A NOTE ON REARRANGEMENT POINCARÉ INEQUALITIES AND THE DOUBLING CONDITION

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ABSTRACT. We introduce Poincaré type inequalities based on rearrangement invariant spaces in the setting of metric measure spaces and analyze when they imply the doubling condition on the underline measure.

1. Introduction

An important topic of study in the rich theory of Sobolev spaces defined on metric measure spaces, are the so-called spaces that support Poincaré inequalities introduced in [14] (for more information, see for example [9], [10], [11], [12], [17], [16], [24], [7], [3], [20], [21], [22] and [23] the references quoted therein).

In order to define this notion, let us recall that a metric-measure space (Ω, d, μ) is a metric space (Ω, d) with a Borel measure μ such that $0 < \mu(B) < \infty$, for every ball B in Ω, we denote by σB the dilation of a ball B by the factor σ, i.e. if $B = B(x,r) = \{y \in \Omega : d(x,y) < r\}$, then $\sigma B := B(x, \sigma r)$. A Borel μ-measurable function $g \geq 0$ is an upper gradient of f, if for all rectifiable curves γ joining points x, y in Ω, we have

$$|f(x) - f(y)| \leq \int_\gamma g ds.$$

This definition of upper gradient is due to Heinonen and Koskela in [14] (see also [15] for a more detailed exposition).

A metric measure space (Ω, d, μ) is said to support a (q,p) $-$Poincaré inequality, $q, p \in [1, \infty)$, if there is a constant $c > 0$ and $\sigma \geq 1$ such that

$$\left(\frac{1}{\mu(B)} \int_B |f - f_B|^q \, d\mu \right)^{1/q} \leq cr \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} |g|^p \, d\mu \right)^{1/p} \tag{1}$$

whenever B is a ball of radius $r > 0$, $f \in L^1_{loc}(\Omega)$ and g is an upper gradient of f (here f_B denotes the integral average: $f_B = \frac{1}{\mu(B)} \int_B f d\mu$).

Usually, at the root of analysis in this field, doubling measures play a central role since they provide a homogeneous space structure, which makes it possible to

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adapt many classical tools available in the Euclidean space. Recall that μ is said to be doubling provided there exists a constant $C = C_\mu > 0$ such that

$$\mu(2B) \leq C\mu(B) \text{ for all balls } B \subset \Omega.$$

For example it was proved in [12, Theorem 5.1] that a $(1,p)$ −Poincaré inequality self-improves in the sense that it implies a (q,p) −Poincaré inequality for some $q \in (p,\infty)$.

Recently, the study of when the family of inequalities (1) implies that the underlying measure is doubling has begun to be widely considered (see for example [11], [24], [10], [15], [12] and the references quoted therein, for more applications).

In [16], it has been shown that, if a Borel measure μ in the Euclidean space $(\mathbb{R}^n, d)$, satisfies the following $(q,p)-$ Poincaré type inequality $(q > p \geq 1)$: There exists a constant $C > 0$ such that for every Euclidean ball B and every Lipschitz function φ compactly supported on B it holds true that

$$\left(\frac{1}{\mu(B)}\int_B |\varphi|^q \, d\mu\right)^{1/q} \leq Cr\left(\frac{1}{\mu(B)}\int_B |\nabla\varphi|^p \, d\mu\right)^{1/p} + C\left(\frac{1}{\mu(B)}\int_B |\varphi|^p \, d\mu\right)^{1/q},$$

then μ is doubling. In [1], a perusal at the proof of the previous result, allowed the authors to translate this beautiful result to the metric setting without assuming the balls had the same radius on both sides.

It is know (see for example [17, Example 2.2] and [1, Example 4]) that there exits metric-measure spaces endowed with a non-doubling measure that supports a $(p,p)-$Poincaré inequality (1) for all $1 \leq p < \infty$, therefore the result obtained in [1] is sharp when considering Poincaré inequalities in which the functions norms involved are given by L^p-spaces.

