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Symmetric comet-type periodic orbits in the elliptic three-dimensional restricted (N+1)-body problem

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ABSTRACT

For $N \geq 3$, we show the existence of symmetric periodic orbits of very large radii in the elliptic three-dimensional restricted $(N+1)$-body problem when the N primaries have equal masses and are arranged in a N-gon central configuration. These periodic orbits are close to very large circular Keplerian orbits lying nearly a plane perpendicular to that of the primaries. They exist for a discrete sequence of values of the mean motion, no matter the value of the eccentricity of the primaries.

1. Introduction

The study of the motion of a system of particles is a subject of great interest, among other things, due to many applications in several physical and chemical systems, and even more so in the three-dimensional case. A particular example is the restricted $(N+1)$-body problem, where one body is considered to have infinitesimal mass and the other N bodies, called primaries, follows a solution of the N-body problem. This model for instance can be used to describe the dynamics of many situations in the Solar System, such that the motion of an artificial satellite or the motion of asteroids.

In the case of non-integrable dynamical systems, the only solutions that can be known for all time are those of a periodic or asymptotic nature. A classical tool in order to show the existence of periodic orbits in a non-integrable systems is the well-known Poincare's continuation method, where the implicit function theorem is applied to solve a system of equations with a small parameter.

In [1] the authors show the existence of some symmetric periodic solutions of the elliptic three-dimensional restricted three-body problem when the primaries are of equal mass, and without any restriction imposed on the eccentricity. The solutions found are slightly perturbed circular Keplerian orbits lying in a plane perpendicular to that of the primaries, with very large semiaxes, and the inverse of the semiaxis itself is taken as the small parameter used in the continuation. In [2], the same authors remove the restriction that the primaries have equal mass and consider any possible value of the mass parameter, with the result that a new degeneracy appears in the system due to the loss of a symmetry. These two papers together cover what would correspond to the case $N = 2$ of the results in the present paper; note nevertheless that the arguments that we provide here require $N \geq 3$ and therefore, in a strict sense, we do not obtain a generalization of their results.

After that, in [3] a double averaging procedure was used to search for periodic solutions that are not necessarily perturbed circular Keplerian orbits, but have any eccentricity. However, periodic orbits were only proved to exist in the truncated doubly-reduced system, and their persistence in the original system would require further study.

Besides, periodic orbits in restricted $(N+1)$-body problems have already been studied for the circular case in [4–8].

We will adapt the previous results to the case where N primaries with equal masses are located at the vertices of a regular N-gon and move in elliptic orbits in the same plane, for $N \geq 3$. Our approach follows closely what was done in [2]. Note that, like in that paper, here we search for periodic orbits located approximately in a plane perpendicular to the q_1q_2-plane where the primaries are located. This is convenient to do, since polar orbits make it easier to address the

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problem of the precession of the line of nodes (i.e., the line where the orbital plane of the infinitesimal mass and the q_1q_2-plane of the primaries intersect). Essentially, the fact that the orbits of the primaries are elliptical and not circular causes a slow rotation of the orbital plane (of the infinitesimal mass) around the vertical axis q_3. This hinders the search for periodic orbits, since a whole turn of the precession would be required for the orbit to return to the original position. The precession is minimal for orbits near polar ones, and can hopefully be compensated with the variation of the orbital plane due to the perturbation.

The work is structured as follows: first we introduce the equations of motion of the elliptic three-dimensional restricted $(N+1)$-body problem, and introduce a small parameter that is roughly the inverse of the distance to the primaries. Then a new difficulty shows up due to the fast motion of the primaries: the equations are no longer analytic when the parameter is equal to zero, which precludes the use of the standard implicit function theorem. This difficulty is readily overcome in Section 3.4 by resorting to Arenstorf's version of that theorem [9], where weaker assumptions on differentiability are needed. This turns out to be fundamental in the proof of the result. In Section 4 doubly-symmetric periodic orbits in the case of even N are also established. Finally we discuss in Section 5 the extension of the present results when the N primaries follow other homographic solutions of the N-body problem.

2. The elliptic three-dimensional restricted (N+1)-body problem

The elliptic three-dimensional restricted $(N+1)$-body problem describes the motion in $\mathbb{R}^3$ of a body of infinitesimal mass under the Newtonian gravitational field created by N bodies called *primaries*. We will consider the case where there are at least three primaries; i.e., $N \geq 3$.

In our study, the primaries are supposed to be arranged in a N-gon central configuration and move in elliptic orbits, all of them contained in the same plane, around their center of mass which remains fixed at the origin. The primaries have the same mass M, and the elliptic orbits have the same eccentricity $\eta \in [0,1)$ and the same semi-major axis a.

The equations of motion of the elliptic three-dimensional restricted $(N+1)$-body problem can be derived from the non-autonomous Hamiltonian

$$\mathcal{H}(t,\mathbf{q},\mathbf{p}) = \frac{1}{2}\,\|\mathbf{p}\|^2 - \frac{1}{N}\sum_{k=1}^{N}\frac{1}{R_k(t,\mathbf{q})}\,, \tag{1}$$

where $\mathbf{q} = (q_1,q_2,q_3)$ and $\mathbf{p} = (p_1,p_2,p_3)$ are, respectively, the position and momentum of the infinitesimal mass and R_k, $k = 1,\dots,N$, are the distances from the infinitesimal mass to each of the N primaries, which move in the q_1q_2-plane. Notice that a normalization of time, distance and masses is applied in such a way that $a = 1$ and $G\,M = 1/N$, where G is the gravitational constant. In these units, the period of the elliptic orbits of the primaries is 2π.

Letting $\mathbf{u}_k(t)$ denote the position of the kth primary at time t, we have that

$$\mathbf{u}_k(t) = (\rho\,\cos(\varphi + 2((k-1)/N)\pi)\,,\ \rho\,\sin(\varphi+2((k-1)/N)\pi)\,,\ 0)\,, \tag{2}$$

and then

$$\begin{aligned}\big(R_k(t,\mathbf{q})\big)^2 &= \|\mathbf{q}-\mathbf{u}_k(t)\|^2\\ &= \big(q_1 - \rho\,\cos(\varphi+2((k-1)/N)\pi)\big)^2\\ &\quad+\big(q_2-\rho\,\sin(\varphi+2((k-1)/N)\pi)\big)^2 + q_3^2\,,\end{aligned}$$

where $\rho = \rho(t)$ and $\varphi = \varphi(t)$ are defined through the expressions (see [10, p. 195 and p. 203]):

$$\rho = \frac{1-\eta^2}{1+\eta\cos\varphi}\,, \tag{3}$$

$$\begin{cases}\dfrac{d\varphi}{dt} = \dfrac{(1+\eta\cos\varphi)^2}{(1-\eta^2)^{3/2}},\\ \varphi(0) = 0.\end{cases} \tag{4}$$

Lemma 1. *The function φ has the following properties:*

(a) The maximal domain of definition of φ is $\mathbb{R}$.
(b) $\varphi : \mathbb{R} \to \mathbb{R}$ is bijective.
(c) $\varphi(-t) = -\varphi(t)$, $\forall t \in \mathbb{R}$.
(d) For any $\kappa \in \mathbb{Z}$, we have that $\varphi(t+\kappa\pi) = \varphi(t)+\kappa\pi$, $\forall t \in \mathbb{R}$.

Note that in particular, taking $\kappa = 2$ in (d), we have that $\varphi(t+2\pi) = \varphi(t)+2\pi$, $\forall t \in \mathbb{R}$. It is then immediate to see that Hamiltonian (1) is 2π-periodic in the time variable t.

Our aim is to show the existence of periodic solutions using a discrete symmetry, in a similar way that, for instance, in [11].

Next lemma shows that the equations of motion are invariant by a symmetry. In essence, it is a consequence of the primaries being arranged in the q_1q_2-plane in a configuration (the regular N-gon) which is symmetric with respect to the q_1-axis.

