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Energy Function of 2D and 3D Dynamical Systems

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It is far well-known that *energy function* of a two-dimensional autonomous dynamical system can be simply obtained by multiplying its corresponding second-order ordinary differential equation, i.e., its *equation of motion* by the first time derivative of its *state variable*. In the nineties, one of us (J.C.S.) stated that a three-dimensional autonomous dynamical system can be also transformed into a third-order ordinary differential *equation of motion* todays known as *jerk equation*. Although a method has been developed during these last decades to provide the *energy function* of such three-dimensional autonomous dynamical systems, the question arose to determine by which type of term, i.e., by the first or second time derivative of their *state variable*, the corresponding *jerk equation* of these systems should be multiplied to deduce their *energy function*. We prove in this work that the *jerk equation* of such systems must be multiplied by the second time derivative of the *state variable* and not by the first like in dimension two. We then provide an interpretation of the new term appearing in the *energy function* and called *jerk energy*. We thus established that it is possible to obtain the *energy function* of a three-dimensional dynamical system directly from its corresponding *jerk equation*. Two and three-dimensional Van der Pol models are then used to exemplify these main results. Applications to Lorenz and Chua's models confirms their validity.

I. INTRODUCTION

Dynamical systems are generally used for modeling the dynamics of a given physical, mechanical, electrical, biological, economical system from the point of view of a deterministic process which is expressed in terms of *state* variables. As an example, let's consider the classical *damped pendulum* also called *damped harmonic oscillator* using the small amplitude approximation, i.e. when the *state variable* $\theta(t)$ represents here the angle from the vertical to the pendulum as a function of time is such that $\theta \ll 1$ and so $\sin\theta \simeq \theta$. Then, starting from Newton's second law, the linear second-order ordinary differential equation of its motion reads:

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\theta = 0 \quad \Leftrightarrow \quad \ddot{\theta} + \gamma\dot{\theta} + \omega^2\theta = 0 \tag{1}$$

where γ is the damping coefficient and ω the angular frequency. This *equation of motion* can be also written like this:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\gamma y - \omega^2 x, \end{aligned} \tag{2}$$

where $x(t) = \theta(t)$ and $y(t) = \dot{\theta}(t)$. In the middle of the seventeenth century, French mathematician Jean le Rond de d'Alembert [5] stated that any n^{th}-order ordinary differential equation can be transformed into a set of n simultaneous first-order ordinary differential equation, i.e., into a *velocity vector field*, like in the above example. Then, a *classical method* for obtaining the *energy function* consists in multiplying the *equation of motion* (1), that we will call in what follows, *acceleration equation*, by the time derivative of the *state variable*, i.e., by $\dot{\theta}(t)$. Thus, we have:

$$\dot{\theta}\ddot{\theta} + \gamma\dot{\theta}^2 + \omega^2\theta\dot{\theta} = 0. \tag{3}$$

By considering that $\dot{\theta}\ddot{\theta} = \dfrac{d}{dt}\left(\dfrac{\dot{\theta}^2}{2}\right)$ and $\theta\dot{\theta} = \dfrac{d}{dt}\left(\dfrac{\theta^2}{2}\right)$, we find that:

$$\frac{d}{dt}\left[\frac{\dot{\theta}^2}{2} + \omega^2\frac{\theta^2}{2}\right] = -\gamma\dot{\theta}^2, \tag{4}$$

where $\frac{\dot{\theta}^2}{2}$ and $\omega^2\frac{\theta^2}{2}$ represents respectively the *kinetic* and *potential* energy of the damped pendulum. Thus, its *energy function* H reads:

$$H = \frac{\dot{\theta}^2}{2} + \omega^2\frac{\theta^2}{2} \tag{5}$$

So, we have:

$$\frac{dH}{dt} = -\gamma\dot{\theta}^2, \tag{6}$$

where $-\gamma\dot{\theta}^2$ represents the *energy dissipation* or rate of change of this energy due to the damping.

