Triebel-Lizorkin regularity and bi-Lipschitz maps: composition operator and inverse function regularity

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Abstract

We study the stability of Triebel-Lizorkin regularity of bounded functions and Lipschitz functions under bi-Lipschitz changes of variables and the regularity of the inverse function of a Triebel-Lizorkin bi-Lipschitz map in Lipschitz domains. To obtain the results we provide an equivalent norm for the Triebel-Lizorkin spaces with fractional smoothness in uniform domains in terms of the first-order difference of the last weak derivative available averaged on balls.

1 Introduction

Let Ω_1, Ω_2 be Lipschitz domains in $\mathbb{R}^d$ and let $f : \Omega_1 \to \Omega_2$ be a bi-Lipschitz homeomorphism belonging to the non-homogeneous Triebel-Lizorkin space $F_{p,q}^s(\Omega_j)$, where $\|f\|_{F_{p,q}^s(\Omega_j)} := \inf_{g|_{\Omega_j} \equiv f} \|g\|_{F_{p,q}^s(\mathbb{R}^d)}$. In this paper we give sufficient conditions for f^{-1} to be in $F_{p,q}^s(\Omega_2)$ and conditions to ensure that the composition operator $T_f : g \mapsto g \circ f$ maps the function space $F_{p,q}^s(\Omega_2)$ into $F_{p,q}^s(\Omega_1)$.

As it turns out, if $(s-1)p > d$ and $f \in F_{p,q}^s(\Omega_1)$, then the inverse function has the same regularity, and the composition operator map leaves the Triebel-Lizorkin regularity invariant as well. If $(s-1)p \leqslant d$, with $s > 1$, a positive answer is also provided but we have to substitute $F_{p,q}^s$ by $\mathbf{F}_{p,q}^s = F_{p,q}^s \cap C^{0,1}$. The reason for this to happen is that the chain rule involves products of the derivatives of two mappings, so we require an algebra structure for the function spaces to grant that the indices remain invariant.

The precise result is the following:

Theorem 1.1. *Let $0 < s < \infty$, $s \notin \mathbb{N}$, let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $d \in \mathbb{N}$. Given bounded Lipschitz domains $\Omega_j \subset \mathbb{R}^d$ and a bi-Lipschitz function f with $f(\Omega_1) = \Omega_2$, then*

$$f \in \mathbf{F}_{p,q}^s(\Omega_1) \text{ and } g \in \mathbf{F}_{p,q}^s(\Omega_2) \implies g \circ f \in F_{p,q}^s(\Omega_1), \tag{1.1}$$

(see Figure 2.2) and

$$f \in \mathbf{F}_{p,q}^s(\Omega_1) \implies f^{-1} \in F_{p,q}^s(\Omega_2). \tag{1.2}$$

Note that if $(s-1)p > d$ then $\mathbf{F}_{p,q}^s = F_{p,q}^s$.

The reason behind the rather unnatural assumption $s \notin \mathbb{N}$ in Theorem 1.1 is the use of first-order differences to characterize the function space, since otherwise one needs to use second-order differences and the techniques used throughout this paper are not enough. However, the

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composition result holds for $s \in \mathbb{N}$ in the one-dimensional case at least (see [BMS10]) and the inverse function result holds also in the one-dimensional case for $p = q$ and $sp > 1$ (see [APS17, Lemma 2.11]), and for $s \in \mathbb{N}$ and $q = 2$ in arbitrary dimensions, that is, in the Sobolev scale (see Lemma 2.10). For the Sobolev scale ($q = 2$, $s \in \mathbb{N}$) in the higher dimensional case we refer the reader to [CHK15], and Lemma 2.10 below. The result also holds for non-integer Hölder spaces, see Lemma 2.5.

We can conjecture that Theorem 1.1 remains true in uniform domains, see Remark 2.8 for a discussion. Recall that bounded Lipschitz domains are uniform.

To obtain the preceding result, we use elementary techniques such as Hölder inequalities, but we need to build on first-order differences to be able to use the change of variables. We will use the following characterization, which is proven for uniform domains, see Section 3 for the definitions.

Theorem 1.2. *Let Ω be a uniform domain, let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $s = k + \sigma$ with $0 < \sigma < 1$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and consider the auxiliary index $1 \leqslant u \leqslant \infty$ so that $\sigma > \frac{d}{p \wedge q} - \frac{d}{u}$. Then*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_p + \left(\int_\Omega \left(\int_0^1 \frac{\left(\fint_{B(x,t) \cap \Omega} |\nabla^k f(x) - \nabla^k f(y)|^u dy \right)^{\frac{q}{u}}}{t^{(\sigma + \frac{d}{u})q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \qquad (1.3)$$

Theorem 1.2 is proven in Section 4. The idea is to check that this norm is equivalent to the restriction of the usual Fourier definition for the Triebel-Lizorkin scale for $\Omega = \mathbb{R}^d$ (see [Tri06]). Thus, it is enough to find a suitable extension operator such that the Triebel-Lizorkin norm of the extended function is bounded by the right-hand side of (1.3). As a matter of fact, in Theorem 4.7 below we will see that the Peter-Jones extension operator for the Sobolev scale is also an extension operator for the relevant function spaces, following a similar reasoning to [PS17].

The introduction of uniform domains at this step is quite natural. At the end of the day, most extension operators are defined by a discretization of the complementary domain by associating to a given patch the value defined at a certain symmetrization of this patch (see [Ryc99] for a different approach). Usually, this patches are Whitney cubes. The connectivity properties of this symmetrizations for neighboring patches are used to ensure that no new jumps are created when gluing together the discretized reflections. In Lipschitz domains one can use the trivial symmetrization (at least locally), but as showed by Peter Jones in [Jon81], this procedure can be extended quite naturally to uniform domains and the proof does not get essentialy more involved.

At this point we want to remark that the Peter Jones extension operator defined [Jon81] for Sobolev spaces with smoothness one in uniform domains is also an extension operator for Triebel-Lizorkin spaces on domains with smoothness below one in interior corkscrew domains. This fact was unnoticed in [PS17], although the proof there can be easily modified to cover this quite general setting. See Theorem 4.3 and Remark 4.4 below for a discussion on this matter.

The issue of stability of the composition operators has already been discussed thoroughly in the literature. See [CFR10, HK13, OP17] for results concerning the linear composition operator for quasiconformal mappings, and [HK08] for mappings of finite distortion, in both cases the authors study the case of smoothness smaller or equal than one. It is interesting to note that for critical Bessel potential spaces i.e. $F^{d/p}_{p,2}$ with $\frac{d}{p} \leqslant 1$, every quasiconformal mapping preserves the function space. Quasiconformal mappings are known to have Hölder regularity below one, determined by their distortion. This is much weaker than bi-Lipschitz, and it is natural to wonder whether Theorem 1.2 can be also weakened in such a way for function spaces with regularity greater than one.

We also refer to [Dah79, Vod89, BS99, Bou00, BMS14, BMS20] for the study of the non-linear composition operator $\widetilde{T}_f g = f \circ g$ and regularity in the Besov and Triebel-Lizorkin scales. It

is worthy to note the result in [Dah79], where it is seen that for $d \geqslant 3$ and $f \in C^\infty(\mathbb{R})$ then $\widetilde{T}_f : W^{s,p}(\mathbb{R}^d) \to W^{s,p}(\mathbb{R}^d)$ implies that $f = c\mathrm{Id}$ whenever $s \in \mathbb{N}$, $s \geqslant 2$, $1 \leqslant p < \infty$ with $s < \frac{d}{p}$.

The author of the present paper is unaware of any study concerning the Triebel-Lizorkin regularity of the inverse of bi-Lipschitz mappings, see [CHK15] for the inverse mapping theorem in the Sobolev case.

1.1 Notation

Throughout this paper we will write C for constants which may change from one occurrence to the next. If we want to make clear in which parameters C depends, we will add them as a subindex. In the same spirit, when comparing two quantities x_1 and x_2, we may write $x_1 \lesssim x_2$ instead of $x_1 \leqslant Cx_2$, and $x_1 \lesssim_{p_1,\ldots,p_j} x_2$ for $x_1 \leqslant C_{p_1,\ldots,p_j} x_2$, meaning that the constant depends on all these parameters.

Given $1 \leqslant p \leqslant \infty$ we write p' for its Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

Given $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x,r)$ or $B_r(x)$ for the open ball centered at x with radius r and $Q(x,r)$ for the open cube centered at x with sides parallel to the axis and side-length $2r$. Given any cube Q, we write $\ell(Q)$ for its side-length, and rQ will stand for the cube with the same center but enlarged by a factor r. We will use the same notation for one dimensional balls and cubes, that is, intervals. Given two bounded sets A and B, we write $\mathrm{D}(A,B) := \mathrm{diam}(A) + \mathrm{diam}(B) + \mathrm{dist}(A,B)$, and we call it the *long distance* of A and B.

Definition 1.3. *Let $\delta, R > 0$, $d \geqslant 2$. We say that a domain $\Omega \subset \mathbb{R}^d$ is a (δ, R)-Lipschitz domain (or just a Lipschitz domain when the constants are not important) if for every point $z \in \partial\Omega$, there exists a cube $\mathcal{Q} = Q(0,R)$ and a Lipschitz function $A_z : \mathbb{R}^{d-1} \to \mathbb{R}$ supported in $[-4R, 4R]^{d-1}$ such that $\|A_z'\|_{L^\infty} \leqslant \delta$ and, possibly after a translation that sends z to the origin and a rotation, we have that*

$$\mathcal{Q} \cap \Omega = \{(x,y) \in \mathcal{Q} : y > A_z(x)\}.$$

If $d = 1$ we say that $\Omega \subset \mathbb{R}$ is a Lipschitz domain if Ω is an open interval.

The natural numbers are denoted by $\mathbb{N}$ if 0 is not included, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The multiindex notation for exponents and derivatives will be used: for $\alpha \in \mathbb{Z}^d$ its *modulus* is $|\alpha| = \sum |\alpha_i|$ and its *factorial* is $\alpha! = \prod(\alpha_i!)$. Given two multiindices $\alpha, \gamma \in \mathbb{Z}^d$ we write $\alpha \leqslant \gamma$ if $\alpha_i \leqslant \gamma_i$ for every i. We say $\alpha < \gamma$ if, in addition, $\alpha \neq \gamma$. For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}^d$ we write $x^\alpha := \prod x_i^{\alpha_i}$. A similar notation is used for directional weak derivatives: $D^\alpha := \prod \partial_{x_i}^{\alpha_i}$.

2 Composition and inverse function theorems

In this section we will show that the function spaces considered are stable under composition with bi-Lipschitz mappings of the same space and they satisfy an inverse function theorem.

First we need a lemma on a generalized chain rule. For this purpose we recover the multivariate version of Faà di Bruno's formula (see [KP92, Lemma 1.3.1] for the one-dimensional case), whose proof is a mere exercise on induction. Given a multiindex $\vec{i} \in \mathbb{N}_0^D$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define $m(\vec{i}) \in \{1, \cdots, D\}^{|\vec{i}|}$ as the vector whose components are non-decreasing (i.e, $m(\vec{i})_\ell \leqslant m(\vec{i})_{\ell+1}$), and such that

$$\#\{j : m(\vec{i})_\ell = j\} = \vec{i}_j.$$

For instance, $m(3,2) = (1,1,1,2,2)$, and $m(4,0,1) = (1,1,1,1,3)$.

Lemma 2.1 (Chain rule). *Given* $f = (f^1, \cdots, f^D) : \mathbb{R}^d \to \mathbb{R}^D$, $g : \mathbb{R}^D \to \mathbb{R}$ *with* $f^i \in W^{M,\infty}(\mathbb{R}^d)$ *and given a bi-Lipschitz function* $g \in W^{M,\infty}(\mathbb{R}^D)$ *and* $\vec{k} \in \mathbb{N}_0^d$ *with* $|\vec{k}| = M$, *there exist appropriate constants such that*

$$D^{\vec{k}}(g \circ f) = \sum_{\substack{1 \leqslant |\vec{i}| \leqslant M \\ \{\alpha_j\}_{j=1}^{|\vec{i}|} \subset \mathbb{N}_0^d \setminus \{\vec{0}\} : \sum |\alpha_j| = M}} C_{\vec{k}, \vec{i}, \{\alpha_j\}} D^{\vec{i}} g(f) \prod_{\ell=1}^{|\vec{i}|} D^{\alpha_\ell} f^{m(\vec{i})_\ell} \tag{2.1}$$

almost everywhere.

Remark 2.2. *The chain rule (2.1) can be applied also to functions with weaker a-priori conditions. Note that given a bi-Lipschitz function f and $g \in W_{\mathrm{loc}}^{M,1}$, for $|\vec{i}| \leqslant M - 1$ we have that $D(D^{\vec{i}}g(f)) = D(D^{\vec{i}}g)(f) \cdot Df$ by [Zie89, Theorem 2.2.2]. Thus, to prove (2.1) by induction for functions in $W_{\mathrm{loc}}^{M,1}$, one only needs to check that the product rule for the derivatives applies at each step. For this to hold it is enough that for $|\vec{k}| \leqslant M$ the right-hand side of (2.1) is locally in L^1, see [GT01, (7.18)].*

2.1 Toy case: Hölder continuity

Definition 2.3. *Given an open set $U \subset \mathbb{R}^d$, a measurable function $f : U \to \mathbb{R}$, and $0 < s < 1$, we say that $f \in \dot{\mathcal{C}}^s(U)$ if*

$$\|f\|_{\dot{\mathcal{C}}^s(U)} := \sup_{x,y \in U} \frac{|f(x) - f(y)|}{|x - y|^s} < \infty.$$

For $k \in \mathbb{N}$ and $k < s < k + 1$, we say that $f \in \dot{\mathcal{C}}^s(U)$ if $\nabla^k f := (\partial_1^k f, \partial_1^{k-1}\partial_2 f, \cdots, \partial_d^k f)$ (that is, a vector with all the partial derivatives of order k) is in $\dot{\mathcal{C}}^{s-k}(U)$, with

$$\|f\|_{\dot{\mathcal{C}}^s(U)} := \|\nabla^k f\|_{\dot{\mathcal{C}}^{s-k}(U)}.$$

One can define Banach spaces of functions modulo polynomials using the previous seminorms. However, the standard non-homogeneous Hölder-Zygmund spaces are more suitable for our purposes:

Definition 2.4. *For $0 < s < \infty$ with $s \notin \mathbb{N}$, we say that $f \in \mathcal{C}^s(U)$ if $f \in L^\infty \cap \dot{\mathcal{C}}^s(U)$. We define the norm*

$$\|f\|_{\mathcal{C}^s(U)} := \|f\|_{L^\infty(U)} + \|f\|_{\dot{\mathcal{C}}^s(U)}.$$

Most likely the following results appear in the literature, but we were not able to locate them, so we include these results for the sake of completeness. Moreover, the main steps of the proof of Theorem 1.1 appear already in the Hölder scale:

Lemma 2.5. *Let $1 < s$, $s \notin \mathbb{N}$ and $d \in \mathbb{N}$. Assume that $\Omega_j \subset \mathbb{R}^d$, $j = 1, 2$, are open sets. Let $f : \Omega_1 \twoheadrightarrow \Omega_2$ be bi-Lipschitz with $f \in \mathcal{C}^s(\Omega_1.)$ Then for any $g \in \mathcal{C}^s(\Omega_2)$ we have*

$$g \circ f \in \mathcal{C}^s(\Omega_1), \tag{2.2}$$
$$\nabla g \circ f \in \mathcal{C}^{s-1}(\Omega_1),$$

and

$$f^{-1} \in \mathcal{C}^s(\Omega_2). \tag{2.3}$$

Proof. Let us check (2.2). According to (2.1), for $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$, we get

$$|\nabla^k (g \circ f)(x) - \nabla^k (g \circ f)(y)|$$

$$\lesssim \sum_{1 \leqslant i \leqslant k} |\nabla^i g(f(x)) - \nabla^i g(f(y))| \sum_{\alpha \in \mathbb{N}^i : |\alpha| = k} \prod_{j=1}^{i} |\nabla^{\alpha_j} f(x)| \qquad (2.4)$$

$$+ \sum_{1 \leqslant i \leqslant k} |\nabla^i g(f(y))| \sum_{\alpha \in \mathbb{N}^i : |\alpha| = k} \sum_{\ell=1}^{i} |\nabla^{\alpha_\ell} f(x) - \nabla^{\alpha_\ell} f(y)| \prod_{j=1}^{\ell-1} |\nabla^{\alpha_j} f(y)| \prod_{j=\ell+1}^{i} |\nabla^{\alpha_j} f(x)|,$$

where we assume always $\alpha_j \geqslant 1$. This implies that

$$\|g \circ f\|_{\dot{\mathcal{C}}^s} \lesssim \sum_{1 \leqslant i \leqslant k} C_f \|\nabla^i g\|_{\dot{\mathcal{C}}^\sigma} \sum_{\alpha \in \mathbb{N}^i : |\alpha| = k} \prod_{j=1}^{i} \|\nabla^{\alpha_j} f\|_{L^\infty}$$

$$+ \sum_{1 \leqslant i \leqslant k} \|\nabla^i g\|_{L^\infty} \sum_{\alpha \in \mathbb{N}^i : |\alpha| = k} \sum_{\ell=1}^{i} \|\nabla^{\alpha_\ell} f\|_{\dot{\mathcal{C}}^\sigma} \prod_{j \neq \ell} \|\nabla^{\alpha_j} f\|_{L^\infty},$$

so

$$\|g \circ f\|_{\dot{\mathcal{C}}^s} \leqslant C_f \|g\|_{\mathcal{C}^s(\Omega_2)}, \qquad (2.5)$$

with C_f depending polynomially on the $\mathcal{C}^\sigma$ norm of the derivatives of f and its bi-Lipschitz constant. The second inequality follows from the first one. In fact, $h \circ f \in \mathcal{C}^{s-1}(\Omega_1)$ whenever $h \in \mathcal{C}^{s-1}(\Omega_2)$.

