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Contributions to the characterization of the Schema using APOS theory: Graphing with derivative

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Abstract
This study contributes to Action, Process, Object, Schema (APOS) theory research by showing two approaches used by advanced mathematics students to construct relations between higher-order derivatives to solve complex problems. We show evidence of students' ability to perform Actions on their graphing derivative Schema, that is, of its thematization. It also contributes to the literature on the learning of differential calculus by showing how advanced students use their knowledge to construct relations between concepts when facing complex situations. The work of three graduate students on transforming complex graphs and determining their properties and their relation to the domain structure is analyzed to determine their solution approaches. Their graphing derivative Schema is analyzed in depth in terms of the construction of relations among the Schema structures and assimilation and accommodation mechanisms involved in thematization in APOS theory. These findings are important in informing and developing didactic strategies to foster university students' understanding of derivatives, which can smoothe the transition to the study of advanced mathematics courses.

Keywords Differential calculus · Derivative · APOS theory · Schema · Schema thematization · Relations

1 Introduction

The learning and teaching of differential calculus has received much attention from mathematics education researchers (Artigue, 2021) and research results have influenced the way it is taught. Some fundamental issues, such as students' understanding of the relations among different concepts and the conceptual tools needed by students to be prepared for the transition to higher mathematics, have received less attention (Trigueros et al., 2021). Advanced mathematics courses require a deeper understanding of differential calculus courses and the development of abilities to analyze complex mathematical and application problems (Hotchmuth et al., 2021). Research on students' preparation to face the demands of advanced mathematics topics has not received the needed attention. Moreover, results obtained from students who have finished early differential and integral calcutlus courses indicate that a lot of work is needed for students to develop a rich understanding of calculus concepts (Hotchmuth et al., 2021).

Research has shown the potential of APOS theory as a framework for modeling variation and as a conceptual tool for prediction in different areas of knowledge such as mathematics, physics, engineering, social sciences, and biology, among others; it has also provided tools that help to study students' understanding of problems related to applications (Pepin et al., 2021). APOS is a cognitive theory describing changes in individuals learning through its structures: Action is considered as a tranformation on previous knowledge guided by external stimuli; Process is an internal transformation enabling individuals to generalize and predict their results; Object is the result from considering the Process in its totality so that new Actions can be performed on it. Schema is defined as a framework that individuals may evoque when facing a new problem. It is formed by different APOS structures and the relations among them (Dubinsky & McDonald, 2001) and is thematized when Actions can be applied to it.

This study contributes to the understanding of the development of the graphing derivative Schema (GDS) and,

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particularly, on its thematization. Results obtained are important since this relevant information plays an important role in the design of teaching approaches focusing on a deeper understanding of differential calculus that may prepare students to succeed in advanced mathematics courses.

1.1 Literature review

Understanding the role of derivatives is fundamental in the study of phenomena involving the change or variation of magnitudes. Thus, for example, certain physical phenomena have been explained more rigorously thanks to the development of mathematics, which has led to studies focused on its application and use in physics (i.e. Carli et al., 2020; Christensen & Thompson, 2012; Hu & Rebello, 2013; Susac et al., 2018). This fact is the reason why the concept is part of all university curricula in both mathematics and related areas (i.e. Rasmussen, et al., 2014).

Bressoud et al. (2016) present a general revision of research studies in calculus. They begin with a description of the main theoretical approximations dealing with calculus concepts. They focus on different research studies dealing with students' difficulties with limits, derivatives and integrals. They also describe how calculus is taught in Europe and the United States and summarize research advances in this topic. Rasmussen et al. (2014), in their survey, affirm that the research on calculus is continually developing, and findngs contribute fundamental knowledge about its teaching and learning process. However, they underline that although we have in-depth knowledge about how students learn particular calculus concepts, as well as the potential of digital technologies and different teaching approaches to promoting a better understanding and application of those concepts to interpret different phenomena, there is still a need for explanation about how researchers can coordinate the diversity of fundamental and informed results from different theoretical perspectives. Thompson and Harel (2021), after revising different research studies on calculus learning, particularly those related to the derivative function concept, found that the cognitive roots below the significative understanding of this concept are quantitative reasoning, functions expressed dynamically and in different representations, rate as a multiplicative relation and the coordination of and between covariations.

A result shared by different studies indicates that research studies focusing on the learning and teaching of derivatives, both at high school and in university, is the complexity associated with its understanding when focusing on the mastery of algebraic techniques and algorithms (Rasmussen et al., 2014). Moreno-Armella (2014) underlines the historical tension between the intuitive and formal approaches to calculus. For example, the graphical interpretation of the derivative is discussed by coordinating local and global approaches and their formalization by combining formal aspects of limit with intuitive aspects of movement. This practice causes a large number of students to use such techniques to solve certain types of tasks, and yet not use them when faced with tasks that require an understanding of the meaning of the derivative (García et al., 2011).

The results of these investigations have led to a better understanding of the complexity of the concept of the derivative and have provided guidelines to promote deeper student comprehension of the derivative and its role in mathematics. In this sense, research on understanding differential calculus in terms of APOS theory has provided important results with respect to the characterization of the derivative Schema. In a study intended to determine how students use their differential calculus knowledge to draw the graph of a continuous function, given information about its derivatives in terms of intervals, Baker et al. (2000) used the Schema structure from APOS theory and introduced the notion of interaction between Schemas, which they called the "graphing property Schema" and we call GDS, to analyze new data. They showed that students faced many obstacles when graphing a function, and they highlighted the importance of the role played by the intersection of intervals in the domain of the function to find the graph of that function. They found that only a few students from a very large sample were able to show the construction of relations among Schema components. In a follow-up study with very good students Cooley et al. (2007) studied the thematization of this Schema by analyzing students' ability to perform Actions on the same function to find changes in the graph when some of the given properties were removed from the problem. Their findings showed that none of the selected students was able to accomplish these tasks and that only a very advanced mathematics student showed thematization of the GDS. A later study (Trigueros & Escandón, 2008) used questions related to the description of the properties of a function when given its graph, and graphing functions given their properties in terms of intervals, which included the question from the Baker et al. study. They used implicative and cohesitive statistics to analyze the obtained data, and concluded that most of the students were able to draw the graph of a function when its analytic representation was given, and that they guided their graphing Actions by using only the function's first derivative, but showed serious difficulties in understanding the geometric meaning of the second derivative in their interrelation, due to changes in the intervals where both properties overlapped, and determining inflection points. They also called attention to the interrelation of continuity and differentiability of functions.

Since Schoenfeld et al.'s (1990) study on students' understanding of the concept of slope and its graphical interpretation it has been noticed that what may seem a straightforward concept for individuals at different mathematical levels can pose significant challenges, even for students with a solid mathematical knowledge. In this regard, we believe it relevant to continue analyzing how advanced mathematics students use their knowledge and construct relationships between higher-order derivatives to solve complex problems.

In this study, the role of the interactions of interval and property in terms of the mechanisms of assimilation and accommodation (Piaget, 1975) will be introduced. In addition, the construction of relationships between Schema structures in the understanding of the graph of functions will be analyzed in depth.