Lemma 2. *The equations of motion of the elliptic three-dimensional restricted $(N+1)$-body problem are invariant by the symmetry*

$$S : (t,q_1,q_2,q_3,p_1,p_2,p_3) \mapsto (-t,q_1,-q_2,-q_3,-p_1,p_2,p_3)\,. \tag{5}$$

Proof. First, using that $\varphi(-t) = -\varphi(t)$ (see (c) of Lemma 1), we have that

$$\begin{aligned}\cos\left(\varphi(-t)+2\left(\frac{k-1}{N}\right)\pi\right) &= \cos\left(-\varphi(t)+2\left(\frac{k-1}{N}\right)\pi\right) =\\ &= \cos\left(\varphi(t)+2\left(\frac{N-(k-1)}{N}\right)\pi\right)\\ &= \cos\left(\varphi(t)+2\left(\frac{(N-k+2)-1}{N}\right)\pi\right)\,,\end{aligned}$$

and similarly,

$$\begin{aligned}\sin\left(\varphi(-t)+2\left(\frac{k-1}{N}\right)\pi\right) &= \sin\left(-\varphi(t)+2\left(\frac{k-1}{N}\right)\pi\right) =\\ &= -\sin\left(\varphi(t)+2\left(\frac{N-(k-1)}{N}\right)\pi\right)\\ &= -\sin\left(\varphi(t)+2\left(\frac{(N-k+2)-1}{N}\right)\pi\right)\,.\end{aligned}$$

Then, taking into account that $\rho(-t) = \rho(t)$, we obtain that $R_k \circ S = R_{f(k)}$, where

$$f : \{1,\dots,N\} \to \{1,\dots,N\}$$

$$f(k) = \begin{cases}1, & \text{if } k = 1,\\ N-k+2, & \text{otherwise.}\end{cases}$$

Note that f is a bijection. Therefore $\sum_{k=1}^{N}\frac{1}{R_k} = \sum_{k=1}^{N}\frac{1}{R_k\circ S}$, and $\mathcal{H}\circ S = \mathcal{H}$. From here it is immediate to check that the equations of motions are invariant by the symmetry S. □

Due to this symmetry, and taking into account the Uniqueness Theorem for Ordinary Differential Equations, we have the following result.

Proposition 1. *Let $(\mathbf{q}(t),\mathbf{p}(t)) = (q_1(t),q_2(t),q_3(t),p_1(t),p_2(t),p_3(t))$ be a solution of the equations of motion for the infinitesimal mass, and $\varphi(t)$ a solution of (4). If, for a certain $T \in \mathbb{R}^+$, the following conditions are satisfied:*

- $q_2(0) = 0$, $q_3(0) = 0$, $p_1(0) = 0$, $\varphi(0) = 0$.
- $q_2(T/2) = 0$, $q_3(T/2) = 0$, $p_1(T/2) = 0$, $\varphi(T/2) = \kappa\pi$, *for some* $\kappa \in \mathbb{N}$.

Then $(\mathbf{q}(t),\mathbf{p}(t))$ is a symmetric periodic solution of period T.

Remark 1. Note that in order to have $\varphi(T/2) = \kappa\pi$ we must necessarily take $T = 2\kappa\pi$ (see (b) and (d) of Lemma 1).

2.1. Scaling of the spatial variables

If the infinitesimal body moves far away from the primaries, in a so-called *comet-like orbit*, its trajectory will be close to a Keplerian orbit, although in the limit the orbit would be of infinite radius. We can then take a suitable system of units in which the infinitesimal body is at distance one from the origin and the N primaries are very close to one another. In this situation, the semi-major axis of the primaries is chosen as the small parameter. When it takes very small values, the orbit of the infinitesimal mass tends to a Keplerian circular orbit of radius one; moreover, the perturbation originates a very fast periodic forcing, which has as a consequence a loss of differentiability. We introduce a small parameter $\varepsilon > 0$ and scale the variables by:

$$\tilde{\mathbf{q}} = \varepsilon^2 \mathbf{q}\,,\ \ \tilde{\mathbf{p}} = \varepsilon^{-1} \mathbf{p}\,. \tag{6}$$

This is a symplectic transformation with multiplier ε; i.e., $\mathrm{d}\tilde{\mathbf{q}} \wedge \mathrm{d}\tilde{\mathbf{p}} = \varepsilon\,\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p}$. Thus, Hamiltonian $\mathcal{H}(t, \mathbf{q}, \mathbf{p})$ in (1) becomes

$$\begin{aligned} \widetilde{\mathcal{H}}(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}; \varepsilon) &= \varepsilon\, \mathcal{H}(t, \varepsilon^{-2}\tilde{\mathbf{q}}, \varepsilon\, \tilde{\mathbf{p}}) = \varepsilon \left(\frac{1}{2} \left\| \varepsilon\, \tilde{\mathbf{p}}^2 \right\| - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{R_k(t, \varepsilon^{-2}\tilde{\mathbf{q}})} \right) \\ &= \varepsilon^3 \left(\frac{1}{2} \left\| \tilde{\mathbf{p}}^2 \right\| - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\widetilde{R}_k(t, \tilde{\mathbf{q}})} \right), \end{aligned} \tag{7}$$

where $\widetilde{R}_k(t, \tilde{\mathbf{q}}) := \varepsilon^2 R_k(t, \varepsilon^{-2}\tilde{\mathbf{q}}) = \left\| \tilde{\mathbf{q}} - \varepsilon^2\, \mathbf{u}_k(t) \right\|$ and $\mathbf{u}_k$ is given in (2).

2.2. Expansion in terms of Legendre polynomials

Using Proposition 5 in Appendix A, we expand each $1/\widetilde{R}_k$ in (7) in terms of Legendre polynomials, arriving at:

$$\begin{aligned} \frac{1}{\widetilde{R}_k(t, \tilde{\mathbf{q}})} &= \frac{1}{\| \tilde{\mathbf{q}} - \varepsilon^2\, \mathbf{u}_k(t) \|} = \sum_{j=0}^{\infty} \frac{\left\| \varepsilon^2\, \mathbf{u}_k(t) \right\|^j}{\| \tilde{\mathbf{q}} \|^{j+1}}\, L_j(\cos(\alpha_k)) \\ &= \sum_{j=0}^{\infty} \frac{\varepsilon^{2j} \rho^j}{\| \tilde{\mathbf{q}} \|^{j+1}}\, L_j(\cos(\alpha_k))\,, \end{aligned}$$

where α_k is the angle between the infinitesimal body and the kth primary, which satisfies:

$$\begin{aligned} \cos(\alpha_k) &= \frac{\langle\, \tilde{\mathbf{q}},\, \varepsilon^2\, \mathbf{u}_k(t)\, \rangle}{\| \tilde{\mathbf{q}} \|\ \| \varepsilon^2\, \mathbf{u}_k(t) \|} \\ &= \frac{\tilde{q}_1\, \varepsilon^2 \rho\, \cos\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) + \tilde{q}_2\, \varepsilon^2 \rho\, \sin\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right)}{\| \tilde{\mathbf{q}} \|\ \varepsilon^2 \rho} \\ &= \frac{\tilde{q}_1\, \cos\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) + \tilde{q}_2\, \sin\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right)}{\| \tilde{\mathbf{q}} \|}\,. \end{aligned} \tag{8}$$

We now add together the terms in the jth position of the Legendre expansions, and define:

$$\begin{aligned} \widetilde{\mathcal{H}}_0(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}; \varepsilon) &:= \varepsilon^3 \left(\frac{1}{2} \left\| \tilde{\mathbf{p}}^2 \right\| - \frac{1}{N} \sum_{k=1}^{N} \frac{\varepsilon^{2\cdot 0} \rho^0}{\| \tilde{\mathbf{q}} \|^{0+1}}\, L_0(\cos(\alpha_k)) \right) \\ &= \varepsilon^3 \left(\frac{1}{2} \left\| \tilde{\mathbf{p}}^2 \right\| - \frac{1}{\| \tilde{\mathbf{q}} \|} \right), \\ \widetilde{\mathcal{H}}_j(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}; \varepsilon) &:= \varepsilon^3 \left(-\frac{1}{N} \sum_{k=1}^{N} \frac{\varepsilon^{2j} \rho^j}{\| \tilde{\mathbf{q}} \|^{j+1}}\, L_j(\cos(\alpha_k)) \right) \\ &= \varepsilon^{3+2j} \left(-\frac{1}{N} \frac{\rho^j}{\| \tilde{\mathbf{q}} \|^{j+1}} \sum_{k=1}^{N} L_j(\cos(\alpha_k)) \right), \ \text{for } j \geq 1. \end{aligned}$$

It is then clear that we can express Hamiltonian $\widetilde{\mathcal{H}}$ in (7) in the form

$$\widetilde{\mathcal{H}}(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{3+2j}\ \widetilde{\mathrm{H}}_j(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}})\,, \tag{9}$$

Expressions for the first terms in (9) are given in the next proposition.

Proposition 2. *The functions* $\widetilde{\mathrm{H}}_0$, $\widetilde{\mathrm{H}}_1$ *and* $\widetilde{\mathrm{H}}_2$ *are given by:*

$$\begin{aligned} \widetilde{\mathrm{H}}_0(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) &= \frac{1}{2} \left\| \tilde{\mathbf{p}}^2 \right\| - \frac{1}{\| \tilde{\mathbf{q}} \|}\,, \\ \widetilde{\mathrm{H}}_1 &= 0\,, \\ \widetilde{\mathrm{H}}_2(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}) &= -\frac{1}{2} \rho^2 \frac{1}{\| \tilde{\mathbf{q}}^3 \|} \left(\frac{3}{2} \frac{(\tilde{q}_1^2 + \tilde{q}_2^2)}{\| \tilde{\mathbf{q}}^2 \|} - 1 \right). \end{aligned} \tag{10}$$

Proof. The expression for $\widetilde{\mathrm{H}}_0$ was already obtained.