Finally, let us prove (2.3). Applying the inverse function theorem,

$$D(f^{-1})(x) = (Df)^{-1}(f^{-1}(x)).$$

That is, the first-order derivatives of the inverse can be expressed using Cramer's rule as

$$(D(f^{-1}))_{ij} = g_{ij} \circ (f^{-1}), \qquad \text{where} \qquad g_{ij} = \frac{P_{ij}(Df)}{\det(Df)} \qquad (2.6)$$

for certain homogeneous polynomials $P_{ij} : \mathbb{R}^{d \times d} \to \mathbb{R}$ of degree $d - 1$. By induction we can assume $f^{-1} \in \mathcal{C}^{s-1}$ (note that the starting point of the induction is obtained from the bi-Lipschitz assumption), and by (2.2) it is enough to check that $g_{ij} \in C^{s-1}(\Omega_1)$. But every derivative of degree $k - 1$ of g_{ij} is a polynomial of degree $kd - 1$ on the derivatives of f with $k - 1$ new derivations taken at each summand, possibly taking more than one of these new derivations to some of the derivatives of f, divided by the k-th power of the Jacobian determinant, i.e., for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k - 1$ we have

$$D^\alpha g_{ij} = \sum_{\substack{\beta \in \mathbb{N}_0^{d \times d} : |\beta| = (d-1)k \\ \gamma \in (\mathbb{N}_0^d)^{k-1} : |\gamma_\ell| \geqslant 1 \, \& \, \sum |\gamma_\ell| = 2k-2 \\ \mu \in \{1, \dots, d\}^{k-1}}} \frac{C_{i,j,\alpha,\beta,\gamma,\mu}(Df)^\beta \prod_{\ell=1}^{k-1} D^{\gamma_\ell} f_{\mu_\ell}}{\det(Df)^k}. \qquad (2.7)$$

Applying the argument in (2.4) to each of these derivatives we obtain (2.3). $\qquad \square$

2.2 Justification of the chain rule: the Sobolev scale

Next we adapt the approach above to show a counterpart to Theorem 1.1 for Sobolev spaces. To adapt the argument above to the Sobolev setting we need to add a restriction that allows us to take appropriate Hölder inequalities. This is based on the following interpolation inequalities:

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Proposition 2.6 (see [RS96, Theorem 2.2.5]). *Let* $0 < t < \infty$, $0 < p < \infty$, $0 < r, \ell \leqslant \infty$ *and* $0 < \Theta < 1$. *Then every locally integrable function* g *satisfies that*

$$\|g\|_{F^{\Theta t}_{\frac{p}{\Theta}, r}} \leqslant C_{t,p,r,\ell,\Theta} \|g\|^{\Theta}_{F^t_{p,\ell}} \|g\|^{1-\Theta}_{L^{\infty}}. \tag{2.8}$$

We also need the following property of the Rychkov extension operator:

Theorem 2.7 (see [APS17, Appendix]). *Given a bounded Lipschitz domain* Ω *and* $s \in \mathbb{N}$, *there exists an operator* $\mathcal{E} := \mathcal{E}_s$ *defined in* $\mathcal{D}'(\Omega)$ *that is an extension operator from* $L^{\infty}(\Omega)$ *to* L^{∞} *and from* $F^{\sigma}_{p,q}(\Omega)$ *to* $F^{\sigma}_{p,q}$ *for every* $\sigma \leqslant s$, *every* $1/s < p < \infty$ *and every* $1/s \leqslant q \leqslant \infty$.

Remark 2.8. *It would be highly appreciated to have the same result for uniform domains. In the Sobolev scale this is proven in [Rog06]. It seems natural to think that the same operator may work in the Triebel-Lizorkin scale and may include also* L^{∞}. *If that was true, all the results in this paper could be extended to uniform domains.*

Lemma 2.9. *Let* $k \in \mathbb{N}$, $0 < \sigma \leqslant 1$ *and* $s := k + \sigma$, *let* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ *and* $d \in \mathbb{N}$ *and let* $f \in F^s_{p,q}(\Omega) \cap C^{0,1}(\Omega)$ *where* $\Omega \subset \mathbb{R}^d$ *is a bounded Lipschitz domain. Then, for every positive index* $j \leqslant k$

$$\left\|\nabla^j f\right\|_{L^{p\frac{s-1}{j-1}}(\Omega)} \lesssim_{s,p,q,j,\Omega} \|f\|^{\frac{j-1}{s-1}}_{F^s_{p,q}(\Omega)} \|\nabla f\|^{\frac{s-j}{s-1}}_{L^{\infty}(\Omega)}$$

(here, and throughout the paper, we make the convention $\frac{t}{0} = \infty$ *for any* $t > 0$*). Moreover, for every* $1 \leqslant r \leqslant \infty$ *and* $M \geqslant 1$ *with* $j + \sigma/M < s$, *we have*

$$\left\|\nabla^j f\right\|_{F^{\sigma/M}_{\frac{p(s-1)}{j+\sigma/M-1}, r}(\Omega)} \lesssim_{s,p,q,r,j+\sigma/M,\Omega} \|f\|^{\frac{j+\sigma/M-1}{s-1}}_{F^s_{p,q}(\Omega)} \|\nabla f\|^{\frac{s-j-\sigma/M}{s-1}}_{L^{\infty}(\Omega)}.$$

Note that $j + \sigma/M < s$ excludes only the case when both $M = 1$ and $j = k$.

Proof. For the first embedding, note that the case $j = 1$ holds trivially:

$$\|\nabla f\|_{L^{\infty}(\Omega)} = \|f\|^0_{F^s_{p,q}(\Omega)} \|\nabla f\|^1_{L^{\infty}(\Omega)}.$$

Otherwise, use Proposition 2.6 choosing $g = \mathcal{E}_s(\nabla f)$ where $\mathcal{E}_s$ is the Rychkov extension operator from Theorem 2.7, $t = s - 1$, $r = 2$, $\ell = q$ and $\Theta = \frac{j-1}{s-1}$. Then

$$\|\mathcal{E}_s(\nabla f)\|_{F^{j-1}_{\frac{p(s-1)}{j-1}, r}} \leqslant C_{s,p,q,j} \|\mathcal{E}_s(\nabla f)\|^{\frac{j-1}{s-1}}_{F^{s-1}_{p,q}} \|\mathcal{E}_s(\nabla f)\|^{\frac{s-j}{s-1}}_{L^{\infty}},$$

and the first statement follows.

For the second inequality, we do the same trick but we take instead r given and set $\Theta = \frac{j+\sigma/M-1}{s-1}$. In this way we obtain

$$\|\mathcal{E}_s(\nabla f)\|_{F^{j+\sigma/M-1}_{\frac{p(s-1)}{j+\sigma/M-1}, r}} \leqslant C_{s,p,q,r,j+\sigma/M} \|\mathcal{E}_s(\nabla f)\|^{\frac{j+\sigma/M-1}{s-1}}_{F^{s-1}_{p,q}} \|\mathcal{E}_s(\nabla f)\|^{\frac{s-j-\sigma/M}{s-1}}_{L^{\infty}},$$

and the second statement follows as well. $\qquad\square$

According to the previous result, we will prove some properties for subspaces of $W^{s,p}$ whose functions have bounded first derivatives. Namely, we define the space $\mathbf{W}^{s,p}(\Omega) := W^{s,p}(\Omega) \cap C^{0,1}(\Omega)$. By the Sobolev embedding theorem, when $sp > d$ we have that $\mathbf{W}^{s+1,p}(\Omega) = W^{s+1,p}(\Omega)$.

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Figure 2.1: On the first graphic, $f \in W^{5,p} \cap W^{1,\infty}$, so $\nabla^2 f \in L^{4p}$, $\nabla^3 f \in L^{2p}$ and $\nabla^4 f \in L^{\frac{4p}{3}}$. On the second we depict the case $f \in F^s_{p,q} \cap W^{1,\infty}$, with $4 < s = 4 + \sigma < 5$; in that case $\nabla f \in F^{\sigma/M}_{\frac{(s-1)p}{\sigma/M},r}$, $\nabla^2 f \in F^{\sigma/M}_{\frac{(s-1)p}{1+\sigma/M},r}$, $\nabla^3 f \in F^{\sigma/M}_{\frac{(s-1)p}{2+\sigma/M},r}$, and $\nabla^4 f \in F^{\sigma/M}_{\frac{(s-1)p}{3+\sigma/M},q}$ (q can be replaced by r if $M > 1$). The circular dots describe the case $M = 1$, the squares describe the case $M = 2$. See Lemma 2.9.
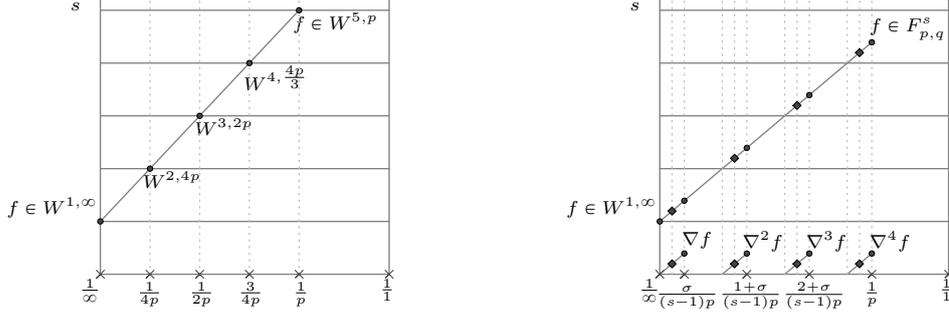
Lemma 2.10. *Let $s, d \in \mathbb{N}$, and $1 < p < \infty$. Given bounded Lipschitz domains $\Omega_j \subset \mathbb{R}^d$ and functions f, g with $f(\Omega_1) = \Omega_2$ and f bi-Lipschitz, then*

$$f \in \mathbf{W}^{s,p}(\Omega_1) \text{ and } g \in \mathbf{W}^{s,p}(\Omega_2) \implies g \circ f \in \mathbf{W}^{s,p}(\Omega_1), \tag{2.9}$$

$$f \in \mathbf{W}^{s,\frac{ps}{s-1}}(\Omega_1) \text{ and } g \in W^{s,p} \cap L^{\infty}(\Omega_2) \implies g \circ f \in W^{s,p}(\Omega_1)$$

(see Figure 2.2) and the chain rule (2.1) applies (for $M = s$). Moreover,

$$f \in \mathbf{W}^{s,p}(\Omega_1) \implies f^{-1} \in W^{s,p}(\Omega_2), \tag{2.10}$$

and (2.7) holds.

Proof. To check (2.9), the case $s = 1$ is [Zie89, Theorem 2.2.2], so let us assume that $s \geqslant 2$. Since both f_1 and f_2 are in $W^{s,p}(\Omega_j)$, all their derivatives satisfy that $\nabla^i f_j \in L^{p\frac{s-1}{i-1}}$ in view of Lemma 2.9.

By Remark 2.2, we only need to check that the right-hand side of (2.1) is in L^p, and then by induction it follows that the chain rule applies. By Hölder's inequality,

$$\textcircled{1} := \left\| \sum_{1 \leqslant i \leqslant s} \sum_{\alpha \in \mathbb{N}^i : |\alpha| = s} \nabla^i f_2(f_1) \prod_{\ell=1}^i \nabla^{\alpha_\ell} f_1 \right\|_{L^p} \lesssim_{d,s} \sum_{1 \leqslant i \leqslant s} \sum_{\alpha \in \mathbb{N}^i : |\alpha| = s} \left\| \nabla^i f_2(f_1) \right\|_{p_0} \prod_{\ell=1}^i \left\| \nabla^{\alpha_\ell} f_1 \right\|_{p_\ell}, \tag{2.11}$$

where $\sum_0^i \frac{1}{p_\ell} = \frac{1}{p}$. This can be achieved by letting $p_0 = \frac{p(s-1)}{i-1}$ and $p_\ell = \frac{p(s-1)}{\alpha_\ell - 1}$. Note that both i and α_ℓ may equal 1 in some occasions, in which case we consider the L^∞ norm of a derivative of order one, which is finite by hypothesis. Thus, Lemma 2.9 applies and using Young's inequality for products we get

$$\textcircled{1} \lesssim \sum_i \left\| \nabla f_1^{-1} \right\|_{\infty}^{\frac{d(i-1)}{p(s-1)}} \| f_2 \|_{W^{s,p}(\Omega_2)}^{\frac{i-1}{s-1}} \| \nabla f_2 \|_{L^\infty(\Omega_2)}^{\frac{s-i}{s-1}} \| f_1 \|_{W^{s,p}(\Omega_1)}^{\frac{s-i}{s-1}} \| \nabla f_1 \|_{L^\infty(\Omega_1)}^{\frac{is-s}{s-1}} \tag{2.12}$$

$$\lesssim C_{f_1} \left(\| f_2 \|_{W^{s,p}(\Omega_2)} \| \nabla f_1 \|_{L^\infty(\Omega_1)}^s + \| \nabla f_2 \|_{L^\infty(\Omega_2)} \| f_1 \|_{W^{s,p}(\Omega_1)} \right),$$

with the constant C_{f_1} depending on the bi-Lipschitz character of f_1.

The second inclusion in (2.9) can be shown analogously, setting $p_0 = \frac{ps}{i}$ and $p_\ell = \frac{ps}{\alpha_\ell - 1}$ in (2.11). We leave the details to the reader.

The inverse function bound (2.10) can be proven by the same methods using (2.7). Indeed, (2.6) holds for every bi-Lipschitz function by the chain rule, and arguing as in Remark 2.2, we only need to check that the right-hand side in (2.7) belongs to L^p for every $|\alpha| = s$ in order to prove that (2.7) holds and that $g_{ij} \in W^{s-1,p}(\Omega_1)$. Indeed,

$$
\left\| \sum_{\substack{\gamma \in (\mathbb{N}_0^d)^{s-1} \\ |\gamma_\ell| \geqslant 1 \, \& \, \sum |\gamma_\ell| = 2s-2}} |Df|^{(d-1)s} \prod_{\ell=1}^{s-1} |D^{\gamma_\ell} f| |D(f^{-1})|^{ds} \right\|_{L^p}
$$

$$
\lesssim_{d,s} \sum_{\substack{\gamma \in (\mathbb{N}_0^d)^{s-1} \\ |\gamma_\ell| \geqslant 1 \, \& \, \sum |\gamma_\ell| = 2s-2}} \|Df\|_\infty^{(d-1)s} \prod_{\ell=1}^{s-1} \|D^{\gamma_\ell} f\|_{p_\ell} \|D(f^{-1})\|_{L^\infty(\Omega)}^{ds}
$$

$$
\lesssim \|f\|_{W^{s,p}(\Omega)} \|Df\|_{L^\infty(\Omega)}^{ds-2} \|D(f^{-1})\|_{L^\infty(\Omega)}^{ds}
$$

by choosing $p_\ell = \frac{p(s-1)}{|\gamma_\ell|-1}$ and applying Lemma 2.9 with $q = 2$. We obtain that $g_{ij} \in W^{s-1,p}(\Omega_1)$ and (2.7) holds.

On the other hand, if $s = 2$ then $f \in C^{0,1}$ and $f^{-1} \in C^{0,1}$. By (2.6) and (2.9) we get $(D(f^{-1}))_{ij} \in W^{s-1,p}(\Omega_2)$. If, instead, $s > 2$ then $f \in \mathbf{W}^{s-1, \frac{p(s-1)}{s-2}}(\Omega_1)$ by Lemma 2.9 and therefore, by induction, we can assume that (2.10) holds in this case, so $f^{-1} \in W^{s-1, \frac{p(s-1)}{s-2}}(\Omega_2)$ and applying the second estimate in (2.9) to the composition in (2.6) we get that $(D(f^{-1}))_{ij} \in W^{s-1,p}(\Omega_2)$. $\square$

2.3 Proof of Theorem 1.1: the Triebel-Lizorkin scale

Finally we will verify that Triebel-Lizorkin spaces have the same properties. Again, we define $\mathbf{F}_{p,q}^s(\Omega) := F_{p,q}^s(\Omega) \cap C^{0,1}(\Omega)$ for $k < s < k+1$. Note that when $sp > d$ we have that $\mathbf{F}_{p,q}^{s+1}(\Omega) = F_{p,q}^{s+1}(\Omega)$.

Proof of (1.1). Let us write $\Omega = \Omega_1$. We begin by showing (1.1). Since bounded Lipschitz domains are uniform, by Theorem 1.2, it is enough to check that

$$
\text{①} := \int_\Omega \left(\int_0^1 \frac{\left(\fint_{B(x,t) \cap \Omega} |\nabla^k(g \circ f)(x) - \nabla^k(g \circ f)(y)| dy \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \leqslant C_f^p \|g\|_{\mathbf{F}_{p,q}^s(\Omega_2)}^p, \quad (2.13)
$$

where $\sigma = s - k$.

The case $k = 0$ follows using f as a bi-Lipschitz change of variables in the integrals in x and y:

$$
\text{①} \leqslant C_f \int_{f(\Omega)} \left(\int_0^1 \frac{\left(\fint_{f(B(x,t) \cap \Omega)} |g(x) - g(y)| dy \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx
$$

$$
\leqslant C_f \int_{\Omega_2} \left(\int_0^1 \frac{\left(\fint_{B(x,C_f t) \cap \Omega_2} |g(x) - g(y)| dy \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx.
$$

The right hand-side expression corresponds to the norm of a rescaling of g in a rescaled domain, and therefore it is bounded by $C_{d,p,q,\sigma} C_f \|g\|_{\mathbf{F}_{p,q}^s(\Omega_2)}^p$.