The natural question, in view of the above, is whether we can obtain weaker versions of Poincaré's inequalities that still would imply the doubling property. In other words, we wonder if it is possible to replace the norm on the left-hand side by one smaller than any power bump and still obtain that μ is doubling. A first attempt in this direction was done by L. Korobenko in [17, Theorem 2.4], where she considered in the left hand side an Orlicz norm[1] and a L^1 norm in the right hand side and was able to prove the following result: Let (Ω, d, μ) be a metric measure space and let Φ be a Young function (see Section 2 below) such that there exists an $\alpha > 1$ satisfying

$$\Phi(t) \geq t(1+\ln t)^\alpha, \quad t > 1, \tag{2}$$

for some $\alpha > 1$. Assume that there is a constant $C > 0$, such that

$$\begin{aligned} \|w\|_{L^\Phi\left(B;\frac{d\mu}{\mu(B)}\right)} \quad : \quad &= \inf\left\{\lambda > 0; \int_B A\left(\frac{|w(x)|}{\lambda}\right)\frac{d\mu}{\mu(B)} \leq 1\right\} \\ &\leq \quad Cr\int_B |w|^p \,\frac{d\mu}{\mu(B)}. \end{aligned} \tag{3}$$

whenever B is a ball, w is a μ-measurable function supported on B with zero boundary values and g is an upper-gradient of w, then μ is doubling.

The above two results raise the question of what kind of function spaces are suitable in Poincaré's inequalities definition in order obtain the doubling condition. The aim of this paper will try to answer this question, to this end, we will enlarge

[1] See Section 2 below.

the class of function spaces, and corresponding norms, which define Poincaré's inequalities and will analyze when still imply the doubling condition. Since the norm of functions in Lebesgue or Orlicz spaces just only depends on its integrability properties, the natural class seems to be the class of rearrangement invariant function spaces (*r.i. spaces* for short). Roughly speaking, a r.i. space is a Banach function space where the norm of a function depends only on the μ-measure of its level sets. Lebesgue and Orlicz spaces are examples of r.i. spaces (see Section 2 below).

The paper is organized as follows. In Section 2 we introduce the notation, the standard assumptions, give a brief overview in the theory of r.i. spaces used in the paper and state our main result. Section 3 is dedicated to the proof of this result and we make further comments and remarks. Finally, the last section contains several examples.

2. Preliminaries

In this section we establish some further notation and background information, we provide more details about metrics spaces and r.i. spaces in will be working with and state our main result.

For measurable functions $f:\Omega\to\mathbb{R}$, the distribution function of f is given by

$$\mu_f(t)=\mu\{x\in\Omega:|f(x)|>t\}\qquad (t>0).$$

The **decreasing rearrangement** f_μ^* of f is the right-continuous non-increasing function from $[0,\infty)$ into $[0,\infty)$ which is equimeasurable with f. Namely,

$$f_\mu^*(s)=\inf\{t\geq 0:\mu_f(t)\leq s\}.$$

We say that a Banach function space with the Fatou property $X=X(\Omega)$ on (Ω,d,μ) is a **r.i. space**, if $g\in X$ implies that all equimeasurable function f, i.e. $f_\mu^*=g_\mu^*$, also belong to X, and $\|f\|_X=\|g\|_X$.

A basic example of r.i. spaces are the standard Lebesgue spaces $L^p(\Omega)$, for $p\geq 1$. A generalization of the Lebesgue spaces is provided by the Orlicz spaces.

Let $A\colon[0,\infty)\to[0,\infty)$ a Young function, namely a convex (non trivial), left-continuous function vanishing at 0, the **Orlicz space** $L^A(\Omega,\mu)$ is the collection of all $\mu-$measurable functions f for which there exists a λ such that

$$\int_\Omega A\left(\frac{|f(x)|}{\lambda}\right)d\mu<\infty.$$

The Orlicz space $L^A(\mu)$ is endowed with the Luxemburg norm

$$\|f\|_{L^A(\Omega,\mu)}=\inf\left\{\lambda>0;\int_\Omega A\left(\frac{|f(x)|}{\lambda}\right)d\mu\leq 1\right\}.$$

A r.i. space $X=X(\Omega)$ on (Ω,d,μ) can be represented by a r.i. space on the interval $(0,\mu(\Omega))$, with Lebesgue measure, $\bar{X}=\bar{X}(0,\mu(\Omega))$ such that $\|f\|_X=\|f_\mu^*\|_{\bar{X}}$, for every $f\in X$.

The space $\bar{X}$ is called the representation spaces of X, a characterization of the norm $\|\cdot\|_{\bar{X}}$ is given in [4, Theorem 4.10 and subsequent remarks].