2 Theoretical framework

This study uses APOS theory (Arnon et al., 2014; Dubinsky, 1991) as a theoretical framework. It provides elements to analyze students' mathematical learning. Following Piaget's epistemology, APOS theory posits that learning occurs when students reflect on their previous knowledge. Abstract reflection mechanism from Piaget's theory is also considered in APOS theory as the mechanism involved in the construction of a concept, so in APOS theory abstract reflection fosters the construction of new structures related to new knowledge. However, it is referred to in different ways according to the constructed APOS structures; Actions, guided by memorization or external stimuli on previously constructed Objects, are the basic structures in this construction. Through the interiorization mechanism, a Process is constructed. It can be recognized by the ability of students to predict the result of an Action or a chain of Actions without explicitly performing them. A Process can be encapsulated into an Object on which new Actions can be performed. Processes can be reversed into the Actions they derive from. Objects can be de-encapsulated into the Process from which they originated.

The reversal and de-encapsulation mechanisms play a fundamental role in APOS theory. Reversal is involved in the construction of a new Process which includes the inverse Actions of the original Process. For example, once the derivative Process is constructed, the reversal Process enables the student to construct the anti-derivative Process. The construction of both Processes plays a very important role in mathematics.

Processes can be coordinated with other Processes to construct new Processes (Dubinsky, 1991). A Schema is defined, in APOS theory, as a collection of Actions, Processes, Objects and other previously constructed related Schemas. A Schema is a structure that students bring to bear when faced with a specific mathematical problem (Arnon et al., 2014). It is a complex structure, which is in constant development through the construction and change of relations among its structures. When used in research, the data analysis focuses on its development, that is, on the possible changes in the structures of the Schema and particularly on the relations that each student constructs among these structures. According to APOS theory, a Schema is coherent when the individual can determine, explicitly or implicitly, which phenomena are in the scope of the Schema and which are not (Dubinsky & McDonald, 2001). Thematization is the mechanism involved in the transformation of a Schema into an Object. When this happens, new Actions can be applied on the Object, and the construction of new concepts is possible (Arnon et al., 2014). This study analyzes the thematization of GDS.

Relations constructed between the structures of a Schema can be classified as correspondence, transformation, and equivalence or conservation depending on the characteristics of the links between Schema structures. Correspondence relations are those that are used in the comparison of structures in terms of similarities or differences. They may arise from the repeated observation of pairs of structures that appear jointly in problem-solving situations, but where students are not able to justify the relation. Transformation relations are developed when the individual discovers that some structures in the Schema can be grouped or related to each other in terms of changes in the other. Students are able to argue and justify these relations. A relation between two structures that plays a role in explaining or justifying such interrelation between other structures will be considered a transformation relation. Transformation relations can be distinguished when students show that related structures can be used interchangeably, in other words, that one can be used as a substitute for another. Conservation relations can also involve the maintenance of properties in which one structure of a Schema is dependent upon the others. Students can explain and justify this reliance. Conservation relations can be distinguished when one structure is interchangeably used as a substitute for another (Trigueros, 2019).

Schema development is characterized by three stages: Intra-, Inter- and Trans-. We consider that a Schema is at an Intra- stage of development when its structures are mainly isolated from one another and the relations that exist between them are for the most part correspondence relations. The Inter- stage of Schema development appears when students start grouping different structures and describing relations between structures of the Schema by explaining their changes. These relations can be considered transformation relations. The Trans- stage of Schema development is characterized by the construction of relations among all the Schema structures and the appearance of conservation relations. Another characteristic of a Trans- stage Schema is that students evidence the coherence of the Schema by being able to determine, when facing a problem, whether it involves the concept described by the Schema independently of the concept being explicitly mentioned (Fig. 1) (Trigueros, 2019).

Derivative / Interval	Intra-Derivative	Inter-Derivative	Trans-Derivative
Intra-Interval	Correspondence relations between function and first derivative in mostly isolated intervals (Limits, and continuity not considered).	Transformation relations between function, first and second derivative on isolated intervals. Some correspondence relation with limits and continuity.	Conservation relations between function, first and second derivative, correspondence relations with limits and continuity. A few correspondence relations between intervals appear.
Inter-Interval	Correspondence relations between function and first derivative. Transformation relation between contiguous intervals through unions, subdivision, and intersections, in terms of the graph's properties (Limits, and continuity not considered).	Transformation relations between function, first and second derivative, limit, and continuity. Transformation relations between contiguous intervals through unions, subdivision, and intersections, in terms of the graph's properties.	Conservation relations between function, first and second derivative on all intervals. Transformation relations between contiguous intervals through unions, subdivision, and intersections, in terms of the graph's properties.
Trans-Interval	Correspondence relations between function and first derivative, correspondence relations with limits and continuity. Consideration of all intervals in the domain of the function.	Transformation relations between function, first and second derivative, limit, and continuity. Conservation relations between contiguous intervals through unions, subdivision, and intersections, in terms of the graph's properties.	Conservation relations between function, first and second derivative on all intervals. Conservation relations between contiguous intervals through unions, subdivision, and intersections, in terms of the graph's properties.

Fig. 1 Genetic decomposition of the graphing derivative Schema

The development of a Schema is also related to changes in its structure through the inclusion of new structures. When students face a challenging problem, they need to reorganize their Schema structures. APOS theory takes from Piaget's Schema definition the mechanisms involved in Schema restructure, which can be accomplished through two mechanisms: accommodation and assimilation. The accommodation mechanism involves a restructuring of the relations among structures in order to incorporate a new structure. It is evidenced by students' need to deeply reconsider their previous work or the use of auxiliary tools to equilibrate the Schema. The assimilation mechanisms act when new structures or relations can be smoothly incorporated to equilibrate the Schema, such as when students need to perform Actions on a Schema that is in the Trans- stage of development. For example, when students are required to find the properties of the Schema, apply it to a new situation, analyze its properties or its scope, the Schema is thematized into an Object.

In some studies of the Schema related to sketching graphs of a function using derivatives, the analysis has focused on the logical relations between the properties of the derivative Schema to describe students' different approaches (Fuentealba et al., 2017, 2019; García et al., 2011). In another study, the GDS was considered to describe how the various approaches of students were related to the different stages of two Schemas (Baker et al., 2000). The authors referred to this analysis as "the double triad". They found that the interaction between these Schemas was useful in describing different reasoning patterns in students' constructions as each of them follows its own evolution path (Arnon et al., 2014). The present study contributes to the literature by focusing on the thematization of the GDS as a new Schema.

Schema thematization has also been previously studied in the case of the derivative Schema (Fuentealba et al., 2017; García et al., 2011) and in the GDS case (Cooley, et al., 2007). This study contributes to the literature on Schema thematization by introducing the relations involved in the development of the GDS and analyzing the mechanisms used by students in its thematization.

The research questions addressed in this study are:

- How do advanced students address a differential calculus complex problem related to the graphs of functions?
- How does the interaction between the properties and intervals Schemas support GDS thematization?

3 Methodology

The study presented is qualitative and descriptive in character. According to this study's objectives, we looked for evidence in students' approach to different complex problems where they could show thematization of the GDS together with some information related to the mechanisms they applied to do so. To accomplish this, we used the theoretical and analytical tools from APOS theory related to the construction and development of a GDS (Fig. 1).

