Hopf bifurcation takes place at invariant plane $x = 0$

Without loss of generality we take $b_2 = 4$, $b_5 = 4/3$, $c_2 = 1/5$ to show that a Hopf bifurcation occurs in the invariant $x = 0$ plane. With these values, from Proposition 2.1 the corresponding predator-prey system is

$$\frac{dy}{dt} = \frac{434}{285}\left(1 - \frac{y}{7}\right) y - \frac{105yz}{19(y+4)}, \tag{4.1}$$

$$\frac{dz}{dt} = \frac{14895dyz}{1721(y+4)} - dz - \frac{1628dz^2}{1721}, \tag{4.2}$$

where d is our bifurcation parameter. If we solve to find the Hopf conditions, we have that

$$z_* = -\frac{-461090\sqrt[3]{439102601739935 - 23832\gamma} + \sqrt[3]{35}(439102601739935 - 23832\gamma)^{2/3} + 416075987\ 35^{2/3}}{85470\sqrt[3]{439102601739935 - 23832\gamma}},$$

$$y_* = \frac{14245z_*^2}{20522} - \frac{230545z_*}{41044} + 7, \; d_* = -\frac{929628355271z_*^2}{23980899029310} + \frac{3694595933728z_*}{11990449514655} - \frac{2286110195874995}{7808180723943336},$$

where $\gamma = \sqrt{335038740911780872505}$. The transversality condition for this two–dimensional Hopf bifurcation is

$$-\frac{1}{2}\,\mathrm{D}_d\,\mathrm{tr}\,\mathrm{J}\,(y_*, z_*, d_*) > 0,$$

Then, the equilibrium point (y_*, z_*) is locally stable for $d > d_*$ and a two dimensional stable limit cycle appear for $d < d_*$, and additionally it is fulfilled that $d_* < d_0$. The last mentioned condition suggests that there is a value for the natural mortality rate of predators at which the coexistence in the three-dimensional limit cycle passes to coexistence in a two-dimensional limit cycle in the invariant plane $x = 0$ when d varies smoothly for $d < d_0$, in particular, a value of this rate at which the prey x reaches extinction. In Figure 4, this continuous transition can be easily visualized

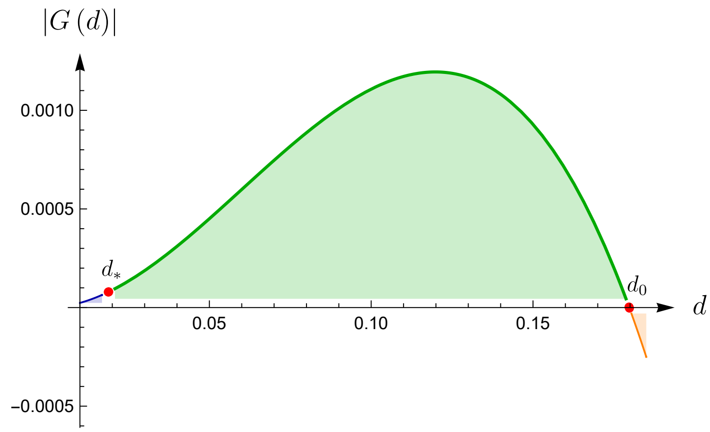

Figure 4: Stability regions in terms of the parameter d.

In Figures 5 and 6, we can observe the local stability of (y_*, z_*) and the loss of it, considering for this purpose the values $d = d_0 - 1/7$ and $d = d_0 - 1/6$, and the initial condition X_0 as well.

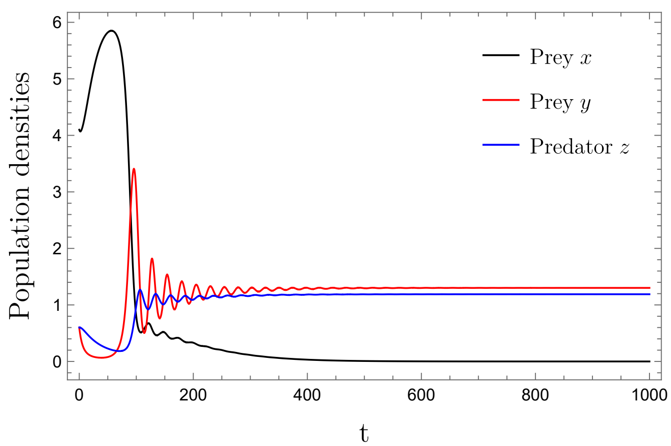

Figure 5: Local behaviour $d > d_*$.

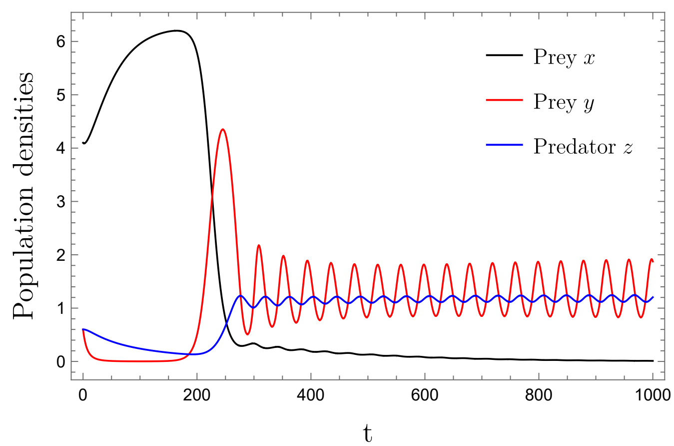

Figure 6: Local behaviour $d < d_*$.

The two–dimensional stable limit cycle is visualized in a clear fashion in Figure 7, using the initial condition

$$X_{03} = \left(\frac{13}{3}, \frac{5}{6}, \frac{5}{6}\right)$$

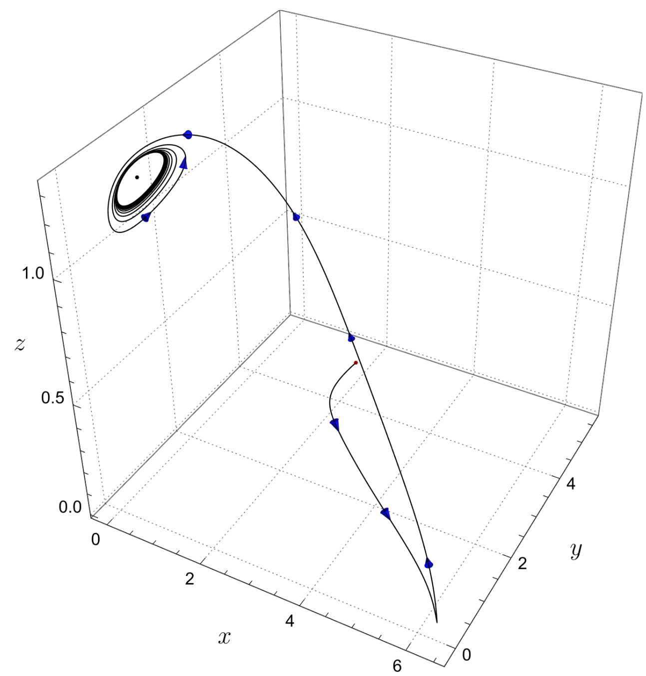

Figure 7: An orbit tending to the stable limit cycle for $d < d_*$.

4.3. Detecting a chaotic regime with the half–saturation constant b_2

In this subsection we show numerically the evidence of a chaotic regime when the half–saturation constant b_2, i.e. the resource availability at which half of the maximum intake is reached, is greatly increased.