Remark 1. *Since (see* [4]*) for any nonnegative, increasing left continuous function* $\psi:[0,\mu(\Omega))$ *such that* $\psi(0^+)=0$ *we have that*

$$\int_\Omega\psi(|f|)d\mu=\int_0^{\mu(\Omega)}\psi(f_\mu^*(s))ds,$$

it can be easily seen that the representation space of an Orlicz space $L^{\Phi}(\Omega,\mu)$ is $L^{\Phi}((0,\mu(\Omega)))$, the Orlicz space defined on $(0,\mu(\Omega))$ with respect then Lebesgue measure, in particular, the representation space of $L^{p}(\Omega)$ is $L^{p}((0,\mu(\Omega)))$.

We now look for a form of Poincaré's inequalities defined by r.i. spaces.

Definition 2. *Let (Ω,d,μ) be a metric measure space, let $B\subset\Omega$ be a ball , denote $\mu_B=\mu(B)^{-1}\mu$. Given a r.i. space X on Ω we define*

$$X(B,\mu_B)=\left\{f\in L^1_{loc}(\Omega):\|f\|_{X(B,\mu_B)}:=\left\|(f\chi_B)^*_{\mu}(s\mu(B))\right\|_{\bar{X}(0,1)}<\infty\right\}.$$

($\bar{X}(0,1)$ denotes the representation spaces of X where its function norm is restricted to $(0,1)$)[2].

Definition 3. *Let (Ω,d,μ) be as above. Given two X,Y two r.i. spaces on Ω, we say that the triple (Ω,d,μ) admits a $(X,Y)-$Poincaré inequality if there exist constants C_S, $\sigma\geq 1$ such that*

$$\|f-f_B\|_{X(B,\mu_B)}\leq C_S r\,\|g\|_{Y(\sigma B,\mu_{\sigma B})}\,, \tag{4}$$

whenever B is a ball of radius $r\in(0,\infty)$, $f\in L^1_{loc}(\Omega)$ and $g:\Omega\to[0,\infty]$ is an upper gradient of f.

Remark 4. *Let Φ be a Young function an $h\in L^1_{loc}(\Omega)$. By remark 1 we get*

$$\begin{aligned}\int_B\Phi(|h|)\frac{d\mu}{\mu(B)} &= \int_\Omega\Phi(h\chi_B)\frac{d\mu}{\mu(B)}\\ &= \int_0^{\mu(B)}\Phi\left((h\chi_B)^*_\mu(s)\right)\frac{ds}{\mu(B)}\\ &= \int_0^1\Phi\left((h\chi_B)^*_\mu(s\mu(B))\right)ds.\end{aligned}$$

Thus

$$\|h\|_{L^{\Phi}(B,\mu_B)}=\left\|(h\chi_B)^*_\mu(s\mu(B))\right\|_{L^{\Phi}(0,1)}.$$

In Particular, our $(L^q,L^p)-$Poincaré inequality coincides with the classical $(q,p)-$Poincaré inequality given by (1). Similarly for an Orlicz space L^{Φ}, the $\left(L^{\Phi},L^1\right)-$Poincaré inequality is the same that (3).

Obviously, we cannot expect that a $(X,X)-$Poincaré inequality implies the doubling property, we shall need to replace the space in the left-hand side in order to obtain "a gain", to this end we shall consider the fundamental function of a r.i. space X on Ω.

Given a r.i. space X on Ω, its **fundamental function** is defined by $\varphi_X(s)=\|\chi_E\|_X$, where $E\subset\Omega$ is an arbitrary measurable subset with $\mu(E)=t$. By renorming, if necessary (see [4]), we can always assume that φ_X is concave and $\varphi_X(1)=1$. We also assume in what follows that $\varphi_X(0)=0$.

Our main result is the following theorem.

[2]The norm $\|\cdot\|_{X(B,\mu_B)}$ is closely related with the average norm introduced in [6].