3.1 Participants and context

Twenty-six students who had finished their Mathematics undergraduate program and were enrolled in a Master's Program in either Mathematics Teacher Education in Spain (19) or Mathematics Education in Mexico (7) participated voluntarily in this study. They agreed to continue participating whenever it was necessary either by responding to questionnaires or through being interviewed.

3.2 Instruments

Three instruments were designed to obtain data for this study: two questionnaires and a semi-structured interview. They were applied at three different moments. All participants responded to the first questionnaire: it consisted of the two tasks taken from the Baker et al. (2000) study to determine the evolution of their GDS. Students were asked to graph a function h given its behavior in different intervals in terms of derivatives, and then they were explicitly asked about the implications on the graph of the function they had drawn if the continuity condition was removed (Fig. 2). Students individually responded to the questions on paper. They had enough time to reflect and complete the graph and answer the question. All the questions in the instruments were analyzed in terms of the genetic decomposition of the GDS (Fig. 1). The goal of this instrument was to determine the level of Schema evolution shown by each student. We were particularly interested in selecting those students who evidenced the construction of the GDS at the Trans-Intervals-Trans-Properties level, and those who also demonstrated the construction of a coherent Schema. The coherence of the Schema is demonstrated when students can determine which properties of the graph of the function change, and which remain the same when the continuity condition is taken away.

The first questionnaire was analyzed and nine students, six from Spain and three from Mexico, who demonstrated the construction of conservation relations and the construction of the GDS at a Trans-Properties-Trans-Intervals level were invited to respond to a second questionnaire consisting of three tasks (Fig. 3) adapted from previous studies related to the analysis of graphs of functions using derivatives (Fuentealba, 2017; García et al., 2011; Sánchez-Matamoros et al., 2006). Students' responses to this second questionnaire were analyzed in terms of evidence of thematization of the GDS.

Four students, two from Spain and two from México, responded correctly to all the questions of the second questionnaire and were selected to participate in a semi-structured interview. All these students showed they had again constructed a Trans-Properties-Trans-Intervals Schema. They also showed that they were able to perform Actions on the GDS, thereby providing evidence of thematization of this Schema. Only three of these students agreed to participate in this phase of the study, one from Spain and two from México (we will call them Juan, Luis and Tomás from now on). The interviews enabled us to deeply explore the relations involved in the development of the GDS and the implicit mechanisms involved in thematization of the GDS through students' ability to act on it as an Object.

3.3 Analysis of two interview questions

Two tasks were used during the interview and analyzed in terms of the genetic decomposition of the GDS (Fig. 1) (Baker et al., 2000; Cooley et al., 2007). Only the two tasks from the interview that required the use of successive

Sketch a graph of a function h that satisfies the following conditions:

h is continuous;

$h(0) = 2$, $h'(-2) = h'(3) = 0$, and $\lim_{x \to 0} h'(x) = \infty$;

$h'(x) > 0$ when $-4 < x < -2$ and when $-2 < x < 3$;

$h'(x) < 0$ when $x < -4$ and when $x > 3$;

$h''(x) < 0$ when $x < -4$, when $-4 < x < -2$, and when $0 < x < 5$;

$h''(x) > 0$ when $-2 < x < 0$ and when $x > 5$;

$\lim_{x \to -\infty} h(x) = \infty$ and $\lim_{x \to -\infty} h(x) = -2$.

What will happen to the graph of the function h if the continuity condition is removed?

Fig. 2 Questionnaire 1. What will happen to the graph of the function h if the continuity condition is removed?

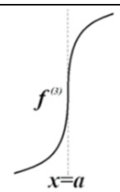
Task 1

Draw a graph of a function f satisfying the following conditions:

a) f is continuous in its domain
b) $f(2) = 0$
c) $f'(3) = f'(5) = 0$
d) $\lim_{x \to -\infty} f(x) = -4$
e) $\lim_{x \to 8^-} f(x) = -\infty$
f) $f'(x) < 0$ when $5 < x < 8$
g) $f'(x) > 0$ when $x < 5$
h) $f''(x) < 0$ when $3 < x < 8$
i) $f''(x) > 0$ when $x > 3$

Task 2

Consider the graph of the third-order derivative of a function f on an interval around $x = a$, shown in the following figure:



Describe and explain the graphs of the derivatives f'' and $f^{(4)}$ in the same interval around $x = a$.

Task 3

Given the following graph for the second derivative of a function, draw, describe and explain the graph of the third derivative of the function and that of its first derivative. Explain.

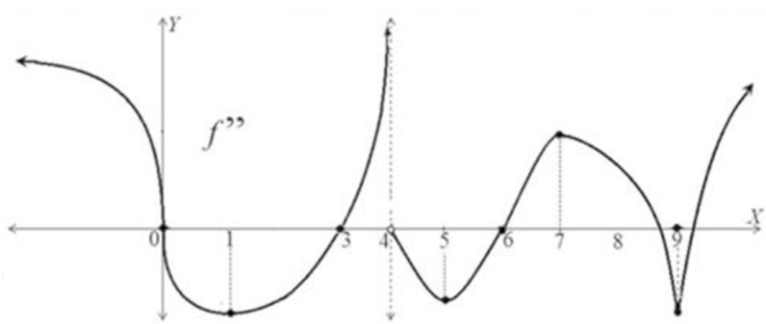
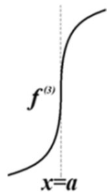


Fig. 3 Questionnaire 2

derivatives are analyzed; the complexity of solving these tasks is an indicator of students understanding the derivative as a linear operator and is what allows them to interpret critical points of the function.

3.3.1 Task 2

Consider the graph of the third-order derivative of a function f on an interval around $x = a$, shown in the following figure:

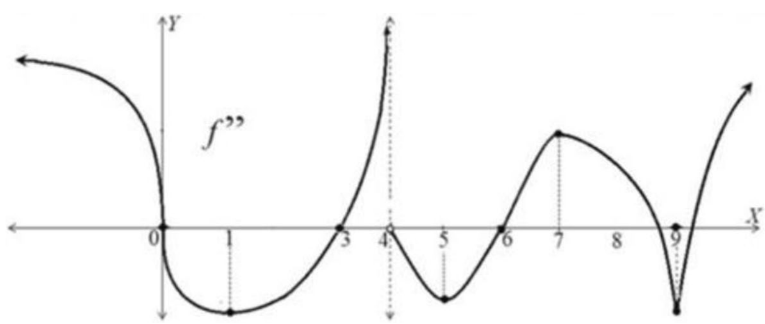


Describe and explain the graphs of the derivatives f'' and $f^{(4)}$ in the same interval around $x = a$.

