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ON THE INTEGRABILITY OF A SPROTT CUBIC CONSERVATIVE JERK SYSTEM

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Abstract. We consider the Sprott cubic conservative jerk differential equation $\dddot{x} - a(1-x^2)x + x^2\dot{x} = 0$ with $a \in \mathbb{R}$. It is known that this differential equation exhibits chaotic motion for some values of the parameter a. Here we study when this differential equation has no chaotic motion, i.e. when it has first integrals, and then we describe its dynamics.

1. Introduction and statement of the main results

J.C. Sprott showed numerically that the following third-order differential equation

$$\dddot{x} - a(1-x^2)x + x^2\dot{x} = 0 \tag{1}$$

is conservative and chaotic for some values of the parameter $a \in \mathbb{R}$, see its differential equation (23) in [11]. Note that this differential equation is a particular jerk differential system, see again [11]. These last years many authors have studied different kinds of jerk differential systems, see for instance [3, 4, 7, 10, 12].

In this paper we study when the differential equation (1) is non-chaotic. More precisely, when this equation is integrable, and in this case we describe its dynamics.

We write the third-order differential equation as the following differential system of first order in $\mathbb{R}^3$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a(1-x^2)x - x^2y, \end{aligned} \tag{2}$$

with $a \in \mathbb{R}$. We denote by

$$X = X(x,y,z) = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (a(1-x^2)x - x^2y)\frac{\partial}{\partial z}, \tag{3}$$

the *vector field* associated to the differential system (2).

Since the differential system (2) is invariant under the symmetry

$$\tau(x,y,z) = (-x,-y,-z), \tag{4}$$

its phase portrait is symmetric with respect to the origin of coordinates.

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Let U be an open subset of $\mathbb{R}^3$ dense in $\mathbb{R}^3$. A C^1 function $H : U \to \mathbb{R}$ is a *first integral* of the differential system (2) if it is not locally constant but it is constant on the solutions of system (2) contained in U. So a necessary and sufficient condition in order that H be a first integral of system (2) is that

$$\frac{dH}{dt} = XH = 0,$$

restricted to the orbits of system (2) contained in U.

Two first integrals of system (2) defined in U are *independent* if their gradients are independent in all points of U except perhaps in a zero Lebesgue measure set.

System (2) is *completely integrable* if it has two independent first integrals.

Our results are the following ones.

Proposition 1. *If $a = 0$ then the differential system (2) is completely integrable with the two independent first integrals*

$$H(x, y, z) = z + \frac{1}{3}x^3 \quad \text{and} \quad F(x, y, z) = -\frac{1}{2}\,x^4 - 2\,xz + y^2,$$

Proposition 1 is proved in section 3.

An equilibrium point p of a 2-dimensional differential system is a *center* if there is a neighborhood N of p such that $N \setminus \{p\}$ is filled with periodic orbits. If we have a differential system in $\mathbb{R}^2$ with a center p such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits, then we say that p is a *global center.*

Roughly speaking the *Poincaré disc* is the closed unit disc with its interior identified with the whole plane $\mathbb{R}^2$, and its boundary (the circle $\mathbb{S}^1$) identified with the infinity of $\mathbb{R}^2$, see subsection 2.1 for more details on the Poincaré disc.

Theorem 2. *The phase portrait on the Poincaré disc of the differential system (2) with $a = 0$ restricted to the invariant surface $H(x, y, z) = h$ for all $h \in \mathbb{R}$ is topologically equivalent to the one of Figure 1, i.e. on every invariant surface $H(x, y, z) = h$ the differential system (2) has a global center, see its phase portrait in Figure 1.*

Corollary 3. *The differential system (2) with $a = 0$ has the x-axis filled with equilibria, and all the other orbits are periodic orbits surrounding the x-axis.*

Theorem 2 and its Corollary 3 are proved in section 3.

From Corollary 3 the dynamics of the differential system (2) with $a = 0$ is very easy, only equilibrium points and periodic orbits, while the dynamics of system (2) with $a = 0.01$ consists of two sets of nested tori, one at positive x and the other at negative x, coupled in such a way that trajectories near their intersection are chaotic and encircle both tori, see for details [11, page 542]. To describe this change in the dynamics of system (2) is a challenge problem.

Consider a function of the form

$$H(x, y, z) = f_1(x, y, z)^{\lambda_1} \cdots f_k(x, y, z)^{\lambda_k} e^{\mu_1 g_1(x,y,z)/h_1(x,y,z)} \cdots e^{\mu_\ell g_\ell(x,y,z)/h_\ell(x,y,z)},$$

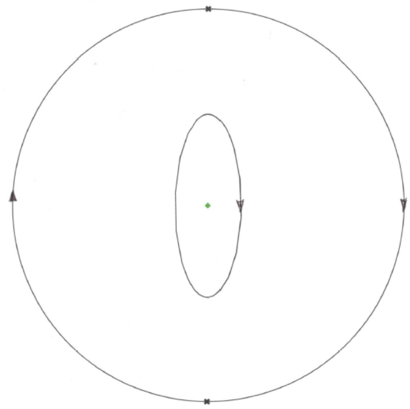

FIGURE 1. The phase portrait in the Poincaré disc of the differential system (2) with $a = 0$ restricted to the invariant surface $H(x, y, z) = h$ for all $h \in \mathbb{R}$.

where $k \geq 0$ and $\ell \geq 0$ are integers such that $k + \ell > 0$, $f_i(x, y, z)$ for $i = 1, \dots, k$, $g_j(x, y, z)$ and $h_j(x, y, z)$ for $j = 1, \dots, \ell$ are polynomials. If this function $H(x, y, z)$ is a first integral, then we say that it is a *Darboux first integral.* See for more details on these kind of first integrals [6, Chapter 8] and [5].