For $\widetilde{\mathrm{H}}_1$, taking into account that $L_1(x) = x$, we get:

$$\widetilde{\mathrm{H}}_1(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = -\frac{1}{N} \frac{\rho}{\| \tilde{\mathbf{q}}^2 \|} \sum_{k=1}^{N} \cos(\alpha_k).$$

Then, from (8) and Lemma 5 in Appendix B, we obtain that $\sum_{k=1}^{N} \cos(\alpha_k) = 0$.

In order to get the expression for $\widetilde{\mathrm{H}}_2$, we first compute:

$$\begin{aligned} \sum_{k=1}^{N} \cos^2(\alpha_k) =& \frac{\tilde{q}_1^2}{\| \tilde{\mathbf{q}}^2 \|} \sum_{k=1}^{N} \cos^2\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) \\ &+ \frac{\tilde{q}_2^2}{\| \tilde{\mathbf{q}}^2 \|} \sum_{k=1}^{N} \sin^2\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) + \\ &+ \frac{2\, \tilde{q}_1\, \tilde{q}_2}{\| \tilde{\mathbf{q}}^2 \|} \sum_{k=1}^{N} \cos\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) \sin\left(\varphi + 2 \left(\frac{k-1}{N} \right) \pi \right) \\ =& \frac{\tilde{q}_1^2}{\| \tilde{\mathbf{q}}^2 \|} \frac{N}{2} + \frac{\tilde{q}_2^2}{\| \tilde{\mathbf{q}}^2 \|} \frac{N}{2} = \frac{N}{2} \frac{(\tilde{q}_1^2 + \tilde{q}_2^2)}{\| \tilde{\mathbf{q}}^2 \|}\,, \end{aligned}$$

where for the penultimate equality we have used identities (B.4), (B.5) and (B.6) of Lemma 6 in Appendix B. Notice that the used identities do not work when $N = 2$. In conclusion, recalling that $L_2(x) = \frac{1}{2}(3x^2 - 1)$, we obtain:

$$\begin{aligned} \widetilde{\mathrm{H}}_2(t, \tilde{\mathbf{q}}, \tilde{\mathbf{p}}) &= -\frac{1}{N} \frac{\rho^2}{\| \tilde{\mathbf{q}}^3 \|} \sum_{k=1}^{N} \frac{1}{2} \left(3\, \cos^2(\alpha_k) - 1 \right) \\ &= -\frac{1}{N} \frac{\rho^2}{\| \tilde{\mathbf{q}}^3 \|} \frac{1}{2} \left(-N + 3 \sum_{k=1}^{N} \cos^2(\alpha_k) \right) \\ &= -\frac{1}{N} \frac{\rho^2}{\| \tilde{\mathbf{q}}^3 \|} \frac{1}{2} \left(-N + 3\, \frac{N}{2} \frac{(\tilde{q}_1^2 + \tilde{q}_2^2)}{\| \tilde{\mathbf{q}}^2 \|} \right) \\ &= -\frac{\rho^2}{\| \tilde{\mathbf{q}}^3 \|} \frac{1}{2} \left(-1 + \frac{3}{2} \frac{(\tilde{q}_1^2 + \tilde{q}_2^2)}{\| \tilde{\mathbf{q}}^2 \|} \right). \quad \square \end{aligned}$$

Remark 2. The function $\mathrm{H}_0(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ is the Hamiltonian of the Kepler problem and therefore Hamiltonian (9) can be seen as a small perturbation of the Kepler problem.

3. Continuation of symmetric periodic orbits

In this section we show that circular orbits of the unperturbed Kepler problem can be continued to symmetric periodic orbits of the elliptic three-dimensional restricted $(N + 1)$-body problem for small values of ε.

3.1. Poincaré-Delaunay variables

According to Remark 2, Hamiltonian $\widetilde{\mathcal{H}}$ can be seen as a perturbed Kepler problem, and therefore it is convenient to use Poincaré–Delaunay variables. The change of variables from $(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ to the new set of variables, which we denote by $(\mathbf{Q}, \mathbf{P}) = (Q_1, Q_2, Q_3, P_1, P_2, P_3)$, can be expressed in a abbreviated manner by resorting to a series of magnitudes of the unperturbed Kepler problem; namely,

$$\begin{aligned} Q_1 &= l + g\,, & P_1 &= L\,, \\ Q_2 &= -\sqrt{2(L-G)}\sin(g)\,, & P_2 &= \sqrt{2(L-G)}\cos(g)\,, \\ Q_3 &= \xi\,, & P_3 &= G\,\cos(i)\,, \end{aligned} \tag{11}$$

where $a = a(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ is the semi-major axis of the infinitesimal body, $G = G(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ its angular momentum, $i = i(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ the inclination of its orbital plane to the $\tilde{q}_1\tilde{q}_2$-plane, $l = l(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ the mean anomaly, $g = g(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ the argument of periapsis and $\xi = \xi(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ the longitude of the ascending node; moreover, $L := \sqrt{a}$. The eccentricity of the infinitesimal body is given by $e = \sqrt{1 - G^2/L^2}$.

We emphasize that the quantities in the previous paragraph refer to magnitudes of the infinitesimal body in the unperturbed Kepler problem. For example, the angular momentum G is simply given by the formula $G = \|\tilde{\mathbf{q}}\| \times \tilde{\mathbf{p}}$.

The new variables $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$ are defined on a neighborhood of the direct circular Keplerian orbits, which occur at $L = G$, or equivalently, at $Q_2 = 0$, $P_2 = 0$. If $P_3 = 0$ the orbit lies in a plane perpendicular to the $\tilde{q}_1\tilde{q}_2$-plane.

In the new variables defined in (11), the periodicity conditions given in Proposition 1 state that at time $t = 0$ we must have

$$Q_1(0) = 0 \bmod \pi\,,\; Q_2(0) = 0\,,\; Q_3(0) = 0 \bmod \pi\,,\; \varphi(0) = 0\,, \tag{12}$$

and at time $t = T/2$

$$Q_1(T/2) = 0 \bmod \pi\,,\; Q_2(T/2) = 0\,,\; Q_3(T/2) = 0 \bmod \pi\,,\; \varphi(T/2) = \kappa\pi\,. \tag{13}$$

The condition $Q_2 = 0$ implies that either $g = 0 \bmod \pi$ or $L = G$, so that the infinitesimal body is on an elliptic orbit with its periapsis on the $\tilde{q}_1$-axis or on a circular orbit.

Moreover, the change of variables in (11) is symplectic, and Hamiltonian $\widetilde{\mathcal{H}}$ in (9) becomes

$$\begin{aligned} \mathfrak{H}(t, \mathbf{Q}, \mathbf{P}; \varepsilon) &= \sum_{j=0}^{\infty} \varepsilon^{3+2j}\, \mathfrak{H}_j(t, \mathbf{Q}, \mathbf{P}) \\ &= \varepsilon^3 \mathfrak{H}_0(\mathbf{Q}, \mathbf{P}) + \varepsilon^7 \mathfrak{H}_2(t, \mathbf{Q}, \mathbf{P}) + \varepsilon^9 \mathfrak{H}_R(t, \mathbf{Q}, \mathbf{P}; \varepsilon)\,, \end{aligned} \tag{14}$$

where

$$\mathfrak{H}_R(t, \mathbf{Q}, \mathbf{P}; \varepsilon) := \frac{1}{\varepsilon^9} \sum_{j=3}^{\infty} \varepsilon^{3+2j}\, \mathfrak{H}_j(t, \mathbf{Q}, \mathbf{P})\,,$$

$$\mathfrak{H}_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2P_1^2}\,,$$

$$\mathfrak{H}_2(t, \mathbf{Q}, \mathbf{P}) = -\frac{1}{2}\rho^2 \frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right),$$

$r = r(\mathbf{Q}, \mathbf{P})$ being the distance of the infinitesimal body to the origin, and $G = G(\mathbf{Q}, \mathbf{P})$ its angular momentum. Here we have taken into account (10), and in particular, for the expression of $\mathfrak{H}_2$, we have used that

$$\frac{(\tilde{q}_1^2 + \tilde{q}_2^2)}{\|\tilde{\mathbf{q}}^2\|} = \frac{\|(\tilde{q}_1, \tilde{q}_2, 0)\|^2}{\|\tilde{\mathbf{q}}^2\|} = \cos^2(i) = \frac{P_3^2}{G^2}\,.$$

Notice that the expression for $\mathfrak{H}_2(t, \mathbf{Q}, \mathbf{P})$ is different from the one in [2].

3.2. Lie transform

We aim to find a symplectic transformation that removes the time-dependent component in $\mathfrak{H}_2$.