Taking $b_2 = 20$, $c_2 = 1/5$, $b_5 = 4/3$ and $d = d_0 - 1105/10000$, we observe in Figure 8 the appearance of a chaotic attractor around the CEP p_2 using the following initial condition

$$X_{04} = \left(\frac{401}{100}, \frac{51}{100}, \frac{51}{100}\right)$$

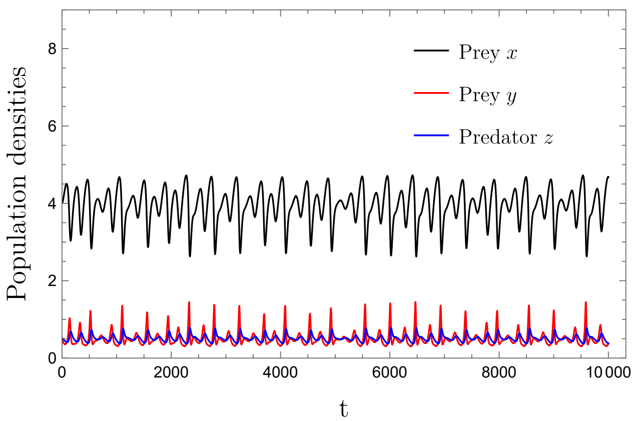

Figure 8: Time series for a chaotic dynamics.

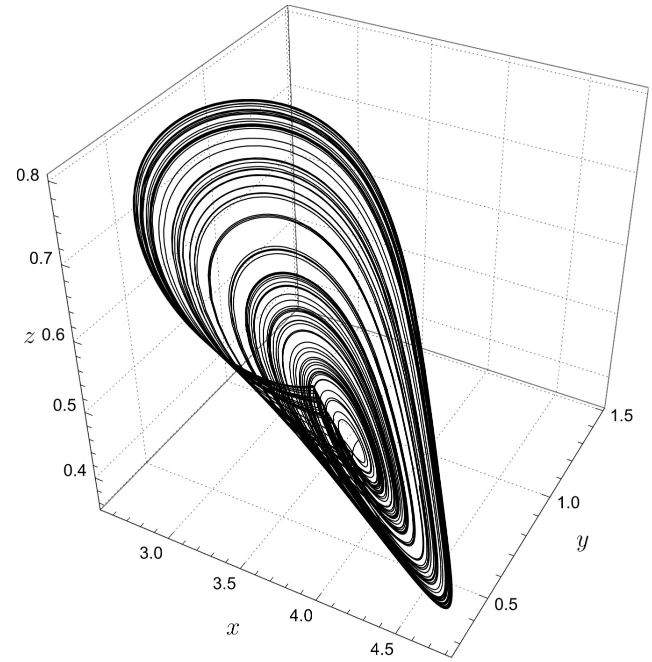

Figure 9: Phase portrait for a chaotic dynamics.

Using critical Lyapunov exponents, we show in Figure 10 that a chaotic attractor indeed appears (see Ref. [34]). In this sense, the convergence plot is the follow

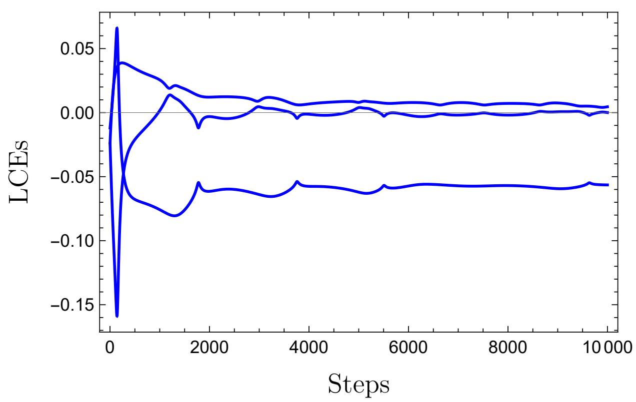

Figure 10: Convergence of the critical Lyapunov exponents.

According to the classification provided by Klein and Baier (see Ref. [35]) regarding the spectrum of critical Lyapunov exponents of continuous–time attractors, it is confirmed that evidence of a chaotic regime emerges within ecological system (2.1) when the half-saturation rate b_2 exceeds its value at the Hopf bifurcation by a considerable margin.

5. Conclusions

The task of computing coexistence equilibria in ecological systems involving three populations is inherently complex. However examining the dynamics within invariant planes offers a promising avenue for conditioning the system's behavior and facilitating the discovery of coexistence equilibria. In the model under investigation, the invariant planes $x = 0$ and $y = 0$ exhibit characteristics akin to Bazykin-type predator-prey systems (see Ref. [36]), in which the Hopf bifurcation has a restriction on the zero–isocline of the prey population, as occurs in the host-parasitoid model analysed in [26]. In this way, we managed to find conditions on the parameters that allow us to detect an unreported scenario, that is, a continuous transition between two limit cycles (varying the natural mortality rate of the predator), one in three dimensions and the other in the invariant plane $x = 0$, for which the principle of competitive exclusion is confirmed (see Ref. [37]).

The inclusion of predator competition terms renders the prey competition system more amenable to stabilization. However a noteworthy observation arises as the half-saturation constant b_2 increases: chaos begins to manifest. More precisely, a transition unfolds between a three-dimensional limit cycle and a chaotic attractor, occurring gradually as b_2 smoothly escalates beyond a threshold of four. It becomes evident that the half-saturation constant significantly dictates the chaotic dynamics of these models, a trend consistent with prior research (see Ref. [38]). Thus, it is unsurprising that the most pronounced instabilities emerge during fluctuations in the value of b_2.

Acknowledgement

The first and second authors are partially supported by CONACYT, grant A–1–S–47710.

The third author is partially supported by the Agencia Estatal de Investigación grant PID2022-136613NB-100, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Real Acadèmia de Ciències i Arts de Barcelona.

A. Hopf bifurcations and bialternate sum matrix theorems

A.1. Hopf bifurcation theorem

Following Guckenheimer and Kuznetsov [39, 40] we state a simple Hopf bifurcation using the bialternate sum matrix criterion.

Consider a system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_\mu(x), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}, \tag{A.1}$$

with an equilibrium point (x^*, μ^*), and $f_\mu \in \mathbb{C}^\infty$. Assume that

(i) The Jacobian matrix $D_x f_\mu(x)|_{(x=x^*,\mu=\mu^*)}$ has a simple pair of purely imaginary eigenvalues, and all the other eigenvalues have negative real parts.

Then there is a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu^*) = x^*$. The eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ of $J(\mu) = D_x f_\mu(x)$ which are purely imaginary at $\mu = \mu^*$ vary smoothly with μ. Furthermore, if

(ii) $\dfrac{d}{d\mu}(\mathrm{Re}\lambda(\mu^*)) \neq 0$

then there is a simple Hopf bifurcation.

On the other hand, to use the bialternate sum matrix in the analysis of the supercritical Hopf bifurcation, we establish a definition and a theorem given in [27] by Stéphanos.

Definition A.1 (Stéphanos)**.** *Let A be an n–dimensional matrix (a_{ij}) and let B be an n–dimensional matrix (b_{ij}). Let F be a matrix $m = n(n-1)/2$–dimensional $(f_{pq,rs})$ whose rows are labeled by the indices pq $(p = 2, 3, \ldots, n; q = 1, 2, \ldots, p-1)$, whose columns are labeled by the indices rs $(r = 2, 3, \ldots, n; s = 1, 2, \ldots, r-1)$, and whose elements are*

$$f_{pq,rs} = \frac{1}{2}\left[\left|\begin{array}{cc} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{array}\right| + \left|\begin{array}{cc} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{array}\right|\right]. \tag{A.2}$$

Then, F is the bialternate product of the matrices A and B, and is written as $A \odot B$.

Theorem A.1 (Stéphanos)**.** *The characteristic roots of the matrix*

$$G = 2A \odot I_n, \tag{A.3}$$

where A is an n–dimensional matrix (a_{ij}) and I_n is the n–dimensional identity matrix, are the $n(n-1)/2$ values

$$\lambda_i + \lambda_j, \quad (i = 2, 3, \ldots, n; j = 1, 2, \ldots, i-1) \tag{A.4}$$

where λ_i are the eigenvalues of A.