The solution to this task requires students to make Actions on the graph of the third derivative of a function which gives evidence of thematization of the Schema. It also intends to analyze the type of relations between the properties, as well as intervals evidenced by students while interpreting the function's fourth and second derivatives. We focused on how students work on an interval around the given point corresponding to a vertical tangent to the curve representing the third-order derivative of the function. Particularly, we were interested in the method they used and the way they explained the properties of the derivative in the surroundings of $x = a$ when there is not an explicit expression for the function. Students needed to pay attention to an interval and relate it to the given function's properties to be able to find the properties of the fourth derivative and its graph on the same interval. Then, they are asked to do the reversion of the Process that led to the third derivative of the function to find the graph and properties of the second derivative. It can be noticed that the graph is not drawn on a Cartesian plane, so students may also think about what changes when the position of the graph in relation to the x-axis changes. The approximations used by students are analyzed in terms of the different relations constructed and the mechanisms associated with thematization of the GDS (Fuentealba et al., 2022).

3.3.2 Task 3

Given the following graph for the second derivative of a function, draw, describe and explain the graph of the third derivative of the function and that of its first derivative.



This task shows a very complex graph where several conflicting points appear: inflection points, cusps, asymptotes, etc. When looking for the previous and following derivatives, it is necessary to carefully analyze the graph in terms of the relations between the properties of successive derivatives and changes in the domain's intervals, in order to be able to determine where they come from and what will be the behavior of their graphs. Again, their use of relations, reversion, accommodation, and assimilation can give information about these mechanisms and thematization.

3.4 Data analysis methods

A qualitative analysis of an inductive nature was carried out individually by the four authors in order to analyze the obtained data from each of the three instruments. Results from each researcher were discussed among them until agreement was reached. Researchers started by analyzing a small sample from each instrument and discussed the codes and the relations between them and the evidence to reach a consensus on the final code used to analyze all it. Once a consensus was obtained, new data was added in order to review the initial coding system and to confirm its validity (Strauss & Corbin, 1994). This analysis was done in three phases. During the first phase the data obtained from the first questionnaire was analyzed individually. Each researcher codified the type of relations constructed in terms of the properties Schema and of the intervals Schema described in the GDS. Then, the structures and relations evidenced by students when solving the tasks were coded in terms of the development of the GDS. This information was discussed among the researchers until consensus was reached. Results of this phase of the analysis enabled them to identify those students who demonstrated the construction of a Trans-Intervals-Trans-Properties GDS level and its coherence, in terms of being able to determine when it was appropriate to analyze the proposed problems (Fig. 1).

This same procedure was used to analyze the second questionnaire. In this case, each researcher codified those structures and relations between successive derivatives (indicating the thematization of the GDS) shown by the participants' approach to the tasks. Through collective discussion, consensus was achieved. Results of this phase enabled researchers to choose those students who showed, through their answers, that they had thematized the GDS. These students were selected according to two criteria: their work clearly showed the construction at the Trans-Intervals-Trans-Properties Schema so that it could have been thematized, and their different approaches to the problem which could be related to different mechanisms involved in thematization of the Schema.

In order to obtain such evidence, we used the theoretical and analytical tools afforded by APOS theory to characterize Schema development and thematization (Baker et al., 2000; Cooley et al., 2007; García et al., 2011; Trigueros, 2019). The analysis of the mechanisms used by students when they show evidence of thematization of a Schema can provide information about possible differences in the Schema construction as an Object. Such analysis also provides information useful in the design of activities that can either account for thematization of the Schema or for the possibility of thematization while students work on them. Evidence of similarities and differences in students' thematization of the Schema provide information about different mechanisms used by students to thematize the Schema.

In the final phase, students' ability to perform Actions on the given function evidence students' thematization of the Schema through their construction of transformation and conservation relations among the Schema structures and evidences the use of assimilation or accommodation mechanisms in doing so.

4 Results

In this section we describe the main results obtained through the analysis of students' responses, highlighting the type of relations evidenced by their explanations and the mechanism used in thematization of the GDS, taking into account all their responses.

4.1 Approximation through higher derivatives: Juan's case

For Task 2, Juan started by describing the given graph for the third derivative paying attention to its behavior and linking it to the function's corresponding properties. He considered its domain and $x = a$ as an important point. He recognized its role as the graph's inflection point determining how the function's properties change. Then, while working on the graph for $f^{(4)}(x)$ he wrote on his sketch "as this is $f^{(3)}$ derivative, it is necessary to find out what happens with its slope when $x = a$ is approached", underlining that "the slope tends to infinity and when it passes through $x = a$ it softens, it decreases". Then he explained that "at $x = a$ the given function is not derivable" and continued sketching $f^{(4)}(a)$ while justifying: "it is positive around $x = a$ since $f^{(3)}$ increases all around". Next, he focused on analyzing the behavior of the slopes of the function. Through this analysis Juan showed the construction of transformation relations while drawing a vertical line at $x = a$ to separate both intervals and by drawing the fourth derivative at the two surrounding intervals to the left and right of $x = a$. He also made clear the important changes in the properties of the function in different parts in its domain. He continued by quickly sketching the

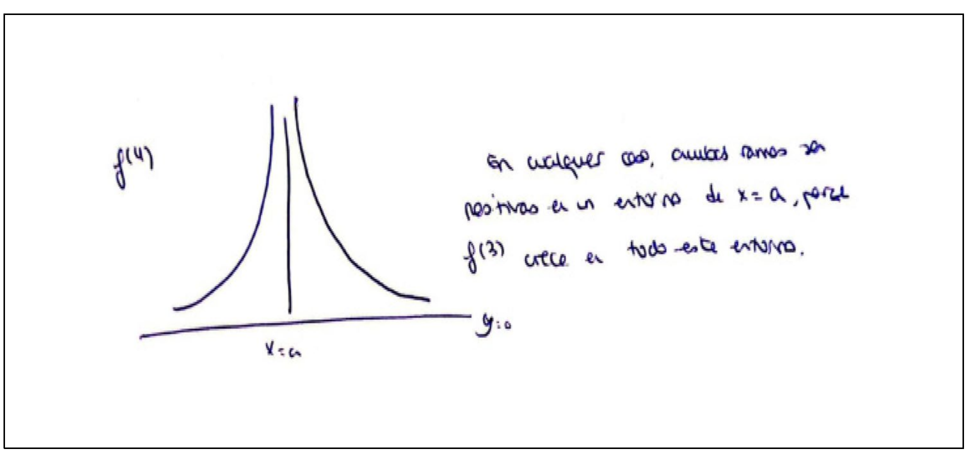


Fig. 4 Juan's answer to Task 2: Graph of the fourth derivative "in any case, before $x = a$ for $f^{3)}$, it grows in all this interval"

function's fourth derivative correctly (Fig. 4). Juan's actions and explanations evidenced the construction of the necessary transformation relations. These were demonstrated by his interpretation of the graph's changes through the function's domain and also by his construction of conservation relations shown in his consideration of the inflection point and its relation with the continuity of the fourth derivative. The interviewer asked him if he was sure of the graph's behavior around $x = a$, and he responded, "Yes, I am thinking about an asymptotic behavior around the point $x = a$".

Juan went on to find the antiderivative by doing an inverse Process. Juan drew a vertical line at $x = a$, but did not include the x-axis (Fig. 5). He drew the graph considering the third derivative of the function as positive. After sketching the graph, he explained that "the function has a growing slope, which is larger around the point $x = a$ and then it softens, although it continues growing all the time". By "softens", he means that its growth is slower as the points are farther from $x = a$.