More specifically, the function $\mathfrak{H}_2(t, \mathbf{Q}, \mathbf{P})$ is 2π-periodic in t and can be expanded as a Fourier series:

$$\mathfrak{H}_2(t, \mathbf{Q}, \mathbf{P}) = -\frac{1}{2}\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right) \sum_{\nu=-\infty}^{\infty} a_\nu(\mathbf{Q}, \mathbf{P})\, \mathrm{e}^{i\nu t}\,,$$

where

$$a_\nu(\mathbf{Q}, \mathbf{P}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} (\rho(t))^2\, \mathrm{e}^{-i\nu t}\, \mathrm{d}t\,.$$

In particular

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\rho(t))^2\, \mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1-\eta^2}{1+\eta\cos(\varphi)}\right)^2 \mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1-\eta^2}{1+\eta\cos(\varphi)}\right)^2 \frac{\left(1-\eta^2\right)^{3/2}}{(1+\eta\cos(\varphi))^2}\, \mathrm{d}\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1-\eta^2\right)^{7/2}}{(1+\eta\cos(\varphi))^4}\, \mathrm{d}\varphi = \frac{3}{2}\eta^2 + 1\,. \end{aligned}$$

Remark 3. In fact, in the end we will only need that a_0 does not vanish for any value of the eccentricity $\eta \in [0, 1[$, which can be already deduced from its definition as the integral of a strictly positive function, without calculating its explicit expression.

Note also that a_0 does not depend on $(\mathbf{Q}, \mathbf{P})$, only on the parameter η, again in contrast to the one in [2].

We are going to find the suitable symplectic transformation using the Lie transform method. First, given two functions $H = H(t, \mathbf{Q}, \mathbf{P})$ and $F = F(t, \mathbf{Q}, \mathbf{P})$, their Poisson bracket is defined as

$$\{H, F\} = \sum_{\varkappa=1}^{3}\left(\frac{\partial H}{\partial Q_\varkappa}\frac{\partial F}{\partial P_\varkappa} - \frac{\partial H}{\partial P_\varkappa}\frac{\partial F}{\partial Q_\varkappa}\right) - \frac{\partial F}{\partial t}\,. \tag{15}$$

Let Φ_F^s be the time s map of the Hamiltonian vector field associated to F. Applying the corresponding Lie transform to the Hamiltonian $\mathfrak{H}$ in (14), we obtain

$$\mathfrak{H}\circ\Phi_F^1 = \mathfrak{H} + \{\mathfrak{H}, F\} + \int_0^1 (1-s)\{\{\mathfrak{H}, F\}, F\}\circ\Phi_F^s\, \mathrm{d}s. \tag{16}$$

Let $F = F(t, \mathbf{Q}, \mathbf{P})$ be such that

$$\frac{\partial F}{\partial t} = \varepsilon^7\left(\frac{1}{2}\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)a_0 + \mathfrak{H}_2\right).$$

Notice that such F is $\mathcal{O}(\varepsilon^7)$. Then, taking into account (15), (16) and using the same letters $\mathbf{Q}, \mathbf{P}$ for the new variables, applying the transformation given by Φ_F^1 we arrive at a new Hamiltonian

$$K(t, \mathbf{Q}, \mathbf{P}; \varepsilon) = \varepsilon^3 K_0(\mathbf{Q}, \mathbf{P}) + \varepsilon^7 K_2(\mathbf{Q}, \mathbf{P}) + \varepsilon^9 K_R(t, \mathbf{Q}, \mathbf{P}; \varepsilon)\,,$$

where

$$K_0(\mathbf{Q}, \mathbf{P}) = \mathfrak{H}_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2P_1^2}\,,$$

$$K_2(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2}\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)a_0\,.$$

Thus the new Hamiltonian K does not depend on time up to order ε^9.

We now scale the time variable by $\tau = \varepsilon^3 t$, obtaining a new Hamiltonian $\mathfrak{K}(\tau, \mathbf{Q}, \mathbf{P}; \varepsilon) = \varepsilon^{-3} K(\varepsilon^{-3}\tau, \mathbf{Q}, \mathbf{P}; \varepsilon)$ of the form

$$\mathfrak{K}(\tau, \mathbf{Q}, \mathbf{P}; \varepsilon) = \mathfrak{K}_0(\mathbf{Q}, \mathbf{P}) + \varepsilon^4 \mathfrak{K}_1(\mathbf{Q}, \mathbf{P}) + \varepsilon^6 \mathfrak{K}_R(\tau, \mathbf{Q}, \mathbf{P}; \varepsilon)\,, \tag{17}$$

where

$$\mathfrak{K}_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2P_1^2}\,, \tag{18}$$

$$\mathfrak{K}_1(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2}\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)a_0\,, \tag{19}$$

$$\mathfrak{K}_R(\tau, \mathbf{Q}, \mathbf{P}; \varepsilon) = K_R(\varepsilon^{-3}\tau, \mathbf{Q}, \mathbf{P}; \varepsilon)\,. \tag{20}$$

From now on we will work with τ as the new time variable and we will continue using the notation $\dot{\mathbf{Q}}(\tau) := \frac{\mathrm{d}\mathbf{Q}(\tau)}{\mathrm{d}\tau}$, $\dot{\mathbf{P}}(\tau) := \frac{\mathrm{d}\mathbf{P}(\tau)}{\mathrm{d}\tau}$.

3.3. Approximation of solutions of the perturbed system

The function $\mathfrak{K}_R(\tau, \mathbf{Q}, \mathbf{P}; \varepsilon)$ given in (20) is bounded by a constant independent of ε, since the term $\varepsilon^{-3}\tau$ only appears in the argument of trigonometric functions. Note that $\varepsilon^6\mathfrak{K}_R$ is continuous at $\varepsilon = 0$, but $\mathfrak{K}_R$ is not so because, as ε goes to zero, the frequency of the oscillations tends to infinity. Moreover, the loss of differentiability with respect to ε at $\varepsilon = 0$ prevents the use of expansions in power series in ε, and instead we are going to apply the results in Appendix C.

Let us denote $\mathbf{z} = (\mathbf{Q}, \mathbf{P})$, then the equations of motion derived from the $2\pi\epsilon^3$-periodic Hamiltonian (17) can be written as

$$\dot{\mathbf{z}} = \mathcal{F}_0(\mathbf{z}) + \epsilon^4 \mathcal{F}_1(\mathbf{z}) + \epsilon^6 \mathcal{F}_R(\tau, \mathbf{z}; \epsilon), \tag{21}$$

where

$$\mathcal{F}_0(\mathbf{z}) = \left(P_1^{-3}, 0, 0, 0, 0, 0\right),$$
$$\mathcal{F}_1(\mathbf{z}) = \left(\frac{\partial \mathfrak{K}_1}{\partial P_1}, \frac{\partial \mathfrak{K}_1}{\partial P_2}, \frac{\partial \mathfrak{K}_1}{\partial P_3}, -\frac{\partial \mathfrak{K}_1}{\partial Q_1}, -\frac{\partial \mathfrak{K}_1}{\partial Q_2}, -\frac{\partial \mathfrak{K}_1}{\partial Q_3}\right).$$

The solution of Kepler problem (i.e., of (21) for $\epsilon = 0$) with initial conditions $\zeta = (Q_{10}, Q_{20}, Q_{30}, P_{10}, P_{20}, P_{30})$ at $\tau = 0$ is given by

$$\mathbf{z}^{(0)}(\tau, \zeta) = (Q_{10} + P_{10}^{-3}\,\tau, Q_{20}, Q_{30}, P_{10}, P_{20}, P_{30}). \tag{22}$$

In particular, for $\mathbf{z}_0^* = (0,0,0,1,0,0)$ we have that

$$\mathbf{z}^{(0)}(\tau, \mathbf{z}_0^*) = (\tau, 0, 0, 1, 0, 0). \tag{23}$$

We will look for initial conditions, in a neighborhood of $\mathbf{z}_0^*$, of the form $\mathbf{z}_0 = (0,0,0,P_{10}, P_{20}, P_{30})$ (see (12)), in such a way that the solution $\mathbf{z}(\tau, \mathbf{z}_0; \epsilon)$ of system (21), with $\epsilon \neq 0$ small enough, is a symmetric periodic solution.