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Let us assume $k \geqslant 1$. Note that $\mathbf{F}_{p,q}^s(\Omega_j) \subset \mathbf{W}^{s,p}(\Omega_j)$, and by Lemma 2.10 we can use the chain rule almost everywhere and in particular (2.4) applies. However, after considering (2.4), the reader will note that there are functions on x and functions on y in the integrand, and this is an obstruction for using Hölder inequalities as it was done in the previous proof. Instead, we need to write all the functions depending on x and then address all the terms appearing in a telescopic summation. To keep the notation compact, we write $\Delta_h g(x) := g(x+h) - g(x)$ for $h = y - x$ in (2.4). It follows that $\Delta_h(g_1 g_2) = \Delta_h g_1 \Delta_h g_2 + \Delta_h g_1 g_2 + g_1 \Delta_h g_2$ and, by induction,

$$\prod_{i=1}^{\ell} g_i(x+h) = \sum_{\nu \in \{0,1\}^{\ell}} \prod_{r \leqslant \ell : \nu_r = 1} \Delta_h g_r(x) \prod_{e \leqslant \ell : \nu_e = 0} g_e(x). \tag{2.14}$$

Combining with (2.4) we get

$$|\Delta_h \nabla^k(g \circ f)(x)| \tag{2.15}$$

$$\lesssim \sum_{1 \leqslant i \leqslant k} |\Delta_h[(\nabla^i g) \circ f](x)| \sum_{\alpha \in \mathbb{N}^i : |\alpha| = k} \prod_{j=1}^i |\nabla^{\alpha_j} f(x)|$$

$$+ \sum_{1 \leqslant i \leqslant k} |\nabla^i g(f(x))| \sum_{\substack{\alpha \in \mathbb{N}^i \\ |\alpha| = k}} \sum_{\ell=1}^i \sum_{\substack{\nu \in \{0,1\}^i \\ \nu_\ell = 1 \\ \nu_r = 0 \, \forall r > \ell}} \prod_{r \leqslant i : \nu_r = 1} |\Delta_h(\nabla^{\alpha_r} f)(x)| \prod_{e \leqslant i : \nu_e = 0} |\nabla^{\alpha_e} f(x)|$$

$$+ \sum_{1 \leqslant i \leqslant k} |\Delta_h[(\nabla^i g) \circ f](x)| \sum_{\substack{\alpha \in \mathbb{N}^i \\ |\alpha| = k}} \sum_{\ell=1}^i \sum_{\substack{\nu \in \{0,1\}^i \\ \nu_\ell = 1 \\ \nu_r = 0 \, \forall r > \ell}} \prod_{r \leqslant i : \nu_r = 1} |\Delta_h(\nabla^{\alpha_r} f)(x)| \prod_{e \leqslant i : \nu_e = 0} |\nabla^{\alpha_e} f(x)|.$$

Plugging this decomposition in the numerator of the integrand in (2.13) and applying the triangle inequality, we get an expression with several terms which we summarize as follows:

$$\boxed{1} \lesssim \sum_{1 \leqslant i \leqslant k} \sum_{\substack{\alpha \in \mathbb{N}^i \\ |\alpha| = k}} \boxed{\boxed{2i\alpha}} + \sum_{1 \leqslant i \leqslant k} \sum_{\substack{\alpha \in \mathbb{N}^i \\ |\alpha| = k}} \sum_{\ell=1}^i \sum_{\substack{\nu \in \{0,1\}^i \\ \nu_\ell = 1 \\ \nu_r = 0 \, \forall r > \ell}} \left(\boxed{\boxed{3i\alpha\ell\nu}} + \boxed{\boxed{4i\alpha\ell\nu}} \right). \tag{2.16}$$

The precise definition of each summand is what the reader may expect, and will be provided in due time. To clarify concepts, one can understand for instance $\boxed{\boxed{4i\alpha\ell\nu}}$ as the term of *fourth kind* which depends on particular values of i, α, ℓ and ν, which are indices with the restrictions specified in the summation. Note that the coefficients α_j are all strictly positive natural numbers. Regarding the first term, we have

$$\boxed{\boxed{2i\alpha}} = \int_\Omega \left(\int_0^1 \frac{\left(\fint_{(B(x,t) \cap \Omega) - x} |\Delta_h[(\nabla^i g) \circ f](x)| dh \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} \prod_{j=1}^i |\nabla^{\alpha_j} f(x)|^p dx$$

$$\lesssim \|\nabla f^{-1}\|_\infty^{\frac{dp}{p_0} + dp} \|\nabla f\|_\infty^{(\sigma+d)p} \|\nabla^i g\|_{F_{p_0,q}^\sigma(\Omega_2)}^p \prod_{j=1}^i \|\nabla^{\alpha_j} f\|_{L^{p_j}(\Omega_1)}^p,$$

where $\sum_0^i \frac{1}{p_j} = \frac{1}{p}$. Note that we have used f as a bi-Lipschitz change of variables to obtain the $F_{p_0,q}^\sigma$ norm in the right-hand side of the last inequality above. Letting $p_0 = \frac{p(s-1)}{i+\sigma-1}$ and $p_j = \frac{p(s-1)}{\alpha_j - 1}$

so that we can apply Lemma 2.9, and noting that $\sum_1^i \alpha_j - 1 = k - i$ and $\sum_1^i s - \alpha_j = is - k$, we get

$$\boxed{2i\alpha}^{\frac{1}{p}} \lesssim \|\nabla f^{-1}\|_{\infty}^{\frac{d}{p_0}+d} \|\nabla f\|_{\infty}^d \|g\|_{F_{p,q}^s(\Omega_2)}^{\frac{i+\sigma-1}{s-1}} \|\nabla g\|_{\infty}^{\frac{s-i-\sigma}{s-1}} \|f\|_{F_{p,q}^s(\Omega_1)}^{\sum_1^i \frac{\alpha_j-1}{s-1}} \|\nabla f\|_{\infty}^{\sigma+\sum_1^i \frac{s-\alpha_j}{s-1}}$$

$$= C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \|\nabla f\|_{\infty}^s\right)^{\frac{i+\sigma-1}{s-1}} \left(\|\nabla g\|_{\infty} \|f\|_{F_{p,q}^s(\Omega_1)}\right)^{\frac{k-i}{s-1}}$$

$$\leqslant C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \|\nabla f\|_{\infty}^s + \|\nabla g\|_{\infty} \|f\|_{F_{p,q}^s(\Omega_1)}\right),$$

where $C_f = \|\nabla f^{-1}\|_{\infty}^{\frac{d}{p_0}+d} \|\nabla f\|_{\infty}^d$ depends only on the bi-Lipschitz character of f.

For the second term in the right-hand side of (2.16), we need to apply Hölder inequality three times, once for each variable. Namely, writing $U_x^t := B(x,t) \cap \Omega - x$,

$$\boxed{3i\alpha\ell\nu} = \int_{\Omega} \left(\int_0^1 \frac{\left(\int_{U_x^t} \prod_{r \leqslant i:\nu_r=1} |\Delta_h(\nabla^{\alpha_r} f)(x)|dh\right)^q}{t^{(\sigma+d)q}} \frac{dt}{t}\right)^{\frac{p}{q}} |\nabla^i g(f(x))|^p \prod_{e \leqslant i:\nu_e=0} |\nabla^{\alpha_e} f(x)|^p dx$$

$$\leqslant \int_{\Omega} \prod_{r \leqslant i:\nu_r=1} \left(\int_0^1 \frac{\left(\int_{U_x^t} |\Delta_h(\nabla^{\alpha_r} f)(x)|^{u_r} dh\right)^{\frac{q_r}{u_r}}}{t^{(\sigma+d)q}} \frac{dt}{t}\right)^{\frac{p}{q_r}} |\nabla^i g(f(x))|^p \prod_{e \leqslant i:\nu_e=0} |\nabla^{\alpha_e} f(x)|^p dx$$

where we assume that $\sum_r \frac{1}{u_r} = \sum_r \frac{q}{q_r} = 1$. In particular, let us fix $q_r := u_r q$ so that $(\sigma+d)q = (\frac{\sigma q}{q_r} + \frac{d}{u_r})q_r$. Take also $\sum_0^i \frac{1}{p_j} = \frac{1}{p}$ and apply Hölder's inequality again to get

$$\boxed{3i\alpha\ell\nu} \lesssim \|\nabla f^{-1}\|_{\infty}^{\frac{dp}{p_0}} \|\nabla^i g\|_{L^{p_0}(\Omega_2)}^p \prod_{r \leqslant i:\nu_r=1} \|\nabla^{\alpha_r} f\|_{F_{p_r,q_r}^{\sigma q/q_r}(\Omega_1)}^p \prod_{e \leqslant i:\nu_e=0} \|\nabla^{\alpha_e} f\|_{L^{p_e}(\Omega_1)}^p,$$

as long as $1 \leqslant u_r \leqslant \min\{p_r, q_r\}\frac{d+\sigma}{d}$.

The fact that $u_r \leqslant q_r$ is clear from the definition of q_r. Let us write $M = \sum_{r:\nu_r=1}(\alpha_r - 1)$ and define $u_r := \frac{M}{\alpha_r-1}$, $p_0 := \frac{p(s-1)}{i-1}$, $p_r = \frac{p(s-1)}{\alpha_r+\sigma/u_r-1}$ and $p_e = \frac{p(s-1)}{\alpha_e-1}$. Note that $\sum \frac{1}{u_r} = 1$ and $1 \leqslant u_r$ trivially, while the condition $u_r \leqslant p_r$ is equivalent to $u_r(\alpha_r - 1) \leqslant p(s-1) - \sigma$, that is, equivalent to $M \leqslant p(s-1) - \sigma$. But $M \leqslant |\alpha| - 1 = k - 1 = s - 1 - \sigma \leqslant p(s-1) - \sigma$, and thus it follows that $u_r \leqslant p_r$.

Thus, we can apply Lemma 2.9 again to get

$$\boxed{3i\alpha\ell\nu}^{\frac{1}{p}} \lesssim C_f \|g\|_{F_{p,q}^s(\Omega_2)}^{\frac{i-1}{s-1}} \|\nabla g\|_{\infty}^{\frac{s-i}{s-1}} \|f\|_{F_{p,q}^s(\Omega_1)}^{\sum_r \frac{\alpha_r+\sigma/u_r-1}{s-1}+\sum_e \frac{\alpha_e-1}{s-1}} \|\nabla f\|_{\infty}^{\sum_r \frac{s-\alpha_r-\sigma/u_r}{s-1}+\sum_e \frac{s-\alpha_e}{s-1}}$$

$$\leqslant C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \|\nabla f\|_{\infty}^s + \|\nabla g\|_{\infty} \|f\|_{F_{p,q}^s(\Omega_1)}\right),$$

where $C_f = \|\nabla f^{-1}\|_{\infty}^{\frac{d}{p_0}}$ depends only on the bi-Lipschitz character of f.

For the last term in the right-hand side of (2.15), we argue analogously to get

$$\boxed{4i\alpha\ell\nu} = \int_\Omega \left(\int_0^1 \frac{\left(\int_{U_x^t} |\Delta_h[(\nabla^i g) \circ f](x)| \prod_{\substack{r \leq i \\ \nu_r = 1}} |\Delta_h(\nabla^{\alpha_r} f)(x)| dh \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} \prod_{\substack{e \leq i \\ \nu_e = 0}} |\nabla^{\alpha_e} f(x)|^p dx$$

$$\leq \int_\Omega \left(\int_0^1 \frac{\left(\int_{U_x^t} |\Delta_h[(\nabla^i g) \circ f](x)|^{u_0} dh \right)^{\frac{q_0}{u_0}}}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q_0}}$$

$$\cdot \left(\int_0^1 \frac{\left(\int_{U_x^t} \prod_{\substack{r \leq i \\ \nu_r = 1}} |\Delta_h(\nabla^{\alpha_r} f)(x)|^{u_r} dh \right)^{\frac{q_r}{u_r}}}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q_r}} \prod_{\substack{e \leq i \\ \nu_e = 0}} |\nabla^{\alpha_e} f(x)|^p dx,$$

where we assume $\frac{1}{u_0} + \sum_r \frac{1}{u_r} = \frac{q}{q_0} + \sum_r \frac{q}{q_r} = 1$. In particular, let us fix $q_0 := u_0 q$ and $q_r := u_r q$ so that $(\sigma+d)q = (\frac{\sigma q}{q_r} + \frac{d}{u_r})q_r$. Take also $\sum_0^i \frac{1}{p_j} = \frac{1}{p}$ and apply Hölder's inequality again to get

$$\boxed{4i\alpha\ell\nu} \lesssim \|\nabla f^{-1}\|_\infty^{\frac{dp}{p_0} + \frac{dp}{u_0}} \|\nabla f\|_\infty^{\frac{(\sigma+d)p}{u_0}} \|\nabla^i g\|_{F_{p_0,q_0}^{\sigma q/q_0}(\Omega_2)}^p \prod_{\substack{r \leq i \\ \nu_r = 1}} \|\nabla^{\alpha_r} f\|_{F_{p_r,q_r}^{\sigma q/q_r}(\Omega_1)}^p \prod_{\substack{e \leq i \\ \nu_e = 0}} \|\nabla^{\alpha_e} f\|_{L^{p_e}(\Omega_1)}^p,$$

as long as $1 \leq u_r \leq \min\{p_r, q_r\} \frac{d+\sigma}{d}$.

The fact that $u_r \leq q_r$ is clear from the definition of q_r. Let us write $M = \sum_r (\alpha_r - 1)$ and define $u_0 := \frac{M+i-1}{i-1}$, $u_r := \frac{M+i-1}{\alpha_r - 1}$, $p_0 := \frac{p(s-1)}{i+\sigma/u_0 - 1}$, $p_r = \frac{p(s-1)}{\alpha_r + \sigma/u_r - 1}$ and $p_e = \frac{p(s-1)}{\alpha_e - 1}$. Note that $\sum \frac{1}{u_r} = 1$ and $1 \leq u_r$ trivially, while the condition $u_r \leq p_r$ is equivalent to $M + i - 1 \leq p(s-1) - \sigma$. But

$$M + i - 1 \leq \sum_{j=1}^i (\alpha_j - 1) + i - 1 = |\alpha| - 1 = k - 1 = s - 1 - \sigma \leq p(s-1) - \sigma,$$

and thus it follows that $u_r \leq p_r$.

Thus, we can apply Lemma 2.9 again to get

$$\boxed{4i\alpha\ell\nu}^{\frac{1}{p}} \lesssim C_f \|g\|_{F_{p,q}^s(\Omega_2)}^{\frac{i+\sigma/u_0-1}{s-1}} \|\nabla g\|_\infty^{\frac{s-i-\sigma/u_0}{s-1}} \|f\|_{F_{p,q}^s(\Omega_1)}^{\sum_r \frac{\alpha_r + \sigma/u_r - 1}{s-1} + \sum_e \frac{\alpha_e - 1}{s-1}} \|\nabla f\|_\infty^{\frac{\sigma}{u_0} + \sum_r \frac{s-\alpha_r - \sigma/u_r}{s-1} + \sum_e \frac{s-\alpha_e}{s-1}}$$

$$\leq C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \|\nabla f\|_\infty^s + \|\nabla g\|_\infty \|f\|_{F_{p,q}^s(\Omega_1)} \right),$$

where $C_f = \|\nabla f^{-1}\|_\infty^{\frac{d}{p_0} + \frac{d}{u_0}} \|\nabla f\|_\infty^{\frac{d}{u_0}}$ depends only on the bi-Lipschitz character of f.

Combining these estimates with (2.16), we obtain (1.1). In particular, we obtain (2.13), where the constant C_f is affine with respect to the $F_{p,q}^s(\Omega)$ norm of f and depends polynomially on its bi-Lipschitz constants, as well as on d, s, p, q and the extension constants of the domains for all the different indices p_j, q_j, u_j appearing in the proof. $\qquad\square$

Lemma 2.11. *Let $0 < s < \infty$, $s \notin \mathbb{N}$, let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $d \in \mathbb{N}$. Given bounded Lipschitz domains $\Omega_j \subset \mathbb{R}^d$ and a bi-Lipschitz function f with $f(\Omega_1) = \Omega_2$, then*

$$f \in \mathbf{F}_{\frac{ps}{s-1},q}^s(\Omega_1) \text{ and } g \in F_{p,q}^s \cap L^\infty(\Omega_2) \implies g \circ f \in F_{p,q}^s(\Omega_1) \tag{2.17}$$

(see Figure 2.2)

Figure 2.2: Composition rule for Sobolev and Triebel-Lizorkin scales of a bounded continuous function g and a bi-Lipschitz function f in Lemmata 2.10 and 2.11.
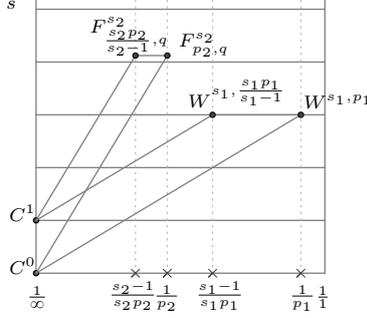
Proof. The proof is just a modification of the proof of (1.1). One has to set $p_0 = \frac{ps}{i+\sigma}$ and $p_j = \frac{ps}{\alpha_j - 1}$ in $\boxed{2i\alpha}$, $p_0 = \frac{ps}{i}$, $p_r = \frac{ps}{\alpha_r + \sigma/u_r - 1}$, $p_e = \frac{ps}{\alpha_e - 1}$ and $u_r = \frac{M}{\alpha_r - 1}$ in $\boxed{3i\alpha\ell\nu}$ and $p_0 = \frac{ps}{i+\sigma/u_0}$, $p_r = \frac{ps}{\alpha_r + \sigma/u_r - 1}$, $p_e = \frac{ps}{\alpha_e - 1}$, $u_0 = \frac{M+i}{i}$ and $u_r = \frac{M+i}{\alpha_r - 1}$ in $\boxed{4i\alpha\ell\nu}$. We leave the details to the reader. $\qquad\square$

Remark 2.12. *The precise dependence obtained in the preceeding proofs is*

$$\|g \circ f\|_{F_{p,q}^s(\Omega_1)} \lesssim_{d,p,q,s} C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \|\nabla f\|_\infty^s + \|\nabla g\|_\infty \|f\|_{F_{p,q}^s(\Omega_1)} \right),$$

where

$$C_f = \left(1 + \|\nabla f^{-1}\|_\infty^{\frac{d}{p}} \right) \left(1 + \|\nabla f^{-1}\|_\infty^d \|\nabla f\|_\infty^d \right),$$

and

$$\|g \circ f\|_{F_{p,q}^s(\Omega_1)} \lesssim_{d,p,q,s} C_f \left(\|g\|_{F_{p,q}^s(\Omega_2)} \left(\|\nabla f\|_\infty^{\frac{s}{s-1}} \right)^s + \|g\|_\infty \|f\|_{F_{\frac{ps}{s-1},q}^{s-1}(\Omega_1)}^{\frac{s}{s-1}} \right),$$

where

$$C_f = \|\nabla f\|_\infty^{\frac{-s}{s-1}} \left(\|\nabla f^{-1}\|_\infty^{\frac{d}{ps}} + \|\nabla f^{-1}\|_\infty^{\frac{d}{p}} \right) \left(1 + \|\nabla f^{-1}\|_\infty^d \|\nabla f\|_\infty^d \right).$$

Proof of (1.2). Inequality (1.2) is proven by analogous techniques using (2.6) and (2.7) which apply by Lemma 2.10. We claim that it is enough to check that $g_{ij} \in F_{p,q}^{s-1}(\Omega_1)$. Indeed, in case $1 < s < 2$, then we have that f^{-1} is a bi-Lipschitz change of variables and, therefore, $g_{ij} \circ (f^{-1}) \in F_{p,q}^{s-1}(\Omega_2)$ if and only if $g_{ij} \in F_{p,q}^{s-1}(\Omega_1)$. Otherwise, by Lemma 2.9 we have that $f \in F_{\frac{p(s-1)}{s-2},q}^{s-1}(\Omega_1)$. Inductively we can assume that $f^{-1} \in F_{\frac{p(s-1)}{s-2},q}^{s-1}(\Omega_1)$, and by (2.17), if $g_{ij} \in F_{p,q}^{s-1}(\Omega_1)$ then we get $g_{ij} \circ (f^{-1}) \in F_{p,q}^{s-1}(\Omega_2)$ and the claim follows.