Theorem 4. *The following statements hold for the differential system* (2) *with* $a \neq 0$.

(a) *The system has no polynomial first integrals.*
(b) *The system has no Darboux first integrals.*

Theorem 4 is proved in section 4.

2. Preliminary Results

2.1. **Equililbrium points of 2-dimensional differential systems.** Let (x_0, y_0) be an equilibrium point of a differential system in $\mathbb{R}^2$, and denote by X the vector field associated to this system. Let λ_1 and λ_2 be the eigenvalues of the Jacobian matrix $DX(x_0, y_0)$. It is said that

(a) (x_0, y_0) is *hyperbolic* if λ_1 and λ_2 have no zero real parts;
(b) (x_0, y_0) is *semi-hyperbolic* if $\lambda_1\lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 \neq 0$;
(c) (x_0, y_0) is *nilpotent* if $\lambda_1 = \lambda_2 = 0$ and the matrix $DX(x_0, y_0)$ is not the zero matrix;
(d) (x_0, y_0) is *linearly zero* if the matrix $DX(x_0, y_0)$ is the zero matrix.

The hyperbolic and semi-hyperbolic equilibrium points are also called *elementary equilibrium points*, and their local phase portraits are well-known, see for instance Theorems 2.15 and 2.19 of [6]. Also the local phase portraits of the nilpotent singular points are well-known, see for example Theorem 3.5 of [6].

2.2. **The vertical homogeneous blow-up.** In the following we present a technique for determining the local phase portrait around an equilibrium point of a 2-dimensional differential system when it is linearly zero. This method determine the local phase portrait of an equilibrium point using changes of variables called vertical blow-ups. The idea of a blow-up is to turn an equilibrium point into the whole vertical axis and study the phase portrait in a neighborhood of this axis instead of studying it in the neigborhood of the equilibrium point, and repeating this process as many times if linearly zero equilibria appear on the vertical axes. In general, such equilibrium points are less degenerate. For more details see [6, chapter 3].

Consider the following analytical differential system

$$\dot{x} = P(x,y) = P_m(x,y) + \dots, \qquad \dot{y} = Q(x,y) = Q_n(x,y) + \dots, \tag{5}$$

where P_m and Q_n are homogeneous polynomials of degree $m \geq 1$ and $n \geq 1$ respectively, and the dots mean higher order terms in the variables x and y of m in $P(x,y)$ and of n in $Q(x,y)$. Consider the polynomial

$$\mathcal{F}(x_1,x_2) = \begin{cases} xQ_m(x_1,x_2) - yP_m(x_1,x_2) & \text{if } m = n \\ -yP_m(x_1,x_2) & \text{if } m < n \\ xQ_n(x_1,x_2) & \text{if } n < m \end{cases}.$$

The homogeneous polynomial $\mathcal{F}$ is called the *characteristic polynomial* at the origin of system (5) and the straight lines through the origin defined by the real linear factors of the polynomial $\mathcal{F}$ are called the *characteristic directions* at the origin. It is known that if there are orbits starting or ending at the origin of coordinates of system (5) these at the origin are tangent to a characteristic direction. see for more details [1].

The *vertical blow-up* is the change of variables $(x_1,x_2) \to (u_1,u_2)$ where $(x_1,x_2) = (u_1,u_1u_2)$. The new system in the variables u_1 and u_2 is

$$\dot{u}_1 = P(u_1,u_1u_2), \qquad u_2' = \frac{Q(u_1,u_1u_2) - u_2P(u_1,u_1u_2)}{u_1}. \tag{6}$$

We only do a vertical blow-up when the vertical axis $x_1 = 0$ is not a characteristic direction of system (5), otherwise we can loss information on the orbits of system (5) tangent to the vertical axis.

The following result establishes relationships between the equilibrium at the origin of system (5) and the equilibrium points on the vertical axis $u_1 = 0$ of system (6), for more details see [1].

Theorem 5. *Let φ be an orbit of the differential system* (5) *tending to origin when $t \longrightarrow +\infty$ (or $t \longrightarrow -\infty$) tangent to one of the two directions θ determined by* $\tan\theta = w \neq \pm\infty$. *Assume that $\mathcal{F} \not\equiv 0$. Then*

(i) *the straight line* (x_1, wx_1) *is a characteristic direction;*
(ii) *the point* $(u_1, u_2) = (0, w)$ *is an equilibrium point of system* (6) *and*
(iii) *an orbit* φ *as in the hypothesis is in biunivocal correspondence with an orbit of system* (6) *tending to the equilibrium point* $(0, w)$.