Juan's work on this task shows the construction of transformation relations between the properties of the second and first derivative on the two considered intervals, and the construction conservation relations, as he was aware of the changes in the properties on both sides of the vertical line. His ability to perform Actions on the given function to construct the demanded function evidences his GDS thematization.

While working on Task 3, Juan used the same strategies as those he used in the previous question. He started by considering the sketch of the graph for $f^{(3)}$ using information from the given graph, taking into account all its complexities. He first determined the singular points on the domain of the given function: $x = 0$, $x = 1, x = 3x = 4, x = 5, x = 6, x = 7 \text{and} x = 9$. He then analyzed the behavior of the function on the intervals surrounding each of them.

When focusing on the intervals around $x = 0$, he said, while sketching the graph: "$f''(0)$ has a vertical tangent line, $f^{(3)}$ does not exist, and there should be a vertical asymptote towards minus infinity" (Fig. 7b). He continued:

> at the intervals around zero, when approaching zero, these slopes of the tangent lines will tend to infinity, in this case minus infinity as the curve goes down, I understood going down where $f^{(3)}$ has to be negative, in all of them, and as at one it has a relative extremum, a minimum in this case, then there $f^{(3)}$ should have a root at 3; […] as f'' started with a horizontal slope and little by little it approaches the verticality, I did the same to $f^{(3)}$ but I sent it to infinity since I can see that the tangent line of f at zero (referring to $f''(0)$) has to be vertical.

Juan continued drawing the graph for $f^{(3)}$ and explained his decisions, taking into account the given function's behavior on each interval between different singular points. He carefully considered the changes inside those intervals (Fig. 7b).

When Juan started looking for the first derivative he used the same approach: he explained and drew the graph at the same time while paying attention to the surrounding intervals of the same singular points. When analyzing the graph of f' around $x = 0$, Juan followed the behavior of f'' starting with the intervals around $x = 0$, and explained that "at $x = 0 f'$ should have a maximum since it was growing and then decreasing [taking the graph of f'' as a referent which is positive before $x = 0$ and negative after $x = 0$]". He also considered the curvature: "I wanted it to be first less slanted and then its slope was larger, and then the slope decreases it is less and less. I started with a certain slope and then I reduced it until here, it reached horizontality at $x = 0$. So, at $(0, f'(0))$ there is a relative extreme […] at 1, so we have an inflection point".

Juan then changed his description approach. He started considering the changes involved in both the first and the third derivatives at singular points. He used a table he designed "to guide my reflection" (Fig. 6).

He continued considering f' and $f^{(3)}$ graphs at $x = 3$ and $x = 4$, interpreting f'' around those points:

> as it has a root at $x = 3$, f'' has to have a relative extreme at $(3, f'(3))$, when it changes from being negative to being positive f'', for f' changing from decreasing to increasing, so it is a minimum, there it does not grow anymore and then it will end up with the slope going to infinity [referring to the intervals surrounding $x = 0$ at f'], because this is the slope of

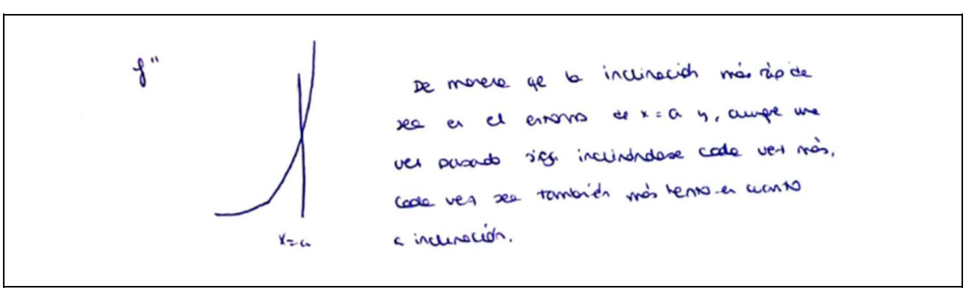
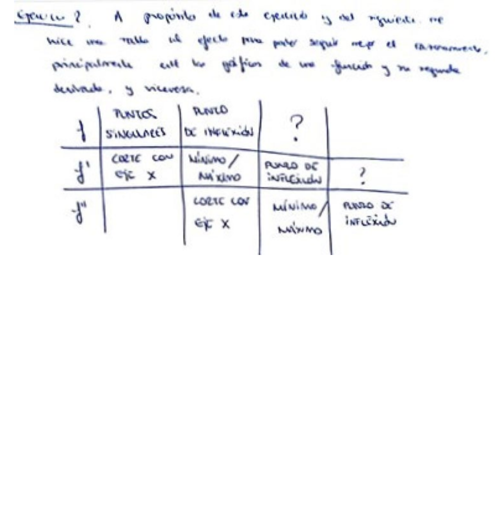


Fig. 5 Juan's answer in Task 2 for the second derivative. "In such a way that the maximum slope is in the surroundings of $x = a$ and even when it is passing it continues growing more each time, at the same time, more slowly regarding the inclination."



For this exercise and the next one I made a diagram to better follow the reasoning, mainly between the graphs of a function and its second derivative, and vice versa.

f	Singular Points	Inflection Points	?	
f'	Cutting *x*-axis	Minimum /Maximum	Inflection Point	?
f''		Cutting *x*-axis	Minimum /Maximum	Inflection Point

Fig. 6 Table used by Juan to support his reasoning

> the tangent to f'' […] at $x = 4$ in turn, we have that for $f(3)$ we are going down to a minimum in this case. And as I noted in my table, if it had a minimum in f' at the next derivative [f''] we will have a point crossing the x-axis. (see Fig. 7a).

He continued his explanation: "for f'' at $x = 5$, at $x = 7$ and $x = 1$, so at $x = 5$ there is a minimum in f", a crossing point with the x-axis for $f(3)$ and an inflection point for $f^{(3)}$". He repeated the same argumentation for $x = 6$: "as in $x = 6$ there is an inflection point for f'', in $f'(6)$ there will be a maximum [he draws a cusp at this point]". Finally, he identified $x = 9$ as a point where the derivative is not defined for f'' and looked for the slope of the graph for $f^{(3)}$ and its meaning for the graph of f'. He successfully sketched both graphs around $x = 9$: "for the third, at $x = 9$ there is a non-derivable point, so there it is not derivable. I considered it as an asymptote […] to the left. I think it could go down to minus infinity since it decreases, and it tends to be vertical". However, he struggled when interpreting the behavior of $f'(9)$:

> f' is more difficult for me as we had a minimum at that point, I am trying to find how the graph will look as [f''], its slope starts decreasing a lot […] but as there is a minimum at 9 in f'' in f' there should be an inflection point at $x = 9$. So we would change the graph for f' [from concavity down in the interval (7, 9)] to a smiling face [concave up in the interval (4, 7)] and we will follow with a smiling face, as I said, until 9 and something where there would be a minimum […] the curvature will not change […], but that angled point [(9,f''(9))] would have to go down and make an inflection point at $x = 9$ [in f'] then decreases a little bit more to reach the minimum. (see Fig. 7b).