Now, we want to prove

$$①:= \left(\int_{\Omega_2} \left(\int_0^1 \frac{\left(\fint_{U_{x,t}} |\nabla^{k-1} g_{ij}(x) - \nabla^{k-1} g_{ij}(y)| dy \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leqslant C_f \|f\|_{\mathbf{F}_{p,q}^s(\Omega_1)}, \qquad (2.18)$$

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where $U_{x,t} := B(x,t) \cap \Omega_2$. Again we use first-order differences, and write $h := y - x$. For $|\alpha| = k + 1$, we have

$$
|\Delta_h D^\alpha g_{ij}(x)| \lesssim \sum_{\beta,\gamma,\mu} \left| \frac{\prod_{\ell=1}^{k-1} D^{\gamma_\ell} f_{\mu_\ell}(x)}{\det(Df)^k(x)} \Delta_h (Df)^\beta(x) \right|
$$
$$
+ \left| (Df)^\beta(x+h) \prod_{\ell=1}^{k-1} D^{\gamma_\ell} f_{\mu_\ell}(x) \Delta_h \left(\frac{1}{\det(Df)^k} \right)(x) \right|
$$
$$
+ \left| \frac{(Df)^\beta(x+h)}{\det(Df)^k(x+h)} \Delta_h \left(\prod_{\ell=1}^{k-1} D^{\gamma_\ell} f_{\mu_\ell}(x) \right) \right|
$$

Now we use some trivial properties of first order differences, together with (2.14) to get

$$
|\Delta_h D^\alpha g_{ij}| \lesssim \sum_{\gamma,\mu} \|\nabla f\|_\infty^{(d-1)k-1} \|\nabla f^{-1}\|_\infty^{dk} |\Delta_h (Df)| \prod_{\ell=1}^{k-1} |D^{\gamma_\ell} f_{\mu_\ell}|
$$
$$
+ \|\nabla f\|_\infty^{(d-1)k} \|\nabla f^{-1}\|_\infty^{d(k+1)} |\Delta_h \det(Df)| \prod_{\ell=1}^{k-1} |D^{\gamma_\ell} f_{\mu_\ell}|
$$
$$
+ \|\nabla f\|_\infty^{(d-1)k} \|\nabla f^{-1}\|_\infty^{dk} \sum_{\nu \in \{0,1\}^{k-1}:|\nu|\geqslant 1} \prod_{r\leqslant k-1:\nu_r=1} |\Delta_h D^{\gamma_r} f_{\mu_r}| \prod_{e\leqslant k-1:\nu_e=0} |D^{\gamma_e} f_{\mu_e}|.
$$

To end, note that $|\Delta_h \det(Df)| \leqslant c\|\nabla f\|_\infty^{d-1} |\Delta_h(Df)|$, so

$$
|\Delta_h D^\alpha g_{ij}| \lesssim \sum_{\gamma,\mu} \left(\|\nabla f\|_\infty^{(d-1)k-1} \|\nabla f^{-1}\|_\infty^{dk} + \|\nabla f\|_\infty^{(d-1)k+d-1} \|\nabla f^{-1}\|_\infty^{d(k+1)} \right) |\Delta_h(Df)| \prod_{\ell=1}^{k-1} |\nabla^{|\gamma_\ell|} f|
$$
$$
+ \|\nabla f\|_\infty^{(d-1)k} \|\nabla f^{-1}\|_\infty^{dk} \sum_{\nu \in \{0,1\}^{k-1}:|\nu|\geqslant 1} \prod_{r\leqslant k-1:\nu_r=1} |\Delta_h \nabla^{|\gamma_r|} f| \prod_{e\leqslant k-1:\nu_e=0} |\nabla^{|\gamma_e|} f|.
$$

Therefore, we write

$$
\boxed{1} \lesssim C_f \sum_{\substack{\gamma\in(\mathbb{N}_0^d)^{k-1} \\ |\gamma_\ell|\geqslant 1 \,\&\, \sum|\gamma_\ell|=2k-2}} \left(\left(1 + \|\nabla f\|_\infty^d \|\nabla f^{-1}\|_\infty^d \right) \boxed{2\gamma} + \|\nabla f\|_\infty \sum_{\nu\in\{0,1\}^{k-1}:|\nu|\geqslant 1} \boxed{3\gamma\nu} \right),
$$

with $C_f = \|\nabla f\|_\infty^{(d-1)k-1} \|\nabla f^{-1}\|_\infty^{dk}$, with $\boxed{2\gamma}$ and $\boxed{3\gamma\nu}$ as defined below.

Regarding the first term, by Hölder's inequality we have

$$
\boxed{2\gamma} := \left(\int_{\Omega_2} \left(\int_0^1 \frac{\left(\fint_{U_x^t} |\Delta_h(Df)(x)| dh \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} \prod_{\ell=1}^{k-1} |\nabla^{|\gamma_\ell|} f(x)|^p dx \right)^{\frac{1}{p}}
$$
$$
\lesssim \|Df\|_{F_{p_0,q}^\sigma(\Omega_1)} \prod_{\ell=1}^{k-1} \left\| \nabla^{|\gamma_\ell|} f \right\|_{L^{p_\ell}(\Omega_1)},
$$

where $\sum_0^{k-1} \frac{1}{p_\ell} = \frac{1}{p}$ and $U_x^t := B(x,t) \cap \Omega - x$. In particular choose $p_0 = \frac{p(s-1)}{\sigma}$ and $p_\ell = \frac{p(s-1)}{|\gamma_\ell|-1}$.

By Lemma 2.9 we get

$$\boxed{2\gamma} \lesssim \|f\|_{F^s_{p,q}(\Omega_1)}^{\frac{\sigma}{s-1}+\sum_\ell \frac{|\gamma_\ell|-1}{s-1}} \|\nabla f\|_\infty^{\frac{s-1-\sigma}{s-1}+\sum_\ell \frac{s-|\gamma_\ell|}{s-1}}$$

$$= \|f\|_{F^s_{p,q}(\Omega_1)}^{\frac{\sigma+k-1}{s-1}} \|\nabla f\|_\infty^{\frac{s-1-\sigma+s(k-1)-(2k-2)}{s-1}} = \|f\|_{F^s_{p,q}(\Omega_1)} \|\nabla f\|_\infty^{k-1}$$

On the other hand, by Hölder's inequality again

$$\boxed{3\gamma\nu} := \left(\int_{\Omega_2} \left(\int_0^1 \frac{\left(\fint_{U_{x,t}} \prod_{r \leqslant k-1:\nu_r=1} |\Delta_h \nabla^{|\gamma_r|} f(x)|dh \right)^q}{t^{(\sigma+d)q}} \frac{dt}{t} \right)^{\frac{p}{q}} \prod_{e \leqslant k-1:\nu_e=0} |\nabla^{|\gamma_e|} f(x)|^p dx \right)^{\frac{1}{p}}$$

$$\lesssim \prod_r \left\| \nabla^{|\gamma_r|} f \right\|_{F^{\sigma/u_r}_{p_r,q_r}(\Omega_1)} \prod_e \left\| \nabla^{|\gamma_e|} f \right\|_{L^{p_e}(\Omega_1)},$$

where we assume that $\sum_r \frac{1}{u_r} = \sum_1^{k-1} \frac{p}{p_j} = 1$, $q_r := u_r q$, as long as $1 \leqslant u_r \leqslant \min\{p_r, q_r\} \frac{d+\sigma}{d}$.

The fact that $u_r \leqslant q_r$ is clear from the definition of q_r. Let us write $M = \sum_r (|\gamma_r| - 1)$ and define $u_r := \frac{M}{|\gamma_r|-1}$, $p_r = \frac{p(s-1)}{|\gamma_r|+\sigma/u_r-1}$ and $p_e = \frac{p(s-1)}{|\gamma_e|-1}$. Note that $\sum \frac{1}{u_r} = 1$ and $1 \leqslant u_r$ trivially, while the condition $u_r \leqslant p_r$ is equivalent to $u_r(|\gamma_r| - 1) \leqslant p(s-1) - \sigma$, that is, equivalent to $M \leqslant p(s-1) - \sigma$. But $M \leqslant |\gamma| - (k-1) = k - 1 = s - 1 - \sigma \leqslant p(s-1) - \sigma$, and thus it follows that $u_r \leqslant p_r$.

By Lemma 2.9 we get

$$\boxed{3\gamma\nu} \lesssim \|f\|_{F^s_{p,q}(\Omega_1)}^{\sum_r \frac{|\gamma_r|+\sigma/u_r-1}{s-1}+\sum_e \frac{|\gamma_e|-1}{s-1}} \|\nabla f\|_\infty^{\sum_r \frac{s-|\gamma_r|-\sigma/u_r}{s-1}+\sum_e \frac{s-|\gamma_e|}{s-1}}$$

$$= \|f\|_{F^s_{p,q}(\Omega_1)}^{\frac{\sigma+k-1}{s-1}} \|\nabla f\|_\infty^{\frac{s(k-1)-(2k-2)-\sigma}{s-1}} = \|f\|_{F^s_{p,q}(\Omega_1)} \|\nabla f\|_\infty^{k-2}.$$

All in all,

$$\boxed{1} \lesssim \|\nabla f\|_\infty^{dk-2} \|\nabla f^{-1}\|_\infty^{dk} \left(\left(1 + \|\nabla f\|_\infty^d \|\nabla f^{-1}\|_\infty^d \right) \right) \|f\|_{F^s_{p,q}(\Omega_1)}.$$

$\square$

3   Corkscrew and uniform domains

Next we provide the necessary background to work with uniform and corkscrew domains. We begin by defining a Whitney covering and then we provide definitions of interior corkscrew domains, chains of cubes and uniform domains. Here, uniform domains are defined in terms of the existence of this chains, which can be understood as discretized *cigar paths* or Harnack chains with a uniform bound on the number of balls of each scale. Finally we define the shadow of a cube, which plays the role of Carleson boxes, and we provide some well-known results relating the Hardy-Littlewood maximal operator to Whitney coverings.

The definitions will be given in terms of dyadic cubes, which is a usual tool in harmonic analysis. We define $\mathcal{D}_0$ as the collection of open cubes with side length 1 whose vertices have integer coordinates. Then its rescalings $\mathcal{D}_j := 2^{-j} \mathcal{D}_0$ are the cubes of generation j, whose side-length is 2^{-j}, and $\mathcal{D} := \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$ is the whole collection of *dyadic cubes*. Note that each generation is formed by disjoint cubes, and the union of all the cubes in a generation is dense in $\mathbb{R}^d$.

Definition 3.1. *Given a domain Ω, we say that a collection of open dyadic cubes $\mathcal{W}$ is a* Whitney covering *of Ω if they are disjoint, the union of the cubes and their boundaries is Ω, there exists a constant $C_{\mathcal{W}}$ such that*

$$C_{\mathcal{W}}\ell(Q) \leqslant \operatorname{dist}(Q, \partial\Omega) \leqslant 4C_{\mathcal{W}}\ell(Q),$$

and the family $\{50Q\}_{Q \in \mathcal{W}}$ has finite superposition i.e., $\sum_Q \chi_{50Q} \leqslant C < \infty$. Moreover,

$$S \subset 5Q \implies \ell(S) \geqslant \frac{1}{2}\ell(Q) \qquad \text{for every } S, Q \in \mathcal{W}. \tag{3.1}$$

We call $Q \in \mathcal{W}$ a Whitney cube. *We say that two cubes $Q, S \in \mathcal{W}$ are neighbors if $\overline{Q} \cap \overline{S} \neq \varnothing$.*

The existence of such a covering is granted for any open set different from $\mathbb{R}^d$ and in particular for any domain as long as $C_{\mathcal{W}}$ is big enough (see [Ste70, Chapter 1] for instance).

Definition 3.2. *We say that a domain $\Omega \subset \mathbb{R}^d$ is an* interior (resp. exterior) (ε, δ)-corkscrew domain *if there is a Whitney covering of Ω (resp. $\overline{\Omega}^c$) such that given any ball $B(x, r)$ centered at $\partial\Omega$ with $0 < r \leqslant \delta$ there exists a Whitney cube $Q \subset B(x, r)$ such that $\ell(Q) \geqslant \varepsilon r$, see Figure 3.1.*
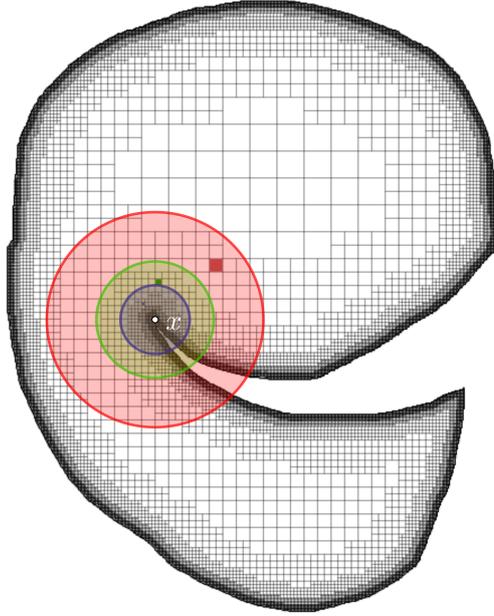

Figure 3.1: Interior corkscrew domains may have cusps, in opposition to uniform domains.

Definition 3.3. *Let Ω be a domain, $\mathcal{W}$ a Whitney decomposition of Ω and $Q, S \in \mathcal{W}$. Given M cubes $Q_1, \ldots, Q_M \in \mathcal{W}$ with $Q_1 = Q$ and $Q_M = S$, the M-tuple $\mathcal{V} = (Q_1, \ldots, Q_M) \in \mathcal{W}^M$ is a* chain *connecting Q and S if the cubes Q_j and Q_{j+1} are neighbors for $j < M$.*

Let $\varepsilon \in \mathbb{R}$. We say that the chain $\mathcal{V}$ is ε-admissible if

- *the* length *of the chain is bounded by*

$$\ell(\mathcal{V}) := \sum_{j=1}^{M} \ell(Q_j) \leqslant \frac{1}{\varepsilon}\mathrm{D}(Q, S), \tag{3.2}$$

(recall that D stands for the long distance, and $\mathrm{D}(Q, S) \approx \ell(Q) + \ell(S) + \operatorname{dist}(Q, S)$)

- *and there exists $j_\mathcal{V} < M$ such that the cubes in the chain satisfy*

$$\ell(Q_j) \geqslant \varepsilon \mathrm{D}(Q_1, Q_j) \text{ for all } j \leqslant j_\mathcal{V} \qquad \text{and} \qquad \ell(Q_j) \geqslant \varepsilon \mathrm{D}(Q_j, Q_M) \text{ for all } j \geqslant j_\mathcal{V}. \quad (3.3)$$

The $j_\mathcal{V}$-th cube, which we call central, *satisfies that $\ell(Q_{j_\mathcal{V}}) \gtrsim_d \varepsilon \mathrm{D}(Q, S)$ by (3.3) and the triangle inequality.*

We write (abusing notation) $\mathcal{V}$ also for the set $\{Q_j\}_{j=1}^M$. Thus, we will write $P \in \mathcal{V}$ if P appears in a coordinate of the M-tuple $\mathcal{V}$.

Definition 3.4. *We say that a domain $\Omega \subset \mathbb{R}^d$ is a* uniform domain *if there exists a Whitney covering $\mathcal{W}$ of Ω and $\varepsilon, \delta \in \mathbb{R}$ such that for any pair of cubes $Q, S \in \mathcal{W}$ with $\mathrm{D}(Q, S) \leqslant \delta$, there exists an ε-admissible chain connecting Q and S which we denote as $[Q, S]$, see Figure 3.2. Sometimes we will write (ε, δ)-uniform domain to fix the constants.*

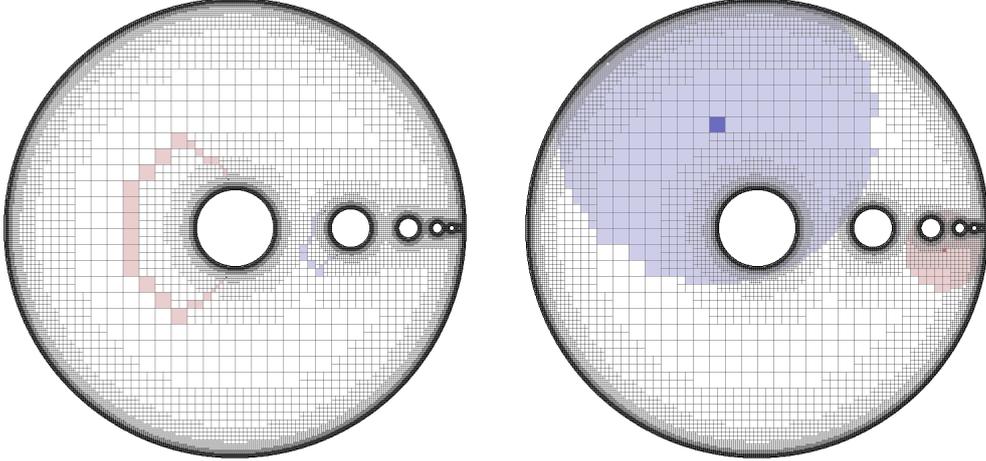*We will write $Q_S = Q_{j_{[Q,S]}}$ for its central cube.*

Figure 3.2: In uniform domains we can join cubes by admissible chains, which play the role of cigar paths. Shadows are the cubes *under the influence* of a given cube, and they play the role of Carleson boxes.

Consider a uniform domain Ω with covering $\mathcal{W}$ and two cubes $Q, S \in \mathcal{W}$ with $\mathrm{D}(Q, S) \leqslant \delta$. From Definition 3.3 it follows that

$$\mathrm{D}(Q, S) \approx_{\varepsilon, d} \ell([Q, S]) \approx_{\varepsilon, d} \ell(Q_S). \quad (3.4)$$

Note that any bounded Lipschitz domain is an (ε, ∞)-uniform domain, and every uniform domain is also an interior corkscrew domain, perhaps with smaller parameters. Note also that there may be several admissible chains joining Q and S, but we are assuming that a choice is made for every pair of cubes when the domain is uniform, as soon as Q, S are close enough.

Now we can define the shadows:

Definition 3.5. *Let Ω be an (ε, δ)-uniform domain with Whitney covering $\mathcal{W}$. Given a cube $P \in \mathcal{W}$ centered at x_P and a real number ρ, the ρ-shadow of P is the collection of cubes*

$$\mathbf{SH}_\rho(P) = \{Q \in \mathcal{W} : Q \subset B(x_P, \rho\,\ell(P))\},$$

and its "realization" is the set

$$\mathbf{Sh}_\rho(P) = \bigcup_{Q \in \mathbf{SH}_\rho(P)} Q,$$

see Figure 3.2.

By the previous remark and the properties of the Whitney covering, we can define $\rho_\varepsilon > 1$ such that the following properties hold:

- For every ε-admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$ we have that $Q \in \mathbf{SH}_{\rho_\varepsilon}(P)$.

- Moreover, every cube P belonging to an ε-admissible chain $[Q, S]$ belongs to the shadow $\mathbf{SH}_{\rho_\varepsilon}(Q_S)$.