2.3. **The Poincaré compactification of polynomial differential systems in $\mathbb{R}^2$.** In order to study the dynamics of a polynomial differential system in the plane $\mathbb{R}^2$ near infinity we need its Poincaré compactification. This tool was created by Poincaré in [9].

Consider the polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{7}$$

where P and Q are polynomial being d the maximum of the degrees of the polynomials P and Q.

We consider the plane $\mathbb{R}^2 \equiv \{(x_1, x_2, 1);\ x_1, x_2 \in \mathbb{R}\}$, the 2-dimensional sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 + x_3^2 = 1\}$, the northern hemisphere $H_+ = \{(x_1, x_2, x_3) \in \mathbb{S}^2;\ x_3 > 0\}$, the southern hemisphere $H_- = \{(x_1, x_2, x_3) \in \mathbb{S}^2;\ x_3 < 0\}$ and the equator $\mathbb{S}^1 \equiv \{(x_1, x_2, x_3) \in \mathbb{S}^2;\ x_3 = 0\}$ of the sphere $\mathbb{S}^2$.

In order to study a vector field over $\mathbb{S}^2$ we consider six local charts that cover the whole sphere $\mathbb{S}^2$. So, for $i = 1, 2, 3$, let

$$U_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2;\ x_i > 0\} \text{ and } V_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2;\ x_i < 0\}.$$

Consider the diffeomorphisms $\varphi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ given by

$$\varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3) = \left(\frac{x_j}{x_i}, \frac{x_k}{x_i}\right)$$

with $j, k \neq i$ and $j < k$. The sets (U_i, φ_i) and (V_i, ψ_i) are called the *local charts* over $\mathbb{S}^2$.

Let $f^{\pm} : \mathbb{R}^2 \longrightarrow H_{\pm}$ be the central projections from $\mathbb{R}^2$ to $\mathbb{S}^2$ given by

$$f^{\pm}(x_1, x_2) = \pm\left(\frac{x_1}{\Delta(x_1, x_2)}, \frac{x_2}{\Delta(x_1, x_2)}, \frac{1}{\Delta(x_1, x_2)}\right)$$

where $\Delta(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + 1}$. In other words $f^{\pm}(x_1, x_2)$ is the intersection of the straight line through the points $(0, 0, 0)$ and $(x_1, x_2, 1)$ with $H_{\pm}$. Note that $f^+ = \varphi_3^{-1}$ and $f^- = \psi_3^{-1}$. Moreover, the maps $f^{\pm}$ induces over $H_{\pm}$ vector fields analytically conjugate with the differential system (7). Indeed, f^+ induces on $H_+ = U_3$ the vector field $X_1(y) = Df^+(\varphi_3(y))X(\varphi_3(y))$, and f^- induces on $H_- = V_3$ the vector field $X_2(y) = Df^-(\psi_3(y))X(\psi_3(y))$. Thus we obtain a vector field on $\mathbb{S}^2 \backslash S^1$ that admits an analytic extension $p(X)$ on $\mathbb{S}^2$, see for more details [6, chapter 5]. The vector field $p(X)$ on $\mathbb{S}^2$ is called the *Poincaré compactification.*

Denote $(u, v) = \varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3)$. Then the expression of the differential system associated to the vector field $p(X)$ in the chart U_1 is

$$u' = v^d \left[Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right],\ v' = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right).$$

The expression of $p(X)$ in U_2 is

$$u' = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \; v' = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

The expression of $p(X)$ in U_3 is

$$u' = P(u, v), \; v' = Q(u, v).$$

For $i = 1, 2, 3$ the expression of $p(X)$ in the chart V_i differs of the expression in U_i only by the multiplicative constant $(-1)^{d-1}$.

Note that we can identify the infinity of $\mathbb{R}^2$ with the circle $\mathbb{S}^1$. Two points for each direction in $\mathbb{R}^2$ provide two antipodal points of $\mathbb{S}^1$. An equilibrium point of $p(X)$ on $\mathbb{S}^1$ is called *infinite equilibrium point* and an equilibrium point on $\mathbb{S}^2 \backslash \mathbb{S}^1$ is called a *finite equilibrium point*. Observe that the coordinates of the infinite equilibrium points are of the form $(u, 0)$ on the charts U_1, V_1, U_2 and V_2. Thus, if $(x_1, x_2, 0) \in \mathbb{S}^1$ is an infinite equilibrium point, then its antipode $(-x_1, -x_2, 0)$ is also a infinite equilibrium point.

The image of the closed northern hemisphere of $\mathbb{S}^2$ under the projection $(x_1, x_2, x_3) \to (x_1, x_2, 0)$ is the *Poincaré disc*, denoted by $\mathbb{D}^2$.