Considering his work, we determined he showed thematization of the GDS through the assimilation mechanism. It is clear that Juan showed the construction of transformation relations when describing all the properties changes in both graphs, and conservation relations among the structures of his thematized GDS. He proved to be capable of performing Actions on the given graph to deduce its properties according to the different intervals on its domain. The table he constructed to support his reasoning helped to direct his Actions. We consider he showed his ability to perform the needed Actions on the original graph by fluidly interpreting and explaining the relation needed to de- thematize the behavior of the third and first derivative of a function and to perform new Actions on it in order to draw and explain their behavior. Although in the interview Juan acknowledges the fact that it takes more effort for him to think in terms of the antiderivative than the derivatives, his decision to create the table was appropriate to guide his analysis.

4.2 Approximation through antiderivatives: Luis and Tomás's case

In contrast to Juan, Luis and Tomás started their analysis of both tasks by considering the behavior of the antiderivative. Luis initiated his analysis of the second task by underlining that "as in the figure the position of the graph is not detailed, I can consider two different cases". He first considered the case where the $f^{(3)}$ curve is located above the x-axis on the plane and then he located the curve below it, while Tomás

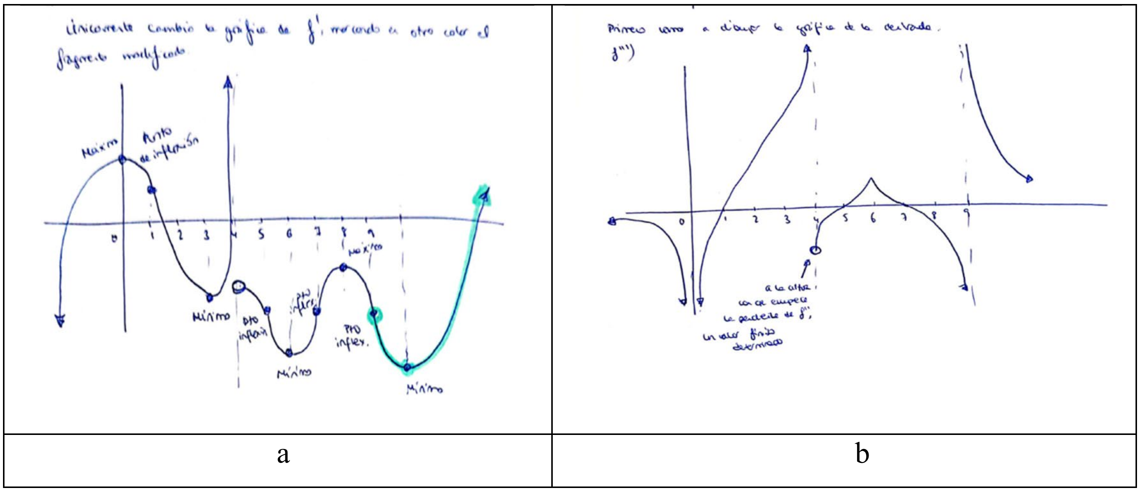


Fig. 7 Juan's answers during interview for Task 3

only took into account the case where the curve for $f^{(3)}$ is located above the x-axis, although he mentioned that the response would be different according to the position of the x-axis.

Luis continued by easily drawing the curve for $f\varepsilon$ as an increasing curve for the first case, and a decreasing curve for the second case (Fig. 8a). He included a vertical line at the point $x = a$ in both cases and drew its tangent at that point. While doing this, he commented that "these tangent lines are supposed to be vertical, although they don't look really vertical, they are meant to be". Then he went on to explain:

> Each point on the graph of the third derivative represents the value of the slope of the tangent line for the second derivative … then it is important to see what happens at point a. … I can see that to the left of a, the function values keep growing, which means that the second derivative function has growing slopes.

He continued describing the second case: "In this case, the idea is the same, the slopes in the case where the derivative takes negative values, so the second derivative would also be negative but decreasing in this case". Then he considered the behavior of the curve on a small interval around $x = a$: "We see here that $f^{(3)}$ is growing in the neighborhood of a, so the second derivative function's slopes are growing and there is a vertical inflection point on $x = a$". He drew it and he continued reasoning in the same way for the negative case for $f^{(3)}$: "Here, the second derivative is also negative and now it is decreasing" (Fig. 8a).

Tomás used an approach similar to that of Luis when working with the second derivative. He also explained

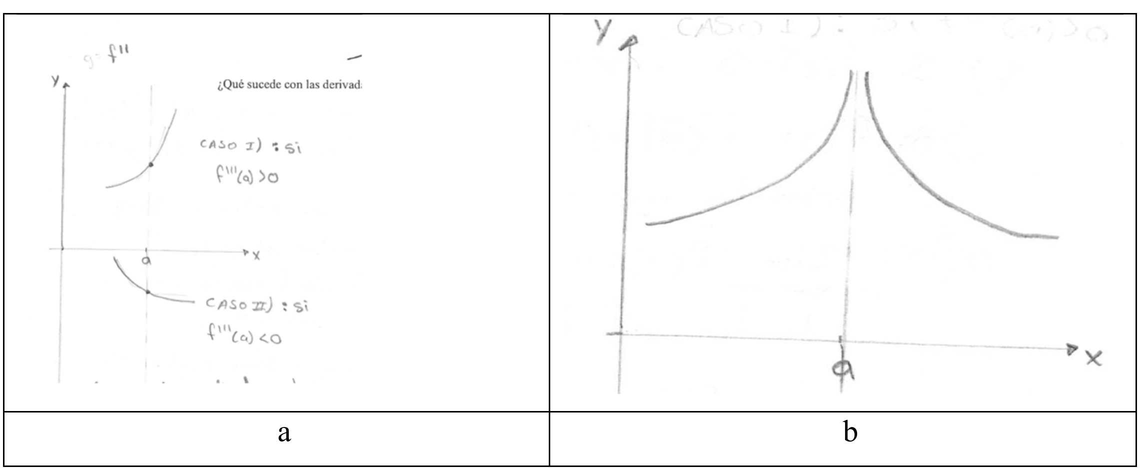


Fig. 8 Luis's answer to Task 2: Two cases for the second derivative and two cases for the fourth derivative

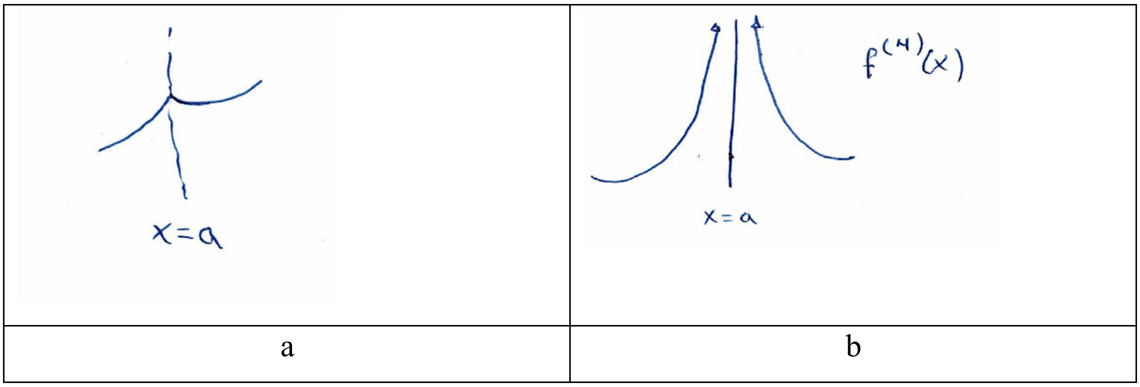


Fig. 9 Tomás's answer, second and fourth derivative, in Task 2

it with ease, although he only considered the positive case, and easily obtained the graph of the corresponding derivative(Figure 9a).