From Lemma 7, we have that $\mathbf{z}(\tau, \mathbf{z}_0; \epsilon) = \mathbf{z}^{(0)}(\tau, \mathbf{z}_0) + \epsilon^4 \mathbf{z}^{(1)}(\tau, \mathbf{z}_0) + \mathbf{z}_R(\tau, \mathbf{z}_0; \epsilon)$, where $\mathbf{z}_R(\tau, \mathbf{z}_0; \epsilon) = \mathcal{O}(\epsilon^6)$, $\mathbf{z}^{(0)}(\tau, \mathbf{z}_0) = \left(P_{10}^{-3}\,\tau, 0, 0, P_{10}, P_{20}, P_{30}\right)$ and $\mathbf{z}^{(1)}(\tau, \mathbf{z}_0)$ is given by

$$\mathbf{z}^{(1)}(\tau, \mathbf{z}_0) = \mathcal{Z}(\tau, \mathbf{z}_0) \int_0^\tau \left(\mathcal{Z}(s, \mathbf{z}_0)\right)^{-1} \mathcal{F}_1(\mathbf{z}^{(0)}(s, \mathbf{z}_0))\,\mathrm{d}s, \tag{24}$$

where $\mathcal{Z}(\tau, \mathbf{z}_0)$ is the matrix

$$\mathcal{Z}(\tau, \mathbf{z}_0) = \left.\frac{\partial \mathbf{z}^{(0)}(\tau, \zeta)}{\partial \zeta}\right|_{\zeta = \mathbf{z}_0} = \begin{pmatrix} 1 & 0 & 0 & -3P_{10}^{-4}\tau & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & I_5 & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}.$$

Note that

$$\left(\mathcal{Z}(\tau, \mathbf{z}_0)\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 3P_{10}^{-4}\tau & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & I_5 & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}.$$

Then we have, correct to order ϵ^4

$$Q_1(\tau, \mathbf{z}_0; \epsilon) = P_{10}^{-3}\,\tau + \mathcal{O}(\epsilon^4),$$

and correct to order ϵ^6

$$Q_2(\tau, \mathbf{z}_0; \epsilon) = \epsilon^4 Q_2^{(1)}(\tau, \mathbf{z}_0) + \mathcal{O}(\epsilon^6),$$
$$Q_3(\tau, \mathbf{z}_0; \epsilon) = \epsilon^4 Q_3^{(1)}(\tau, \mathbf{z}_0) + \mathcal{O}(\epsilon^6),$$

where $Q_2^{(1)}(\tau, \mathbf{z}_0)$ and $Q_3^{(1)}(\tau, \mathbf{z}_0)$ are given in the following lemma.

Lemma 3. *Let* $\mathbf{x} := (\Delta P_1, P_2, P_3) := (P_1 - 1, P_2, P_3)$ *and* $\mathbf{x}_0 = (\Delta P_{10}, P_{20}, P_{30}) = (P_{10} - 1, P_{20}, P_{30})$. *Then*

$$Q_2^{(1)}(\tau, \mathbf{z}_0) = \frac{1}{2} a_0\, 3\sin(\tau) + \frac{1}{2} a_0 \left(-\frac{21}{2}\sin(\tau) - 9\tau\cos(\tau)\right) \Delta P_{10} + \frac{1}{2} a_0 \left(3\tau + 9\sin(\tau)\cos(\tau)\right) P_{20} + \mathcal{O}(\|\mathbf{x}_0\|^2),$$
$$Q_3^{(1)}(\tau, \mathbf{z}_0) = -\frac{1}{2} a_0\, 3\tau P_{30} + \mathcal{O}(\|\mathbf{x}_0\|^2),$$

for τ in a finite interval of time.

Proof. From Eqs. (19) and (24) we have, for $\varkappa = 2, 3$,

$$Q_\varkappa^{(1)}(\tau, \mathbf{z}_0) = \int_0^\tau \left(\frac{\partial \mathfrak{K}_1}{\partial P_\varkappa}\right)_{\mathbf{z}^{(0)}(s, \mathbf{z}_0)} \mathrm{d}s = -\frac{1}{2} a_0 \int_0^\tau \left(\frac{\partial}{\partial P_\varkappa}\left(\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)\right)\right)_{\mathbf{z}^{(0)}(s, \mathbf{z}_0)} \mathrm{d}s.$$

We aim to expand G and r as power series in $\mathbf{x} = (\Delta P_1, P_2, P_3)$ up to second order. Since we are interested in a neighborhood of the circular periodic orbit $\mathbf{z}^{(0)}(\tau, \mathbf{z}_0^*)$ given in (23), the expansions provided in [2, Appendix] are convergent. For instance, we have that

$$\begin{aligned} r = {} & 1 + 2\Delta P_1 - P_2\cos(Q_1) + \Delta P_1^2 + P_2^2\sin^2(Q_1) \\ & - \frac{3}{2}\Delta P_1 P_2\cos(Q_1) + Q_2\sin(Q_1) + Q_2^2\cos^2(Q_1) \\ & + \frac{3}{2} Q_2 \Delta P_1 \sin(Q_1) + Q_2 P_2 \sin(2Q_1) + \mathcal{O}(\|\mathbf{x}\|^3). \end{aligned} \tag{25}$$

On the other hand, from the definition of the Poincaré–Delaunay variables in (11), it is straightforward that

$$G = P_1 - \frac{1}{2}\left(Q_2^2 + P_2^2\right) = 1 + \Delta P_1 - \frac{1}{2}P_2^2 - \frac{1}{2}Q_2^2. \tag{26}$$

Moreover, for an initial condition of the form $\mathbf{z}_0 = (0,0,0,P_{10}, P_{20}, P_{30})$, all the points in the periodic orbit $\mathbf{z}^{(0)}(s, \mathbf{z}_0) = \left(P_{10}^{-3}\,s, 0, 0, P_{10}, P_{20}, P_{30}\right)$ clearly satisfy that $Q_2 = Q_3 = 0$.

We shall therefore omit the terms in (25) and (26) which vanish for $Q_2 = Q_3 = 0$, obtaining respectively

$$\begin{aligned} r = {} & 1 + 2\Delta P_1 - \cos(Q_1) P_2 + \Delta P_1^2 - \frac{3}{2}\cos(Q_1)\Delta P_1 P_2 \\ & + \sin^2(Q_1) P_2^2 + \mathcal{O}(\|\mathbf{x}\|^3) \end{aligned} \tag{27}$$

and

$$G = 1 + \Delta P_1 - \frac{1}{2}P_2^2. \tag{28}$$

Using (27) and (28), we get that

$$\begin{aligned} \frac{1}{r^3} = {} & 1 - 6\Delta P_1 + 3\cos(Q_1) P_2 + 21\Delta P_1^2 - \frac{39}{2}\cos(Q_1)\Delta P_1 P_2 \\ & + \left(6\cos^2(Q_1) - 3\sin^2(Q_1)\right) P_2^2 + \mathcal{O}(\|\mathbf{x}\|^3), \\ \frac{1}{G^2} = {} & 1 - 2\Delta P_1 + 3\Delta P_1^2 + P_2^2 + \mathcal{O}(\|\mathbf{x}\|^3). \end{aligned}$$

Thus

$$\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right) = -1 + 6\Delta P_1 - 3\cos(Q_1) P_2 - 21\Delta P_1^2 + \frac{39}{2}\cos(Q_1)\Delta P_1 P_2 + \left(3\sin^2(Q_1) - 6\cos^2(Q_1)\right) P_2^2 + \frac{3}{2}P_3^2 + \mathcal{O}(\|\mathbf{x}\|^3).$$

When $\varkappa = 2$, first we calculate

$$\frac{\partial}{\partial P_2}\left(\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)\right) = -3\cos(Q_1) + \frac{39}{2}\cos(Q_1)\Delta P_1 + \left(6\sin^2(Q_1) - 12\cos^2(Q_1)\right) P_2 + \mathcal{O}(\|\mathbf{x}\|^2).$$

Taking into account that on the periodic orbit $\mathbf{z}^{(0)}(s, \mathbf{z}_0)$ we have that $Q_1^{(0)}(s, \mathbf{z}_0) = P_{10}^{-3}\,s = (\Delta P_{10} + 1)^{-3}\,s = (1 - 3\Delta P_{10} + \mathcal{O}(\|\mathbf{x}_0\|^2))\,s$, we arrive at

$$\begin{aligned} & \left(\frac{\partial}{\partial P_2}\left(\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)\right)\right)_{\mathbf{z}^{(0)}(s, \mathbf{z}_0)} \\ = {} & -3\cos((1 - 3\Delta P_{10})\,s) + \frac{39}{2}\cos(s)\Delta P_{10} \\ & + \left(6\sin^2(s) - 12\cos^2(s)\right) P_{20} + \mathcal{O}(\|\mathbf{x}_0\|^2) \\ = {} & -3\cos(s) + \left(\frac{39}{2}\cos(s) - 9s\sin(s)\right)\Delta P_{10} \\ & + \left(6\sin^2(s) - 12\cos^2(s)\right) P_{20} + \mathcal{O}(\|\mathbf{x}_0\|^2). \end{aligned}$$

And finally we obtain

$$\begin{aligned} & \int_0^\tau \left(\frac{\partial}{\partial P_2}\left(\frac{1}{r^3}\left(\frac{3}{2}\frac{P_3^2}{G^2} - 1\right)\right)\right)_{\mathbf{z}^{(0)}(s, \mathbf{z}_0)} \mathrm{d}s \\ = {} & -3\sin(\tau) + \left(\frac{21}{2}\sin(\tau) + 9\tau\cos(\tau)\right)\Delta P_{10} \\ & + (-3\tau - 9\sin(\tau)\cos(\tau)) P_{20} + \mathcal{O}(\|\mathbf{x}_0\|^2). \end{aligned}$$

The expression of $Q_3^{(1)}(\tau, \mathbf{z}_0)$ can be derived similarly. □

Recall now the periodicity conditions (13) that our solution had to satisfy at $t = T/2$ (from Remark 1 we in fact know that $T/2 = \kappa\pi$ for some $\kappa \in \mathbb{N}$). Since $\tau = \varepsilon^3 t$, in the new time variable τ it corresponds to the value $\varepsilon^3\kappa\pi$.