Remark 3.6 (see [PS17, Remark 2.6])**.** *Given an (ε, δ)-uniform domain Ω we will write* $\mathbf{Sh}$ *for* $\mathbf{Sh}_{\rho_\varepsilon}$*. We will write also* $\mathbf{SH}$ *for* $\mathbf{SH}_{\rho_\varepsilon}$*.*

For $Q \in \mathcal{W}$ and $s > 0$, we have that

$$\sum_{L: Q \in \mathbf{SH}(L)} \ell(L)^{-s} \lesssim \ell(Q)^{-s} \qquad and \qquad \sum_{\substack{L: Q \in \mathbf{SH}(L) \\ \ell(L) \leqslant \rho}} \ell(L)^s \lesssim \rho^s \qquad (3.5)$$

and, moreover, if $Q \in \mathbf{SH}(P)$ and $\mathrm{D}(Q, P) \leqslant \delta$, then

$$\sum_{L \in [Q, P]} \ell(L)^s \lesssim \ell(P)^s \qquad and \qquad \sum_{L \in [Q, P]} \ell(L)^{-s} \lesssim \ell(Q)^{-s}. \qquad (3.6)$$

Note that the property (3.5) is not a consequence of uniformity, but of the definition of shadow.

We recall the definition of the non-centered Hardy-Littlewood maximal operator. Given $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define $Mf(x)$ as the supremum of the mean of f in cubes containing x, that is,

$$Mf(x) = \sup_{Q: x \in Q} \frac{1}{|Q|} \int_Q f(y)\, dy.$$

It is a well known fact that this operator is bounded in L^p for $1 < p < \infty$. The following lemma is proven in [PT15] and will be used repeatedly along the proofs contained in the present text.

Lemma 3.7. *Let Ω be a domain with Whitney covering $\mathcal{W}$. Assume that $g \in L^1(\Omega)$ and $r > 0$. For every $\eta > 0$, $Q \in \mathcal{W}$ and $x \in \mathbb{R}^d$, we have*

1) *The non-local inequalities for the maximal operator*

$$\int_{|y-x|>r} \frac{g(y)\, dy}{|y-x|^{d+\eta}} \lesssim_d \frac{Mg(x)}{r^\eta} \qquad and \qquad \sum_{S: \mathrm{D}(Q,S)>r} \frac{\int_S g(y)\, dy}{\mathrm{D}(Q,S)^{d+\eta}} \lesssim_d \frac{\inf_{y \in Q} Mg(y)}{r^\eta}. \qquad (3.7)$$

2) *The local inequalities for the maximal operator*

$$\int_{|y-x|<r} \frac{g(y)\, dy}{|y-x|^{d-\eta}} \lesssim_d r^\eta Mg(x) \qquad and \qquad \sum_{S: \mathrm{D}(Q,S)<r} \frac{\int_S g(y)\, dy}{\mathrm{D}(Q,S)^{d-\eta}} \lesssim_d \inf_{y \in Q} Mg(y)\, r^\eta. \qquad (3.8)$$

3) *In particular, if Ω is a uniform domain, we have*

$$\sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{\mathrm{D}(Q,S)^{d+\eta}} \lesssim_d \frac{1}{\ell(Q)^\eta} \qquad and \qquad \sum_{S \in \mathbf{SH}_\rho(Q)} \ell(S)^d \lesssim_{d,\rho} \ell(Q)^d \qquad (3.9)$$

and, by Definition 3.5,

$$\sum_{S \in \mathbf{SH}_\rho(Q)} \int_S g(x)\, dx \lesssim_{d,\rho} \inf_{y \in Q} Mg(y)\, \ell(Q)^d. \qquad (3.10)$$

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4 Extension operators

In this section we prove Theorem 1.2. First we reduce Theorem 1.2 to finding an extension operator. Then, in Section 4.1 we define this extension operator for smoothness parameter below one in corkscrew domains and then we show that this operator is indeed an extension operator, establishing Theorem 1.2 for $s < 1$. As a matter of fact we obtain a much better result than in the statement of the theorem, since we can substitute the hypothesis of the domain being uniform by just being interior corkscrew. Finally, in Section 4.2 we establish the remaining cases.

The proof of Theorem 4.3, that is, the extension operator when $s < 1$, follows a similar structure to [PS17, Theorem 1.4]. The novelty here is that we use a norm with means on balls, that allows us to drop the restriction $\sigma > \frac{d}{p} - \frac{d}{q}$, so we get results with full generality. The use of means in balls limiting the radius to a constant depending on the corkscrew character allows us to avoid the error terms $\overset{\frown}{(a2)}$, $\overset{\frown}{(b2)}$ and $\overset{\frown}{(c4)}$ from [PS17]. In this way we relate the extended function to the homogeneous norm in all terms except for $\overset{\frown}{(c3)}$.

The proof of Theorem 4.7, that is, the boundedness of the extension operator when $s > 1$, follows a similar structure to [Pra19, Theorem 1.5]. Again the main difference is the usage of means in balls allowing us to drop the restriction $\sigma > \frac{d}{p} - \frac{d}{q}$. We use a similar notation to [Pra19] so that it is easy for the reader to spot the differences. Here $\boxed{\boxed{0}}$ corresponds to $\boxed{\boxed{\beta}}$ in the aforementioned paper. Using means on balls allows us to drop the error terms $\boxed{\boxed{\beta.a.2}}$, $\boxed{\boxed{\beta.b.2}}$ and $\boxed{\boxed{\beta.c.3}}$ in [Pra19]. In the present paper, we need to distinguish the local part $\boxed{\boxed{c.1}}$ from the non-local part $\boxed{\boxed{c.2}}$. Finally, the non-homogeneous contribution is obtained from $\boxed{\boxed{c.3}}$, which corresponds to $\boxed{\boxed{\beta.c.2}}$ in [Pra19]. Following also the scheme in that paper, we defer the main estimates to a couple of technical lemmata. Lemma 4.8 is the natural counterpart to [Pra19, Lemma 3.4], and Lemma 4.9 is the counterpart to [Pra19, Lemma 3.5]. The proof of the first lemma is much easier this time, thanks to a clever reduction to Lemma 4.9. The latter, however, deals with the main technical difficulty in this setting.

Definition 4.1. *Consider* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $1 \leqslant u \leqslant \infty$ *and* $0 < \sigma < 1$ *so that* $\sigma > \frac{d}{\min\{p,q\}} - \frac{d}{u}$. *Let* U *be an open set in* $\mathbb{R}^d$. *We say that a locally integrable function* $f \in F_{p,q,u}^{\sigma,\rho}(U)$ *if*

- *The function* $f \in L^p(U)$, *and*

- *the seminorm*

$$\|f\|_{\dot{F}_{p,q,u}^{\sigma,\rho}(U)} := \left(\int_U \left(\int_0^\rho \frac{\left(\fint_{U_{x,t}} |f(x) - f(y)|^u \right)^{\frac{q}{u}}}{t^{\sigma q + \frac{dq}{u}}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \tag{4.1}$$

is finite, where we denote $U_{x,t} := B(x,t) \cap U$.

We define the norm

$$\|f\|_{F_{p,q,u}^{\sigma,\rho}(U)} := \|f\|_{L^p(U)} + \|f\|_{\dot{F}_{p,q,u}^{\sigma,\rho}(U)}.$$

For $s = k + \sigma$ *with* $k \in \mathbb{N}$, *we write*

$$\|f\|_{F_{p,q,u}^{s,\rho}(U)} := \|f\|_{W^{k,p}(U)} + \left\|\nabla^k f\right\|_{\dot{F}_{p,q,u}^{\sigma,\rho}(U)}.$$

In order to prove that $F_{p,q}^s(\Omega) = F_{p,q,u}^{s,1}(\Omega)$ for a given domain Ω, it suffices to find an extension operator $\mathcal{E} : F_{p,q,1}^{s,\rho}(\Omega) \to F_{p,q,1}^{s,1}(\mathbb{R}^d)$ with $\rho < 1$. Once this is established, using the equivalence of norms in the ambient space (see [Tri06, Theorem 1.116]) we obtain the equivalence of norms in the domain by classical arguments:

First note that

$$\|g\|_{F_{p,q}^s(\mathbb{R}^d)} \approx \|g\|_{F_{p,q,u}^{s,1}(\mathbb{R}^d)} \approx \|g\|_{F_{p,q,u}^{s,\rho}(\mathbb{R}^d)}. \tag{4.2}$$

The first comparison comes from [Tri06, Theorem 1.116]. The second can be obtained easily by using the change of variables $\widetilde{x} = \rho^{-1}x$, $\widetilde{t} = \rho^{-1}t$, $\widetilde{y} = \rho^{-1}y$ in the last norm and then compare the norms of g and its rescaling $g(\rho\cdot)$ in $F_{p,q}^s$.

Thus,

$$\inf_{g|_\Omega \equiv f} \|g\|_{F_{p,q}^s(\mathbb{R}^d)} \lesssim \|\mathcal{E}f\|_{F_{p,q}^s(\mathbb{R}^d)} \approx \|\mathcal{E}f\|_{F_{p,q,1}^{s,1}(\mathbb{R}^d)} \lesssim \|f\|_{F_{p,q,1}^{s,\rho}(\Omega)} \lesssim \|f\|_{F_{p,q,u}^{s,\rho}(\Omega)} \lesssim \inf_{g|_\Omega \equiv f} \|g\|_{F_{p,q,u}^{s,\rho}(\mathbb{R}^d)}.$$

Since the first and the last are comparable it follows that all the quantities are comparable and, in particular,

$$F_{p,q}^s(\Omega) = F_{p,q,u}^{s,\rho}(\Omega).$$

To end, since $\rho < 1$ then

$$\|f\|_{F_{p,q,u}^{s,\rho}(\Omega)} \lesssim \|f\|_{F_{p,q,u}^{s,1}(\Omega)} \lesssim \|\mathcal{E}f\|_{F_{p,q,u}^{s,1}(\mathbb{R}^d)} \lesssim \|f\|_{F_{p,q,u}^{s,\rho}(\Omega)}.$$

4.1 Corkscrew domains and smoothness below one

Let Ω be an interior corkscrew domain and consider given Whitney coverings $\mathcal{W}_0$ of Ω and $\mathcal{W}_2$ of $\overline{\Omega}^c$. To define the extension operator we will take averages on certain 'reflected' cubes, but we need to avoid large cubes, say with side-length bounded by a constant ℓ_0 which depends on the corkscrew constants. For this reason we define some related collections of cubes as follows:

- In case Ω is unbounded and $\delta = \infty$, choose $\ell_0 = 1$. Otherwise, fix ℓ_0 depending on the corkscrew parameters of the domain, so that $c_0\ell_0 << \delta \wedge 1$. Note that c_0 is a constant which will be fixed during the proof of Theorem 4.7 below, right before (4.14).

- We define $\mathcal{W}_3$ to be the collection of cubes in $\mathcal{W}_2$ with side-lengths smaller than $10\ell_0$.

- We define $\mathcal{W}_1$ to be the collection of cubes in $\mathcal{W}_0$ with side-length smaller than $c_0\ell_0$.

- Choose ℓ_0 so that for any $Q \in \mathcal{W}_3$ there is a $S \in \mathcal{W}_1$ with

$$\mathrm{D}(Q,S) \leqslant C\ell(Q) \quad \text{and} \quad \ell(Q) = \ell(S) \quad \text{(see [Jon81, Lemma 2.4]).} \tag{4.3}$$

For $Q \in \mathcal{W}_3$ we define the symmetrized cube $Q^* \in \mathcal{W}_1$ as one of the cubes satisfying (4.3). Note that the number of possible choices for Q^* is uniformly bounded.

Lemma 4.2. *[see [Jon81, Section 2]] Let Ω be an interior corkscrew domain. For cubes $Q_1, Q_2 \in \mathcal{W}_3$ and $S \in \mathcal{W}_1$ we have that*

- *The symmetrized cubes have finite overlapping: there exists a constant C depending on the parameter ε and the dimension d such that $\#\{Q \in \mathcal{W}_3 : Q^* = S\} \leqslant C$.*

- *The long distance is invariant in the following sense:*

$$\mathrm{D}(Q_1^*, Q_2^*) \approx \mathrm{D}(Q_1, Q_2) \quad \text{and} \quad \mathrm{D}(Q_1^*, S) \approx \mathrm{D}(Q_1, S) \tag{4.4}$$

We define the family of bump functions $\{\psi_Q\}_{Q\in\mathcal{W}_2}$ to be a partition of the unity associated to $\left\{\frac{11}{10}Q\right\}_{Q\in\mathcal{W}_2}$, that is, their sum $\sum \psi_Q \equiv 1$, they satisfy the pointwise inequalities $0 \leqslant \psi_Q \leqslant \chi_{\frac{11}{10}Q}$ and $\|\nabla\psi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$. We can define the operator

$$\Lambda_0 f(x) = f(x)\chi_\Omega(x) + \sum_{Q\in\mathcal{W}_3} \psi_Q(x)f_{Q*} \text{ for any } f \in L^1_{loc}(\Omega)$$

(here f_U stands for the mean of a function f in a set U). This function is defined almost everywhere because the boundary of the domain Ω has zero Lebesgue measure (see [Jon81, Lemma 2.3]).

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^d$ be an interior (ε,δ)-corkscrew domain, let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < 1$. Then, $\Lambda_0 : F^{s,C\ell_0}_{p,q,1}(\Omega) \to F^{s,1}_{p,q,1}(\mathbb{R}^d)$, with C depending only on d and ε while ℓ_0 depends also on δ.*

Proof. In light of (4.2), it is enough to check $\Lambda_0 : F^{s,C\ell_0}_{p,q,1}(\Omega) \to F^{s,\ell_0}_{p,q,1}(\mathbb{R}^d)$ for ℓ_0 small enough, that is

$$\|\Lambda_0 f\|_{F^{s,\ell_0}_{p,q,1}(\mathbb{R}^d)} = \|\Lambda_0 f\|_{L^p} + \left\|\left\|\frac{\|\Lambda_0 f(x) - \Lambda_0 f(y)\|_{L^1_y(B_{x,t})}}{t^{s+d+\frac{1}{q}}}\right\|_{L^q_t(0,\ell_0)}\right\|_{L^p_x(\mathbb{R}^d)} \lesssim \|f\|_{F^{s,C\ell_0}_{p,q,1}(\Omega)},$$

where $B_{x,t} := B(x,t)$.

First, note that $\|\Lambda_0 f\|_{L^p} \leqslant \|f\|_{L^p(\Omega)} + \|\Lambda_0 f\|_{L^p(\Omega^c)}$. By Jensen's inequality, we have that

$$\|\Lambda_0 f\|^p_{L^p(\Omega^c)} \lesssim_p \sum_{Q\in\mathcal{W}_3} |f_{Q*}|^p \|\psi_Q\|^p_{L^p} \leqslant \sum_{Q\in\mathcal{W}_3} \frac{1}{\ell(Q)^d}\|f\|^p_{L^p(Q*)}\left(\frac{11}{10}\ell(Q)\right)^d.$$

By the finite overlapping of the symmetrized cubes,

$$\|\Lambda_0 f\|^p_{L^p(\Omega^c)} \lesssim \|f\|^p_{L^p(\Omega)}. \tag{4.5}$$

It remains to check that

$$\|\Lambda_0 f\|_{\dot{F}^{s,\ell_0}_{p,q,1}(\mathbb{R}^d)} = \left\|\left\|\frac{\|\Lambda_0 f(x) - \Lambda_0 f(y)\|_{L^1_y(B_{x,t})}}{t^{s+d+\frac{1}{q}}}\right\|_{L^q_t(0,\ell_0)}\right\|_{L^p_x(\mathbb{R}^d)} \lesssim \|f\|_{F^{s,C\ell_0}_{p,q,1}(\Omega)}.$$

By a decomposition of the integration domains $x, y \in \Omega \cup \Omega^c$, we can reduce the preceding estimate to proving that

$$\text{(a)} + \text{(b)} + \text{(c)} \lesssim \|f\|_{F^{s,C\ell_0}_{p,q,1}(\Omega)},$$

where

$$\text{(a)} := \left\|\left\|\frac{\|f(x) - \Lambda_0 f(y)\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}}\right\|_{L^q_t(0,\ell_0)}\right\|_{L^p_x(\Omega)},$$

$$\text{(b)} := \left\|\left\|\frac{\|\Lambda_0 f(x) - f(y)\|_{L^1_y(\Omega_{x,t})}}{t^{s+d+\frac{1}{q}}}\right\|_{L^q_t(0,\ell_0)}\right\|_{L^p_x(\Omega^c)}$$

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and

$$\text{(c)} := \left\| \left\| \frac{\|\Lambda_0 f(x) - \Lambda_0 f(y)\|_{L_y^1(\Omega_{x,t}^c)}}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega^c)}.$$

Here, and for the rest of the paper, $\Omega_{x,t} := \Omega \cap B_{x,t}$ and $\Omega_{x,t}^c := \overline{\Omega}^c \cap B_{x,t}$.