2.4. **The differential system** (2) **with** $a = 0$ **restricted to** $H(x, y, z) = h$ **is Hamiltonian.** The expression of the differential system (2) with $a = 0$ restricted to the invariant surface $H(x, y, z) = h$, i.e. $z = h - x^3/3$, is

$$\dot{x} = y = \frac{\partial G}{\partial y}; \qquad \dot{y} = h - \frac{x^3}{3} = -\frac{\partial G}{\partial x}, \tag{8}$$

with the Hamiltonian

$$G = G(x, y) = \frac{1}{2} y^2 + \frac{x^4}{12} - hx.$$

The fact that the flow of the Hamiltonian systems preserves the area (see for details [2]) implies that system (8) has no limit cycles, and that its finite equilibrium points are either centers, or union of hyperbolic sectors.

2.5. **On the topological indices of the equilibrium points of** 2**-dimensional polynomial differential systems.** It is known that the local phase portrait of any equilibrium point of an analytic differential system in the plane $\mathbb{R}^2$ is either a focus, a center, or finite union of hyperbolic, elliptic and parabolic sectors, see [1] or [6].

The *topological index* or simply the *index* of an equililbrium point of an analytic differential system is an integer number which can be computed using the Poincaré Index Formula, i.e. if h, e and p are the number of hyperbolic, elliptic and parabolic sectors, respectively, of the local phase portrait of an equilibrium point its index is given by the formula

$$\frac{e - h}{2} + 1.$$

For a proof of this formula see for instance [6, Chapter 5]. Thus the index of a saddle is -1, the index of a center is 1 because it has no sectors.

The next theorem shows that the sum of the indices of all equilibria of a compactified polynomial vector field $p(X)$ in the Poincaré sphere $\mathbb{S}^2$, having finitely many equilibria, does not depend on the polynomial vector field X.

Theorem 6 (Poincaré-Hopf Theorem)**.** *The sum of the indices of all equilibria of a compactified polynomial vector field $p(X)$ in the Poincaré sphere $\mathbb{S}^2$, having finitely many equilibria, is two.*

For a simple proof of the Poincaré-Hopf Theorem on the sphere $\mathbb{S}^2$ see [6, Chapter 5].

2.6. **The Darboux theory of integrability for 3-dimensional polynomial differential systems.** A *Darboux polynomial* of the polynomial differential system (2) is a polynomial $f \in \mathbb{C}[x, y, z] \backslash \mathbb{C}$ such that

$$y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + \left(a(1-x^2)\,x - x^2 y\right)\frac{\partial f}{\partial z} = Kf, \tag{9}$$

for some polynomial $K \in \mathbb{C}[x, y, z]$ called the *cofactor* of the polynomial f and its degree at most two. From (9) it follows that $f = 0$ is an *invariant algebraic surface* for the flow of system (2).

Note that if in (9) the cofactor $K = 0$, then $f = f(x, y, z)$ is a *polynomial first integral.*

A nonconstant function $F = \exp(g/h)$ where $g, h \in \mathbb{C}[x, y, z]$ are coprime polynomials it is an *exponential factor* of system (2) if it satisfies

$$y\frac{\partial F}{\partial x} + z\frac{\partial F}{\partial y} + \left(a(1-x^2)\,x - x^2 y\right)\frac{\partial F}{\partial z} = LF, \tag{10}$$

for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree at most two, called the *cofactor* of F.

For more information on invariant algebraic surfaces and exponential factors see [6, 8].

The next result is proved in [6, Theorem 8.7].

Theorem 7. *Suppose that the differential polynomial system* (2) *defined in* $\mathbb{R}^3$ *admits* p *invariant algebraic surfaces* $f_i = 0$ *with cofactors* K_i *for* $i = 1, \dots p$ *and* q *exponential factors* $F_j = \exp(g_j/h_j)$ *with cofactors* L_j *for* $j = 1, \dots, q$*. Then, there exist* λ_j, $\mu_j \in \mathbb{C}$*, not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_i = 0$ *if and only if the following real (multi-valued) function of Darboux type*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}$$

substituting $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbb{R}$ *is a first integral of system* (2).

3. Proofs of Proposition 1, Theorem 2 and Corollary 3

Proof of Proposition 1. Let X be the vector field associated to the differential system (2) with $a = 0$. Then two easy computations show that $XH = 0$ and $XF = 0$, hence H

and F are polynomial first integrals of the differential system (2) with $a = 0$. Moreover, clearly that the two first integrals H and F are independent because H depends on z and F is independent of z. Therefore system (2) is completely integrable. □

3.1. **Infinite equilibria of the differential system** (2) **with** $a = 0$ **restricted to** $H(x, y, z) = h$**.** From subsection 2.3 the expression of system (8) in the local chart U_1 is

$$\dot{u} = \frac{1}{3}\left(3hv^3 - 3u^2v^2 - 1\right), \qquad \dot{v} = -uv^3.$$

Since $\dot{u}|_{v=0} = -1/3$ there are no infinite equilibria on the chart U_1. Then the unique possible infinite equilibria can be the origins of the local charts U_2 and V_2.