In dimension three, the problem of determining the *equation of motion* or *jerk equation* of any autonomous dynamical system starting from its set of three first-order ordinary differential equations or *velocity vector field* is not trivial nor guaranteed. That's the reason why many methods have been proposed these three last decades by Gottlieb [10], Sprott [17], Linz [12], Eichhorn *et al.* [7], Sprott [18–23], Buscarino *et al.* [2] and more recently by Xu *et al.* [25] to provide the third-order ordinary differential equation, i.e., *jerk equation* of three-dimensional autonomous dynamical systems. Another problem then arose, that of determining by which type of term, i.e., by the first or second time derivative of their *state variable*, the corresponding *jerk equation* of these systems should be multiplied to deduce their *energy function*. To overcome this difficulty, Sarasola *et al.* [16], Wang *et al.* [28], Ma *et al.* [14] and Yu *et al.* [27] to name but a few have developed a method based on the Helmholtz's theorem which enables to deduce the *energy function* of any n-dimensional autonomous dynamical systems and more precisely that of chaotic dynamical systems such as Lorenz, Rossler and Chua's models [3, 13, 15]. Of course, Helmholtz's theorem can be applied to two-dimensional autonomous dynamical systems. Nevertheless, in this case, it led us to an expression of an *energy function* which seems different from the one obtained with the *classical method* presented above. In fact, we prove in this work that they are actually identical. By using Helmholtz's theorem, Sarasola *et al.* [16] and Wang *et al.* [28] have provided the *energy function* of three-dimensional autonomous dynamical systems such as Lorenz and Chua's models. Then, starting from the *jerk equation* of these models given by Ginoux *et al.* [9], Buscarino *et al.* [2] and Xu *et al.* [25], we multiplied their respective *jerk equations* by the second time derivative of their *state variable* and thus deduced their *energy functions*. Finally, by comparing these *energy functions* with those obtained by Sarasola *et al.* [16] and Wang *et al.* [28], we proved that the *jerk equation* of three-dimensional autonomous dynamical systems must be multiplied by the second time derivative of the *state variable*, i.e., by the *acceleration* and not by its first one, i.e., by the *velocity* like in dimension two.

The paper is organized as follows. In the next section 2, we briefly recall definitions and expressions of two and three-dimensional autonomous dynamical systems. Hemholtz's theorem providing the *energy function* is summarized in Section 3. Then, in section 4, we first prove that *energy function* of two-dimensional autonomous dynamical systems given either by the *classical method* or while using Helmholtz's theorem are identical. Then, we prove that the *jerk equation* of the corresponding version in dimension three of the famous dissipative Van der Pol system [24] must be multiplied by the second time derivative of the *state variable* and not by its first one. In Section 5, these new results will be applied to Lorenz and Chua's models, confirming thus their validity. Interpretations of these results and this new term in the *energy function* called *jerk energy* as well as perspectives to be given to this work are presented in the discussion.

II. DYNAMICAL SYSTEMS

A. Dimension two

Two-dimensional autonomous dynamical systems are generally represented by a set of two first-order ordinary differential equations (ODE) expressing the time evolution of its *state* variables (x, y) as follows:

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \end{aligned} \tag{7}$$

where f and g are supposed to be continuous and infinitely differentiable with respect to x, y and t, i.e. are C^∞ functions in $E \subset \mathbb{R}^2$ and with values included in $\mathbb{R}$, satisfy the assumptions of the Cauchy-Lipschitz theorem. For more details, see for example Coddington & Levinson [4]. However, such a dynamical system (7) can be also represented by a single second-order ODE that we called for consistency *acceleration equation* and which reads:

$$\ddot{x} = F\left(x, \dot{x}\right) \tag{8}$$

Let's notice that in dimension two, the transformation from the autonomous dynamical system (7) to its corresponding *acceleration equation* (8) is generally easy although not always possible.