Both students' analysis shows their ability to determine a specific interval where the behavior of the curve has an important change. Through the analysis of the behavior of the function on both intervals around the conflicting point $x = a$, both students showed the construction of transformation relations between the slopes of the third derivative and the implication of changes for the second derivative. Tomás also evidenced the construction of conservation relations when determining that the point $x = a$ plays an important role in determining the relations between the intervals where changes occur (Fig. 9b). His graph shows that the fourth derivative of the function exhibits a vertical asymptote at that point.

When Luis worked on the graph of f's fourth derivative, he considered an auxiliary function:

> Let $g(x) = f^{(3)}(x), g'(x) = f^{(4)}(x)$, then $f^{(4)}(x)$ would be the slopes of the tangent lines over all the domain for $f^{(4)}(x)$. As when $x = a$ the third derivative has an infinite slope, then, when looking at $g'(x)$, for the fourth derivative, when one approaches a from the left, it becomes very big, and the same happens when approaching it from the right … also we see that when it is far from a, either from its left or from its right, slope is almost zero so $f^{(4)}(x)$ graph will be like this asymptote (Fig. 9b) on the intervals on both sides of $x = a$.

When the interviewer asks him if he is sure about the graph's behavior around $x = a$, his explanation is not that clear: "I am looking here around a and then f'' is increasing a lot". It is thus not clear if he is thinking of an asymptotic behavior around $x = a$ or a cusp at point $x = a$, and the interviewer did not ask about this. In any case, he was able to consider the properties of the fourth derivative on the intervals surrounding $x = a$ in association with the derivative of the auxiliary function g.

In contrast with Luis, Tomás continued working on the fourth derivative of the function he was working on. He accounted for the discontinuity: "Here, the fourth derivative has a discontinuity … if we look closely to a very small interval around $x = a$ we can see that the slopes are always growing faster and faster, so the graph has two asymptotes that go to infinity" (Fig. 9b).

In this episode both students give evidence of the construction of transformation relations through their interpretation of the changes to the graph's properties on the corresponding intervals. Although Luis apparently did not take into account the transformation relation implied in the continuity of the function at the point $x = a$, his descriptions of the transformation relations between the given function derivative and its fourth derivative, as well as the transformation relation shown when he analyzed the slopes' change for the fourth derivative case, are enough evidence of that construction. Another difference in these students' work is Tomás's ability to find directly the fourth derivative of the function, while Luis considered the given derivative as a new function to be able to find the graph related to the fourth derivative of the given function.

When working on Task 3, Luis and Tomás started again by focusing on the antiderivative of the given derivative function. Tomás began, as Juan did, dividing the domain of the function into intervals related to the behavior of the given second derivative of the function, while Luis concentrated on the conflicting points on the domain of the second derivative's graph and then used the intervals.

Tomas worked from left to right to sketch the graph of the first derivative function, and Luis defined an auxiliary function $g(x) = f'(x), g'(x) = f''(x)$ to support his reasoning. In spite of this difference, both students stated that "$f''(x)$ values correspond to the tangent lines' slopes for all x in the domain of $f'(x)$" and were able to relate the properties of the second derivative function to the behavior of the first derivative, demonstrating the construction of transformation relations for properties and intervals. Both students showed difficulties when dealing with some conflicting points. Tomás, for example, struggled with the behavior of the function around $x = 4$ and decided that "there is a need for an inflection point". Even when questioned by the interviewer, he did not consider the possibility of the function having a discontinuity there. Both of them also faced difficulties at $x = 9$. In this case, while sketching the graph, Tomás considered:

> The derivative is not defined for the second derivative and there is a change in concavity so, I think there is a discontinuity there, and the function goes to minus infinity, then it starts again after 9 and there is a minimum, and then it increases.

In his reflection, Tomás did not consider the possibility of the function having a vertical inflection point at $x = 9$. In Luis's case this dialogue took place:

> I: But, in $f''(9)$ you had a cusp. How did you consider this fact when working on f'?
> L: To the left of 9, values are less that $f''(9)$. Well, I used the slopes and I see how they change. I can relate that with the behavior of the graph.
> I:Did you take into account the information about concavity, or just slopes in your analysis?
> L: No, I only used slopes, maybe I used it implicitly when working on f'.
> I:But you did not mention it.
> L:Maybe I used it implicitly but for me it is easier to consider the slopes at each point. … I just think that the slope gives me all the information I need.
> Figure 10 shows the sketch Luis developed for the first derivative. Tomás's sketch is similar.

When working on the third derivative, Luis considered again a new function g corresponding to the second derivative, while Tomás worked directly on the given graph. Through the analysis of this question, Luis and Tomás showed more difficulties when deciding the third derivative behavior. Both described this graph in terms of discontinuities and used slopes with ease to analyze the given function at all the important intervals to draw the graph of the third derivative. Tomás showed less confidence in his Actions, while Luis was confident and said that he worked on this task as he did Task 2. He said:

> To the right of $x = 1$, the graph of the second derivative has positive slopes, so $f^{(3)}(x)$ is positive there, and then, I consider it tends to infinity to the left of $x = 4$, since $f''(x)$ also has an asymptote, this means that the slopes grow and become almost vertical, and in $f^{(3)}(x)$ slopes grow and tend to infinity at the left of the four.

In general, Luis's explanation shows that he had constructed all the transformation and conservation relations involved among the GDS structures. He also showed that he was able to de-thematize the Schema to work on the second derivative and deduce the properties of the third derivative by moving from one interval to the next.

In spite of their difficulties, both Tomás and Luis were able to reconsider their decisions when they needed to determine all the properties of the functions of the first and third derivatives. They evidenced the construction of transformation relations through their explanations of changes in the properties for both functions, and the construction of conservation relations when considering the role of slopes as a way to work with derivatives and to determine antiderivatives. They clearly showed they were able to perform Actions on the presented graphs, and to give solid arguments for their decisions about what changed and why, using slopes. They both demonstrated the thematization of the GDS: Luis through the accommodation mechanism, since he needed support from an auxiliary function to make decisions about the properties of higher-order derivatives and antiderivatives, and Tomás, able to do all the transformations directly, through the assimilation mechanism.