Again from subsection 2.3 the expression of the differential system (8) in the local chart U_2 is

$$\dot{u} = \frac{1}{3}\left(-3huv^3 + u^4 + 3v^2\right), \qquad \dot{v} = \frac{1}{3}v\left(u^3 - 3hv^3\right). \tag{11}$$

Hence the origin of the chart U_2 is an infinite equilibrium. In the next proposition we characterize the local phase portrait at this equilibrium.

Proposition 8. *The local phase portrait at the origin of the local chart* U_2 *of the differential system* (8) *is formed by two hyperbolic sectors, whose two separatrices are contained on the circle of the infinity, see Figure* 2*(f).*

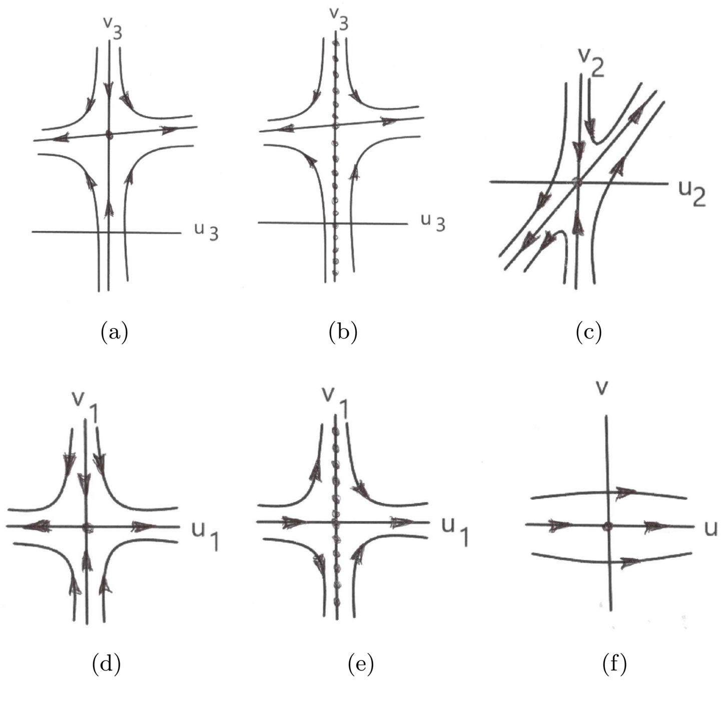

FIGURE 2. The blow up of the origin of the local chart U_2.

Proof. The origin of the local chart U_2 is a linearly zero equilibrium point, see system (11), we shall study its local phase portrait doing the changes of variables called vertical blow ups, see subsection 2.2. Since $u = 0$ is not a characteristic direction at the origin of system (11) we do the vertical blow up $(u, v) = (u_1, u_1 v_1)$, then system (11) writes

$$\dot{u}_1 = \frac{1}{3}u_1^2\left(3u_1^2v_1^2 - 3hu_1^2v_1^3\right), \qquad \dot{v}_1 = -u_1v_1^3. \tag{12}$$

Rescaling the time by u_1 the previous differential becomes

$$\dot{u}_1 = \frac{1}{3}u_1\left(3u_1^2v_1^2 - 3hu_1^2v_1^3\right), \qquad \dot{v}_1 = -v_1^3. \tag{13}$$

The unique equilibrium point of this differential system on the straight line $u_1 = 0$ is the origin of coordinates, that again it is a linearly zero equilibrium. Since $u_1 = 0$ is a characteristic direction at the origin of system (13) we do the following twist $(u_1, v_1) = (u_2, u_2 - v_2)$. Thus in the variables (u_2, v_2) system (13) writes

$$\begin{aligned} \dot{u}_2 = &\ \tfrac{1}{3}u_2\left(-3hu_2^5 + 9hu_2^4v_2 - 9hu_2^3v_2^2 + 3hu_2^2v_2^3 + 4u_2^2 - 6u_2v_2 + 3v_2^2\right), \\ \dot{v}_2 = &\ \tfrac{1}{3}\left(7u_2^3 - 3hu_2^6 + 9hu_2^5v_2 - 9hu_2^4v_2^2 + 3hu_2^3v_2^3 - 15u_2^2v_2 + 12u_2v_2^2 - 3v_2^3\right). \end{aligned} \tag{14}$$

Now we do the blow $(u_2, v_2) = (u_3, u_3v_3)$ to system (14) obtaining the system

$$\begin{aligned} \dot{u}_3 = &\ \tfrac{1}{3}u_3^3\left(3hu_3^3v_3^3 - 9hu_3^3v_3^2 + 9hu_3^3v_3 - 3hu_3^3 + 3v_3^2 - 6v_3 + 4\right), \\ \dot{v}_3 = &\ \tfrac{1}{3}u_3^2(1 - v_3)\left(3hu_3^3v_3^3 - 9hu_3^3v_3^2 + 9hu_3^3v_3 - 3hu_3^3 + 6v_3^2 - 12v_3 + 7\right). \end{aligned} \tag{15}$$