The matrix elements of G are

$$g_{pq,rs} = \left|\begin{array}{cc} a_{pr} & a_{ps} \\ \delta_{qr} & \delta_{qs} \end{array}\right| + \left|\begin{array}{cc} \delta_{pr} & \delta_{ps} \\ a_{qr} & a_{qs} \end{array}\right|, \tag{A.5}$$

where δ_{ij} is the Kronecker delta, $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ij} = 1$ for $i = j$. With $p > q$ and $r > s$, Eq. (A.5) is

$$g_{pq,rs} = \begin{cases} -a_{ps} & \text{if } r = q \\ a_{pr} & \text{if } r \neq p \text{ and } s = q \\ a_{pp} + a_{qq} & \text{if } r = p \text{ and } s = q \\ a_{qs} & \text{if } r = p \text{ and } s \neq q \\ -a_{qr} & \text{if } s = p \\ 0 & \text{otherwise} \end{cases} \tag{A.6}$$

Fuller takes up the definition together with the previous theorem [28] to establish an alternative criterion to the well-known Routh-Hurwitz criterion to study stability, as follows.

Theorem A.2 (Fuller)**.** *Let $A = (a_{ij})$ be a real square matrix of dimension $n > 1$. Let $G = (g_{ij})$ be the square matrix of dimension $m = n(n-1)/2$ defined by Eq. (A.3), with elements given by Eq. (A.5), or equivalently by Eq. (A.6). Then for the characteristic roots of A to have all their real parts negative, it is necessary and sufficient that in the characteristic polynomial of A, namely, $|\lambda I_n - A|$, and in the polynomial of G, namely $|\mu I_m - G|$ the coefficients of λ^i $(i = 0, 1, \ldots, n-1)$ y μ^i $(i = 0, 1, \ldots, m-1)$ should all be positive.*

As a final step, according with the ideas of Stéphanos and Fuller, we denote the characteristic equation of the Jacobian matrix $J(\mu)$, given at the beginning of this section in the Hopf bifurcation theorem, as the following polynomial equation depending on μ:

$$|\lambda I_n - J(\mu)| = 0 \tag{A.7}$$
$$p(\lambda; \mu) = p_n(\mu)\lambda^n + p_{n-1}(\mu)\lambda^{n-1} + \ldots + p_0(\mu) = 0, \quad p_n(\mu) > 0. \tag{A.8}$$

Now suppose that the roots of the equation (A.7) are $\lambda_1(\mu), \lambda_2(\mu), \ldots, \lambda_n(\mu)$. Then there is an equation

$$q(\zeta; \mu) = q_m(\mu)\zeta^m + q_{m-1}(\mu)\zeta^{m-1} + \ldots + q_0(\mu) = 0, \quad q_m(\mu) > 0. \tag{A.9}$$

whose roots are given by

$$\zeta(\mu) = \lambda_i(\mu) + \lambda_j(\mu), \quad (i = 2, 3, \ldots, n; j = 1, 2, \ldots, i-1), \tag{A.10}$$

where every $q_i(\mu)$ is a smooth function of μ and $m = n(n-1)/2$. We define the following biproduct of matrices, known as bialternate sum matrix

$$G(\mu) = 2J(\mu) \odot I_n, \tag{A.11}$$

where $J(\mu)$ is the Jacobian matrix of the differential system (A.1), I_n is an n–dimensional identity matrix, and $\odot$ denotes the biproduct. Then $G(\mu)$ is the matrix whose characteristic equation is (A.9).

To show that system (A.1) undergoes a Hopf bifurcation, it is necessary to calculate the bialternate sum matrix. Then we stablish the Hopf bifurcation theorem for the supercritical case as follows.

Theorem A.3 (Hopf bifurcation Theorem)**.** *Assume there is a smooth curve of equilibria (x, μ) with $x(\mu^*) = x^*$ for the differential system (A.1). Conditions (i) and (ii) for a simple Hopf bifurcation are equivalent to the following conditions in the characteristic polynomial $P(\lambda; \mu)$ and the associated polynomial $q(\zeta; \mu)$:*

(i^*) $p_0(\mu^*) > 0, q_0(\mu^*) = 0, q_1(\mu^*), q_2(\mu^*), \ldots, q_{m-1}(\mu^*) > 0,$

(ii^*) $\dfrac{\mathrm{d}}{\mathrm{d}\mu}(q_0(\mu))|_{\mu=\mu^*} \neq 0,$

where $q_0(\mu)$ is given by $|G(\mu)|$.

B. Bialternate sum matrix computations

This section contains all the explicit computations for our main results using a Mathematica implementation to obtain the bialternate sum matrix needed for the Hopf bifurcation analysis.

The expression a_{10} is

$$a_{10} = \frac{\alpha_1}{\alpha_2},$$

where α_1 and α_2 are

$$\begin{aligned}\alpha_1 =& b_2^4\left(30720b_5^5 + 261312b_5^4 + 888096b_5^3 + 1507404b_5^2 + 1277808b_5 + 432768\right)d + \\ & b_2^3\left(130560b_5^5 + 1109856b_5^4 + 3769776b_5^3 + 6395334b_5^2 + 5418840b_5 + 1834560\right)d + \\ & b_2^2\left(136960b_5^5 + 1165376b_5^4 + 3961744b_5^3 + 6726076b_5^2 + 5702816b_5 + 1931776\right)d + \\ & b_2\left(43520b_5^5 + 370912b_5^4 + 1262768b_5^3 + 2146622b_5^2 + 1822072b_5 + 617792\right)d,\end{aligned}$$

$$\begin{aligned}\alpha_2 =& b_2^4\left(4608b_5^6 + 46496b_5^5 + 193880b_5^4 + 427160b_5^3 + 523712b_5^2 + 338144b_5 + 89600\right) + \\ & b_2^3\left(17664b_5^6 + 174400b_5^5 + 708936b_5^4 + 1515096b_5^3 + 1789448b_5^2 + 1102064b_5 + 274432\right) + \\ & b_2^2\left(23808b_5^6 + 236632b_5^5 + 971410b_5^4 + 2105390b_5^3 + 2536412b_5^2 + 1606608b_5 + 416640\right) + \\ & b_2\left(14976b_5^6 + 154104b_5^5 + 658840b_5^4 + 1497910b_5^3 + 1910024b_5^2 + 1295096b_5 + 364800\right) + \\ & 4224b_5^6 + 45076b_5^5 + 200291b_5^4 + 474357b_5^3 + 631570b_5^2 + 448232b_5 + 132480.\end{aligned}$$

The expressions a_{20} and b_{10} are

$$a_{20} = \frac{(b_2+1)(2b_2+1)(6b_2+11)c_2}{2b_2(b_2+3)+1}, \quad b_{10} = \frac{2(2(b_5+1)+1)}{b_5+2},$$

The expression c_{10} is

$$c_{10} = \frac{\alpha_3}{\alpha_4},$$

where α_3 and α_4 are

$$\begin{aligned}\alpha_3 =& b_2^4 \left(6912b_5^5 + 54840b_5^4 + 172362b_5^3 + 267786b_5^2 + 205176b_5 + 61824\right) d+\\ & b_2^3 \left(29376b_5^5 + 232908b_5^4 + 731589b_5^3 + 1136037b_5^2 + 870060b_5 + 262080\right) d+\\ & b_2^2 \left(30816b_5^5 + 244576b_5^4 + 768922b_5^3 + 1194906b_5^2 + 915712b_5 + 275968\right) d+\\ & b_2 \left(9792b_5^5 + 77852b_5^4 + 245129b_5^3 + 381417b_5^2 + 292604b_5 + 88256\right) d,\end{aligned}$$

$$\begin{aligned}\alpha_4 =& b_2^4 \left(4608b_5^5 + 37280b_5^4 + 119320b_5^3 + 188520b_5^2 + 146672b_5 + 44800\right) +\\ & b_2^3 \left(17664b_5^5 + 139072b_5^4 + 430792b_5^3 + 653512b_5^2 + 482424b_5 + 137216\right) +\\ & b_2^2 \left(23808b_5^5 + 189016b_5^4 + 593378b_5^3 + 918634b_5^2 + 699144b_5 + 208320\right) +\\ & b_2 \left(14976b_5^5 + 124152b_5^4 + 410536b_5^3 + 676838b_5^2 + 556348b_5 + 182400\right) +\\ & 4224b_5^5 + 36628b_5^4 + 127035b_5^3 + 220287b_5^2 + 190996b_5 + 66240.\end{aligned}$$