Theorem 5. *Let X, Y be two r.i. spaces on Ω. Let $\Psi(t) = \dfrac{\varphi_X(t)}{\varphi_Y(t)}$. If there exists a continuous increasing function $g : [1, \infty) \to [1, \infty)$ with $g(1) = 1$, such that*

$$\Psi\left(\frac{1}{t}\right) \geq g(t), \quad (t \geq 1),$$

satisfying Ermakoff's condition[3]

$$\lim_{t\to\infty} \frac{tg(t)}{g(e^t)} = 0. \tag{5}$$

Then, if the triple (Ω, d, μ) admits an $(X, Y)-$ Poincaré inequality, then measure μ is doubling.

Condition (5) controls "the gain" needed on the left-hand side norm that allows us to deduce the doubling condition. For example, if a $\left(L^{\Phi}, L^1\right)$ −Poincaré inequality holds and the Young function satisfies $\Phi(t) \geq t\,(1 + \ln t)^{\alpha}$ for $t > 1$, then $\Psi(1/t) \geq \left(1 + \ln \frac{1}{t}\right)^{\alpha}$. Whence, in case that $\alpha > 1$ (5) holds and the Theorem applies (in particular we recover the main result of [17]). When $\alpha = 1$ it is not possible to conclude the doubling property (see [17, Remark 3.1]) (notice that in that case condition (5) fails).

3. The Proof of Theorem

Let us write $h(t) = tg(t)$. Ermakoff's condition (5) implies that the series $\sum_{j=1}^{\infty} \frac{1}{h(j)}$ is convergent, let us denote by c_1 its sum. Given $B := B(y, r) \subset \Omega$ set $2B := B(y, 2r)$ and define the family of Lipschitz functions $\{f_j\}_{j\in\mathbb{N}}$ in the following way: for $j = 1$, set $r_1 = r$, and

$$r_j - r_{j+1} = \frac{r}{2c_1 h(j)}.$$

Let

$$f_j(x) := \begin{cases} 1, & if\ x \in B_{j+1} \\ \dfrac{r_j - d(x,y)}{r_j - r_{j+1}}, & if\ x \in B_j \setminus B_{j+1} \\ 0, & if\ x \in \Omega \setminus B_j. \end{cases}$$

Also for $j \in \mathbb{N}$, define the balls B_j as

$$\frac{1}{2}B \subset B_j := \{x \in \Omega : d(x,y) \leq r_j\} \subset B \subset 2B.$$

(the first inclusion follows from the fact that $\lim_{j\to\infty} r_j = r/2$). Then, for each j,

$$g_j = \frac{1}{r_j - r_{j+1}} \chi_{B_j(x)} = \frac{2c_1 h(j)}{r} \chi_{B_j(x)}$$

is an upper gradient of f_j. By the (X, Y) −Poincaré inequality (4) applied to each f_j on $2B$ we get

$$\left\| f_j - (f_j)_{2B} \right\|_{X(2B,\mu_{2B})} \leq c2r\, \|g\|_{Y(2\sigma B,\mu_{2\sigma B})}. \tag{6}$$

[3]V. P. Ermakoff [8] gave in 1872 a test for convergence of positives series based on the exponential function and the integral test. Namely, given a continuous, positive increasing function $g : [1, \infty) \to [1, \infty)$, such that $\lim_{t\to\infty} \dfrac{tg(t)}{g(e^t)} = a$, then the series $\sum_{n=1}^{\infty} \frac{1}{ng(n)}$ converges when $a < 1$ and diverges when $a > 1$ (a proof can be found in [18, p. 296 and p. 298]).

We do not know any example where the limit exists but is different from $0, 1$ or ∞.

By the triangle inequality,

$$\begin{aligned} \|f_j\|_{X(2B,\mu_{2B})} &\leq \left\|f_j-(f_j)_{2B}\right\|_{X(2B,\mu_{2B})}+\left\|(f_j)_{2B}\right\|_{X(2B,\mu_{2B})} \\ &= \left\|f_j-(f_j)_{2B}\right\|_{X(2B,\mu_{2B})}+\left|(f_j)_{2B}\right| \\ &\leq c2r\,\|g\|_{Y(2\sigma B,\mu_{2\sigma B})}+\left|(f_j)_{2B}\right| \quad \text{(by (6))} \end{aligned} \tag{7}$$