B. Dimension three

Three-dimensional autonomous dynamical systems are generally represented by a set of three first-order ordinary differential equations (ODE) expressing the time evolution of its *state* variables (x, y, z) as follows:

$$\begin{aligned} \frac{dx}{dt} &= f\left(x, y, z\right), \\ \frac{dy}{dt} &= g\left(x, y, z\right), \\ \frac{dz}{dt} &= h\left(x, y, z\right), \end{aligned} \tag{9}$$

where f, g and h are supposed to be continuous and infinitely differentiable with respect to x, y, z and t, i.e. are C^∞ functions in $E \subset \mathbb{R}^3$ and with values included in $\mathbb{R}$, satisfy the assumptions of the Cauchy-Lipschitz theorem. For more details, see for example Coddington & Levinson [4]. However, such a dynamical system (9) can be also represented by a single third-order ODE called *jerk equation* and which reads:

$$\dddot{x} = F\left(x, \dot{x}, \ddot{x}\right) \tag{10}$$

In dimension three, the transformation from the autonomous dynamical system (9) into its corresponding *jerk equation* (10) is generally not trivial and not always possible. That's the reason why many methods have been proposed these three last decades by Gottlieb [10], Sprott [17], Linz [12], Eichhorn *et al.* [7], Sprott [18–23], Buscarino *et al.* [2] and more recently by Xu *et al.* [25] in an attempt to provide the single third-order ODE or *jerk equation* (10) of any three-dimensional autonomous dynamical system (9).

III. ENERGY FUNCTION OF DYNAMICAL SYSTEMS

A. Dimension Two

In dimension two, a *classical method* used to deduce the *energy function* of any two-dimensional autonomous dynamical system (7) consists in multiplying its second-order differential equation or *acceleration equation* (8) by the time derivative of its *state variable* namely by $\dot{x}$. This process enables to disclose on the left hand side of this equation a term analogous to the time derivative of a *kinetic energy*:

$$\frac{d}{dt}\left[\frac{\dot{x}^2}{2}\right] = \dot{x}F\left(x, \dot{x}\right) \tag{11}$$

and, on the right hand side a term analogous to the time derivative of a *potential energy*. The famous dissipative Van der Pol system [24] will exemplify this classical method in the next Section 4.

B. Dimension Three

In dimension three, Sarasola *et al.* [16], Wang *et al.* [28], Ma *et al.* [14] and Yu *et al.* [27] to name but a few have developed a method based on the Helmholtz's theorem which enables to deduce the *energy function* of any n-dimensional autonomous dynamical systems (9). This method is briefly recalled below. A n-dimensional autonomous dynamical system can be written as follows:

$$\frac{d\vec{X}}{dt} = \overrightarrow{\Im}(\vec{X}), \tag{12}$$

where $\vec{X} = \{x_1, x_2, ..., x_n\}$ and $\overrightarrow{\Im}(\vec{X}) = \{f_1(\vec{X}), f_2(\vec{X}), ..., f_n(\vec{X})\}$. According to Helmholtz's theorem, such dynamical system (12) can be decomposed into gradient and rotational fields, i.e., as the sum of conservative and dissipative vector fields $\overrightarrow{\Im}(\vec{X}) = \overrightarrow{\Im}_c(\vec{X}) + \overrightarrow{\Im}_d(\vec{X})$. Thus, it can be expressed in a generalized Hamiltonian form:

$$\frac{d\vec{X}}{dt} = \left[J(\vec{X}) + R(\vec{X})\right] \nabla H(\vec{X}), \tag{13}$$

where ∇H is the gradient vector of smooth *energy function* $H(\vec{X})$, $J(\vec{X})$ is a skew-symmetric matrix and $R(\vec{X})$ is a symmetric matrix. The Hamiltonian *energy function* can thus be calculated by:

$$\begin{aligned} \frac{dH}{dt} &= \nabla H^T R(\vec{X}) \nabla H, \\ \nabla H^T J(\vec{X}) \nabla H &= 0, \end{aligned} \tag{14}$$

Since the vector field (12) can be decomposed into gradient and rotational fields, it follows that the *energy function* can be deduced by:

$$\begin{aligned} \frac{dH}{dt} &= \nabla H^T R(\vec{X}) \nabla H = \nabla H^T \overrightarrow{\Im}_d(\vec{X}), \\ \nabla H^T J(\vec{X}) \nabla H &= 0 = \nabla H^T \overrightarrow{\Im}_c(\vec{X}), \end{aligned} \tag{15}$$

In the next Section 4, this Helmholtz's theorem will be applied to two-dimensional Van der Pol system [24] and its corresponding version in dimension three.