The expression r_{10} is

$$r_{10} = \frac{\alpha_5\alpha_6}{\alpha_7\alpha_8},$$

where α_5, α_6, α_7 and α_8 are given by

$$\alpha_5 = 3b_2 (b_5 + 2) (4b_5 + 7) (7b_5 + 12) d,$$

$$\begin{aligned}\alpha_6 =& 8 (b_2 (6b_2 (4b_2 + 17) + 107) + 34) b_5^2+\\ & (b_2 (3b_2 (218b_2 + 925) + 2918) + 931) b_5+\\ & 4b_2 (3b_2 (46b_2 + 195) + 616) + 788,\end{aligned}$$

$$\alpha_7 = 2 (2b_2 (b_5 + 2) (16b_5 + 25) + b_5 (32b_5 + 109) + 92),$$

$$\begin{aligned}\alpha_8 =& 2b_2^2 (b_5 (b_5 (204b_5 + 875) + 1181) + 480) + b_5 (b_5 (132b_5 + 695) + 1223)\\ & 2b_2 (7b_5 + 12) (b_5 (24b_5 + 71) + 50) + 4 (b_5 + 1) (4b_5 + 7) (9b_5 + 16) b_2^3 + 720.\end{aligned}$$

The expressions r_{20}, k_{10}, k_{20} are

$$r_{20} = \frac{2 (b_2 + 3) (7b_2 + 3) c_2}{2b_2 (b_2 + 3) + 1}, \quad k_{10} = \frac{7b_5 + 12}{b_5 + 2}, \quad k_{20} = b_2 + 3.$$

The expression e_{10} is

$$e_{10} = \frac{\alpha_9}{\alpha_{10}},$$

where α_9 and α_{10} are

$$\begin{aligned}\alpha_9 =& b_2^3\left(144b_5^4+940b_5^3+2260b_5^2+2360b_5+896\right)+\\ & b_2^2\left(408b_5^4+2566b_5^3+5862b_5^2+5684b_5+1920\right)+\\ & b_2\left(336b_5^4+2242b_5^3+5544b_5^2+6008b_5+2400\right)+\\ & 132b_5^4+959b_5^3+2613b_5^2+3166b_5+1440,\end{aligned}$$

$$\begin{aligned}\alpha_{10} =& b_2^3\left(768b_5^3+3960b_5^2+6786b_5+3864\right)+\\ & b_2^2\left(3264b_5^3+16812b_5^2+28785b_5+16380\right)+\\ & b_2\left(3424b_5^3+17664b_5^2+30282b_5+17248\right)+\\ & 1088b_5^3+5628b_5^2+9669b_5+5516.\end{aligned}$$

The expressions e_{20} and σ_0 are

$$e_{20}=\frac{\left(2b_2\left(b_2+3\right)+1\right)\left(b_5\left(32b_5+109\right)+92\right)d}{\left(6b_2+11\right)\left(2b_2\left(b_5+2\right)\left(16b_5+25\right)+b_5\left(32b_5+109\right)+92\right)c_2},$$

$$\sigma_0=\frac{2b_2\left(4\left(b_5+1\right)+3\right)\left(5\left(b_5+1\right)+3\right)d}{2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15}.$$

In all the above expressions and in those that follow, b_5 is a strictly positive free parameter.

The matrix elements $j_{1,1}$, $j_{1,2}$ and $j_{1,3}$ of the linear approximation A_1 are

$$j_{1,1}=\frac{\beta_1}{\beta_2},\ j_{1,2}=\frac{\beta_3}{\beta_4},\ j_{1,3}=\frac{\beta_5}{\beta_6},$$

where β_1, β_2, β_3, β_4, β_5 and β_6 are

$$\begin{aligned}\beta_1 =& -d\left(b_2^4\left(b_5+2\right)\left(4b_5+7\right)\left(192b_5^2+654b_5+552\right)\left(b_5\left(15b_5+38\right)+20\right)+\right.\\ & b_2^3\left(b_5+2\right)\left(4b_5+7\right)\left(816b_5^2+2775b_5+2340\right)\left(b_5\left(15b_5+38\right)+20\right)+\\ & b_2^2\left(b_5+2\right)\left(4b_5+7\right)\left(856b_5^2+2918b_5+2464\right)\left(b_5\left(15b_5+38\right)+20\right)+\\ & \left.b_2\left(b_5+2\right)\left(4b_5+7\right)\left(272b_5^2+931b_5+788\right)\left(b_5\left(15b_5+38\right)+20\right)\right),\end{aligned}$$

$$\begin{aligned}\beta_2 =& 16b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(5b_5+8\right)\left(9b_5+16\right)\left(16b_5+25\right)+\\ & 16b_2^3\left(5b_5+8\right)\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+\\ & 4b_2^2\left(5b_5+8\right)\left(11904b_5^5+94508b_5^4+296689b_5^3+459317b_5^2+349572b_5+104160\right)+\\ & 4b_2\left(5b_5+8\right)\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+\\ & \left(5b_5+8\right)\left(4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240\right),\end{aligned}$$

$$\begin{aligned}\beta 3 =& -d\left(b_2^4\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(192b_5^2+654b_5+552\right)+\right.\\ & b_2^3\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(816b_5^2+2775b_5+2340\right)+\\ & b_2^2\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(856b_5^2+2918b_5+2464\right)+\\ & \left.b_2\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(272b_5^2+931b_5+788\right)\right),\end{aligned}$$

$$\begin{aligned}\beta_4 =& 8b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(16b_5+25\right)+\\ & 8b_2^3\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+\\ & b_2^2\left(23808b_5^5+189016b_5^4+593378b_5^3+918634b_5^2+699144b_5+208320\right)\\ & 2b_2\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+\\ & 4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240,\end{aligned}$$

$$\begin{aligned}\beta_5 =& -d\left(2b_2^4\left(4b_5+7\right)^2\left(192b_5^2+654b_5+552\right)+\right.\\ & 2b_2^3\left(4b_5+7\right)^2\left(816b_5^2+2775b_5+2340\right)+\\ & 2b_2^2\left(4b_5+7\right)^2\left(856b_5^2+2918b_5+2464\right)+\\ & \left.b_2\left(4b_5+7\right)^2\left(272b_5^2+931b_5+788\right)\right),\end{aligned}$$

$$\begin{aligned}\beta_6 =& 8b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(16b_5+25\right)\\ & 8b_2^3\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+\\ & b_2^2\left(23808b_5^5+189016b_5^4+593378b_5^3+918634b_5^2+699144b_5+208320\right)+\\ & 2b_2\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+\\ & 4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240.\end{aligned}$$

The matrix elements $j_{2,1}$, $j_{2,2}$ and $j_{2,3}$ of the linear approximation A_1 are

$$j_{2,1}=-c_2,\ j_{2,2}=\frac{\left(5-2\left(b_2-4\right)b_2\right)c_2}{\left(b_2+1\right)\left(2b_2\left(b_2+3\right)+1\right)},\ j_{2,3}=-\frac{\left(2b_2+1\right)\left(6b_2+11\right)c_2}{2b_2\left(b_2+3\right)+1}.$$

The matrix elements $j_{3,1}$, $j_{3,2}$ and $j_{3,3}$ of the linear approximation A_1 are

$$j_{3,1}=\frac{4b_2\left(b_5+2\right)\left(2\left(b_5+1\right)+1\right)\left(4\left(b_5+1\right)+3\right)d}{\left(5\left(b_5+1\right)+3\right)\left(2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)},$$

$$j_{3,2}=\frac{b_2\left(2b_2+1\right)\left(\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)d}{\left(b_2+1\right)\left(2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)},$$

$$j_{3,3}=-\frac{2b_2\left(4\left(b_5+1\right)+3\right)\left(5\left(b_5+1\right)+3\right)d}{2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15}.$$