Observe that for each fixed $j\in\mathbb{N}$, we have that

$$\begin{aligned} \|g_j\|_{Y(2\sigma B,\mu_{2\sigma B})} &= \frac{2c_1h(j)}{r}\left\|\chi_{B_j}\right\|_{Y(2\sigma B,\mu_{2\sigma B})} \\ &= \frac{2c_1h(j)}{r}\varphi_Y\left(\frac{\mu(B_j)}{\mu(2\sigma B)}\right) \\ &\leq \frac{2c_1h(j)}{r}\varphi_Y\left(\frac{\mu(B_j)}{\mu(2B)}\right) \quad \text{(since } \sigma\geq 1\text{).} \end{aligned} \tag{8}$$

Using that $f_j\leq 1$ and it is supported on $B_j\subset 2B$, we obtain

$$\begin{aligned} \left|(f_j)_{2B}\right| &\leq \frac{1}{\mu(2B)}\int_{2B}|f_j|\,d\mu=\int_0^1 (f_j)^*\left(s\mu(2B)\right)ds \\ &\leq \left\|(f_j)^*\left(s\mu(2B)\right)\right\|_{\bar{Y}(0,1)} \quad \text{(by Hölder's inequality (see [4]))} \\ &\leq \left\|\chi_{[0,\mu(B_j)]}(s\mu(2B))\right\|_{\bar{Y}(0,1)} \\ &= \varphi_Y\left(\frac{\mu(B_j)}{\mu(2B)}\right). \end{aligned} \tag{9}$$

Moreover, since $f_j=1$ in B_{j+1}, we get

$$\begin{aligned} \|f_j\|_{X(2B,\mu_{2B})} &\geq \left\|\chi_{B_{j+1}}\right\|_{X(2B,\mu_{2B})}=\left\|\chi_{[0,\mu(B_{j+1}))}(s\mu(2B))\right\|_{\bar{X}} \\ &= \varphi_X\left(\frac{\mu(B_{j+1})}{\mu(2B)}\right). \end{aligned} \tag{10}$$

Inserting the information (8), (9) and (10) back in (7) we obtain

$$\varphi_X\left(\frac{\mu(B_{j+1})}{\mu(2B)}\right)\leq 4cr\frac{2c_1h(j)}{r}\varphi_Y\left(\frac{\mu(B_j)}{\mu(2B)}\right). \tag{11}$$

At this point, for $j\in\mathbb{N}$, define

$$P_j(B)=\frac{1}{Ch(j)\varphi_Y\left(\frac{\mu(B_j)}{\mu(2B)}\right)},$$

where $C=8cc_1$.

Using this notation we can write (11) as

$$\varphi_X\left(\frac{\mu(B_{j+1})}{\mu(2B)}\right)\leq\frac{1}{P_j(B)}. \tag{12}$$

On the other hand, (5) ensures that we can pick $D\geq 1$ such that

$$\frac{1}{eC}\frac{g(De^t)}{tg(t)}>1,\ \ t\geq 1. \tag{13}$$

We will show that there is a constant $\tilde{C}$ such that

$$P_1(B) = \frac{1}{C\varphi_Y\left(\frac{\mu(B)}{\mu(2B)}\right)} \leq \tilde{C} \tag{14}$$

for all ball $B \subset \Omega$, which implies that μ is doubling.

Suppose that (14) is not satisfied, then there exists a ball $B \subset \Omega$ such that

$$P_1(B) = \frac{1}{C\varphi_Y\left(\frac{\mu(B)}{\mu(2B)}\right)} > e^2 D. \tag{15}$$

We will show by induction that

$$P_j(B) \geq P_1(B)e^{j-1}. \tag{16}$$

The case $j = 1$ is triviality true. Assume that $P_j(B) \geq P_1(B)e^{j-1}$, for some $j \geq 1$. We claim that

$$\varphi_X^{-1}(t) \leq \varphi_Y^{-1}\left(\frac{t}{g(\frac{1}{t})}\right), \quad 0 < t < 1. \tag{17}$$

Assuming momentarily the validity of (17), and taking into account that $P_j(B) \geq P_1(B)e^{j-1} > 1$, it follows from (12) and (17) that

$$\frac{\mu(B_{j+1})}{\mu(2B)} \leq \varphi_X^{-1}\left(\frac{1}{P_j(B)}\right) \leq \varphi_Y^{-1}\left(\frac{1}{P_j(B)g(P_j(B))}\right).$$