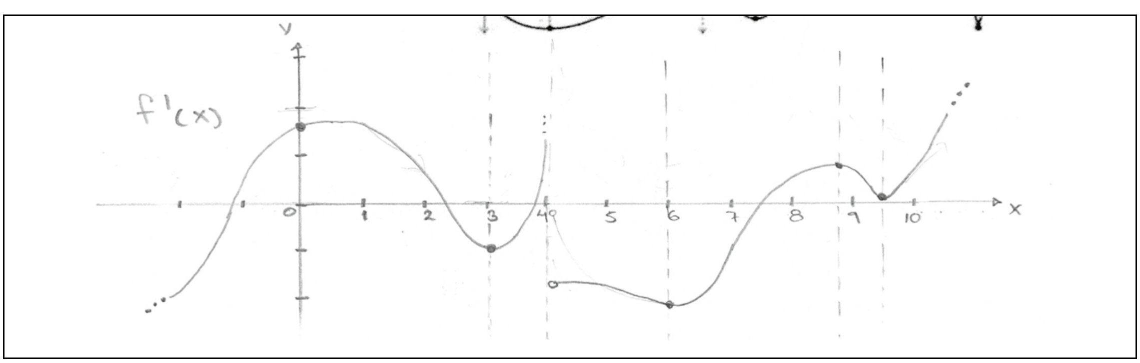


Fig. 10 Luis's answer for the first derivative in Task 3

5 Discussion and conclusion

This study contributes to the development of APOS theory in terms of the thematization of the GDS. This Schema has been described before just as the result of the interaction of the intervals and the derivative Schemas in terms of the notion of coordination among structures (Cooley et al., 2007). In this study the specific relations needed to thematize this Schema have been used in the selection of three students who demonstrated thematization of the GDS as an Object. The subtleties involved in these students' reasoning highlight the subtleties involved in thematization in general and on a double Schema thematization. The two mechanisms involved in Schema thematization are exemplified in the difference in the students' approaches to the problem. One student demonstrated the need to accommodate his previously constructed structures and relations to be able to make decisions about the proposed tasks, thus demonstrating thematization through accommodation. The other two students showed a direct approach in the solution of the tasks showing their use of the assimilation mechanism while performing Actions on the GDS Schema, which has not been studied before.

This study also contributes to the analysis of the complexities involved in a deep understanding of functions' behaviors as described when drawing or understanding their graphs. Understanding the way advanced students use their knowledge in the solution of non-traditional tasks gives information on strategies that can be used when teaching introductory differential calculus to university students. This is not a matter of simply playing with derivatives, but a way to make students aware of derivatives' relation with different intervals or subintervals and conflicting points in the function domain. Looking at the interesting strategies used by graduate students helped us to make sense of the difficulties involved in this interaction. This relevant information about the interrelation between limits, continuity and derivatives of functions and about the serious need to take into account their changing relations with the structure of the function's domain is useful in the development of a deeper understanding of the role of derivatives in calculus. These difficulties would possibly not be apparent when only simple problems were posed to the whole sample of this study and can inform teachers and preservice teachers about the need to take these relations into account early in calculus courses at university.

In terms of the research questions formulated we can appreciate from the results of this study that focusing on how students approach the problems makes it possible to understand the role that transformation and conservation relations play in their thinking process. This study gave researchers the opportunity to analyze how differently succeeding students approach the same tasks from different perspectives and the affordances provided by those approaches. One student, Juan, always started by addressing the higher-order derivatives as they seemed to him to be more direct. The results associated with his approach show his reliance on the analysis of transformation relations related to the slope of the function or to the rate of change. He showed the construction of conservation relation by considering that slopes, rate of change, and derivatives can be used indiscriminately to analyze the given sketches. This enabled him to determine what changes and what remains the same when analyzing the behavior of the function in the vicinity of conflicting points. He struggled with the inverse problems, but again was able to reflect on the relations between the function's properties and the intervals involved in their definition to be able to work on these problems. It is interesting to underline that Juan's use of a table for support demonstrates that he had constructed a conservation relation between a function, its first and second derivative, and its antiderivative that could be used in any problem, independently of the order of the derivative function he had to work on.

The other two students participating in this study always addressed the problems by initially working on the inverse problem sketch, and then continued to work on sketching the higher-order derivatives asked for in the interview tasks. These students also used the slope of the tangent line to help them in their analysis, and both of them, when working on the antiderivative graph, considered the points of the given graph as the value of the tangent slope related to the corresponding antiderivative. They struggled, in particular with the second task, but were able to succeed by determining transformation and conservation relations throughout their work. Although Juan, Luis and Tomás also show the construction of a conservation relation between the slope of the tangent line to the function and the derivative, they did not refer to the rate of change. We consider the analysis of transformation and conservation relations in these calculus problems a contribution of this study to the literature.

The complexity of the designed tasks made the emergence of differences in students' reasoning approaches possible. The analysis of students' detailed explanations gave evidence of their thinking throughout the solution of the problems. It showed that there is not a unique path in the construction of a Schema as an Object, and that these different approaches had in common the constant use of the slope of the tangent line to the function, in order to guide them throughout the analysis of the related graphs and the indispensable relation of changes in domain to the function's structure.

Another difference in the approach to the tasks was evidenced by Luis and Juan following the dynamics of the graph and determining the needed changes with no problem. Their work gives evidence of the use of the assimilation mechanism in the thematization of their Schema. Luis needed on some occasions to revert to the introduction of a new function, an auxiliary function, to support his thinking. The use of this function can be related to Luis's accommodation mechanism in his thematization of the GDS. In general, however, these students' way to accomplish the tasks, through the accommodation or assimilation mechanisms, enabled a richer understanding of the thematization of the GDS and provided valuable information that can be useful to help calculus students to construct a deeper understanding of derivatives.

The analysis of students' approaches to solve both the selection task and the interview tasks also gives information about the construction of the two Schemas involved in the GDS. Throughout the analysis of students' responses we could distinguish some regularities in their approach to work with the given functions.

All students referred constantly to the local slope of the function or its rate of change at a point in order to determine the possible behavior of the new functions on specific domain intervals. They also showed they considered the role of critical points on a given derivative function in underlying possible change of behavior on the former or next derivative. Moreover, they carefully identified those points in the domain where changing the order of a derivative may impact the function's continuity or signal the possible existence of asymptotic behavior around points where the given function was not continuous. Points on the domain that they considered to be "not common or conflicting" were considered "very difficult" but as playing an important role in "strong" changes in functions behavior.

All these regularities can be related to their development of the intervals Schema or the development of the derivative Schema, but in most cases, they worked with both Schemas almost at the same time; they showed they considered derivatives in particular intervals or properties on the intervals to focus on a specific point of the derivative's behavior.

These graduate students showed they had constructed the necessary tools to work but most of the students recruited to work on the selection task showed some specific difficulties in terms particularly of the relation between continuity and the derivative, with the role of inflection points and other points where the derivative was not defined or with the changes on the intervals in the needed function when new properties were taken into account. All these graduate students, who in many cases are calculus teachers at the university, showed difficulties when trying to work together with both Schemas. These difficulties have been already described (Baker et al., 2000; Cooley et al., 2007), but the use of information about Schema thematization found in this study underlines specific information about the development of the intervals Schema and of the derivative Schema by themselves. This information can be used together with the information given by specific strategies used by students in this study in the design of teaching strategies and tasks to be used in introductory differential calculus courses to help the construction of the GDS and foster its thematization as early as possible.

Further studies involving larger student samples are necessary to generalize the conclusions of this study. Research on Schema thematization and on interactions between Schemas is needed to advance our understanding of APOS theory Schemas.

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