Now rescaling the time by u_3^2 system (15) writes

$$\begin{aligned} \dot{u}_3 = &\ \tfrac{1}{3}u_3\left(3hu_3^3v_3^3 - 9hu_3^3v_3^2 + 9hu_3^3v_3 - 3hu_3^3 + 3v_3^2 - 6v_3 + 4\right), \\ \dot{v}_3 = &\ \tfrac{1}{3}(1 - v_3)\left(3hu_3^3v_3^3 - 9hu_3^3v_3^2 + 9hu_3^3v_3 - 3hu_3^3 + 6v_3^2 - 12v_3 + 7\right). \end{aligned} \tag{16}$$

The unique equilibrium point of system (16) on the straight line $u_3 = 0$ is the $(0, 1)$, whose linear part has eigenvalues $\pm 1/3$, so it is a hyperbolic saddle.

The local phase portrait of system (16) in a neighborhood of the straight line $u_3 = 0$ is shown in Figure 2(a). Hence the local phase portrait of system (15) in a neighborhood of the straight line $u_3 = 0$ is shown in Figure 2(b). Going back through the blow up $(u_2, v_2) = (u_3, u_3v_3)$ and taking into account that $\dot{v}_2|_{v_2=0} = 7u_2^3/3 + h.o.t.$ we obtain the local phase portrait at the origin of system (14) in Figure 2(c), as usual "h.o.t." denotes higher order terms. Undoing the twist $(u_1, v_1) = (u_2, u_2 - v_2)$ we get the local phase portrait at the origin of system (13) in Figure 2(d). Therefore the local phase portrait at the origin of system (12) is given in Figure 2(e). Undoing the blow up $(u, v) = (u_1, u_1v_1)$ and taking into account that $\dot{u}|_{u=0} = v^2$ we obtain the local phase portrait at the origin of the local chart U_2 in Figure 2(f). □

Proof of Theorem 2. The differential system (2) with $a = 0$ restricted to $H(x, y, z) = h$ given in (8) has a unique finite equilibrium point, namely $p = ((3h)^{1/3}, 0)$. This system has at infinity only two equilibria, the origins of the local charts U_2 and V_2, that from Proposition 8 and subsection 2.5 they have index zero. Then by the Poincaré-Hopf theorem the index of the finite equilibrium p is one. Since this differential system is Hamiltonian, by subsection 2.4 the finite equilibrium p is a center.

We claim that it does not exist the last periodic orbit surrounding p. Assume that γ is the last periodic orbit surrounging p and we shall arrive to a contradiction. Indeed, consider the Poincaré return map defined in a transversal section to the periodic orbit γ. Since the polynomial differential system (8) is analytic, such a Poincaré map is analytic, and it is the identity on the transversal section that intersect the periodic orbits surrounding the center p, but an analytic map of one variable that is the identity in a piece of the transversal section is the identity in the whole transversal section where the map is defined. So the periodic orbit γ cannot be the last periodic orbit surrounding p. The claim is proved.

Moreover, since by Proposition 8 the unique infinite equilibria are the origins of the local charts U_2 and V_2, and the local phase portraits at these equilibria are formed by two hyperbolic sectors having their two separatrices on the infinite circle, we obtain the center p is global. Hence the phase portrait of system (8) in the Poincaré disc topologically is the one described in Figure 1. □

Proof of Corollary 3. It is clear that the equilibria of the differential system (2) with $a = 0$ are all the points of the x-axis. From Theorem 2 each equilibrium point $(x, 0, 0)$ is on the invariant surface $H(x, y, z) = h = H(x, 0, 0)$, and on this surface it is a global center. Hence the corollary follows. □

4. Proof of Theorem 4

A polynomial $F(x, y, z)$ has *weight degree* d with *weight exponents* $s_1, \ldots, s_n \in \mathbb{Z}$ if

$$F(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n) = \alpha^d F(x_1, \ldots, x_n).$$

Consider the differential system

$$\dot{x}_i = P_i(x_1, \ldots, x_n), \qquad \text{for } i = 1, \ldots, n,$$

in $\mathbb{R}^n$ where P_i are polynomials. This differential system is *quasi-homogeneous* of *weight degree* the positive integer d with *weight exponents* $s_1, \ldots, s_n \in \mathbb{Z}$ if for any $\alpha > 0$ we have that

$$P_i(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n) = \alpha^{s_i+d-1} P_i(x_1, \ldots, x_n), \qquad \text{for } i = 1, \ldots, n.$$

Proof of statement (*a*) *of Theorem* 4. In order to simplify the computations we consider the following change of variables in system (2)

$$x = \mu^{-1} X, \quad y = \mu^{-1} Y, \quad z = \mu^{-2} Z \quad \text{and} \quad t = \mu T, \ \text{with} \ \mu \in \mathbb{R}/\{0\}.$$

Then system (2) becomes

$$\begin{aligned} \dot{X} &= \mu Y \\ \dot{Y} &= Z \\ \dot{Z} &= \mu^2 a X - a X^3 - X^2 Y, \end{aligned} \tag{17}$$

where now the dot denotes derivative with respect to the variable T.