Therefore

$$\varphi_Y\left(\frac{\mu(B_{j+1})}{\mu(2B)}\right) \leq \frac{1}{P_j(B)g(P_j(B))}.$$

and so

$$P_{j+1}(B) \geq \frac{P_j(B)g(P_j(B))}{Ch(j+1)}.$$

Hence,

$$\begin{aligned} P_{j+1}(B) &\geq \frac{P_j(B)g\left(P_j(B)\right)}{Ch(j+1)} \\ &\geq \frac{P_1(B)e^j g(\frac{P_1(B)}{e}e^j)}{Ce(j+1)g(j+1)} \text{ (by (16))} \\ &\geq \frac{P_1(B)e^j g(De^{j+1})}{Ce(j+1)g(j+1)} \text{ (by (15))} \\ &\geq P_1(B)e^j \text{ by (13).} \end{aligned}$$

But now (16) implies that $P_j(B) \to \infty$ as $j \to \infty$, which contradicts the fact that

$$P_j(B) = \frac{1}{Ch(j)\varphi_Y\left(\frac{\mu(B_j)}{\mu(2B)}\right)} \leq \frac{1}{Ch(j)\varphi_Y\left(\frac{\mu(\frac{1}{2}B)}{\mu(2B)}\right)} \to 0, \;\; j \to \infty.$$

Thus, (14) holds true and the proof is complete.

It remains to prove (17). By the concavity of φ_X we get that $t \leq \varphi_X(t)$ if $0 < t < 1$, therefore since $g\left(1/t\right)$ decreases we have that

$$g\left(\frac{1}{\varphi_X(t)}\right) \leq g\left(\frac{1}{t}\right) \leq \frac{\varphi_X(t)}{\varphi_Y(t)}.$$

Letting $t = \varphi_X^{-1}(s)$ we obtain

$$g\left(\frac{1}{s}\right) \leq \frac{s}{\varphi_Y(\varphi_X^{-1}(s))}$$

and so

$$\varphi_Y(\varphi_X^{-1}(s)) \leq \frac{s}{g\left(\frac{1}{s}\right)}$$

i.e.

$$\varphi_Y(t) \leq \frac{\varphi_X(t)}{g\left(\frac{1}{\varphi_X(t)}\right)}.$$

Whence

$$t = \varphi_Y^{-1}(\varphi_Y(t)) \leq \varphi_Y^{-1}\left(\frac{\varphi_X(t)}{g\left(\frac{1}{\varphi_X(t)}\right)}\right),$$

which is equivalent to

$$\varphi_X^{-1}(s) \leq \varphi_Y^{-1}\left(\frac{s}{g\left(\frac{1}{s}\right)}\right),$$

as we wished to show.

Remark 6. *The iterative arguments used in the proof are inspired in the method used in* [17].

Remark 7. *It is plain that if in Theorem 3, the r.i. space X (resp. Y) is replaced by any r.i. space $\hat{X}$ (resp. $\hat{Y}$) such that has equivalent fundamental function, i.e. there is $c \geq 1$ such that $\frac{1}{c}\varphi_X \leq \varphi_{\hat{X}} \leq c\varphi_X$, then the conclusion remains true with X replaced by $\hat{X}$ (resp. Y replaced by $\hat{Y}$).*

Associated with a r.i. space X we get the Lorentz and Marcinkiewicz space defined by the r.i. norms

$$\|f\|_{M(X)} = \sup_t \left(\frac{1}{t}\int_0^t f_\mu^*(s)ds\right)\varphi_X(t), \quad \|f\|_{\Lambda(X)} = \int_0^{\mu(\Omega)} f_\mu^*(t)d\varphi_X(t).$$

Since

$$\varphi_{M(X)}(t) = \varphi_{\Lambda(X)}(t) = \varphi_X(t),$$

and (see [4]*)*

$$\Lambda(X) \subset X \subset M(X),$$

we have that $(X,Y)-$Poincaré's inequalities, in the previous Theorem can be replaced by the weaker $(M(X),\Lambda(Y))-$Poincaré's inequalities.

4. Examples

In this Section we shall give several examples where Theorem 3 can be applied.