Taking into account that our solution $\mathbf{z}(\tau, \mathbf{z}_0; \varepsilon)$ is given at first order by the unperturbed Kepler solution $\mathbf{z}^{(0)}(\tau, \mathbf{z}_0^*)$ in (23), which satisfies those periodicity conditions at $\tau = \pi$, we choose ε such that $\varepsilon^3\kappa\pi = \pi$; i.e., we take $\varepsilon^3 = 1/\kappa$ for some $\kappa \in \mathbb{N}$.

In summary, to guarantee the existence of a periodic orbit, we will impose the following conditions of symmetry at time $\tau = \pi$:

$$Q_1(\pi, \mathbf{z}_0; \varepsilon) = \pi\,, \quad Q_2(\pi, \mathbf{z}_0; \varepsilon) = 0\,, \quad Q_3(\pi, \mathbf{z}_0; \varepsilon) = 0\,.$$

Let us define $f = (f_1, f_2, f_3)$, where $f_1(\mathbf{x}_0, \varepsilon) := Q_1(\pi, \mathbf{z}_0; \varepsilon) - \pi$, $f_\varkappa(\mathbf{x}_0, \varepsilon) := \varepsilon^{-4} Q_\varkappa(\pi, \mathbf{z}_0; \varepsilon)$, $\varkappa = 2, 3$. Taking into account Lemma 3, we have that

$$\begin{aligned} f_1(\mathbf{x}_0, \varepsilon) &= (1 + \Delta P_{10})^{-3}\pi - \pi + \mathcal{O}(\varepsilon^4)\,,\\ f_2(\mathbf{x}_0, \varepsilon) &= \frac{1}{2}\, a_0 \left(9\pi\, \Delta P_{10} + 3\pi\, P_{20}\right) + \mathcal{O}(\|\mathbf{x}_0\|^2) + \mathcal{O}(\varepsilon^2)\,,\\ f_3(\mathbf{x}_0, \varepsilon) &= -\frac{1}{2}\, a_0\, 3\pi P_{30} + \mathcal{O}(\|\mathbf{x}_0\|^2) + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{29}$$

Remark 4. Strictly speaking, the function $f = f(\mathbf{x}_0, \varepsilon)$ is not defined when $\varepsilon = 0$. Nevertheless, taking into account Lemmas 7 and 8, f and its derivative with respect to $\mathbf{x}_0$ admit a continuous extension for $\varepsilon = 0$, which can be obtained by making formally vanish the terms $\mathcal{O}(\varepsilon)$ in (29).

3.4. Continuation method with non-regular dependency on the parameter

In conclusion, to obtain symmetric periodic orbits we need to find solutions $\mathbf{x}_0$ of the system of equations $f(\mathbf{x}_0, \varepsilon) = 0$ for $\varepsilon \neq 0$.

However, the standard implicit function theorem cannot be applied, since it requires differentiability of the function with respect to all variables, and our function f is not differentiable with respect to ε at $\varepsilon = 0$.

Instead, we shall use Arenstorf's theorem [9]. More specifically, we are going to use the following corollary, which was given in [2].

Proposition 3. *Let $U \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ be open domains with $(0,0) \in U \times I$, and let $f : U \times I \to \mathbb{R}^n$, $f = f(\mathbf{x}, \varepsilon)$ be differentiable with respect to $\mathbf{x} \in U$ and such that $f(0,0) = 0$ and $\partial_{\mathbf{x}} f(0,0)$ is nonsingular. Assume that there exist $c > 0$, $C > 0$ such that for $\mathbf{x} \in U$, $\varepsilon \in I$:*

(i) $\|\partial_{\mathbf{x}} f(\mathbf{x}, \varepsilon) - \partial_{\mathbf{x}} f(0,0)\| \le c\,(\|\mathbf{x}\| + |\varepsilon|)$,
(ii) $\|f(0, \varepsilon)\| \le C|\varepsilon|$.

Then there exist an open domain $\widetilde{I}$ such that $0 \in \widetilde{I} \subseteq I$, and a function $\mathbf{x} : \widetilde{I} \to U$ satisfying that $x(0) = 0$ and $f(\mathbf{x}(\varepsilon), \varepsilon) = 0$.

Note that in Proposition 3, and in contrast to the standard implicit function theorem, differentiability with respect to ε is not required.

We are now going to prove our main result:

Theorem 1. *Let $N \ge 3$ and consider the equations of motion for the elliptic three-dimensional restricted $(N+1)$-body problem with the primaries moving around each other on elliptic orbits with semi-major axis one, period 2π and eccentricity $\eta \in [0, 1[$. Then, for κ a positive integer large enough, there exist initial conditions for the infinitesimal body such that its motion is a symmetric periodic solution of period $2\kappa\pi$, near a Keplerian circular orbit on a plane perpendicular to that of the primaries and with a radius of order $\kappa^{2/3}$.*

Proof. We have to check that the function $f(\mathbf{x}_0, \varepsilon)$ defined in (29) satisfies the conditions stated in Proposition 3 that guarantee the existence of solutions of $f(\mathbf{x}_0, \varepsilon) = 0$ in a neighborhood of $(0,0)$. Throughout the proof it will be important to take into account Remark 4.

Note that the system $f(\mathbf{x}_0, \varepsilon) = 0$ has the solution $\mathbf{x}_0 = 0$ for $\varepsilon = 0$.

Let $\partial_{\mathbf{x}_0} f(\mathbf{x}_0, \varepsilon)$ be the Jacobian matrix of $f(\mathbf{x}_0, \varepsilon)$ with respect to $\mathbf{x}_0$. For $\varepsilon = 0$ we have

$$\partial_{\mathbf{x}_0} f(0,0) = -\frac{1}{2}\, a_0 \begin{pmatrix} \frac{-3\pi}{-\frac{1}{2} a_0} & 0 & 0\\ -9\pi & -3\pi & 0\\ 0 & 0 & 3\pi \end{pmatrix}.$$

Note that the matrix $\partial_{\mathbf{x}_0} f(0,0)$ is nonsingular for any value of the eccentricity η, since $a_0 \neq 0$ for any value of η (see Remark 3).

In order to check Condition (i) of Proposition 3, we write

$$\begin{aligned} &\left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, \varepsilon) - \partial_{\mathbf{x}_0} f(0,0)\right\|\\ &\le \left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, \varepsilon) - \partial_{\mathbf{x}_0} f(\mathbf{x}_0, 0)\right\| + \left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, 0) - \partial_{\mathbf{x}_0} f(0,0)\right\|\,. \end{aligned}$$

Then, on the one hand, we have that

$$\left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, \varepsilon) - \partial_{\mathbf{x}_0} f(\mathbf{x}_0, 0)\right\| \le \sum_{\varkappa=1}^{3} \left\|\partial_{\mathbf{x}_0} f_\varkappa(\mathbf{x}_0, \varepsilon) - \partial_{\mathbf{x}_0} f_\varkappa(\mathbf{x}_0, 0)\right\|\,,$$

where the first term of the sum is bounded by $c_2\,\varepsilon^4$ and the second and third are less than $c_3\,\varepsilon^2$ because of (29).

And on the other hand

$$\left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, 0) - \partial_{\mathbf{x}_0} f(0,0)\right\| \le \sum_{\varkappa=1}^{3} \left\|\partial_{\mathbf{x}_0} f_\varkappa(\mathbf{x}_0, 0) - \partial_{\mathbf{x}_0} f_\varkappa(0,0)\right\| \le c_1\,\|\mathbf{x}_0\|\,,$$

where in the last inequality we have used that $f(\mathbf{x}_0, 0)$ is analytic as a function of $\mathbf{x}_0$.

In conclusion, for $\mathbf{x}_0$ small enough, we have that

$$\left\|\partial_{\mathbf{x}_0} f(\mathbf{x}_0, \varepsilon) - \partial_{\mathbf{x}_0} f(0,0)\right\| \le c\left(\|\mathbf{x}_0\| + |\varepsilon|\right)\,.$$

Finally, Condition (ii) of Proposition 3 is a straightforward consequence of (29) . □

4. Doubly-symmetric periodic orbits in the case of even N

When the number of primaries N is even, then the system has an additional symmetry:

Lemma 4. *If N is an even number, then the equations of motion of the elliptic three-dimensional restricted $(N+1)$-body problem are also invariant by the symmetry*

$$\widetilde{S} : (t, q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (-t, -q_1, q_2, q_3, p_1, -p_2, -p_3)\,. \tag{30}$$

In this case, due to the presence of the two symmetries S in (5) and $\widetilde{S}$ in (30), we may prove, as in [1], the existence of doubly-symmetric periodic orbits (i.e., periodic solutions invariant by both symmetries) using the following result

Proposition 4. *Let $(\mathbf{q}(t), \mathbf{p}(t)) = (q_1(t), q_2(t), q_3(t), p_1(t), p_2(t), p_3(t))$ be a solution of the equations of motion for the infinitesimal mass, and $\varphi(t)$ a solution of (4). If, for a certain $T \in \mathbb{R}^+$, the following conditions are satisfied:*

- $q_2(0) = 0$, $q_3(0) = 0$, $p_1(0) = 0$, $\varphi(0) = 0$.
- $q_1(T/4) = 0$, $p_2(T/4) = 0$, $p_3(T/4) = 0$, $\varphi(T/4) = \kappa\pi$, *for some* $\kappa \in \mathbb{N}$.