The expression for p_0 associated with A_1 is

$$p_0=\frac{\beta_7}{\beta_8},$$

where β_7 and β_8 are

$$\begin{aligned}
\beta_7 = -\big(&16700061952b_2^5c_2d^2 + 12441364992b_2^4c_2d^2 + 4527095552b_2^3c_2d^2 + 628029696b_2^2c_2d^2 + \\
&448100352b_2^8c_2d^2 + 3693688320b_2^7c_2d^2 + 11401393920b_2^6c_2d^2 + 2926934432b_5b_2^2c_2d^2 + \\
&50090848320b_5b_2^6c_2d^2 + 74888632192b_5b_2^5c_2d^2 + 56768683104b_5b_2^4c_2d^2 + 20931869760b_5b_2^3c_2d^2 + \\
&1894597632b_5b_2^8c_2d^2 + 15906062208b_5b_2^7c_2d^2 + 42360337552b_5^2b_2^3c_2d^2 + 5967705528b_5^2b_2^2c_2d^2 + \\
&29830562496b_5^2b_2^7c_2d^2 + 96088779936b_5^2b_2^6c_2d^2 + 146888179632b_5^2b_2^5c_2d^2 + 113377436216b_5^2b_2^4c_2d^2 + \\
&3477807360b_5^2b_2^8c_2d^2 + 129466715932b_5^3b_2^4c_2d^2 + 49009722336b_5^3b_2^3c_2d^2 + 6952764852b_5^3b_2^2c_2d^2 + \\
&3616011360b_5^3b_2^8c_2d^2 + 31810976448b_5^3b_2^7c_2d^2 + 105123958680b_5^3b_2^6c_2d^2 + 164621001152b_5^3b_2^5c_2d^2 + \\
&115319646140b_5^4b_2^5c_2d^2 + 92463798200b_5^4b_2^4c_2d^2 + 35458067164b_5^4b_2^3c_2d^2 + 5062777168b_5^4b_2^2c_2d^2 + \\
&115319646140b_5^4b_2^5c_2d^2 + 92463798200b_5^4b_2^4c_2d^2 + 35458067164b_5^4b_2^3c_2d^2 + 5062777168b_5^4b_2^2c_2d^2 + \\
&2325745584b_5^4b_2^8c_2d^2 + 21087792312b_5^4b_2^7c_2d^2 + 71743895592b_5^4b_2^6c_2d^2 + 2359421121b_5^5b_2^2c_2d^2 + \\
&31278731292b_5^5b_2^6c_2d^2 + 51715235160b_5^5b_2^5c_2d^2 + 42297488969b_5^5b_2^4c_2d^2 + 16427596936b_5^5b_2^3c_2d^2 + \\
&945669600b_5^5b_2^8c_2d^2 + 8893760688b_5^5b_2^7c_2d^2 + 4759606792b_5^6b_2^3c_2d^2 + 687240852b_5^6b_2^2c_2d^2 + \\
&2328880512b_5^6b_2^7c_2d^2 + 8507999616b_5^6b_2^6c_2d^2 + 14501442880b_5^6b_2^5c_2d^2 + 12104034004b_5^6b_2^4c_2d^2 + \\
&236740032b_5^6b_2^8c_2d^2 + 1981249952b_5^7b_2^4c_2d^2 + 788488864b_5^7b_2^3c_2d^2 + 114387872b_5^7b_2^2c_2d^2 + \\
&33230592b_5^7b_2^8c_2d^2 + 345886464b_5^7b_2^7c_2d^2 + 1320209472b_5^7b_2^6c_2d^2 + 2325122560b_5^7b_2^5c_2d^2 + \\
&163238400b_2^5b_5^8c_2d^2 + 142033920b_2^4b_5^8c_2d^2 + 57182976b_2^3b_5^8c_2d^2 + 8329728b_2^2b_5^8c_2d^2 + \\
&1990656b_2^8b_5^8c_2d^2 + 22284288b_2^7b_5^8c_2d^2 + 89487360b_2^6b_5^8c_2d^2\big) ,
\end{aligned}$$

$$
\begin{aligned}
\beta_8 =& 7915528192b_2^3 + 3999866880b_2^2 + 1057013760b_2 + 97505280+ \\
& 1144422400b_2^7 + 3856277504b_2^6 + 7549444096b_2^5 + 9569398784b_2^4+ \\
& 143360000b_2^8 + 19232827136b_5b_2^2 + 5000779520b_5b_2 + 457609472b_5+ \\
& 20050526208b_5b_2^6 + 39063062528b_5b_2^5 + 48482705408b_5b_2^4 + 39003434240b_5b_2^3+ \\
& 722380800b_5b_2^8 + 5878772736b_5b_2^7 + 10346227808b_5^2b_2 + 939192288b_5^2+ \\
& 87462749056b_5^2b_2^5 + 106736555584b_5^2b_2^4 + 83835836832b_5^2b_2^3 + 40420580704b_5^2b_2^2+ \\
& 1579686656b_5^2b_2^8 + 13068087936b_5^2b_2^7 + 45036886400b_5^2b_2^6 + 1101012480b_5^3+ \\
& 133465510272b_5^3b_2^4 + 102683639840b_5^3b_2^3 + 48494968528b_5^3b_2^2 + 12226198960b_5^3b_2+ \\
& 1959861248b_5^3b_2^8 + 16445084032b_5^3b_2^7 + 57196289856b_5^3b_2^6 + 110862719552b_5^3b_2^5+ \\
& 78394562584b_2^3b_5^4 + 36327291868b_2^2b_5^4 + 9025623018b_2b_5^4 + 806356150b_5^4+ \\
& 12829298144b_5^4b_2^7 + 44988909168b_5^4b_2^6 + 87119852136b_5^4b_2^5 + 103736823176b_5^4b_2^4+ \\
& 1509955520b_5^4b_2^8 + 17398085704b_5^5b_2^2 + 4262178374b_5^5b_2 + 377796630b_5^5+ \\
& 22469283344b_5^5b_2^6 + 43505380888b_5^5b_2^5 + 51347437264b_5^5b_2^4 + 38205740544b_5^5b_2^3+ \\
& 740189120b_5^5b_2^8 + 6359492256b_5^5b_2^7 + 1257322264b_5^6b_2 + 110581992b_5^6+ \\
& 13492758112b_2^5b_5^6 + 15812646496b_2^4b_5^6 + 11608327072b_2^3b_5^6 + 5202266128b_2^2b_5^6+ \\
& 225563904b_5^6b_2^8 + 1957584000b_5^6b_2^7 + 6964932928b_5^6b_2^6 + 18487808b_5^7+ \\
& 2770907136b_2^4b_5^7 + 2010607616b_2^3b_5^7 + 887931392b_2^2b_5^7 + 211833856b_2b_5^7+ \\
& 39084032b_5^7b_2^8 + 342327296b_5^7b_2^7 + 1225992192b_5^7b_2^6 + 2377633792b_5^7b_2^5+ \\
& 152002560b_2^3b_5^8 + 66232320b_2^2b_5^8 + 15605760b_2b_5^8 + 1351680b_5^8+ \\
& 26050560b_2^7b_5^8 + 93880320b_2^6b_5^8 + 182353920b_2^5b_5^8 + 211599360b_2^4b_5^8+ \\
& 2949120b_2^8b_5^8.
\end{aligned}
$$

The matrix elements $j_{1,1}^*$, $j_{1,2}^*$ and $j_{1,3}^*$ of the linear approximation A_2 are given by the following expressions:

$$
j_{1,1}^* = \frac{\beta_9}{\beta_{10}}, \quad j_{1,2}^* = \frac{\beta_{11}}{\beta_{12}}, \quad j_{1,3}^* = \frac{\beta_{13}}{\beta_{14}},
$$

where β_9, β_{10}, β_{11}, β_{12}, β_{13} and β_{14} are

$$
\begin{aligned}
\beta_9 =& -d\left(b_2^4\left(b_5+2\right)\left(192b_5^2+654b_5+552\right)\left(b_5\left(24b_5+85\right)+76\right)+\right. \\
& b_2^3\left(b_5+2\right)\left(816b_5^2+2775b_5+2340\right)\left(b_5\left(24b_5+85\right)+76\right)+ \\
& b_2^2\left(b_5+2\right)\left(856b_5^2+2918b_5+2464\right)\left(b_5\left(24b_5+85\right)+76\right)+ \\
& \left. b_2\left(b_5+2\right)\left(272b_5^2+931b_5+788\right)\left(b_5\left(24b_5+85\right)+76\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\beta_{10} =& 8b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(16b_5+25\right)+ \\
& 8b_2^3\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+ \\
& b_2^2\left(23808b_5^5+189016b_5^4+593378b_5^3+918634b_5^2+699144b_5+208320\right)+ \\
& 2b_2\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+ \\
& 4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240,
\end{aligned}
$$