IV. APPLICATION TO TWO AND THREE-DIMENSIONAL VAN DER POL SYSTEMS

A. Two-dimensional Van der Pol system

Van der Pol system [24] can be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha \left(y - f\left(x\right)\right), \\ \frac{dy}{dt} &= -x, \end{aligned} \tag{16}$$

where $f(x) = x^3/3 - x$ is a cubic function and α a real positive parameter. By using Helmholtz's theorem recalled in the previous Section 3, Van der Pol system (16) can be rewritten as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\alpha f(x) \\ 0 \end{pmatrix} \tag{17}$$

Then, according to Eq. (15), the *energy function* H associated with the Van der Pol system (16) must satisfy the partial differential equation:

$$\left(\frac{\partial H}{\partial x},\ \frac{\partial H}{\partial y}\right)\begin{pmatrix}\alpha y\\ -x\end{pmatrix}=0 \quad \Leftrightarrow \quad \begin{cases}\dfrac{\partial H}{\partial x}=x,\\ \dfrac{\partial H}{\partial y}=\alpha y,\end{cases} \tag{18}$$

This leads to the *energy function*:

$$H=\frac{x^2}{2}+\alpha\frac{y^2}{2} \tag{19}$$

Thus, according to Eq. (15), the rate of change of this energy reads:

$$\frac{dH}{dt}=-\alpha x f(x). \tag{20}$$

Let's notice that this result can be also obtained by taking the time derivative of the above equation (19) and by replacing the time derivative of each *state* variable by their values given by Van der Pol system (16). Now, let's use the *classical method* to deduce the *energy function*. First of all, it's easy to transform the Van der Pol system (16) into a single second-order nonlinear ordinary differential equation or *acceleration equation* which reads:

$$\ddot{x}+\alpha x=-\alpha\dot{f}(x). \tag{21}$$

Then, by using the *classical method*, i.e., by multiplying this *acceleration equation* (21) by the time derivative of its *state variable* namely by $\dot{x}$ we obtain:

$$\dot{x}\ddot{x}+\alpha x\dot{x}=-\alpha\dot{x}\dot{f}(x). \tag{22}$$

It can be rewritten as follows:

$$\frac{d}{dt}\left[\frac{\dot{x}^2}{2}+\alpha\frac{x^2}{2}\right]=-\alpha\dot{x}\dot{f}(x). \tag{23}$$

And so, the *energy function* and its rate of change read:

$$H=\frac{\dot{x}^2}{2}+\alpha\frac{x^2}{2} \quad \Leftrightarrow \quad \frac{dH}{dt}=-\alpha\dot{x}\dot{f}(x). \tag{24}$$

With such a formulation (24), we find again the expressions of the *kinetic* energy ($\dot{x}^2/2$) and *potential* energy ($\alpha x^2/2$). Nevertheless, at first sight, these Eqs. (24) seem to be completely different from Eqs. (19-20). Let's notice that Eqs. (24) can be directly obtained from the time derivative of the original Van der Pol system (16) which reads:

$$\begin{aligned}\ddot{x}&=\alpha\left(\dot{y}-\dot{f}\left(x\right)\right),\\ \ddot{y}&=-\dot{x},\end{aligned} \tag{25}$$

Moreover, by replacing in the first Eq. (24) $\dot{x}$ by $\alpha(y-f(x))$, i.e by the right hand side of the first equation of (16), it's easy to prove that Eqs. (24) are in fact identical to Eqs. (19-20).