Let us begin with

$$\text{(a)} = \left\| \left\| \frac{\|f(x) - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*}\|_{L_y^1(\Omega_{x,t}^c)}}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega)}.$$

Call $\mathcal{W}_4 := \{S \in \mathcal{W}_3 : \text{ all the neighbors of } S \text{ are in } \mathcal{W}_3\}$. We write $\mathcal{W}_j(Q,t) := \{S \in \mathcal{W}_j : S \cap \bigcup_{x \in Q} B_{x,t} \neq \varnothing\}$. Note that the integral in t is null if $x \in Q \in \mathcal{W}_0 \setminus \mathcal{W}_1$. Moreover, if $S \in \mathcal{W}_3(Q,t)$ with $Q \in \mathcal{W}_1$ and $t < \ell_0$, then $S \in \mathcal{W}_4$. Given $y \in \frac{11}{10}S$, where $S \in \mathcal{W}_4$, we have that $\sum_{P \in \mathcal{W}_3} \psi_P(y) \equiv 1$. Therefore

$$\text{(a)} \leq \left\| \left\| \left\| \frac{\sum_{S \in \mathcal{W}_4(Q,t)} |f(x) - f_{S*}| \int_{\frac{11}{10}S} \psi_S(y)\, dy}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)}$$

Above, $\ell_Q^p(\mathcal{W}_1)$ stands for the ℓ^p of a sequence indexed by the Whitney cubes in $\mathcal{W}_1$. By the choice of the symmetrized cube we have that $\int_{\frac{11}{10}S} \psi_S(y)\, dy \approx \ell(S^*)^d$. Thus,

$$\text{(a)} \lesssim_d \left\| \left\| \left\| \frac{\sum_{S \in \mathcal{W}_4(Q,t)} \int_{S*} |f(x) - f(\xi)|\, d\xi}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)}$$

By (4.4), if $Q \in \mathcal{W}_1$ and $S \in \mathcal{W}_4(Q,t)$ then $S^* \in \mathcal{W}_1(Q,C_\Omega t)$ and using also the finite overlapping of the symmetrized cubes, we get that

$$\text{(a)} \lesssim_{d,\varepsilon} \left\| \left\| \left\| \frac{\sum_{S \in \mathcal{W}_1(Q,C_\Omega t)} \int_S |f(x) - f(\xi)|\, d\xi}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)} \lesssim_{s,d} \|f\|_{\dot{F}_{p,q,1}^{s,C_\Omega}(\Omega)}.$$

Next, note that, using the same reasoning as above and the finite superposition of the rescaled Whitney cubes, we have that

$$\text{(b)} = \left\| \left\| \frac{\left\| \sum_{Q \in \mathcal{W}_3} \psi_Q(x) f_{Q*} - f(y) \right\|_{L_y^1(\Omega_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega^c)}$$

$$\lesssim_{d,p} \left\| \left\| \psi_Q(x) \left\| \frac{\|f_{Q*} - f(y)\|_{L_y^1(\Omega_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\frac{11}{10}Q)} \right\|_{\ell_Q^p(\mathcal{W}_4)}.$$

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Taking absolute values inside and enlarging the integration domain in y and computing the integral in Q, we have that

$$\text{(b)} \lesssim_{d,p} \left\| \left| \ell(Q)^{\frac{d}{p}} \right\| \frac{\left\| \frac{1}{\ell(Q)^d} \|f(\xi) - f(y)\|_{L^1_\xi(Q*)} \right\|_{L^1_y(\Omega_{Q,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0,\ell_0)} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

where $\Omega_{Q,t} = \bigcup_{x \in Q} \Omega_{x,t}$. Thus, applying Fubini's theorem and Minkowsky's integral inequality (see [Ste70, Appendix A1]), we get

$$\text{(b)} \lesssim_{d,p} \left\| \left\| \frac{\ell(Q)^{\frac{d}{p}}}{\ell(Q)^d} \right\| \frac{\|f(\xi) - f(y)\|_{L^1_y(\Omega_{Q,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0,\ell_0)} \right\|_{L^1_\xi(Q*)} \right\|_{\ell^p_Q(\mathcal{W}_4)}.$$

If $Q \in \mathcal{W}_4$ and $y \in \Omega_{Q,t}$, then $y \in \Omega_{\xi, C_\Omega t}$ for every $\xi \in Q*$. By Hölder's inequality and the finite overlapping of symmetrized cubes, we get that

$$\text{(b)} \lesssim_{d,p,\varepsilon} \left\| \left\| \left\| \frac{\|f(\xi) - f(y)\|_{L^1_y(\Omega_{\xi, C_\Omega t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0,\ell_0)} \right\|_{L^p_\xi(Q)} \right\|_{\ell^p_Q(\mathcal{W}_1)} \lesssim \|f\|_{\dot{F}^{s,C\mathcal{W}}_{p,q,1}(\Omega)}.$$

Let us focus on (c). By the triangle inequality, we have that

$$\text{(c)} \leq \left\| \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*} \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10}, \ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

$$+ \left\| \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*} \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0, \frac{\ell(Q)}{10})} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

$$+ \left\| \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*} \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0, \ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_2 \setminus \mathcal{W}_4)}$$

$$=: \text{(c1)} + \text{(c2)} + \text{(c3)}.$$

The first term is bounded using the same techniques as in (a) and (b). Indeed, given $x \in \frac{11}{10}Q$ where $Q \in \mathcal{W}_4$ and $y \in \Omega^c_{x, \ell_0/10}$, then neither x nor y are in the support of any bump function of a cube in $\mathcal{W}_2 \setminus \mathcal{W}_3$, so $\sum_{S \in \mathcal{W}_3} \psi_S(y) \equiv 1$ and $\sum_{P \in \mathcal{W}_3} \psi_P(x) \equiv 1$. Therefore

$$\sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*} = \sum_{P \cap 2Q \neq \varnothing} \sum_{S \in \mathcal{W}_3} \psi_P(x) \psi_S(y) \left(f_{P*} - f_{S*} \right).$$

Using first the triangle inequality and the bounded number of neighboring cubes, and then computing the integral in x, taking absolute values inside the inner integral and changing the order of

summation on Q and P we get

$$\boxed{\text{c1}} = \left\| \left\| \left\| \frac{\left\| \sum_{P \cap 2Q \neq \varnothing} \sum_{S \in \mathcal{W}_3} |\psi_P(x)\psi_S(y)| \, |f_{P*} - f_{S*}| \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10}, \ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

$$\lesssim_{d,p} \left\| \left(\sum_{P \cap 2Q \neq \varnothing} \left\| \psi_P(x) \frac{\sum_{S \in \mathcal{W}_3(P, C_d t)} |f_{P*} - f_{S*}| \, \ell(S)^d}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10}, \ell_0)}^p \right)^{\frac{1}{p}} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

$$\lesssim_d \left\| \ell(P)^{\frac{d}{p}} \left\| \frac{\sum_{S \in \mathcal{W}_3(P, C_d t)} \frac{1}{\ell(P)^d} \left\| \|f(\xi) - f(\zeta)\|_{L^1_\zeta(S*)} \right\|_{L^1_\xi(P*)}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(P)}{20}, \ell_0)} \right\|_{\ell^p_P(\mathcal{W}_3)},$$

and applying Minkowski's and Jensen's inequalities we obtain

$$\boxed{\text{c1}} \lesssim_{d,p} \left\| \left\| \frac{\sum_{S \in \mathcal{W}_3(P, C_d t)} \|f(\xi) - f(\zeta)\|_{L^1_\zeta(S*)}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(P)}{20}, \ell_0)} \right\|_{L^p_\xi(P*)} \right\|_{\ell^p_P(\mathcal{W}_3)}.$$

By Lemma 4.2, we get that

$$\boxed{\text{c1}} \lesssim_{d,p,\varepsilon} \left\| \left\| \frac{\|f(\xi) - f(\zeta)\|_{L^1_\zeta(\Omega_{\xi, C_\Omega t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0, \ell_0)} \right\|_{L^p_\xi(\Omega)} \lesssim \|f\|_{\dot{F}^{s, C_\Omega}_{p,q,1}(\Omega)}.$$

If $x \in \frac{11}{10}Q$ where $Q \in \mathcal{W}_4$ and $y \in \Omega^c_{x, \ell(Q)/10}$, since the points are 'close' to each other, we will use the Lipschitz regularity of the bump functions, so we write

$$\sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S*} = \sum_{P \in \mathcal{W}_3} (\psi_P(x) - \psi_P(y)) f_{P*}. \tag{4.6}$$

Now we use that $\{\psi_Q\}$ is a partition of the unity with ψ_Q supported in $\frac{11}{10}Q$, that is, $\sum_{S \in \mathcal{W}_3} \psi_S(x) = \sum_{S \cap 2Q \neq \varnothing} \psi_S(x) = 1$ if $x \in \frac{11}{10}Q$ with $Q \in \mathcal{W}_4$. Thus,

$$\boxed{\text{c2}} = \left\| \left\| \left\| \frac{\left\| \sum_{S \cap 2Q \neq \varnothing} (\psi_S(x) - \psi_S(y)) f_{S*} \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0, \frac{\ell(Q)}{10})} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}$$

$$= \left\| \left\| \left\| \frac{\left\| \sum_{S \cap 2Q \neq \varnothing} (\psi_S(x) - \psi_S(y)) (f_{S*} - f_{Q*}) \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{s+d+\frac{1}{q}}} \right\|_{L^q_t(0, \frac{\ell(Q)}{10})} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}.$$

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Using the fact that $\|\nabla\psi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$ and computing the integrals in x and t, we have that

$$\boxed{c2} \leq \left\|\left\|\left\|\frac{|\Omega_{x,t}^c|\sum_{S\cap 2Q\neq\varnothing}\|\nabla\psi_S\|_\infty t\,|f_{S*}-f_{Q*}|}{t^{s+d+\frac{1}{q}}}\right\|_{L_t^q(0,\frac{\ell(Q)}{10})}\right\|_{L_x^p(Q)}\right\|_{\ell_Q^p(\mathcal{W}_4)}$$

$$\lesssim_d \left\|\ell(Q)^{\frac{d}{p}}\left\|t^{1-s-\frac{1}{q}}\right\|_{L_t^q(0,\frac{\ell(Q)}{10})}\sum_{S\cap 2Q\neq\varnothing}\frac{|f_{S*}-f_{Q*}|}{\ell(Q)}\right\|_{\ell_Q^p(\mathcal{W}_4)}$$

$$\lesssim_{d,s} \left\|\ell(Q)^{\frac{d}{p}+1-s-1-2d}\left\|\|f(\zeta)-f(\xi)\|_{L_\zeta^1(C_\Omega Q*)}\right\|_{L_\xi^1(Q*)}\right\|_{\ell_Q^p(\mathcal{W}_4)}$$

and using Jensen's inequality and Lemma 4.2 we get

$$\boxed{c2} \lesssim_{d,s,\varepsilon} \left\|\ell(Q)^{-s-d}\left\|\left|f(\zeta)-f(\xi)\right|\fint_{|\xi-\zeta|}^{c_1\ell(Q)}dt\right\|_{L_\zeta^1(C_\Omega Q)}\right\|_{L_\xi^p(Q)}\right\|_{\ell_Q^p(\mathcal{W}_1)}.$$

If c_1 is chosen big enough, depending only on d and the corkscrew constants of Ω, so that $c_1\ell(Q)-C_\Omega\text{diam}(Q)\approx\ell(Q)$, using Fubini's theorem and Hölder's inequality we obtain

$$\boxed{c2} \lesssim_{d,s,\varepsilon} \left\|\left\|\left\|\frac{\|f(\zeta)-f(\xi)\|_{L_\zeta^1(\Omega_{\xi,t})}}{\ell(Q)^{s+d+1}}\right\|_{L_t^1(0,c_1\ell(Q))}\right\|_{L_\xi^p(Q)}\right\|_{\ell_Q^p(\mathcal{W}_1)} \lesssim \|f\|_{\dot{F}_{p,q,1}^{s,C_\Omega}(\Omega)}.$$

Decomposition (4.6) is still valid if $Q\in\mathcal{W}_2\backslash\mathcal{W}_4$ and $y\in\Omega_{x,\ell(Q)/10}^c$. In particular if $y\in\Omega_{x,\ell_0}^c$ we can use the decomposition, but we treat this case apart since we lose the cancellation of the sums of bump functions and we gain instead a uniform lower bound on the side-lengths of the cubes involved:

$$\boxed{c3} = \left\|\left\|\left\|\frac{\left\|\sum_{S\in\mathcal{W}_3:S\cap 2Q\neq\varnothing}(\psi_S(x)-\psi_S(y))f_{S*}\right\|_{L_y^1(\Omega_{x,t}^c)}}{t^{s+d+\frac{1}{q}}}\right\|_{L_t^q(0,\ell_0)}\right\|_{L_x^p(Q)}\right\|_{\ell_Q^p(\mathcal{W}_2\backslash\mathcal{W}_4)}$$

$$\lesssim_d \left\|\left\|\left\|\frac{|\Omega_{x,t}^c|\sum_{S\in\mathcal{W}_3:S\cap 2Q\neq\varnothing}|\frac{t}{\ell_0}f_{S*}|}{t^{s+d+\frac{1}{q}}}\right\|_{L_t^q(0,\ell_0)}\right\|_{L_x^p(Q)}\right\|_{\ell_Q^p(\mathcal{W}_2\backslash\mathcal{W}_4)}.$$

Computing the integrals in x and t and using the triangle inequality we get

$$\boxed{c3} \lesssim_d \frac{1}{\ell_0}\left\|\ell(Q)^{\frac{d}{p}}\sum_{S\in\mathcal{W}_3:S\cap 2Q\neq\varnothing}|f_{S*}|\left\|t^{1-s-\frac{1}{q}}\right\|_{L_t^q(0,\ell_0)}\right\|_{\ell_Q^p(\mathcal{W}_2\backslash\mathcal{W}_4)}$$

$$\lesssim_{d,p,s} \frac{1}{\ell_0^s}\left(\sum_{Q\in\mathcal{W}_2\backslash\mathcal{W}_4}\sum_{S\in\mathcal{W}_3:S\cap 2Q\neq\varnothing}|f_{S*}|^p\ell(S)^d\right)^{\frac{1}{p}} \lesssim_\varepsilon \|f\|_{L^p(\Omega)}.$$

$\square$

Remark 4.4. *It is usual in the literature to define a uniform domain as a domain satisfying the interior corkscrew condition and the so-called Harnack chain condition (this definition can be seen to be equivalent to the one given here). The interior corkscrew condition can be understood as a quantitative openness condition, while the Harnack chain can be understood as a quantitative connectedness condition. It is not quite surprising that we can drop the connectedness condition for smoothness below one, since the norm is completely non-local. That is, the connection between points following paths inside the domain is not needed because the first-order difference is always included in the norm itself.*

The reader may note that the interior corkscrew condition is a bit stronger than the conditions that we have used. Indeed, the proof works for a domain Ω such that $\overline{\Omega}^c$ is an exterior corkscrew domain and such that $\partial\Omega \backslash \partial(\overline{\Omega}^c)$ has null Lebesgue measure. For instance one can remove segments on planar domains without changing the extendability properties for $F_{p,q}^s$ with $0 < s < 1$. [PS17, Theorem 1.4] can also be proven in such a general setting, with the restriction in the indices $s > \frac{d}{p} - \frac{d}{q}$.

4.2 Uniform domains and smoothness above one

Norman G. Meyers introduced a collection of projections $L : W^{k,p}(Q) \to \mathcal{P}^k$ in [Mey78] which allows us to iterate the Poincaré inequality. Peter Jones uses the following particular simple case:

Definition 4.5. *Let $Q \subset \mathbb{R}^d$. Given $f \in L^1(Q)$ with weak derivatives up to order k, we define $\mathbf{P}_Q^k f \in \mathcal{P}^k$ as the unique polynomial of degree smaller or equal than k such that*

$$\int_Q D^\beta \mathbf{P}_Q^k f \, dm = \int_Q D^\beta f \, dm \tag{4.7}$$

for every multiindex $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k$.

Lemma 4.6 (see [PT15, Lemma 4.2]). *Given a cube Q and $f \in W^{k,1}(Q)$, the polynomial $\mathbf{P}_Q^k f \in \mathcal{P}^k$ exists and is unique. Furthermore, this polynomial has the following properties:*

1. *The norm of the polynomial is controlled by*

$$\left\| P_Q^k f \right\|_{L^p(Q)} \leqslant c_k \sum_{j=0}^k \ell(Q)^j \left\| \nabla^j f \right\|_{L^p(Q)} \qquad \text{for } 1 \leqslant p \leqslant \infty. \tag{4.8}$$

2. *Furthermore, if $f \in W^{k,p}(Q)$, for $1 \leqslant p \leqslant \infty$ we have*

$$\left\| f - \mathbf{P}_Q^k f \right\|_{L^p(Q)} \leqslant C\ell(Q)^k \left\| \nabla^k f - (\nabla^k f)_Q \right\|_{L^p(Q)}. \tag{4.9}$$

   Here and through all the text $(f)_Q$ will denote the mean of f in a cube Q, with f possibly vector-valued.

3. *Given a uniform domain Ω with Whitney covering $\mathcal{W}$, given $\beta \in \mathbb{N}_0^d$ with $|\beta| \leqslant k$ and given two Whitney cubes $Q, S \in \mathcal{W}$ and $f \in W^{k,1}(\Omega)$,*

$$\left\| D^\beta (\mathbf{P}_S^k f - \mathbf{P}_Q^k f) \right\|_{L^1(S)} \leqslant \sum_{P \in [S,Q]} \frac{\ell(S)^d \mathrm{D}(P,S)^{k-|\beta|}}{\ell(P)^d} \left\| \nabla^k f - (\nabla^k f)_P \right\|_{L^1(5P)}. \tag{4.10}$$

Estimate (4.10) can be shown just using the approach in [Jon81, Lemma 3.1].

We define the operator $\Lambda_k : W_{\text{loc}}^{k,1}(\Omega) \to W_{\text{loc}}^{k,1}(\Omega \cup \overline{\Omega}^c)$ as

$$\Lambda_k f(x) = f(x)\chi_\Omega(x) + \sum_{Q \in \mathcal{W}_3} \psi_Q(x) P_{Q*}^k f(x).$$

This function is defined almost everywhere because the boundary of the domain Ω has zero Lebesgue measure (see [Jon81, Lemma 2.3]). Note that the operator can be defined in any interior corkscrew domain, but it may fail to be an extension operator if the domain is not uniform (see [Jon81, Shv10, KRZ15] for optimal conditions to grant the existence of an extension operator for $W^{1,p}$).

Theorem 4.7. *Let Ω be a uniform domain and $k \in \mathbb{N}$. For every $0 < \sigma < 1$, $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ with $\sigma > \frac{d}{p} - \frac{d}{q}$ and ℓ_0 small enough, then*

$$\Lambda_k : F_{p,q,1}^{s,C\ell_0}(\Omega) \to F_{p,q,1}^{s,\ell_0}(\mathbb{R}^n)$$

(with $s = \sigma + k$) is a bounded extension operator.