4.1. L^p**-spaces.** Let $X = L^{p\sigma}(\mu)$ and $Y = L^p(\mu)$ where $1 \leq p < \infty$ and $1 < \sigma < \infty$. The function $\Psi(t) = t^{\frac{1}{p}(\frac{1}{\sigma}-1)}$ is decreasing and considering $g(t) = \Psi(1/t)$ an elementary computation shows that $\lim_{t\to\infty} \dfrac{t\Psi(\frac{1}{t})}{\Psi(e^{-t})} = 0$. Thus, if (Ω, d, μ) admits a $(L^{p\sigma}, L^p)-$Poincaré inequality, then measure μ is doubling and we recover the results of [16] and [1].

Remark 8. *Let $p, q \in [1, \infty]$. Assume that either $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, or $p = q = \infty$. Then the functional defined as*

$$\|f\|_{L^{p,q}(\Omega,\mu)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*_\mu(t) \right\|_{L^q((0,\mu(\Omega)),ds)}$$

is equivalent to a r.i. function norm. The corresponding r.i. space is called a ***Lorentz space****.*

Assume now that either $1 < p < \infty$ and $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$, or$p = 1, q = 1$ and $\alpha \geq 0$ or $p = q = \infty$ and $\alpha \leq 0$, or $p = \infty$, $1 \leq q \leq \infty$ and $\alpha + 1/q < 0$. Then also the functional given by

$$\|f\|_{L^{p,q,\alpha}(\Omega,\mu)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*_\mu(t)(1 + \ln^+ \frac{1}{t})^\alpha \right\|_{L^q((0,\mu(\Omega)),ds)}$$

is equivalent to a r.i. function norm. The r.i. space built upon this function norm is called a ***Lorentz-Zygmund*** *space.*

Lorentz spaces and Lorentz-Zygmund spaces has the same fundamental function that Lebesgue spaces, hence the same result holds for these spaces.

4.2. **Orlicz spaces.** When working in Orlicz spaces, it is useful to establish conditions in terms of Young's functions instead of fundamental functions, in the next result we state the version of Theorem 3 in this sense.

Theorem 9. *Let $L^A(\Omega, \mu)$ and $L^{\hat{A}}(\Omega, \mu)$ be two Orlicz spaces on Ω with Young functions A and $\hat{A}$. Let $g : [1, \infty) \to [1, \infty)$ be a continuous increasing function with $g(1) = 1$, such that*

$$A(tg(t)) \leq \hat{A}(t), \quad t > 1. \tag{18}$$

that satisfies (5). Then, if the triple (Ω, d, μ) admits a $\left(L^{\hat{A}}, L^A\right)-$ Poincaré inequality, then measure μ is doubling.

Proof. From $\hat{A}(t) \geq A(tg(t))$ we obtain

$$tg(t) \leq A^{-1}\left(\hat{A}(t)\right).$$

Since $\varphi_{L^A}(t) = \frac{1}{A^{-1}(1/t)}$ (see [4]), we can write the above inequality as

$$tg(t) \leq \frac{1}{\varphi_{L^A}\left(\frac{1}{\hat{A}(t)}\right)}.$$

On the other hand, since $\varphi^{-1}_{L^{\hat{A}}}(\frac{1}{t}) = \frac{1}{\hat{A}(t)}$, we get that

$$tg(t) \leq \frac{1}{\varphi_{L^A}\left(\varphi^{-1}_{L^{\hat{A}}}(\frac{1}{t})\right)},$$

which implies

$$\varphi_{L^{\hat{A}}}^{-1}(t) \leq \varphi_{L^{A}}^{-1}\left(\frac{t}{g\left(\frac{1}{t}\right)}\right),$$

Now with the same argument used in the proof of Theorem 3 from (17) we finish the proof. □

In the particular case that (Ω, d, μ) admits a $\left(L^{A}, L^{1}\right)$ −Poincaré inequality with $A(t) \geq t(1+\ln t)^{\alpha}$ for $t>1$, we recover [17, Theorem 2.4].