B. Three-dimensional Van der Pol system

In 1989, Grasman and Roerdink [11] proposed a three-dimensional version of Van der Pol original system [24]. Starting from their seminal works, we designed the following simple three-dimensional autonomous Van der Pol model by adding a third *state variable*:

$$\begin{aligned}\frac{dx}{dt} &= \alpha\left(y - f\left(x\right)\right),\\ \frac{dy}{dt} &= -x + z,\\ \frac{dz}{dt} &= -\beta y,\end{aligned} \tag{26}$$

where $f(x)$, α are defined as above. Let's notice that when β and z tend to zero, we find again Van der Pol original system (16). This simple three-dimensional autonomous Van der Pol model (26) will be used below to prove that its *jerk equation* must be multiplied by the second time derivative of the *state* variable $x(t)$ and not by the first to deduce the *energy function* of this model. Then, by using Helmholtz's theorem recalled in the previous Section 3, Van der Pol model (26) can be rewritten as:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}0 & \alpha & 0\\ -1 & 0 & 1\\ 0 & -\beta & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}-\alpha f(x)\\ 0\\ 0\end{pmatrix} \tag{27}$$

Thus, according to Eq. (15), the *energy function* H associated with the Van der Pol model (26) must satisfy the partial differential equation:

$$\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right)\begin{pmatrix}\alpha y\\ -x + z\\ -\beta y\end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases}\dfrac{\partial H}{\partial x} = x,\\ \dfrac{\partial H}{\partial y} = \alpha y,\\ \dfrac{\partial H}{\partial z} = \dfrac{\alpha}{\beta} z,\end{cases} \tag{28}$$

This leads to the *energy function*:

$$H = \frac{x^2}{2} + \alpha\frac{y^2}{2} + \frac{\alpha}{\beta}\frac{z^2}{2}. \tag{29}$$

So, according to Eq. (15), the rate of change of this energy reads:

$$\frac{dH}{dt} = -\alpha x f(x). \tag{30}$$

As previously, let's notice that this result can be also obtained by taking the time derivative of the above equation (29) and by replacing the time derivative of each *state* variable by their values given by Van der Pol model (26). Now, let's express the time derivative of this Van der Pol model (26):

$$\begin{aligned}\ddot{x} &= \alpha\left(\dot{y} - \dot{f}\left(x\right)\right),\\ \ddot{y} &= -\dot{x} + \dot{z},\\ \ddot{z} &= -\beta\dot{y},\end{aligned} \tag{31}$$

the *energy function* and its rate of change associated with this Van der Pol system (31) read:

$$H = \frac{\dot{x}^2}{2} + \alpha\frac{\dot{y}^2}{2} + \frac{\alpha}{\beta}\frac{\dot{z}^2}{2} \quad \Leftrightarrow \quad \frac{dH}{dt} = -\alpha\dot{x}\dot{f}(x). \tag{32}$$

Then, by replacing the time derivative of each *state* variable by their values given by Van der Pol system (26), it's easy to prove that Eqs. (32) are in fact identical to Eqs. (29-30). Now, by using one of the methods recalled in the

first section or by using linear combinations of its *state* variables, let's transform the Van der Pol system (26) into a single third-order nonlinear ordinary differential equation or *jerk equation*, we obtain:

$$\dddot{x} + \alpha\ddot{f}(x) + (\alpha+\beta)\dot{x} + \alpha\beta f(x) = 0. \tag{33}$$

Let's notice that when β tends to zero, we find again the *acceleration equation* of Van der Pol original system (21). Then, let's multiply this *jerk equation* (33) by the second time derivative of its *state* variable namely by $\ddot{x}$. This leads to:

$$\frac{d}{dt}\left[\frac{\ddot{x}^2}{2} + (\alpha+\beta)\frac{\dot{x}^2}{2}\right] = -\alpha\ddot{x}\ddot{f}(x) - \alpha\beta\ddot{x}f(x) \tag{34}$$