Assume that $f(x, y, z)$ is a polynomial first integral of system (2). Let $m > 0$ be the minimum integer such that

$$F(X,Y,Z) = \mu^m f(\mu^{-1}X, \mu^{-1}Y, \mu^{-2}Z) = \sum_{i=0}^{m} \mu^i F_i(X,Y,Z),$$

where F_i for $i = 1, \ldots, m$ is the weight homogeneous polynomial of weight degree $m-i$. Then m is the weight degree of the polynomial F with weight exponent $(s_1, s_2, s_3) = (1, 1, 2)$ and the weight homogeneous polynomial F_0 of weight degree m is non-zero.

From the definition of first integral we have

$$\mu Y \sum_{i=0}^{m} \mu^i \frac{\partial F_i}{\partial X} + Z \sum_{i=0}^{m} \mu^i \frac{\partial F_i}{\partial Y} + (\mu^2 aX - aX^3 - X^2Y) \sum_{i=0}^{n} \mu^i \frac{\partial F_i}{\partial Z} = 0. \tag{18}$$

The coefficient of μ^0 in equation (18) is

$$Z \frac{\partial F_0}{\partial Y} - (aX^3 + X^2Y) \frac{\partial F_0}{\partial Z} = 0.$$

Solving this linear partial differential equation we obtain that $F_0 = F_0(X, 2aX^3Y + X^2Y^2 + Z^2)$, i.e. F_0 is an arbitrary polynomial in the variables X and $2aX^3Y + X^2Y^2 + Z^2$. So the polynomial F_0 of weight degree m is of the form

$$F_0(X,Y,Z) = \sum_{k+4\ell=m} a_{k\ell} X^k (2aX^3Y + X^2Y^2 + Z^2)^\ell,$$

with k and ℓ non-negative integers.

The coefficient of μ in equation (18) is

$$Y \frac{\partial F_0}{\partial X} + Z \frac{\partial F_1}{\partial Y} - (aX^3 + X^2Y) \frac{\partial F_1}{\partial Z} = 0.$$

Solving this linear partial differential equation we get that

$$
\begin{aligned}
F_1 =& \frac{2}{3} \sum_{k+4\ell=m-1} a_{k\ell}\, \ell\, X^{k-1} Y^2 Z \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{l-1} \\
&+ \frac{4}{3} \sum_{k+4\ell=m-1} a\, a_{k\ell}\, \ell\, X^k Y Z \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell-1} \\
&- 4 \sum_{k+4\ell=m-1} \ell a^2\, a_{k\ell} X^{k+1} Z \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell-1} \\
&+ \frac{4}{3} \sum_{k+4\ell=m-1} a_{k\ell} \ell\, X^{k-3} Z \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell} \\
&+ \sum_{k+4\ell=m-1} a_{k\ell} k\, Z X^{k-3} \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell} + \\
&+ \frac{1}{|X|} \arctan\left(\frac{|X|\,(aX+Y)}{Z}\right) \sum_{k+4\ell=m-1} \Bigg[k\, a_{k\ell} a \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell} X^k \\
&- 4\, a_{k\ell_1} a^3 \ell \left(2\, aX^3Y + X^2Y^2 + Z^2\right)^{\ell-1} X^{k+4} \Bigg] + G_1(x,y,z).
\end{aligned}
$$

Since $a \neq 0$, and F_1 must be a weight homogeneous polynomial with weight degree $\ell + 4k = m-1$, then $a_{k\ell} = 0$ for all $k + 4\ell = m$. Note that k and ℓ cannot be zero, otherewise $m = 0$ in contradiction with the fact that $m > 0$. But since $a_{k\ell} = 0$ for all $k + 4\ell = m$, it follows that $F_0 = 0$, another contradiction. Hence we have proved that the differential system (2) has no poynomial first integrals. □

To prove statement (b) of Theorem 4 we need to investigate the Darboux polynomials and the exponential factors of system (2) with $a \neq 0$. We start with Darboux polynomials.

Proposition 9. *System* (2) *with* $a \neq 0$ *has no Darboux polynomials with nonzero cofactor.*

Proof. Let f be a Darboux polynomial with a nonzero cofactor K, i.e. $X(f) = Kf$ where X is the vector field of (3). Since K is a polynomial of degree at most 2, K is of the form

$$K = k_0 + k_1x + k_2y + k_3z + k_4x^2 + k_5xy + k_6xz + k_7y^2 + k_8yz + k_9z^2.$$

Due to the symmetry (4) and since f is a Darboux polynomial of system (2), it follows that $\tau(f)$ is also a Darboux polinomial of system (2) with cofactor $\tau(K)$, i.e. $X(\tau(f)) = \tau(K)\tau(f)$. Then $g = f\tau(f)$ is a Darboux polynomial of system (2) with cofactor

$$K_g = K + \tau(K) = 2k_0 + 2k_4x^2 + 2k_5xy + 2k_6xz + 2k_7y^2 + 2k_8yz + 2k_9z^2,$$

because

$$X(g) = X(f)\tau(f) + fX(\tau(f)) = Kf\tau(f) + f\tau(K)\tau(f) = (K + \tau(K))f\tau(f) = K_g g.$$