4.3. **General Case.** The upper and lower Zippin indices associated with a r.i. space X are defined by

$$\overline{\beta}_{X}=\inf_{s>1} \frac{\ln M_{X}(s)}{\ln s} \quad \text{and} \quad \underline{\beta}_{X}=\sup_{s<1} \frac{\ln M_{X}(s)}{\ln s},$$

where

$$M_{X}(s)=\sup_{t>0} \frac{\varphi_{X}(ts)}{\varphi_{X}(t)},\ s>0.$$

It is known that (see [25, p. 272])

$$0 \leq \underline{\beta}_{X} \leq \overline{\beta}_{X} \leq 1,$$

and that for any $\varepsilon>0$ there is $\delta=\delta(\varepsilon)$ such that the following inequalities are satisfied

$$s^{\underline{\beta}_{X}}<M_{X}(s)<s^{\underline{\beta}_{X}-\varepsilon}, \quad 0<s<\delta, \tag{19}$$

$$s^{\overline{\beta}_{X}}<M_{X}(s)<s^{\overline{\beta}_{X}-\varepsilon}, \quad s>1/\delta. \tag{20}$$

Let X, Y two r.i. spaces, then

$$\frac{\varphi_{Y}(t)}{\varphi_{X}(t)} \leq \frac{\varphi_{Y}(st)}{\varphi_{X}(st)} \sup_{t>0} \frac{\varphi_{X}(st)}{\varphi_{X}(t)} \sup_{t>0} \frac{\varphi_{Y}(t)}{\varphi_{Y}(st)}=\frac{\varphi_{Y}(st)}{\varphi_{X}(st)} M_{X}(s) M_{Y}(1/s).$$

Pick $\varepsilon>0$, letting $s=1/t$, and combining (19) and (20) we get (for t small enough)

$$\frac{\varphi_{Y}(t)}{\varphi_{X}(t)} \leq M_{X}(\frac{1}{t}) M_{Y}(t) \leq t^{\overline{\beta}_{X}-\underline{\beta}_{Y}},$$

thus

$$\Psi(t)=\frac{\varphi_{X}(t)}{\varphi_{Y}(t)} \geq t^{\underline{\beta}_{Y}-\bar{\beta}_{X}}.$$

Therefore, if $\underline{\beta}_{Y}<\bar{\beta}_{X}$ and (Ω, d, μ) admits a $(X, Y)-$ Poincaré inequality, then μ is doubling.

In certain sense, Zippin indices (see [25]) indicate us the 'position" of the space, with respecte to L^{p} spaces, notice that $\underline{\beta}_{L^{p}}=\overline{\beta}_{L^{p}}=\frac{1}{p}$.

We can also apply our Theorem when Zippin indices of the involved spaces coincide, but in this case need to ensure that φ_{Y} decreases slightly more rapidly that $\varphi_{Y}(t)$, for example, assuming that there exists $b:(0,1) \rightarrow(0, \infty)$ a slowly varying decreasing function (i.e. for each $\varepsilon>0$, the function $t^{\varepsilon} b(t)$ is equivalent to a increasing function and $t^{-\varepsilon} b(t)$ is equivalent to a decreasing function (see [5])), so that

$$\frac{\varphi_{X}(t)}{\varphi_{Y}(t)} \geq b(t).$$

Then, if $b(1/t)$ satisfies Ermakoff's condition (5) and (Ω, d, μ) admits a $(X,Y)-$ Poincaré inequality, then μ is doubling.

We finish the paper giving some examples of decreasing slowly varying functions that satisfies Ermakoff's condition

Example 10. *Let $n \in \mathbb{N}$. The iterated logarithmic on $(0,1)$ are defined by*

$$\begin{array}{l} L_1(t) = \ell(t) = 1 + \ln \frac{1}{t} \\ L_{n+1}(t) = \ell(L_n(t)), \quad n \geq 2. \end{array}$$

The following functions are decreasing slowly varying functions that satisfy Ermakoff's condition:

(i) *Let $k < m$, $(k, m \in \mathbb{N})$*

$$c_{k,m}(t) = \exp\left(\frac{L_k(t)}{L_m(t)}\right)$$

(ii) *Let $k \geq 1$, and $\hat{\alpha}_k = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ where $0 < \alpha_j < 1$, $(j \leq k)$*

$$d_k(t) = \exp\left((L_1(t))^{\alpha_1} (L_2(t))^{\alpha_2} \cdots (L_k(t))^{\alpha_k}\right).$$

(iii) *Let $m \in \mathbb{N}$ and $\alpha > 1$*

$$b_{m,\alpha}(t) = \left(\prod_{j=1}^{m-1} L_j(t)\right) (L_n(t))^{\alpha}.$$

(Notice that if $\alpha \leq 1$ Ermakoff's condition is not fulfilled).

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