While the meaning of the second term of the left hand side of this expression (34), i.e., $(\alpha+\beta)\dot{x}^2/2$ is well-known since it corresponds to the *kinetic energy*, it does not seem to be the case of the first one the interpretation of which will be provided in the discussion. Then, by replacing in Eq. (34) $\dot{x}$ by $\alpha(y-f(x))$, i.e., by the right hand side of the first equation of (26) and by its time derivative, it's easy to prove that Eq. (34) is in fact identical to Eqs. (32) and also to Eqs. (29-30). This proves that the *jerk equation* (33) must be multiplied by $\ddot{x}(t)$ and not by $\dot{x}(t)$. To confirm such a result, let's multiply the *jerk equation* (33) by the first derivative of the *state* variable, i.e., by $\dot{x}$. We obtain:

$$\dot{x}\dddot{x} + \alpha\dot{x}\ddot{f}(x) + (\alpha+\beta)\dot{x}^2 + \alpha\beta\dot{x}f(x) = 0. \tag{35}$$

The first and third terms of the left hand side of Eq. (35) can be rewritten as:

$$\begin{aligned} \dot{x}\dddot{x} &= \frac{d}{dt}\left[\dot{x}\ddot{x}\right] - \ddot{x}^2, \\ \dot{x}^2 &= \frac{d}{dt}\left[x\dot{x}\right] - x\ddot{x}, \end{aligned} \tag{36}$$

These expressions (36) are not consistent with a *kinetic* and *potential* energy by analogy with *Classical Mechanics*. Moreover, by making several linear combinations of the time derivatives of the *state* variable, it has not been possible to find again the *energy function* (32) or (29).

V. APPLICATION TO LORENZ AND CHUA'S MODELS

A. Chua's model

Chua's continuous model [3] can be written in its original form as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha\left(y - f\left(x\right)\right), \\ \frac{dy}{dt} &= x - y + z, \\ \frac{dz}{dt} &= -\beta y, \end{aligned} \tag{37}$$

where $f(x) = x^3 + cx$ is a cubic function and α and β are real positive parameters. Then, following the works of Sarasola *et al.* [16] and by using Helmholtz's theorem recalled in the previous Section 3, Chua's model (37) can be rewritten as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & 0 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\alpha f(x) \\ -y \\ 0 \end{pmatrix} \tag{38}$$

Thus, according to Eq. (15), the *energy function* H associated with the Chua's model (37) must satisfy the partial differential equation:

$$\left(\frac{\partial H}{\partial x},\ \frac{\partial H}{\partial y},\ \frac{\partial H}{\partial z}\right)\begin{pmatrix}\alpha y\\ x+z\\ -\beta y\end{pmatrix}=0 \quad \Leftrightarrow \quad \begin{cases}\dfrac{\partial H}{\partial x}=-\dfrac{x}{\alpha},\\ \dfrac{\partial H}{\partial y}=y,\\ \dfrac{\partial H}{\partial z}=\dfrac{z}{\beta},\end{cases} \tag{39}$$

This leads to the *energy function*:

$$H=-\frac{x^2}{2\alpha}+\frac{y^2}{2}+\frac{z^2}{2\beta}. \tag{40}$$

So, according to Eq. (15), the rate of change of this energy reads:

$$\frac{dH}{dt}=xf(x)-y^2. \tag{41}$$

Now, let's express the time derivative of this Chua's model (37):

$$\begin{aligned}\ddot{x}&=\alpha\left(\dot{y}-\dot{f}\left(x\right)\right),\\ \ddot{y}&=\dot{x}-\dot{y}+\dot{z},\\ \ddot{z}&=-\beta\dot{y},\end{aligned} \tag{42}$$

the *energy function* and its rate of change associated with this Chua's model (41) read:

$$H=-\frac{\dot{x}^2}{2\alpha}+\frac{\dot{y}^2}{2}+\frac{\dot{z}^2}{2\beta} \quad \Leftrightarrow \quad \frac{dH}{dt}=\dot{x}\dot{f}(x)-\dot{y}^2. \tag{43}$$

Then, by replacing the time derivative of each *state* variable by their values given by Chua's model (37), it's easy to prove that Eqs. (43) are in fact identical to Eqs. (40-41). Now, by using the works of Buscarino *et al.* [2], let's transform the Chua's model (37) into a single third-order nonlinear ordinary differential equation or *jerk equation*, we obtain:

$$\dddot{x}+(\beta-\alpha)\dot{x}=-\ddot{x}-\alpha\beta f(x)-\alpha\dot{f}(x)-\alpha\ddot{f}(x). \tag{44}$$