Therefore

(19)
$$y\frac{\partial g}{\partial x}+z\frac{\partial g}{\partial y}+(ax(1-x^2)-x^2y)\frac{\partial g}{\partial z}=2(k_0+k_4x^2+k_5xy+k_6xz+k_7y^2+k_8yz+k_9z^2)g.$$

We write

$$g=g(x,y,z)=\sum_{j=0}^{n}g_j(x,y,z),$$

where $g_j(x,y,z)$ is a homogeneous polynomial of degree j and $g_n(x,y,z)\neq 0$.

The homogeneous part of degree $n+2$ in (19) is

(20)
$$-(ax^3+x^2y)\frac{\partial g_n}{\partial z}=2(k_4x^2+k_5xy+k_6xz+k_7y^2+k_8yz+k_9z^2)g_n.$$

For solving this partial differential equation we consider two cases.

Case 1: $\frac{\partial g_n}{\partial z}\neq 0$. Then solving the above linear partial differential equation we obtain

$$g_n(x,y,z)=G_n(x,y)e^{-\frac{z\left(6k_4x^2+6y(k_5x+k_7y)+3z(k_6x+k_8y)+2k_9z^2\right)}{6x^2(ax+y)}}$$

where $G_n(x,y)$ is an arbitray function. Since g_n must be a homogeneous polynomial of degree n, it follows that $k_4=k_5=k_6=k_7=k_8=k_9=0$ and $G_n(x,y)$ is a homogeneous polynomial of degree n. Hence the Darboux polynomial $g(x,y,z)$ has cofactor $2k_0$.

Now we want to show that $k_0=0$ and the proposition will be proved in case 1. Let $m>0$ be the minimum integer such that

$$G(X,Y,Z)=\mu^m g(\mu^{-1}X,\mu^{-1}Y,\mu^{-2}Z)=\sum_{i=0}^{m}\mu^i G_i(X,Y,Z),$$

and for $i=1,\ldots,m$ the G_i be the weight homogeneous polynomial with weight degree $m-i$. Then m is the weight degree of the polynomial G with weight exponent $(s_1,s_2,s_3)=(1,1,2)$.

Since $g(x,y,z)$ is a Darbous polynomial of degree n with cofactor $2k_0$ of system (2), we get that $G(X,Y,Z)$ is a Darboux polynomial of weight degree m with cofactor $2k_0$ of system (17). Therefore

(21)
$$\mu Y\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial X}+Z\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial Y}+(\mu^2aX-aX^3-X^2Y)\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial Z}=2k_0\sum_{i=0}^{m}\mu^iG_i.$$

The coefficient of μ^0 in equation (21) is

$$Z\frac{\partial G_0}{\partial Y}-\left(aX^3+X^2Y\right)\frac{\partial G_0}{\partial Z}=2k_0G_0.$$

Solving this linear partial differential equation when $a\neq 0$ we get

$$G_0=e^{2k_0\arctan\left(\frac{|x|(ax+y)}{z}\right)/x}h(x,2ax^3y+x^2y^2+z^2).$$

Since G_0 must be a polynomial it follows that $k_0=0$, and the proposition is proved in case 1.

Case 2: $\frac{\partial g_n}{\partial z} = 0$. Then, from (20) we have that

$$2(k_4x^2 + k_5xy + k_6xz + k_7y^2 + k_8yz + k_9z^2)g_n = 0.$$

Since $g_n \neq 0$ we again obtain that $k_4 = k_5 = k_6 = k_7 = k_8 = k_9 = 0$. So in this case the rest of the proof of the proposition follows as in case 1. □

Proposition 10. *The differential system* (2) *has five exponential factors, namely*
e^x *with cofactor* $L_1 = y$,
e^y *with cofactor* $L_2 = z$,
e^{x^2} *with cofactor* $L_3 = 2xy$,
e^{y^2} *with cofactor* $L_4 = 2yz$, *and*
e^{xy} *with cofactor* $L_5 = xz + y^2$.

Proof. From the definition of exponential factor (10) it easy to check that the five exponentials of the statement of the proposition are exponential factors. Also it is easy to check that the exponential of any other monomial different from the monomials x, y, x^2, y^2 and xy provides cofactors of degree greater than two, and consequently cannot be exponential factors. □

Proof of statement (*b*) *of Theorem* 4. From subsection 2.6 and Proposition 9 we know that the differential system (2) has no invariant algebraic surfaces, and from Proposition 10 this differential system has only the five exponential factors stated there. Since the unique solution for the μ_i's for $i = 1, \ldots, 5$ in the equation $\sum_{i=1}^{5} \mu_i L_i = 0$ is the solution $\mu_i = 0$ for $i = 1, \ldots, 5$, by Theorem 7 it follows that system (2) has no Darboux first integrals. □

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