Proof. Let $f \in F_{p,q,u}^{s,C\ell_0}(\Omega)$. From [Pra19, p.700] we know that

$$\Lambda_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d).$$

Thus, we just need to prove that

$$\|\Lambda_k f\|_{\dot{F}_{p,q,1}^{s,\ell_0}(\mathbb{R}^n)} \leqslant C\|f\|_{F_{p,q,1}^{s,C\ell_0}(\Omega)}.$$

The case $k = 0$ is proven in Theorem 4.3 above. Let us assume that $k \geqslant 1$, and consider $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant k$. We will check that

$$\|D^\alpha \Lambda_k f\|_{\dot{F}_{p,q,1}^{\sigma,\ell_0}(\mathbb{R}^n)} \leqslant C\|f\|_{F_{p,q,1}^{s,C\ell_0}(\Omega)}.$$

Note that

$$D^\alpha \Lambda_k f = D^\alpha f \chi_\Omega + \sum_{Q \in \mathcal{W}_3} D^\alpha(\psi_Q P_{Q*}^k f) = D^\alpha f \chi_\Omega + \sum_{Q \in \mathcal{W}_3} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \psi_Q D^\beta P_{Q*}^k f$$

$$= \Lambda_0(D^\alpha f) + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \sum_{Q \in \mathcal{W}_3} D^{\alpha-\beta} \psi_Q D^\beta P_{Q*}^k f. \tag{4.11}$$

Now, from Theorem 4.3 again, we already have

$$\|\Lambda_0(D^\alpha f)\|_{\dot{F}_{p,q,1}^{\sigma,\ell_0}(\mathbb{R}^n)} \leqslant C\|D^\alpha f\|_{F_{p,q,1}^{\sigma,C\ell_0}(\Omega)} \leqslant C\|f\|_{F_{p,q,1}^{s,C\ell_0}(\Omega)}.$$

Thus, for every $\beta < \alpha$ we need to control

$$
\boxed{0} := \left\| \sum_{P \in \mathcal{W}_3} D^{\alpha-\beta} \psi_P D^\beta P_{P*}^k f \right\|_{\dot{F}_{p,q,1}^{\sigma,\ell_0}(\mathbb{R}^n)}^p
$$

$$
= \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} D^{\alpha-\beta} \psi_P(y) D^\beta P_{P*}^k f(y) \right\|_{L_y^1(\Omega_{x,t}^c)}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega)}
$$

$$
+ \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} D^{\alpha-\beta} \psi_P(x) D^\beta P_{P*}^k f(x) \right\|_{L_y^1(\Omega_{x,t})}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega^c)}
$$

$$
+ \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} \left((D^{\alpha-\beta} \psi_P D^\beta P_{P*}^k f)(x) - (D^{\alpha-\beta} \psi_P D^\beta P_{P*}^k f)(y) \right) \right\|_{L_y^1(\Omega_{x,t}^c)}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega^c)}
$$

$$
= \boxed{a} + \boxed{b} + \boxed{c}. \tag{4.12}
$$

First we study the term $\boxed{a}$. Note that if $Q \in \mathcal{W}_1$ and $S \in \mathcal{W}_3(Q,t)$ for $t < \ell_0$, then $S \in \mathcal{W}_4(Q,t)$ necessarily, where $\mathcal{W}_4 := \{S \in \mathcal{W}_3 : \text{ all the neighbors of } S \text{ are in } \mathcal{W}_3\}$. Thus, if $y \in S$, we have that $\sum_{P \in \mathcal{W}_3} D^{\alpha-\beta} \psi_P(y) = 0$. Therefore,

$$
\boxed{a} \leqslant \left\| \left\| \left\| \frac{\left\| \sum_{P: P \cap 2S \neq \varnothing} D^{\alpha-\beta} \psi_P D^\beta P_{P*}^k f \right\|_{L^1(S)} \right\|_{\ell_S^1(\mathcal{W}_4(Q,t))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)}
$$

$$
\leqslant \left\| \left\| \left\| \frac{\left\| \sum_{P: P \cap 2S \neq \varnothing} \left\| D^{\alpha-\beta} \psi_P (D^\beta P_{P*}^k f - D^\beta P_{S*}^k f) \right\|_{L^1(S)} \right\|_{\ell_S^1(\mathcal{W}_4(Q,t))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)}.
$$

We take absolute values and we use that $\left\| D^{\alpha-\beta} \psi_P \right\|_\infty \lesssim \ell(S)^{-|\alpha-\beta|}$. Moreover we develop the telescopic summation (4.10) along an admissible chain connecting P^* and S^*:

$$
\left\| D^\beta (P_{P*}^k f - P_{S*}^k f) \right\|_{L^1(S)} \lesssim_{d,k} \sum_{L \in [P^*, S^*]} \frac{\ell(S^*)^d \mathrm{D}(L, S^*)^{k-|\beta|}}{\ell(L)^d} \left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^1(5L)} \tag{4.13}
$$

Note that in our summation $2S \cap P \neq \varnothing$, so both cubes have comparable size and $\mathrm{D}(S^*, P^*) \approx \ell(S)$ by (4.4). Thus, combining (3.2) and (3.3), it is clear that all the elements $L \in [P^*, S^*]$ have comparable size and $\mathrm{D}(L, S^*) \approx \ell(S)$. Moreover, by (4.4), it follows that $\mathrm{D}(Q, S) \approx \mathrm{D}(Q, S^*) \approx \mathrm{D}(Q, L)$

$$
\boxed{a} \lesssim_{d,k} \left\| \left\| \left\| \frac{\left\| \sum_{P: P \cap 2S \neq \varnothing} \sum_{L \in [P^*, S^*]} \left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^1(5L)} \right\|_{\ell_S^1(\mathcal{W}_4(Q,t))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)}.
$$

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Fixing c_0 big enough in the definition of $\mathcal{W}_1$ (see Section 4.1), we can ensure that $L \in \mathcal{W}_1$ for every L appearing in the right-hand side term above. To complete the reduction, note that for every $L \in \mathcal{W}_1$ the number of candidates $S \in \mathcal{W}_4$ and $P \cap 2S \neq \varnothing$ such that $L \in [S^*, P^*]$ is uniformly bounded by a constant depending on d and ε. Moreover, for $Q \in \mathcal{W}_1$ and $t < \ell(Q)$ the family $\mathcal{W}_4(Q,t)$ is empty. Therefore, we can use Lemma 4.8 below to get

$$\boxed{\text{a}} \lesssim_{d,k,\varepsilon} \left\| \left\| \left\| \frac{\left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^1(5L)} \right\|_{\ell_L^1(\mathcal{W}_1(Q,Ct))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(\ell(Q),\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_1)} \lesssim C \left\| \nabla^k f \right\|_{\dot{F}_{p,q,1}^{\sigma,C\ell_0}(\Omega)}. \tag{4.14}$$

Next we apply a similar reasoning to deal with $\boxed{\text{b}}$. This case is simpler, because we can use the triangle inequality to get

$$\boxed{\text{b}} \leq \left\| \left\| \sum_{P \in \mathcal{W}_3} D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f \left\| \frac{|\Omega_{x,t}|}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(\ell(Q),\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell^p(\mathcal{W}_4)}$$

$$\lesssim_{d,\sigma} \left(\sum_{Q \in \mathcal{W}_4} \ell(Q)^{-\sigma p} \int_Q \left| \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} D^{\alpha-\beta}\psi_P(x) D^\beta P_{P*}^k f(x) \right|^p dx \right)^{\frac{1}{p}}.$$

As before, we can use the cancellation to obtain

$$\boxed{\text{b}} \lesssim_{d,\sigma} \left(\sum_{Q \in \mathcal{W}_4} \ell(Q)^{-\sigma p - |\alpha-\beta|p} \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \int_Q \left| D^\beta P_{P*}^k f(x) - D^\beta P_{Q*}^k f(x) \right|^p dx \right)^{\frac{1}{p}}. \tag{4.15}$$

We use again (4.10) and the fact that $\ell(P) \approx \ell(Q) \approx \ell(L) \approx \mathrm{D}(Q,L)$ for every $2Q \cap P \neq \varnothing$ and $L \in [Q^*, P^*]$:

$$\boxed{\text{b}} \lesssim_{\varepsilon,k} \left(\sum_{L \in \mathcal{W}_1} \ell(L)^{-\sigma p} \left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^p(5L)}^p \right)^{\frac{1}{p}}.$$

Note that by Jensen's inequality we have that

$$\boxed{\text{b}} \lesssim \left(\sum_{L \in \mathcal{W}_1} \int_{5L} \left(\int_L \frac{|\nabla^k f(x) - \nabla^k f(\xi)| \fint_{|\xi-x|}^{c_1\ell(L)} dt}{\ell(L)^{d+\sigma}} d\xi \right)^p dx \right)^{\frac{1}{p}} \tag{4.16}$$

If c_1 is chosen big enough, depending only on d, so that $c_1\ell(Q) - \mathrm{diam}(Q) \approx \ell(Q)$, using Fubini and Jensen we obtain

$$\boxed{\text{b}} \lesssim \left(\sum_{L \in \mathcal{W}_1} \int_L \left(\fint_0^{c_1\ell(L)} \frac{\fint_{\Omega_{\xi,t}} |\nabla^k f(x) - \nabla^k f(\xi)| d\xi}{\ell(L)^\sigma} dt \right)^p dx \right)^{\frac{1}{p}} \lesssim \left\| \nabla^k f \right\|_{\dot{F}_{p,q,1}^{\sigma,C_d}(\Omega)}. \tag{4.17}$$

Finally we need to deal with the term

$$\boxed{\text{c}} = \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3} \left((D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(x) - (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f) \right) \right\|_{L^1(\Omega_{x,t}^c)}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(\Omega^c)} \tag{4.18}$$

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Here we will use the previous techniques but some additional tools have to be used to tackle the case $\operatorname{dist}(x,y) << \operatorname{dist}(x,\partial\Omega)$, so we separate the integration regions with this idea in mind. We get

$$
\boxed{\text{c}} \leqslant \left\| \left\| \left\| \frac{\left\| \sum_{P\in\mathcal{W}_3} (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(x) - (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(y) \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10},\ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}
$$

$$
+ \left\| \left\| \left\| \frac{\left\| \sum_{P\in\mathcal{W}_3} (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(x) - (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(y) \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(0,\frac{\ell(Q)}{10})} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}
$$

$$
+ \left\| \left\| \left\| \frac{\left\| \sum_{P\in\mathcal{W}_3} (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(x) - (D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f)(y) \right\|_{L^1_y(\Omega^c_{x,t})}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(0,\ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_2\setminus\mathcal{W}_4)}
$$

$$
=: \boxed{\text{c.1}} + \boxed{\text{c.2}} + \boxed{\text{c.3}}. \tag{4.19}
$$

Let us consider the case $x \in Q \in \mathcal{W}_4$, $t > \ell(Q)/10$ and $y \in \Omega^c_{x,t}$. In this case we will bound the numerator in (4.19) above by

$$
\left| \sum_{P\in\mathcal{W}_3} D^{\alpha-\beta}\psi_P(x) D^\beta P_{P*}^k f(x) \right| + \left| \sum_{P\in\mathcal{W}_3} D^{\alpha-\beta}\psi_P(y) D^\beta P_{P*}^k f(y) \right|. \tag{4.20}
$$

We obtain

$$
\boxed{\text{c.1}} \lesssim \left\| \left\| \left\| \frac{\left\| \sum_{P\in\mathcal{W}_3} D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f \right\|_{L^1(\Omega^c_{x,t})}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10},\ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}
$$

$$
+ \left\| \left\| \left\| \sum_{P\in\mathcal{W}_3} D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f \right\| \frac{|\Omega^c_{x,t}|}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10},\ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)}
$$

$$
= \boxed{\text{c.1.1}} + \boxed{\text{c.1.2}}.
$$

Now, $\boxed{\text{c.1.1}}$ is bounded as $\boxed{\text{a}}$: for every $x \in Q \in \mathcal{W}_4$ and $t \in \left(\frac{\ell(Q)}{10},\ell_0\right)$ we write

$$
\left\| \sum_{P\in\mathcal{W}_3} D^{\alpha-\beta}\psi_P D^\beta P_{P*}^k f \right\|_{L^1(\Omega^c_{x,t})} \lesssim \left\| \sum_{P:P\cap 2S\neq\varnothing} \ell(S)^{-|\alpha-\beta|} \left\| D^\beta P_{P*}^k f - D^\beta P_{S*}^k f \right\|_{L^1(S)} \right\|_{\ell^1_S(\mathcal{W}_3(Q,t))}.
$$

Then we use cube chains as in (4.13) and Lemma 4.8 below to get

$$
\boxed{\text{c.1.1}} \lesssim_\varepsilon \left\| \left\| \left\| \frac{\left\| \left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^1(5L)} \right\|_{\ell^1_L(\mathcal{W}_1(Q,Ct))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10},\ell_0)} \right\|_{L^p_x(Q)} \right\|_{\ell^p_Q(\mathcal{W}_4)} \lesssim C \left\| \nabla^k f \right\|_{\dot{F}^{\sigma,Ct}_{p,q,1}(\Omega)}.
$$

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On the other hand, $\boxed{\text{c.1.2}}$ is bounded as $\boxed{\text{b}}$ without much change:

$$\boxed{\text{c.1.2}} \lesssim \left(\sum_{Q \in \mathcal{W}_4} \ell(Q)^{-\sigma p} \int_Q \left| \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} D^{\alpha - \beta} \psi_P(x) D^\beta P_{P*}^k f(x) \right|^p dx \right)^{\frac{1}{p}} \lesssim_{d,s,\varepsilon} \left\| \nabla^k f \right\|_{\dot{F}_{p,q,1}^{\sigma, C_\Omega}(\Omega)}.$$

Combining these estimates we obtain

$$\boxed{\text{c.1}} \lesssim_{d,s,\varepsilon} \left\| \nabla^k f \right\|_{\dot{F}_{p,q,1}^{\sigma, C_\Omega}(\Omega)}. \tag{4.21}$$

If $x \in Q \in \mathcal{W}_4$ and $y \in B(x, \ell(Q)/10)$, then we can use the fact that

$$\sum_{P \in \mathcal{W}_3} D^{\alpha - \beta} \psi_P(y) = \sum_{P \in \mathcal{W}_3} D^{\alpha - \beta} \psi_P(x) = 0.$$

We will bound the numerator of the second term in (4.19) by

$$\left| \sum_{P \in \mathcal{W}_3} (D^{\alpha - \beta} \psi_P(x) - D^{\alpha - \beta} \psi_P(y))(D^\beta P_{P*}^k f(x)) + D^{\alpha - \beta} \psi_P(y)(D^\beta P_{P*}^k f(x) - D^\beta P_{P*}^k f(y)) \right|$$

$$\leq \left| \sum_{P \in \mathcal{W}_3} (D^{\alpha - \beta} \psi_P(x) - D^{\alpha - \beta} \psi_P(y))(D^\beta P_{P*}^k f(x) - D^\beta P_{Q*}^k f(x)) \right| \tag{4.22}$$

$$+ \left| \sum_{P \in \mathcal{W}_3} D^{\alpha - \beta} \psi_P(y) \left((D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f)(x) - (D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f)(y) \right) \right|.$$

We obtain

$$\boxed{\text{c.2}} \lesssim \left\| \left\| \left\| \frac{\sum_{\substack{P \in \mathcal{W}_3 \\ P \cap 2Q \neq \varnothing}} \left\| \nabla D^{\alpha - \beta} \psi_P \right\|_\infty \left| D^\beta P_{P*}^k f(x) - D^\beta P_{Q*}^k f(x) \right|}{t^{\sigma - 1 + d + \frac{1}{q}}} \right\|_{L_y^1(\Omega_{x,t}^c)} \right\|_{L_t^q(0, \frac{\ell(Q)}{10})} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_4)}$$

$$+ \left\| \left\| \left\| \frac{\sum_{\substack{P \in \mathcal{W}_3 \\ P \cap 2Q \neq \varnothing}} \left\| D^{\alpha - \beta} \psi_P \right\|_\infty \left\| \nabla(D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f) \right\|_{L^\infty(P)}}{t^{\sigma - 1 + d + \frac{1}{q}}} \right\|_{L_y^1(\Omega_{x,t}^c)} \right\|_{L_t^q(0, \frac{\ell(Q)}{10})} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_4)}$$

$$= \boxed{\text{c.2.1}} + \boxed{\text{c.2.2}}. \tag{4.23}$$

In the first term above, we integrate on y and t, we use the control on the derivatives of the bump functions and we plug (4.10) in to get

$$\boxed{\text{c.2.1}} \lesssim_{d,\sigma,p} \left(\sum_{Q \in \mathcal{W}_4} \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \ell(P)^{-(|\alpha - \beta| + 1)p} \left\| D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f \right\|_{L^p(Q)}^p \ell(Q)^{(1-\sigma)p} \right)^{\frac{1}{p}}$$

$$\lesssim_\varepsilon \left(\sum_{L \in \mathcal{W}_1} \ell(L)^{-(|\alpha - \beta| + 1)p} \ell(L)^{(1-\sigma)p} \frac{\ell(L)^d \ell(L)^{(|\alpha - \beta|)p}}{\ell(L)^d} \left\| \nabla^k f - (\nabla^k f)_L \right\|_{L^p(5L)} \right)^{\frac{1}{p}}$$

so, as in (4.16) we get

$$\boxed{\text{c.2.1}} \lesssim \left(\sum_{L \in \mathcal{W}_1} \ell(L)^{-\sigma p} \big\| \nabla^k f - (\nabla^k f)_L \big\|_{L^p(5L)} \right)^{\frac{1}{p}} \lesssim \big\| \nabla^k f \big\|_{\dot{F}_{p,q,1}^{\sigma, C_\Omega}(\Omega)}. \tag{4.24}$$

Note that the equivalence of norms of polynomials implies

$$\big\| \nabla (D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f) \big\|_{L^\infty(P)}^p \ell(P)^d \approx_{d,k} \big\| \nabla (D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f) \big\|_{L^p(P)}^p.$$

Thus, in the second term, using the same reasoning as above we get

$$\boxed{\text{c.2.2}} \lesssim_{\sigma,p} \left(\sum_{Q \in \mathcal{W}_4} \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \ell(P)^{-|\alpha-\beta|p} \big\| \nabla (D^\beta P_{P*}^k f - D^\beta P_{Q*}^k f) \big\|_{L^\infty(P)}^p \ell(Q)^{(1-\sigma)p+d} \right)^{\frac{1}{p}}$$

$$\lesssim_{d,k,\varepsilon} \left(\sum_{L \in \mathcal{W}_1} \ell(L)^{-|\alpha-\beta|p} \frac{\ell(L)^d \ell(L)^{(|\alpha-\beta|-1)p}}{\ell(L)^d} \big\| \nabla^k f - (\nabla^k f)_L \big\|_{L^p(5L)} \ell(L)^{(1-\sigma)p} \right)^{\frac{1}{p}}$$

and we get the same case as before. By (4.23) and (4.24) we get

$$\boxed{\text{c.2}} \lesssim_{d,s,p,\varepsilon} \big\| \nabla^k f \big\|_{\dot{F}_{p,q,1}^{\sigma, C_\Omega}(\Omega)}. \tag{4.25}$$

Finally we deal with the term $\boxed{\text{c.3}}$. Whenever $x \in Q \in \mathcal{W}_2 \setminus \mathcal{W}_4$ and $y \in B(x, \ell(Q)/10) \subset \frac{11}{10}Q$, we bound the numerator in (4.19) by the left-hand side of (4.22) above:

$$\boxed{\text{c.3}} \leq \left\| \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \big\| \nabla D^{\alpha-\beta} \psi_P \big\|_\infty D^\beta P_{P*}^k f(x) \right\|_{L_y^1(\Omega_{x,t}^c)}}{t^{\sigma-1+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_2 \setminus \mathcal{W}_4)}$$

$$+ \left\| \left\| \left\| \frac{\left\| \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \big\| D^{\alpha-\beta} \psi_P \big\|_\infty \big\| \nabla D^\beta P_{P*}^k f \big\|_{L^\infty\left(\frac{11}{10}Q\right)} \right\|_{L_y^1(\Omega_{x,t}^c)}}{t^{\sigma-1+d+\frac{1}{q}}} \right\|_{L_t^q(0,\ell_0)} \right\|_{L_x^p(Q)} \right\|_{\ell_Q^p(\mathcal{W}_2 \setminus \mathcal{W}_4)}$$

We write $Q \in \mathcal{W}_3'$ if $Q \in \mathcal{W}_2$ has neighbors $P \in \mathcal{W}_3$. Both terms are controlled by integrating on y and t again and using the control on the derivatives of the bump functions together with (4.8) and the finite overlapping of symmetrized cubes to get

$$\boxed{\text{c.3}} \lesssim_{\sigma,p} \left(\sum_{Q \in \mathcal{W}_3' \setminus \mathcal{W}_4} \sum_{P \in \mathcal{W}_3 : P \cap 2Q \neq \varnothing} \ell_0^{-|\alpha-\beta|p} (\ell_0^{-p} + 1) \big\| |D^\beta P_{P*}^k f| + |\nabla D^\beta P_{P*}^k f| \big\|_{L^p(Q)}^p \ell_0^{(1-\sigma)p} \right)^{\frac{1}{p}}$$

$$\lesssim_k \left(\sum_{P \in \mathcal{W}_3} \|f\|_{W^{k,p}(P*)}^p \right)^{\frac{1}{p}} \lesssim_{\varepsilon,d} \|f\|_{W^{k,p}(\Omega)}. \tag{4.26}$$

Combining (4.19), (4.21), (4.25) and (4.26) we have

$$\boxed{\text{c}} \lesssim \|f\|_{F_{p,q,1}^{s, C_\Omega}(\Omega)},$$

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which combined with (4.12), (4.14) and (4.17), leads to

$$\boxed{0} \lesssim_{d,s,p,\varepsilon} \|f\|_{F^{s,C_\Omega}_{p,q,1}(\Omega)}$$

and the theorem follows.