Then, let's multiply this *jerk equation* (44) by the second time derivative of its *state* variable namely by $\ddot{x}$. This leads to:

$$\frac{d}{dt}\left[\frac{\ddot{x}^2}{2}+(\beta-\alpha)\frac{\dot{x}^2}{2}\right]=-\ddot{x}^2-\alpha\beta\ddot{x}f(x)-\alpha\ddot{x}\dot{f}(x)-\alpha\ddot{x}\ddot{f}(x). \tag{45}$$

Then, by replacing in Eq. (45) $\dot{x}$ by $\alpha(y-f(x))$, i.e., by the right hand side of the first equation of (37) and by its time derivative, it's easy to prove that Eq. (45) is in fact identical to Eqs. (43) and also to Eqs. (40-41). This proves that the *jerk equation* (44) must be multiplied by $\ddot{x}(t)$ and not by $\dot{x}(t)$.

B. Lorenz model

Lorenz model [13] can be written in its original form as follows:

$$\begin{aligned}\frac{dx}{dt} &= \sigma\left(y-x\right),\\ \frac{dy}{dt} &= \rho x - y - xz,\\ \frac{dz}{dt} &= xy - \beta z,\end{aligned} \tag{46}$$

where σ, ρ and β are real positive parameters. Then, following the works of Sarasola *et al.* [16] and by using Helmholtz's theorem recalled in the previous Section 3, *energy function* of Lorenz model (46) reads:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}0 & \sigma & 0\\ \rho & 0 & -x\\ 0 & x & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}-\sigma x\\ -y\\ -\beta z\end{pmatrix} \tag{47}$$

Thus, according to Eq. (15), the *energy function* H associated with the Lorenz model (46) must satisfy the partial differential equation:

$$\left(\frac{\partial H}{\partial x},\ \frac{\partial H}{\partial y},\ \frac{\partial H}{\partial z}\right)\begin{pmatrix}\sigma y\\ \rho x - xz\\ xy\end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases}\dfrac{\partial H}{\partial x} = -\dfrac{\rho}{\sigma}x,\\ \dfrac{\partial H}{\partial y} = y,\\ \dfrac{\partial H}{\partial z} = z,\end{cases} \tag{48}$$

This leads to the *energy function*:

$$H = -\frac{\rho}{\sigma}\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}. \tag{49}$$

So, according to Eq. (15), the rate of change of this energy reads:

$$\frac{dH}{dt} = \rho x^2 - y^2 - \beta z^2. \tag{50}$$

Now, let's express the time derivative of this Lorenz model (46):

$$\begin{aligned}\ddot{x} &= \sigma\left(\dot{y}-\dot{x}\right),\\ \ddot{y} &= \rho\dot{x} - \dot{y} - \dot{x}z - x\dot{z},\\ \ddot{z} &= \dot{x}y + x\dot{y} - \beta\dot{y},\end{aligned} \tag{51}$$

the *energy function* and its rate of change associated with this Lorenz model (51) read:

$$H = -\frac{\rho}{\sigma}\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} \quad \Leftrightarrow \quad \frac{dH}{dt} = \rho\dot{x}^2 - \dot{y}^2 - \beta\dot{z}^2 + \dot{x}\left(y\dot{z} - z\dot{y}\right). \tag{52}$$

Then, by replacing the time derivative of each *state* variable by their values given by Lorenz model (47), it's easy to prove that Eqs. (52) are in fact identical to Eqs. (49-50). Now, by using the works of Ginoux *et al.* [9], let's transform the Lorenz model (46) into a single third-order nonlinear ordinary differential equation or *jerk equation*, we obtain:

$$\dddot{x} + \beta\left(\sigma+1\right)\dot{x} = -\left(\sigma+1+\beta\right)\ddot{x} + \left[\ddot{x} + \left(\sigma+1\right)\dot{x}\right]\frac{\dot{x}}{x} + \beta\sigma\left(\rho-1\right)x - x^2\left(\dot{x}+\sigma x\right). \tag{53}$$