$$\beta_{11} = -d\left(4b_2^4\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(192b_5^2+654b_5+552\right)+\right.$$
$$4b_2^3\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(816b_5^2+2775b_5+2340\right)+$$
$$4b_2^2\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(856b_5^2+2918b_5+2464\right)+$$
$$\left.4b_2\left(b_5+1\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(272b_5^2+931b_5+788\right)\right),$$

$$\beta_{12} = 8b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(16b_5+25\right)+$$
$$8b_2^3\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+$$
$$b_2^2\left(23808b_5^5+189016b_5^4+593378b_5^3+918634b_5^2+699144b_5+208320\right)+$$
$$2b_2\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+$$
$$4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240,$$

$$\beta_{13} = -d\left(4b_2^4\left(4b_5+7\right)\left(5b_5+8\right)\left(192b_5^2+654b_5+552\right)+\right.$$
$$4b_2^3\left(4b_5+7\right)\left(5b_5+8\right)\left(816b_5^2+2775b_5+2340\right)+$$
$$4b_2^2\left(4b_5+7\right)\left(5b_5+8\right)\left(856b_5^2+2918b_5+2464\right)+$$
$$\left.4b_2\left(4b_5+7\right)\left(5b_5+8\right)\left(272b_5^2+931b_5+788\right)\right),$$

$$\beta_{14} = 8b_2^4\left(b_5+1\right)\left(b_5+2\right)\left(4b_5+7\right)\left(9b_5+16\right)\left(16b_5+25\right)+$$
$$8b_2^3\left(2208b_5^5+17384b_5^4+53849b_5^3+81689b_5^2+60303b_5+17152\right)+$$
$$b_2^2\left(23808b_5^5+189016b_5^4+593378b_5^3+918634b_5^2+699144b_5+208320\right)+$$
$$2b_2\left(7488b_5^5+62076b_5^4+205268b_5^3+338419b_5^2+278174b_5+91200\right)+$$
$$4224b_5^5+36628b_5^4+127035b_5^3+220287b_5^2+190996b_5+66240.$$

The matrix elements $j_{2,1}^*$, $j_{2,2}^*$ and $j_{2,3}^*$ of the linear approximation A_2 are

$$j_{2,1}^* = -\frac{c_2}{2},\ j_{2,2}^* = \frac{4\left(-2b_2^2+b_2+2\right)c_2}{\left(2b_2+1\right)\left(2b_2\left(b_2+3\right)+1\right)},\ j_{2,3}^* = -\frac{\left(b_2+1\right)\left(6b_2+11\right)c_2}{2b_2\left(b_2+3\right)+1}.$$

The matrix elements $j_{3,1}^*$, $j_{3,2}^*$ and $j_{3,3}^*$ of the linear approximation A_2 are

$$j_{3,1}^* = \frac{b_2\left(b_5+2\right)\left(2\left(b_5+1\right)+1\right)\left(5\left(b_5+1\right)+3\right)d}{2\left(4\left(b_5+1\right)+3\right)\left(2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)},$$

$$j_{3,2}^* = \frac{2b_2\left(b_2+1\right)\left(\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)d}{\left(2b_2+1\right)\left(2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15\right)},$$

$$j_{3,3}^* = -\frac{b_2\left(4\left(b_5+1\right)+3\right)\left(5\left(b_5+1\right)+3\right)d}{2b_2\left(b_5+2\right)\left(16\left(b_5+1\right)+9\right)+\left(b_5+1\right)\left(32\left(b_5+1\right)+45\right)+15}.$$

The $q_0\left(d\right)$ is given as the following univariate polynomial in the parameter d

$$q_0\left(d\right) = -\left|G\left(d\right)\right| = d^3 s_2 - d^2 s_1 = d^2\left(s_2 d - s_1\right),$$

The critical Hopf bifurcation value d_0 corresponds to the parameter value d where $q_0\left(d\right)$ vanishes, that is

$$d_0\left(b_2, b_5, c_2\right) = \frac{s_1}{s_2},$$

where s_1 and s_2 are given by

$$s_1 = 2c_2 \left(48b_2^7\xi_1 (b_5+2) + 8b_2^6\xi_2 + 4b_2^5\xi_3 + 2b_2^4\xi_4 + 2b_2^3\xi_5 + b_2^2\xi_6 + b_2\xi_7 + \xi_8\right),$$

$$s_2 = b_2^7\xi_9 + b_2^6\xi_{10} + b_2^5\xi_{11} + b_2^4\xi_{12} + b_2^3\xi_{13} + b_2^2\xi_{14} + b_2\xi_{15}.$$

Here, all ξ_i with $i = 1, \ldots, 15$ are the following functions of the free positive parameter b_5

$$\begin{aligned}\xi_1 =& 62208b_5^8 + 972144b_5^7 + 6427612b_5^6 + 23663610b_5^5 + \\ & 53302233b_5^4 + 75436799b_5^3 + 65619972b_5^2 + 32104272b_5 + \\ & 6764800,\end{aligned}$$

$$\begin{aligned}\xi_2 =& 3725568b_5^{10} + 69404928b_5^9 + 575283160b_5^8 + 2796118336b_5^7 + \\ & 8829608957b_5^6 + 18933092811b_5^5 + 27919023868b_5^4 + 27950931068b_5^3 + \\ & 18173752208b_5^2 + 6924989952b_5 + 1173033984,\end{aligned}$$

$$\begin{aligned}\xi_3 =& 35543808b_5^{10} + 631720512b_5^9 + 5025839456b_5^8 + 23562612988b_5^7 + \\ & 72064218959b_5^6 + 150166424825b_5^5 + 215790936800b_5^4 + 211009828608b_5^3 + \\ & 134253027264b_5^2 + 50129417152b_5 + 8329550848,\end{aligned}$$

$$\begin{aligned}\xi_4 =& 175852800b_5^{10} + 3041243136b_5^9 + 23597030992b_5^8 + 108127417596b_5^7 + \\ & 323894768181b_5^6 + 662390544663b_5^5 + 936064503244b_5^4 + 901956372576b_5^3 + \\ & 566658850400b_5^2 + 209393737984b_5 + 34516005888,\end{aligned}$$

$$\begin{aligned}\xi_5 =& 236344320b_5^{10} + 4051485960b_5^9 + 31193642946b_5^8 + 142013473737b_5^7 + \\ & 423245686481b_5^6 + 862563011451b_5^5 + 1216928019704b_5^4 + 1173132597396b_5^3 + \\ & 739206664224b_5^2 + 274776385856b_5 + 45727283200,\end{aligned}$$

$$\begin{aligned}\xi_6 =& 360463104b_5^{10} + 6209841312b_5^9 + 48094708244b_5^8 + 220501086974b_5^7 + \\ & 662656206265b_5^6 + 1363818380569b_5^5 + 1946529004890b_5^4 + 1902199610680b_5^3 + \\ & 1217910126272b_5^2 + 461285441920b_5 + 78472519680,\end{aligned}$$

$$\begin{aligned}\xi_7 =& 152584704b_5^{10} + 2662250040b_5^9 + 20897107802b_5^8 + 97175772101b_5^7 + \\ & 296462115568b_5^6 + 619991979281b_5^5 + 900110715874b_5^4 + 895773366532b_5^3 + \\ & 584807824032b_5^2 + 226163724416b_5 + 39344087040,\end{aligned}$$

$$\begin{aligned}\xi_8 =& 28461312b_5^{10} + 503791848b_5^9 + 4012386970b_5^8 + 18934278365b_5^7 + \\ & 58626923780b_5^6 + 124456235172b_5^5 + 183441032883b_5^4 + 185368884500b_5^3 + \\ & 122901767840b_5^2 + 48277110400b_5 + 8531712000,\end{aligned}$$

$$\begin{aligned}\xi_9 =& 14376960b_5^{10} + 253556736b_5^9 + 2010593280b_5^8 + 9440091712b_5^7 + \\ & 29064405128b_5^6 + 61315460784b_5^5 + 89765898176b_5^4 + 90054499840b_5^3 + \\ & 59250402816b_5^2 + 23087024128b_5 + 4045783040,\end{aligned}$$