Then $(\mathbf{q}(t), \mathbf{p}(t))$ is a doubly-symmetric periodic solution of period T.

Remark 5. Note that in order to have $\varphi(T/4) = \kappa\pi$ we must necessarily take $T = 4\kappa\pi$ (see (b) and (d) of Lemma 1).

In Poincaré–Delaunay variables, these periodicity conditions state that at time $t = 0$ we must have

$$Q_1(0) = 0 \bmod \pi\,,\; Q_2(0) = 0\,,\; Q_3(0) = 0 \bmod \pi\,,\; \varphi(0) = 0\,, \tag{31}$$

and at time $t = T/4$

$$Q_1(T/4) = \frac{\pi}{2} \bmod \pi\,,\; Q_3(T/4) = 0 \bmod \pi\,,\; P_2(T/4) = 0\,,\; \varphi(T/4) = \kappa\pi\,. \tag{32}$$

Akin to Section 3.3, we look for initial conditions of the form $\mathbf{z}_0 = (0,0,0,P_{10},P_{20},P_{30})$, in such a way that the solution $\mathbf{z}(\tau,\mathbf{z}_0;\varepsilon)$ is a doubly-symmetric periodic orbit.

To that end, let $\mathbf{x}_0 = (\Delta P_{10},P_{20},P_{30}) = (P_{10}-1,P_{20},P_{30})$ and compute

$$Q_1(\tau,\mathbf{z}_0;\varepsilon) = (1+\Delta P_{10})^{-3}\,\tau + \mathcal{O}(\varepsilon^4)\,,$$
$$Q_3(\tau,\mathbf{z}_0;\varepsilon) = \varepsilon^4\,Q_3^{(1)}(\tau,\mathbf{z}_0) + \mathcal{O}(\varepsilon^6)\,,$$
$$P_2(\tau,\mathbf{z}_0;\varepsilon) = P_{20} + \mathcal{O}(\varepsilon^4)\,,$$

where, from Lemma 3, we have that

$$Q_3^{(1)}(\tau,\mathbf{z}_0) = -\frac{1}{2}\,a_0\,3\tau P_{30} + \mathcal{O}(\|\mathbf{x}_0\|^2)\,.$$

Recall now the periodicity conditions (32) that our solution had to satisfy at $t = T/4$ (from Remark 5 we in fact know that $T/4 = \kappa\pi$ for some $\kappa \in \mathbb{N}$). Since $\tau = \varepsilon^3 t$, in the new time variable τ it corresponds to the value $\varepsilon^3\kappa\pi$.

Taking into account that our solution $\mathbf{z}(\tau,\mathbf{z}_0;\varepsilon)$ is given at first order by the unperturbed Kepler solution $\mathbf{z}^{(0)}(\tau,\mathbf{z}_0^*)$ in (23), which satisfies those periodicity conditions at $\tau = \pi/2$, we choose ε such that $\varepsilon^3\kappa\pi = \pi/2$; i.e., we take $\varepsilon^3 = \frac{1}{2\kappa}$ for some $\kappa \in \mathbb{N}$.

In summary, to guarantee the existence of a doubly-periodic orbit, we will impose the following conditions of symmetry at time $\tau = \pi/2$:

$$Q_1(\pi/2,\mathbf{z}_0;\varepsilon) = \frac{\pi}{2}\,,\quad Q_3(\pi/2,\mathbf{z}_0;\varepsilon) = 0\,,\quad P_2(\pi/2,\mathbf{z}_0;\varepsilon) = 0\,.$$

Let us define $f = (f_1,f_2,f_3)$, where $f_1(\mathbf{x}_0,\varepsilon) := Q_1(\pi/2,\mathbf{z}_0;\varepsilon) - \frac{\pi}{2}$, $f_2(\mathbf{x}_0,\varepsilon) := \varepsilon^{-4}Q_3(\pi/2,\mathbf{z}_0;\varepsilon)$, $f_3(\mathbf{x}_0,\varepsilon) := P_2(\pi/2,\mathbf{z}_0;\varepsilon)$. Then we have that

$$f_1(\mathbf{x}_0,\varepsilon) = (1+\Delta P_{10})^{-3}\,\frac{\pi}{2} - \frac{\pi}{2} + \mathcal{O}(\varepsilon^4)\,,$$
$$f_2(\mathbf{x}_0,\varepsilon) = -\frac{1}{2}\,a_0\,3\frac{\pi}{2}P_{30} + \mathcal{O}(\|\mathbf{x}_0\|^2) + \mathcal{O}(\varepsilon^2)\,,$$
$$f_3(\mathbf{x}_0,\varepsilon) = P_{20} + \mathcal{O}(\varepsilon^4)\,.$$

Then, it is enough to check that the matrix

$$\partial_{\mathbf{x}_0} f(0,0) = \begin{pmatrix} -3\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\,a_0\,3\frac{\pi}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

is nonsingular in order to apply Arenstorf's theorem as in Section 3.4 and finally obtain

Theorem 2. *Let $N \geq 3$ be an even number and consider the equations of motion for the elliptic three-dimensional restricted $(N+1)$-body problem with the primaries moving around each other on elliptic orbits with semi-major axis one, period 2π and eccentricity $\eta \in [0,1[$. Then, for κ a positive integer large enough, there exist initial conditions for the infinitesimal body such that its motion is a doubly-symmetric periodic solution of period $4\kappa\pi$, near a Keplerian circular orbit on a plane perpendicular to that of the primaries and with a radius of order $(2\kappa)^{2/3}$.*

5. Discussion

The existence of symmetric periodic orbits has been obtained using that the equations of motions of our system are invariant by a discrete symmetry, which in turn follows from the fact that the configuration in which the primaries are arranged (the regular N-gon) is symmetric with respect to a line. Similar results could be obtained when the primaries form other symmetric central configurations; for instance, in the case of three primaries where two of them have equal masses (the case where all the three primaries have equal masses is already covered in the present work). Nevertheless, for each of those cases, the expansion in Legendre polynomials and computations akin to those in Section 3.3 should be addressed in order to check that the Jacobian matrix in the application of Arenstorf's theorem is nonsingular. In some instances it could be much harder to perform those calculations analytically. Moreover, it could be necessary to exclude some values of the eccentricity of the primaries as happened in the elliptic restricted three-body problem in [2].

CRediT authorship contribution statement

Josep M. Cors: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Miguel Garrido:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Legendre polynomials

The jth Legendre polynomial, denoted by L_j, can be given explicitly by the formula:

$$L_j(x) = \sum_{s=0}^{j} \binom{j}{s}\binom{j+s}{s}\left(\frac{x-1}{2}\right)^s$$

For example, the expressions for the first three Legendre polynomials are:

$$L_0(x) = 1,\quad L_1(x) = x,\quad L_2(x) = \frac{1}{2}(3x^2-1)\,.$$

One of the possible uses of Legendre polynomials is stated next.

Proposition 5. *The Newtonian potential can be expanded in terms of Legendre polynomials according to the formula*

$$\frac{1}{\|\mathbf{R}-\mathbf{r}\|} = \sum_{j=0}^{\infty} \frac{\|\mathbf{r}\|^j}{\|\mathbf{R}\|^{j+1}}\,L_j(\cos(\alpha))\,, \tag{A.1}$$

where α is the angle between the vectors $\mathbf{R}$ and $\mathbf{r}$, i.e., $\cos(\alpha) = \frac{\mathbf{R}\cdot\mathbf{r}}{\|\mathbf{R}\|\,\|\mathbf{r}\|}$.

Note that the series in (A.1) converges when $\|\mathbf{R}\| > \|\mathbf{r}\|$.