Then, let's multiply this *jerk equation* (53) by the second time derivative of its *state* variable namely by $\ddot{x}$. This leads to:

$$\frac{d}{dt}\left[\frac{\ddot{x}^2}{2}+\beta\left(\sigma+1\right)\frac{\dot{x}^2}{2}\right]=-\left(\sigma+1+\beta\right)\ddot{x}^2+\left[\ddot{x}+\left(\sigma+1\right)\dot{x}\right]\frac{\dot{x}\ddot{x}}{x}+\beta\sigma\left(\rho-1\right)x\ddot{x}-x^2\left(\dot{x}+\sigma x\right)\ddot{x}. \tag{54}$$

Then, by replacing in Eq. (54) $\dot{x}$ by $\sigma(y-x)$, i.e., by the right hand side of the first equation of (46) and by its time derivative, it's easy to prove that Eq. (54) is in fact identical to Eqs. (52) and also to Eqs. (49-50). This proves that the *jerk equation* (53) must be multiplied by $\ddot{x}(t)$ and not by $\dot{x}(t)$.

VI. DISCUSSION

In this work, we first recalled the *classical method* to obtain the *energy function* of a two-dimensional autonomous dynamical system which consists in multiplying its corresponding second-order ordinary differential equation, i.e., its *equation of motion* that we called for consistency *acceleration equation*, by the first time derivative of its *state variable*. We then briefly presented the method based on Helmholtz's theorem to determine the *energy function* of three-dimensional autonomous dynamical systems. Application of this latter method to two-dimensional autonomous dynamical systems led us to an expression of the *energy function* which seemed to be different from the one obtained with the *classical method*. In fact, we proved in this work that they are identical. In the nineties, one of us (J.C.S.) established that a three-dimensional autonomous dynamical system can be also transformed into a third-order ordinary differential *equation of motion* todays known as *jerk equation*. Although the method based on Helmholtz's theorem enables to provide the *energy function* of such three-dimensional autonomous dynamical systems, the question arose to determine by which type of term, i.e., by the first or second time derivative of their *state variable*, the corresponding *jerk equation* of these systems should be multiplied to deduce their *energy function*. We proved in this work that the *jerk equation* of such systems must be multiplied by the second time derivative of the *state variable* and not by the first like in dimension two. By doing that we highlighted in the *energy function* a term proportional to $\dot{x}^2/2$ which corresponds to the *kinetic energy* and a second one equal to $\ddot{x}^2/2$. According to Desloge [6], this latter term has been introduced by Willard Gibbs in 1879 and twenty years later by Paul Appell. It has been called since the *Gibbs-Appell function* or *jerk energy* and represents the *acceleration energy*. In his notes at the *Comptes Rendus*, Appell [1] has also stated that it can be expressed as the square of Newton's second law of motion as recalled by Yong-fen [26]. Thus, we established in this work that it is possible to obtain the *energy function* of a three-dimensional dynamical system directly from its corresponding *jerk equation*. Application to these main results to the so-called Lorenz and Chua's models have confirmed their validity. In a previous paper [8], we have established a link between the *energy function* and the *curvature* of the trajectory curves integral of two-dimensional generalized Liénard systems. A perspective to be given to this work should be to extend such a result to three-dimensional autonomous dynamical systems.

As recalled previously, Sarasola *et al.* [16], Wang *et al.* [28], Ma *et al.* [14] and Yu *et al.* [27] have developed a method a method based on the Helmholtz's theorem allowing to deduce the *energy function* of the *vector field* of any three-dimensional autonomous dynamical system, i.e. starting from its set of three simultaneous first-order ordinary differential equations. This work has enables to state that it is now possible from its *jerk equation*. Moreover, the method used by these authors, although very useful, provided an *energy function* that precluded any interpretation in terms of *kinetic* and *potential energy*. The approach presented in this work restores in a certain manner their meanings. It also highlights the fact that the rate of change of the energy can now be interpreted according to Sarasola *et al.* [16] as "change in phase space volume in the sense that both go together. Any energy variation cannot occur without a variation in the phase space volume and vice versa."

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