$$\begin{aligned}\xi_{10} =& 103587840b_5^{10} + 1828122624b_5^9 + 14506015488b_5^8 + 68155186176b_5^7 + \\ & 209984731232b_5^6 + 443306806160b_5^5 + 649468635616b_5^4 + 652031563392b_5^3 + \\ & 429313577984b_5^2 + 167407464448b_5 + 29358653440,\end{aligned}$$

$$\begin{aligned}\xi_{11} =& 267540480b_5^{10} + 4732389888b_5^9 + 37638559872b_5^8 + 177258748528b_5^7 + \\ & 547436379694b_5^6 + 1158509915468b_5^5 + 1701430351776b_5^4 + 1712361364288b_5^3 + \\ & 1130267765888b_5^2 + 441844540928b_5 + 77682954240,\end{aligned}$$

$$\begin{aligned}\xi_{12} =& 310947840b_5^{10} + 5526157824b_5^9 + 44160825408b_5^8 + 208971740496b_5^7 + \\ & 648486737282b_5^6 + 1378999859720b_5^5 + 2035085879896b_5^4 + 2058129989472b_5^3 + \\ & 1365117190784b_5^2 + 536252620288b_5 + 94740336640,\end{aligned}$$

$$\begin{aligned}\xi_{13} =& 179619840b_5^{10} + 3207738624b_5^9 + 25758391968b_5^8 + 122481167656b_5^7 + \\ & 381922138657b_5^6 + 816054739100b_5^5 + 1210057992636b_5^4 + 1229559244912b_5^3 + \\ & 819374744384b_5^2 + 323369355008b_5 + 57393162240,\end{aligned}$$

$$\begin{aligned}\xi_{14} =& 48291840b_5^{10} + 865649664b_5^9 + 6977012352b_5^8 + 33297349920b_5^7 + \\ & 104204201492b_5^6 + 223449138512b_5^5 + 332500404592b_5^4 + 339030241728b_5^3 + \quad . \\ & 226699122944b_5^2 + 89767422976b_5 + 15984762880,\end{aligned}$$

$$\begin{aligned}\xi_{15} =& 4400640b_5^{10} + 79052544b_5^9 + 638498832b_5^8 + 3053526960b_5^7 + \\ & 9575526032b_5^6 + 20574280847b_5^5 + 30675344842b_5^4 + 31337857848b_5^3 + \\ & 20994050624b_5^2 + 8328414976b_5 + 1485690880.\end{aligned}$$

The expression $p_0\left(d_0\right)$ associated with the linear approximation at p_2 is

$$p_0\left(d_0\right) = \frac{s_3}{s_4},$$

where s_3 and s_4 are

$$s_3 = \left(b_2^8\xi_{16} + b_2^7\xi_{17} + b_2^6\xi_{18} + b_2^5\xi_{19} + b_2^4\xi_{20} + b_2^3\xi_{21} + b_2^2\xi_{22}\right) cd_0^2,$$

$$s_4 = b_2^8\xi_{23} + b_2^7\xi_{24} + b_2^6\xi_{25} + b_2^5\xi_{26} + b_2^4\xi_{27} + b_2^3\xi_{28} + b_2^2\xi_{29} + b_2\xi_{30} + \xi_{31}.$$

Here all ξ_i with $i = 16, \ldots, 31$ are the following functions of the free positive parameter b_5

$$\begin{aligned}\xi_{16} =& 248832b_5^7 + 3718368b_5^6 + 23085360b_5^5 + 77809320b_5^4 + \\ & 154551888b_5^3 + 181535616b_5^2 + 117038592b_5 + 32007168,\end{aligned}$$

$$\begin{aligned}\xi_{17} =& 3324672b_5^7 + 45607344b_5^6 + 265450392b_5^5 + 850851648b_5^4 + \\ & 1623679896b_5^3 + 1846132992b_5^2 + 1158751872b_5 + 309884928,\end{aligned}$$

$$\begin{aligned}\xi_{18} =& 15073920b_5^7 + 197016984b_5^6 + 1100664540b_5^5 + 3406761666b_5^4 + \\ & 6308922744b_5^3 + 6989896512b_5^2 + 4289891904b_5 + 1125031680,\end{aligned}$$

$$\begin{aligned}\xi_{19} =& 30709440b_5^7 + 389692820b_5^6 + 2119325802b_5^5 + 6401436054b_5^4 + \\ & 11594955676b_5^3 + 12591152928b_5^2 + 7588301376b_5 + 1957549568,\end{aligned}$$

$$\begin{aligned}\xi_{20} =& 30375840b_5^7 + 379695716b_5^6 + 2035217402b_5^5 + 6062594060b_5^4 + \\ & 10836850366b_5^3 + 11621078408b_5^2 + 6921004928b_5 + 1765529600,\end{aligned}$$

$$\begin{aligned}\xi_{21} =& 14381472b_5^7 + 178678756b_5^6 + 951770906b_5^5 + 2817178926b_5^4 + \\ & 5003432532b_5^3 + 5331099344b_5^2 + 3154711488b_5 + 799678976,\end{aligned}$$

$$\begin{aligned}\xi_{22} =& 2620992b_5^7 + 32563852b_5^6 + 173375078b_5^5 + 512707234b_5^4 + \\ & 909393214b_5^3 + 967331048b_5^2 + 571289536b_5 + 144487680,\end{aligned}$$

$$\begin{aligned}\xi_{23} =& 589824b_5^7 + 6873088b_5^6 + 34115840b_5^5 + 93452480b_5^4 + \\ & 152467136b_5^3 + 148024832b_5^2 + 79097600b_5 + 17920000,\end{aligned}$$

$$\begin{aligned}\xi_{24} =& 4915200b_5^7 + 56692736b_5^6 + 278252032b_5^5 + 752676288b_5^4 + \\ & 1210582016b_5^3 + 1156099456b_5^2 + 605889792b_5 + 134092800,\end{aligned}$$

$$\begin{aligned}\xi_{25} =& 16465920b_5^7 + 188528640b_5^6 + 918138752b_5^5 + 2462904352b_5^4 + \\ & 3925200608b_5^3 + 3710414720b_5^2 + 1921873600b_5 + 419468288,\end{aligned}$$

$$\begin{aligned}\xi_{26} =& 29392896b_5^7 + 336223232b_5^6 + 1637303168b_5^5 + 4396393088b_5^4 + \\ & 7022784832b_5^3 + 6664981472b_5^2 + 3473730944b_5 + 765229568,\end{aligned}$$

$$\begin{aligned}\xi_{27} =& 31162368b_5^7 + 358833152b_5^6 + 1762411728b_5^5 + 4784088620b_5^4 + \\ & 7747794284b_5^3 + 7481465952b_5^2 + 3985539200b_5 + 902785536,\end{aligned}$$

$$\xi_{28} = 20398080b_5^7 + 237882368b_5^6 + 1185773504b_5^5 + 3274706844b_5^4 + 5410727808b_5^3 + 5348177696b_5^2 + 2927853888b_5 + 684742912,$$

$$\xi_{29} = 8048640b_5^7 + 95249920b_5^6 + 482500768b_5^5 + 1356245516b_5^4 + 2284663048b_5^3 + 2306554624b_5^2 + 1292274224b_5 + 309960960,$$

$$\xi_{30} = 1695744b_5^7 + 20318976b_5^6 + 104280064b_5^5 + 297149398b_5^4 + 507758880b_5^3 + 520306936b_5^2 + 296050912b_5 + 72157440,$$

$$\xi_{31} = 135168b_5^7 + 1632512b_5^6 + 8446180b_5^5 + 24265775b_5^4 + 41810375b_5^3 + 43204648b_5^2 + 24791792b_5 + 6094080.$$