$\square$

It remains to proof a couple of technical lemmata used during the proof of the boundedness of the extension operator.

Lemma 4.8. *Let $d \geqslant 1$ be a natural number, let $0 < \sigma < 1$, let $1 \leqslant p < \infty$, let $1 \leqslant q \leqslant \infty$ and ℓ_0 small enough. There exists a constant C such that for every $f \in L^p(\Omega)$,*

$$\left\| \left\| \frac{\left\| \|f - f_L\|_{L^1(5L)} \right\|_{\ell^1_L(\mathcal{W}_1(Q,Ct))}}{t^{\sigma+d+\frac{1}{q}}} \right\|_{L^q_t(\frac{\ell(Q)}{10},\ell_0)} \ell(Q)^{\frac{d}{p}} \right\|_{\ell^p_Q(\mathcal{W}_1 \cup \mathcal{W}_4)} \leqslant C\|f\|_{\dot{F}^{\sigma,C_\Omega}_{p,q,1}(\Omega)}.$$

Proof. Writing $h(\xi) := \sum_{L \in \mathcal{W}_1} \chi_{5L}(\xi) \ell(L)^{-\sigma} |f(\xi) - f_L|$ and applying Lemma 4.9 below, we get

$$\boxed{A} = \left(\sum_{Q \in \mathcal{W}_1 \cup \mathcal{W}_4} \left(\int_{\frac{\ell(Q)}{10}}^{\ell_0} \left(\sum_{L \in \mathcal{W}_1(Q,\rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^\sigma}{t^{\sigma+d}} \right)^q \frac{dt}{t} \right)^{\frac{p}{q}} \ell(Q)^d \right)^{\frac{1}{p}} \lesssim \|h\|_{L^p(\Omega)}.$$

By Jensen's inequality, for $q < \infty$, $\xi \in L \in \mathcal{W}_0$ we get

$$h(\xi) = \sum_{P \ni \xi} \ell(P)^{-\sigma} |f(\xi) - f_{5P}| \lesssim_d \sum_{P \ni \xi} \fint_{5P} |f(\xi) - f(\zeta)| \int_{|\xi - \zeta|}^{C\ell(L)} \frac{dt}{t^{\sigma+d+\frac{1}{q}}} \, d\zeta \, \ell(L)^{d+\frac{1}{q}-1},$$

where C is an appropriate dimensional constant. Reordering and applying Jensen's inequality we get

$$h(\xi) \lesssim \fint_0^{C\ell(L)} \frac{\fint_{\Omega_{\xi,t}} |f(\xi) - f(\zeta)| \, d\zeta}{t^\sigma} \frac{dt}{t^{\frac{1}{q}}} \ell(L)^{\frac{1}{q}} \lesssim \left(\int_0^{C\ell_0} \left(\frac{\int_{\Omega_{\xi,t}} |f(\xi) - f(\zeta)| \, d\zeta}{t^{\sigma+d}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

In case $q = \infty$ then also using Fubini's theorem

$$h(\xi) = \ell(L)^{-\sigma} |f(\xi) - f_{5L}| \lesssim \sup_{t \in (0,C\ell_0)} \frac{\int_{\Omega_{\xi,t}} |f(\xi) - f(\zeta)| \, d\zeta}{t^{\sigma+d}}.$$

Therefore,

$$\|h\|_{L^p(\Omega)} \lesssim \|f\|_{\dot{F}^{\sigma,C\ell_0}_{p,q,1}(\Omega)},$$

and the lemma follows.

$\square$

Lemma 4.9. *Let $d \geqslant 1$ be a natural number, let $0 < \sigma < 1$, let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and ℓ_0 small enough. For every constant $\rho > 0$ there exists a constant c_2 such that for every $h \in L^p(\Omega)$,*

$$\left(\sum_{Q \in \mathcal{W}_1 \cup \mathcal{W}_4} \left(\int_{\frac{\ell(Q)}{10}}^{\ell_0} \left(\sum_{L \in \mathcal{W}_1(Q,\rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^\sigma}{t^{\sigma+d}} \right)^q \frac{dt}{t} \right)^{\frac{p}{q}} \ell(Q)^d \right)^{\frac{1}{p}} \leqslant C\|h\|_{L^p},$$

with the usual modifications when $q = \infty$.

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Proof. We write

$$\boxed{\mathbb{B}} := \left(\sum_{Q \in \mathcal{W}_1} \left(\int_{\frac{\ell(Q)}{10}}^{\ell_0} \left(\sum_{L \in \mathcal{W}_1(Q, \rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^\sigma}{t^{\sigma+d}} \right)^q \frac{dt}{t} \right)^{\frac{p}{q}} \ell(Q)^d \right)^{\frac{1}{p}},$$

with the usual modification for $q = \infty$.

First of all let us assume that $1 = p \leqslant q < \infty$. Let us note the following: for $Q \in \mathcal{W}_1$, $t \in \left(\frac{\ell(Q)}{10}, \ell_0 \right)$ and $L \in \mathcal{W}_1(Q, \rho t)$ it follows that $\rho t \gtrsim \mathrm{D}(Q, L)$. Thus, using Minkowski's inequality we get

$$\boxed{\mathbb{B}} \leqslant \sum_{Q \in \mathcal{W}_1} \sum_{L \in \mathcal{W}_1} \|h\|_{L^1(L)} \ell(L)^\sigma \left(\int_{\frac{\mathrm{D}(Q,L)}{C\rho}}^{\ell_0} \frac{dt}{t^{\sigma q + dq + 1}} \right)^{\frac{1}{q}} \ell(Q)^d.$$

Computing, using Hölder's inequality and (3.9) we get

$$\boxed{\mathbb{B}} \lesssim_\rho \sum_{L \in \mathcal{W}_1} \|h\|_{L^1(L)} \ell(L)^\sigma \sum_{Q \in \mathcal{W}_1} \frac{\ell(Q)^d}{\mathrm{D}(Q,L)^{\sigma+d}} \lesssim \sum_{L \in \mathcal{W}_1} \|h\|_{L^1(L)} \ell(L)^{\sigma-\sigma} = \|h\|_{L^1}.$$

If $p = 1$ and $q = \infty$, then

$$\boxed{\mathbb{B}}_\infty := \sum_{Q \in \mathcal{W}_1} \sup_{t \in \left(\frac{\ell(Q)}{10}, \ell_0 \right)} \sum_{L \in \mathcal{W}_1(Q, \rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^\sigma}{t^{\sigma+d}} \ell(Q)^d.$$

Since the supremum of a series is bounded by the series of supremums, we get

$$\boxed{\mathbb{B}}_\infty \leqslant \sum_{Q \in \mathcal{W}_1} \sum_{L \in \mathcal{W}_1} \|h\|_{L^1(L)} \ell(L)^\sigma \ell(Q)^d \sup_{t \in \left(\frac{\mathrm{D}(Q,L)}{C\rho}, \ell_0 \right)} \frac{1}{t^{\sigma+d}} \leqslant \sum_{L \in \mathcal{W}_1} \|h\|_{L^1(L)} \ell(L)^\sigma \sum_{Q \in \mathcal{W}_1} \frac{\ell(Q)^d}{\mathrm{D}(Q,L)^{\sigma+d}},$$

and the lemma follows as in the preceding case.

Next we focus on the case $1 < p$ and $q < \infty$. Consider

$$f(x, t, y) = \sum_{Q \in \mathcal{W}_1} \sum_{L \in \mathcal{W}_1(Q, \rho t)} \chi_Q(x) \chi_{\left(\frac{\ell(Q)}{10}, \ell_0 \right)}(t) \chi_L(y) \|h\|_{L^1(L)} \frac{\ell(L)^{\sigma-d}}{t^{\sigma + \frac{d}{q} + \frac{1}{q}}}.$$

Then by duality we get

$$\|f\|_{L_x^p(L_t^q(L_y^1))} = \sup_{\|g\|_{L_x^{p'}\left(L_t^{q'}\left(L_y^\infty \right) \right)} \leqslant 1} \int_\Omega \int_0^\infty \int_\Omega f(x, t, y) g(x, t, y) \, dy \, dt \, dx.$$

Thus, it is enough to bound

$$\boxed{\mathbb{B}}_g := \sum_{Q \in \mathcal{W}_1} \int_Q \int_{\frac{\ell(Q)}{10}}^{\ell_0} \sum_{L \in \mathcal{W}_1(Q, \rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^{\sigma-d}}{t^{\sigma+d+\frac{1}{q}}} \int_L g(x, t, y) \, dy \, dt \, dx$$

for every given function g such that $\|g\|_{L_x^{p'}\left(L_t^{q'}\left(L_y^\infty \right) \right)} \leqslant 1$.

We will use duality and the boundedness of the maximal operator in the Lebesgue spaces. In particular we will use an extra index r to be fixed later on in order to gain the necessary room for the boundedness of the maximal operator. In the sum above, for every Q and L appearing in

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the sum we consider a chain $[Q, L]$ with central cube $R = Q_L$. If ℓ_0 is small enough, we can grant that $\text{dist}(R, \partial\Omega) << \delta$ and in particular (3.9) and (3.10) hold. We define $\mathcal{W}_1'$ to be the cubes in $\mathcal{W}_0$ that satisfy these properties. Note that $\ell(R) \approx \text{D}(Q, R) \leqslant C\text{D}(Q, L) \leqslant C\rho t$. Then, using (3.4) and reordering we get

$$\boxed{\mathbb{B}}_g \leqslant \sum_{Q \in \mathcal{W}_1} \int_Q \int_{\frac{\ell(Q)}{10}}^{\ell_0} \sum_{\substack{R \in \mathcal{W}_1'(Q, C\rho t) \\ Q \in \mathbf{SH}(R)}} \sum_{L \in \mathbf{SH}(R)} \|h\|_{L^r(L)} \frac{\ell(L)^{\sigma - \frac{d}{r}}}{t^{\sigma + d + \frac{1}{q}}} \int_L g(x, t, y) \, dy \, dt \, dx \qquad (4.27)$$

$$\leqslant \sum_{R \in \mathcal{W}_1'} \sum_{Q \in \mathbf{SH}(R)} \int_Q \int_{C\rho^{-1}\ell(R)}^{\ell_0} \sum_{L \in \mathbf{SH}(R)} \|h\|_{L^r(L)} \frac{\ell(L)^{\sigma - \frac{d}{r}}}{t^{\sigma + d + \frac{1}{q}}} \int_L g(x, t, y) \, dy \, dt \, dx$$

Let $1 < r < p$. We apply the Hölder inequality to get

$$\boxed{\mathbb{B}}_g \leqslant \sum_{R \in \mathcal{W}_1'} \sum_{Q \in \mathbf{SH}(R)} \int_Q \int_{C\rho^{-1}\ell(R)}^{\ell_0} \frac{\sup_{\mathbf{SH}(R)} g(x, t, \cdot) \, dt}{t^{\sigma + d + \frac{1}{q}}} dx$$
$$\left(\sum_{L \in \mathbf{SH}(R)} \|h\|_{L^r(L)}^r \right)^{\frac{1}{r}} \left(\sum_{L \in \mathbf{SH}(R)} \left(\ell(L)^{\sigma + d - \frac{d}{r}} \right)^{r'} \right)^{\frac{1}{r'}}.$$

Let us denote

$$\|g(x, t, \cdot)\|_{L^\infty(\Omega)} =: G_x(t).$$

By (3.10) we get

$$\sum_{L \in \mathbf{SH}(R)} \|h\|_{L^r(L)}^r \leqslant \ell(R)^d \inf_{\zeta \in R} M\left(|h|^r\right)(\zeta).$$

Finally, by (3.9) we get

$$\left(\sum_{L \in \mathbf{SH}(R)} \left(\ell(L)^{\sigma + d - \frac{d}{r}} \right)^{r'} \right)^{\frac{1}{r'}} = \left(\sum_{L \in \mathbf{SH}(R)} \left(\ell(L)^{\sigma r' + d} \right) \right)^{\frac{1}{r'}} \lesssim \ell(R)^{\sigma + \frac{d}{r'}}.$$

All together, we have gotten

$$\boxed{\mathbb{B}}_g \lesssim \sum_{R \in \mathcal{W}_1'} \sum_{Q \in \mathbf{SH}(R)} \int_Q \int_{C\rho^{-1}\ell(R)}^{\ell_0} G_x(t) \frac{dt}{t^{\sigma + d + \frac{1}{q}}} dx \left(\ell(R)^d \inf_{\zeta \in R} M\left(|h|^r\right)(\zeta) \right)^{\frac{1}{r}} \ell(R)^{\sigma + \frac{d}{r'}}.$$

By Hölder's inequality

$$\int_{C\rho^{-1}\ell(R)}^{\ell_0} G_x(t) \frac{dt}{t^{\sigma + d + \frac{1}{q}}} \leqslant \|G_x\|_{L^{q'}} \left(\int_{C\rho^{-1}\ell(R)}^{\ell_0} \frac{dt}{t^{\sigma q + dq + 1}} \right)^{\frac{1}{q}} \lesssim \|G_x\|_{L^{q'}} (\rho\ell(R))^{-\sigma - d},$$

so writing $G(x) := \|G_x\|_{L^{q'}}$ we get

$$\boxed{\mathbb{B}}_g \lesssim_\rho \sum_{R \in \mathcal{W}_1'} \sum_{Q \in \mathbf{SH}(R)} \int_Q G(x) dx \, \ell(R)^{-\sigma - d} \left(\inf_{\zeta \in R} M\left(|h|^r\right)(\zeta) \right)^{\frac{1}{r}} \ell(R)^{\sigma + d}.$$

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Using (3.10) again we get that $\sum_{Q\in\mathbf{SH}(R)}\int_Q G(x)\,dx \lesssim \int_R MG(\zeta)\,d\zeta$ and, computing we get

$$\boxed{\mathbb{B}}_g \lesssim_\rho \sum_{R\in\mathcal{W}_1'} \int_R MG(\zeta)\left(M\left(|h|^r\right)(\zeta)\right)^{\frac{1}{r}} d\zeta \leqslant \|MG\|_{L^{p'}(\Omega)} \|M\left(|h|^r\right)\|_{L^{\frac{p}{r}}(\Omega)}^{\frac{1}{r}}$$

$$\lesssim \|G\|_{L^{p'}(\Omega)} \||h|^r\|_{L^{\frac{p}{r}}(\Omega)}^{\frac{1}{r}} = \|g\|_{L_x^{p'}\left(L_t^{q'}\left(L_y^\infty\right)\right)} \|h\|_{L^p(\Omega)} \leqslant \|h\|_{L^p(\Omega)}.$$

When $1 < p < \infty$ and $q = \infty$ we can perform a similar reasoning avoiding the duality expression:

$$\boxed{\mathbb{B}}_\infty := \left(\sum_{Q\in\mathcal{W}_1} \left(\sup_{t\in\left(\frac{\ell(Q)}{10},\ell_0\right)} \sum_{L\in\mathcal{W}_1(Q,\rho t)} \|h\|_{L^1(L)} \frac{\ell(L)^\sigma}{t^{\sigma+d}}\right)^p \ell(Q)^d\right)^{\frac{1}{p}}$$

$$\leqslant \left(\sum_{Q\in\mathcal{W}_1} \sup_{t\in\left(\frac{\ell(Q)}{10},\ell_0\right)} \left(\sum_{\substack{R\in\mathcal{W}_1'(Q,C\rho t)\\Q\in\mathbf{SH}(R)}} \sum_{L\in\mathbf{SH}(R)} \|h\|_{L^r(L)} \frac{\ell(L)^{\sigma+\frac{d}{r'}}}{t^{\sigma+d}}\right)^p \ell(Q)^d\right)^{\frac{1}{p}}$$

$$\leqslant \left(\sum_{Q\in\mathcal{W}_1} \sup_{t\in\left(\frac{\ell(Q)}{10},\ell_0\right)} \left(\sum_{\substack{R\in\mathcal{W}_1'(Q,C\rho t)\\Q\in\mathbf{SH}(R)}} \|h\|_{L^r(\mathbf{SH}(R))} \frac{\ell(R)^{\sigma+\frac{d}{r'}}}{t^{\sigma+d}}\right)^p \ell(Q)^d\right)^{\frac{1}{p}}.$$

Next we use that $Q \in \mathbf{SH}(R)$ implies that $\|h\|_{L^r(\mathbf{SH}(R))} \lesssim \left(\ell(R)^d \inf_Q M(|h|^r)\right)^{\frac{1}{r}}$, so

$$\boxed{\mathbb{B}}_\infty \leqslant \left(\sum_{Q\in\mathcal{W}_1} \int_Q M(|h|^r)(x)^{\frac{p}{r}} dx \sup_{t\in\left(\frac{\ell(Q)}{10},\ell_0\right)} \left(\sum_{\substack{R\in\mathcal{W}_1'(Q,C\rho t)\\Q\in\mathbf{SH}(R)}} \frac{\ell(R)^{\sigma+\frac{d}{r'}+\frac{d}{r}}}{t^{\sigma+d}}\right)^p\right)^{\frac{1}{p}}.$$

Using (3.5) we get

$$\boxed{\mathbb{B}}_\infty \lesssim_\rho \left(\sum_{Q\in\mathcal{W}_1} \int_Q M(|h|^r)(x)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \lesssim \|h\|_{L^p}.$$

On the other hand, for $Q \in \mathcal{W}_4$, we can just use the finite overlapping of symmetrized cubes to reduce it to the previous situation. $\qquad\square$

Acknowledgements

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