Appendix B. Some trigonometric identities

Lemma 5. *Given an integer $N \geq 2$, the following trigonometric identities hold:*

$$\sum_{k=1}^{N} \cos\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = 0\,,\quad \forall\varphi\in\mathbb{R}\,, \tag{B.1}$$

$$\sum_{k=1}^{N} \sin\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = 0\,,\quad \forall\varphi\in\mathbb{R}\,. \tag{B.2}$$

Proof. We start by checking that

$$\sum_{k=1}^{N} e^{i\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} = 0. \tag{B.3}$$

Indeed, we have that

$$\sum_{k=1}^{N} \mathrm{e}^{i\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} = \mathrm{e}^{i\varphi} \sum_{k=1}^{N} \left(\mathrm{e}^{\frac{i2\pi}{N}}\right)^{k-1}$$

$$= \mathrm{e}^{i\varphi} \left(\frac{1-\left(\mathrm{e}^{\frac{i2\pi}{N}}\right)^{N}}{1-\mathrm{e}^{\frac{i2\pi}{N}}} \right) = \mathrm{e}^{i\varphi} \left(\frac{1-\mathrm{e}^{i2\pi}}{1-\mathrm{e}^{\frac{i2\pi}{N}}} \right) = 0,$$

where, in passing from the first to the second line, we have used the formula for the sum of the first N terms of a geometric series with common ratio $\gamma = \mathrm{e}^{\frac{i2\pi}{N}}$ (note that $\gamma \neq 1$ if $N \geq 2$).

Finally, identities (B.1) and (B.2) are obtained by taking real and imaginary parts, respectively, in (B.3). □

Lemma 6. *Given an integer $N \geq 3$, the following trigonometric identity is verified:*

$$\sum_{k=1}^{N} \cos^2\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = \frac{N}{2}, \quad \forall \varphi \in \mathbb{R}. \tag{B.4}$$

And clearly, also

$$\sum_{k=1}^{N} \sin^2\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = \frac{N}{2}, \quad \forall \varphi \in \mathbb{R}. \tag{B.5}$$

Moreover,

$$\sum_{k=1}^{N} \cos\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) \sin\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = 0, \quad \forall \varphi \in \mathbb{R}. \tag{B.6}$$

Proof. We will start proving (B.4).

Using complex notation,

$$\cos(\theta) = \frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2}, \text{ and consequently } \cos^2(\theta) = \frac{\mathrm{e}^{i2\theta}+\mathrm{e}^{-i2\theta}+2}{4}, \forall \theta \in \mathbb{R}.$$

Therefore,

$$\sum_{k=1}^{N} \cos^2\left(\varphi + 2\left(\frac{k-1}{N}\right)\pi\right) = \sum_{k=1}^{N} \frac{\mathrm{e}^{i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} + \mathrm{e}^{-i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} + 2}{4}$$

$$= \frac{N}{2} + \frac{1}{4}\sum_{k=1}^{N} \mathrm{e}^{i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} + \frac{1}{4}\sum_{k=1}^{N} \mathrm{e}^{-i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)}.$$

We will now show that

$$\sum_{k=1}^{N} \mathrm{e}^{i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} = 0, \text{ and similarly, } \sum_{k=1}^{N} \mathrm{e}^{-i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} = 0.$$

For the first term we have that

$$\sum_{k=1}^{N} \mathrm{e}^{i2\left(\varphi+2\left(\frac{k-1}{N}\right)\pi\right)} = \sum_{k=1}^{N} \mathrm{e}^{i2\varphi}\,\mathrm{e}^{i4\left(\frac{k-1}{N}\right)\pi} = \mathrm{e}^{i2\varphi} \sum_{k=1}^{N} \left(\mathrm{e}^{\frac{i4\pi}{N}}\right)^{k-1}$$

$$= \mathrm{e}^{i2\varphi} \left(\frac{1-\left(\mathrm{e}^{\frac{i4\pi}{N}}\right)^{N}}{1-\mathrm{e}^{\frac{i4\pi}{N}}} \right) = \mathrm{e}^{i2\varphi} \left(\frac{1-\mathrm{e}^{i4\pi}}{1-\mathrm{e}^{\frac{i4\pi}{N}}} \right) = 0,$$

where, in passing from the first to the second line, we have used the formula for the sum of the first N terms of a geometric series with common ratio $\gamma = \mathrm{e}^{\frac{i4\pi}{N}}$ (note that $\gamma \neq 1$ if $N \geq 3$).

The second term can be dealt with analogously. This concludes the proof of identity (B.4).

Formula (B.5) can be deduced from (B.4) simply by noting that $\sin(\theta) = \cos(\theta - \pi/2)$.

Finally, identity (B.6) can be obtained by differentiating both sides of (B.4) with respect to φ. □

This lemma has also been stated in [12] in a different context (central configurations) and with a slightly different proof.

Appendix C. Two lemmas on differential equations

The following two lemmas from [13, Appendix 8.2] (see also [2, Section 3]) state that, under some hypotheses, the solution of a perturbed system can be written as the solution of the unperturbed equation plus higher order terms in the perturbation parameter, and that an analogous result holds for the partial derivatives of the solution with respect to initial conditions.

More specifically, let us consider the differential equation

$$\dot{\mathbf{z}} = \mathcal{F}(t, \mathbf{z}; \varepsilon), \tag{C.1}$$

where $\mathbf{z} \in \mathbb{R}^n$ and

$$\mathcal{F}(t, \mathbf{z}; \varepsilon) = \mathcal{F}_0(\mathbf{z}) + \varepsilon^{\ell}\, \mathcal{F}_1(t, \mathbf{z}; \varepsilon) + \varepsilon^{\ell+r}\, \mathcal{F}_R(t, \mathbf{z}; \varepsilon),$$

where $\ell > 0$, $r > 0$.

Let $\mathbf{z}_0 \in \mathbb{R}^n$ be an initial condition such that $\mathbf{z}^{(0)}(t, \mathbf{z}_0)$ is a solution of

$$\dot{\mathbf{z}}^{(0)}(t, \mathbf{z}_0) = \mathcal{F}_0(\mathbf{z}^{(0)}(t, \mathbf{z}_0)) \tag{C.2}$$

which remains bounded and bounded away from the singularities of $\mathcal{F}$.

Let $C \subset \mathbb{R}^n$ be a compact neighborhood of $\mathbf{z}^{(0)}(t, \mathbf{z}_0)$ without singularities. We assume that the functions $\mathcal{F}_0$, $\varepsilon^{\ell}\,\mathcal{F}_1$, $\varepsilon^{\ell+r}\,\mathcal{F}_R$ are continuous for $\mathbf{z} \in C$, $\varepsilon \in [0, \varepsilon_1]$, $t \in \mathbb{R}$. Furthermore, $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_R$ together with all their derivatives with respect to $\mathbf{z}$ are bounded on C by a constant C_1 independent of ε. In particular, $\mathcal{F}_0$ is Lipschitz with respect to the variable $\mathbf{z}$ with a constant C_2.

The next lemma shows that the solution of (C.1) can be written as the solution of (C.2) plus terms which are of order ε.

Lemma 7. *Let $\mathbf{z}(t, \mathbf{z}_0; \varepsilon)$ be a solution of (C.1) with initial condition $\mathbf{z}_0$ and let $\mathbf{z}^{(1)}(t, \mathbf{z}_0; \varepsilon)$ be given by*

$$\mathbf{z}^{(1)}(t, \mathbf{z}_0; \varepsilon) = \mathcal{Z}(t, \mathbf{z}_0) \int_0^t \left(\mathcal{Z}(s, \mathbf{z}_0)\right)^{-1} \mathcal{F}_1(s, \mathbf{z}^{(0)}(s, \mathbf{z}_0); \varepsilon)\, \mathrm{d}s,$$

where

$$\mathcal{Z}(t, \mathbf{z}_0) = \left.\frac{\partial \mathbf{z}^{(0)}(t, \zeta)}{\partial \zeta}\right|_{\zeta = \mathbf{z}_0}.$$

Then we can write

$$\mathbf{z}(t, \mathbf{z}_0; \varepsilon) = \mathbf{z}^{(0)}(t, \mathbf{z}_0) + \varepsilon^{\ell}\, \mathbf{z}^{(1)}(t, \mathbf{z}_0; \varepsilon) + \mathbf{z}_R(t, \mathbf{z}_0; \varepsilon),$$

where $\mathbf{z}_R(t, \mathbf{z}_0; \varepsilon)$ is $\mathcal{O}(\varepsilon^{\ell+s})$, with $s := \min\{\ell, r\}$, in a finite interval of time.

The next lemma shows that similar bounds hold for the partial derivatives of the solution with respect to the initial conditions.

Lemma 8. *Let $\mathbf{z}_R(t, \mathbf{z}_0; \varepsilon)$ be as in Lemma 7. Then*

$$\partial_{\mathbf{z}_0} \mathbf{z}_R(t, \mathbf{z}_0; \varepsilon) = \mathcal{O}(\varepsilon^{\ell+s}), \quad \textit{where } s := \min\{\ell, r\},$$

for t in a finite interval of time.

Remark 6. The case presented here is slightly different to that in [2, Section 3]; nevertheless, the proofs of the lemmas are completely analogous and follow from reiteratedly applying Grönwall's inequality to bound the needed expressions.

Data availability

No data was used for the research described in the article.

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