The expression for $-\operatorname{tr} G(d_0)$ associated with A_2 is

$$-\operatorname{tr} G(d_0) = \frac{\xi_{32}}{\xi_{33}},$$

where ξ_{32} and ξ_{33} are

$$\begin{aligned}\xi_{32} =& b_2^3\left(2592b_5^5c_2 + 29904b_5^4c_2 + 131160b_5^3c_2 + 277776b_5^2c_2 + 286656b_5c_2 + 115968c_2\right) + \\ & b_2^2\left(23616b_5^5c_2 + 230080b_5^4c_2 + 892276b_5^3c_2 + 1721568b_5^2c_2 + 1652416b_5c_2 + 631168c_2\right) + \\ & b_2\left(45096b_5^5c_2 + 406772b_5^4c_2 + 1472172b_5^3c_2 + 2670560b_5^2c_2 + 2426592b_5c_2 + 882944c_2\right) + \\ & 19272b_5^5c_2 + 173476b_5^4c_2 + 625212b_5^3c_2 + 1127364b_5^2c_2 + 1016704b_5c_2 + 366720c_2,\end{aligned}$$

$$\begin{aligned}\xi_{33} =& b_2^3\left(1920b_5^5 + 17472b_5^4 + 63380b_5^3 + 114584b_5^2 + 103264b_5 + 37120\right) + \\ & b_2^2\left(6720b_5^5 + 61152b_5^4 + 221830b_5^3 + 401044b_5^2 + 361424b_5 + 129920\right) + \\ & b_2\left(3840b_5^5 + 34944b_5^4 + 126760b_5^3 + 229168b_5^2 + 206528b_5 + 74240\right) + \\ & 480b_5^5 + 4368b_5^4 + 15845b_5^3 + 28646b_5^2 + 25816b_5 + 9280.\end{aligned}$$

The transversality condition is derived from the expression that has the determinant of the bialternate sum matrix

$$-D_d\left(|G(d_0)|\right) = -\frac{s_1^2}{s_2} < 0.$$

References

[1] Castellanos, V., Falconi, M., & Llibre, J. (2008). Periodic orbits in predator-prey systems with Holling functional responses. Sci. Math. Jpn., 67(2), 205-218.

[2] Castellanos, V., & Chan-López, R. E. (2017). Existence of limit cycles in a three level trophic chain with Lotka–Volterra and Holling type II functional responses. Chaos Solitons Fractals, 95, 157-167.

[3] Castellanos, V., & Sánchez-Garduño, F. (2019). The existence of a limit cycle in a pollinator–plant–herbivore mathematical model. Nonlinear Anal. Real World Appl., 48, 212-231.

[4] Berestycki, H., & Zilio, A. (2019). Predator-prey models with competition: The emergence of territoriality. Am. Nat., 193(3), 436-446.

[5] Djilali, S., & Ghanbari, B. (2021). Dynamical behavior of two predators–one prey model with generalized functional response and time-fractional derivative. Adv. Differ. Equ., 2021(1), 1-19.

[6] Yousef, A., Thirthar, A. A., Alaoui, A. L., Panja, P., & Abdeljawad, T. (2022). The hunting cooperation of a predator under two prey's competition and fear-effect in the prey-predator fractional-order model. AIMS Math, 7(4), 5463-5479.

[7] Maghool, F. H., & Naji, R. K. (2022). The effect of fear on the dynamics of two competing prey-one predator system involving intra-specific competition. Commun. Math. Biol. Neurosci., 2022, Article-ID.

[8] Jorgensen, S. E., & Fath, B. D. (2008). Encyclopedia of ecology. Elsevier BV.

[9] Choe, J. C. (2019). Encyclopedia of animal behavior. Academic Press.

[10] Jeschke, J. M., Kopp, M., & Tollrian, R. (2004). Consumer-food systems: why type I functional responses are exclusive to filter feeders. Biol. Rev., 79(2), 337-349.

[11] Chiou, J. M., Müller, H. G., & Wang, J. L. (2004). Functional response models. Stat. Sin., 675-693.

[12] Cuthbert, R. N., Dalu, T., Wasserman, R. J., Callaghan, A., Weyl, O. L., & Dick, J. T. (2019). Using functional responses to quantify notonectid predatory impacts across increasingly complex environments. Acta Oecol., 95, 116-119.

[13] Kunegel-Lion, M., Goodsman, D. W., & Lewis, M. A. (2019). When managers forage for pests: Implementing the functional response in pest management. Ecol. Modell., 396, 59-73.

[14] Kohnke, M. C., Siekmann, I., Seno, H., & Malchow, H. (2020). A type IV functional response with different shapes in a predator–prey model. J. Theor. Biol., 505, 110419.

[15] Levin, S. A. (2013). Encyclopedia of Biodiversity. Elsevier Science.

[16] Solomon, M. E. (1949). The natural control of animal populations. J. Anim. Ecol., 1-35.

[17] Holling, C. S. (1959). Some characteristics of simple types of predation and parasitism1. Can. Entomol., 91(7), 385-398.

[18] Hastings, A. (Ed.). (2013). Population biology: concepts and models. Springer Science & Business Media.

[19] May, R. M. (1974). Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos. Science, 186(4164), 645-647.

[20] Schaffer, W. M., & Kot, M. (1986). Differential systems in ecology and epidemiology. Chaos, 8, 158-178.

[21] Klebanoff, A., & Hastings, A. (1994). Chaos in three species food chains. J. Math. Biol., 32, 427-451.

[22] May, R. M. (1976). Simple mathematical models with very complicated dynamics. Nature, 261(5560), 459-467.

[23] Gilpin, M. E. (1979). Spiral chaos in a predator-prey model. Am. Nat., 113(2), 306-308.

[24] Hastings, A., Hom, C. L., Ellner, S., Turchin, P., & Godfray, H. C. J. (1993). Chaos in ecology: is mother nature a strange attractor?. Annu. Rev. Ecol. Evol. Syst., 24(1), 1-33.

[25] Ali, N., & Chakravarty, S. (2015). Stability analysis of a food chain model consisting of two competitive preys and one predator. Nonlinear Dyn., 82, 1303-1316.

[26] Chan-López, E., & Castellanos, V. (2022). Biological control in a simple ecological model via subcritical Hopf and Bogdanov-Takens bifurcations. Chaos Solitons Fractals, 157, 111921.

[27] Stéphanos, C. (1900). Sur une extension du calcul des substitutions linéaires. Journal de Mathématiques pures et Appliquées, 6, 73-128.

[28] Fuller, A. T. (1968). Conditions for a matrix to have only characteristic roots with negative real parts. J. Math. Anal. Appl., 23(1), 71-98.

[29] Liu, W. M. (1994). Criterion of Hopf bifurcations without using eigenvalues. J. Math. Anal. Appl., 182(1), 250-256.

[30] Mustafa, D., & Davidson, T. N. (1995). Block bialternate sum and associated stability formulae. Automatica, 31(9), 1263-1274.

[31] Govaerts, W., & Sijnave, B. (1999). Matrix manifolds and the Jordan structure of the bialternate matrix product. Linear Algebra Appl., 292(1-3), 245-266.

[32] Govaerts, W. J. (2000). Numerical methods for bifurcations of dynamical equilibria. Society for Industrial and Applied Mathematics.

[33] Chan-López, E., & Castellanos, V. (2022). BialternateSum. Wolfram Function Repository. https://resources.wolframcloud.com/FunctionRepository/resources/BialternateSum/

[34] Sandri, M. (1996). Numerical calculation of Lyapunov exponents. Math. J., 6(3), 78-84.

[35] Baier, G., & Klein, M. (Eds.). (1991). A chaotic hierarchy. World Scientific.

[36] Bazykin, A. D. (1998). Nonlinear dynamics of interacting populations. World Scientific.

[37] Gause, G. F. (1932). Experimental studies on the struggle for existence: I. Mixed population of two species of yeast. J. Exp. Biol., 9(4), 389-402.

[38] Mulder, C., & Hendriks, A. J. (2014). Half-saturation constants in functional responses. Glob. Ecol. Conserv., 2, 161-169.

[39] Guckenheimer, J., Myers, M., & Sturmfels, B. (1997). Computing hopf bifurcations i. SIAM J. Numer. Anal., 34(1), 1-21.

[40] Kuznetsov, Y. A. (2004). Elements of Applied Bifurcation, Theory, Third edition (Springer-Verlag).