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CHARACTERIZATION OF GLOBAL CENTERS BY THE MONODROMY AT INFINITY

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ABSTRACT. In this work we focus in the family of real planar polynomial vector fields of arbitrary degree. We are interested in to characterize when a (local) center singularity of these vector fields becomes a global center, that is, its period annulus foliates the punctured real plane. The characterization of any global center is done by blowing-down the polycycle at infinity into a monodromic singular point.

1. Introduction and statement of the main result

A *center* of a real planar polynomial vector field $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_x$, with $P, Q \in \mathbb{R}[x, y]$ polynomials of degree n, is an equilibrium point having a punctured neighborhood foliated by periodic orbits. A *global center* is a center p such that $\mathbb{R}^2 \backslash p$ is foliated by periodic orbits.

The notion of center goes back to the works of Huygens in 1656 about the pendulum clock, see [21, 28]. Some centuries later the definition of center was given in the works of Poincaré [29] in 1881 and Dulac [9] in 1908. To determine if a given differential system has a center at a singular point is in general a difficult problem, see for instance [12, 13, 18, 19] and references therein.

In general it is not easy to determine when a center is global. The method used up to know is based in the blow-up process [3], see for example [23,25]. However using the following result we propose a simple solution of the global center problem based in a well-known established algorithm for determining when a singular point is monodromic.

Theorem 1. *Let the origin be the unique singularity of a real planar polynomial vector field $\mathcal{X}$ of degree n. We consider the Bendixson compactification $\tilde{\mathcal{X}} = \phi_*(\mathcal{X})/(u^2+v^2)^n$ of $\mathcal{X}$ where ϕ_* is the pull-back*

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associated to $\phi(x,y) = (u(x,y), v(x,y)) = (x/(x^2+y^2), y/(x^2+y^2))$. *If the origin is a center of* $\mathcal{X}$ *then it is a global center if and only if the origin of* $\tilde{\mathcal{X}}$ *is a monodromic singularity.*

In order to apply Theorem 1 in a global center classification problem we only need to use one of algorithms developed in the literature for detecting the monodromy of a singularity, see [1, 2, 27].

We consider the associated differential systems to $\mathcal{X}$ given by

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{1}$$

Analysing orbits escaping or coming from infinity, in [11] it was established the following result.

Theorem 2. *System* (1) *with* n *even has no global centers.*

The reader can consult [23] for a recently different proof of Theorem 2. We reprove once more Theorem 2 using a very simple short argument inside the framework of the method proposed in this paper. The first author who systematically study global centers was Conti in [6, 7]. Indeed he proposed the following problem: *To identify all polynomial differential systems (of odd degree* n*) having a global center*, see [7, Problem 14.1]. Last years several authors have study global centers, see for instance [14, 15, 20, 23].

The easiest centers are the linear type centers, that is, the class of polynomial differential systems (1) of the form $\dot{x} = -y + p(x,y)$, $\dot{y} = x + q(x,y)$ where the polynomials p and q have neither constants nor linear terms. The classification of centers at the origin of these differential systems is a difficult problem. The complete classification is only known for polynomial differential systems of degree $n = 2$, see [22] and [5]. When $n > 2$ there are only partial results, see [16, 17]. The centers when p and q are homogeneous polynomials of degree 3 were classified in [26] and [30], and it is just in this family where the unique complete classification of global centers is recently obtained in [15]. The classification of the global centers with p and q homogeneous polynomials of degree 5 is not complete and only partial results are known, see [8]. Other partial results for different families is given in [20, 24, 25].

As an application Theorem 1 we will classify the global centers of the cubic family

$$\dot{x} = y, \qquad \dot{y} = -x + ax^2 + by^2 + cx^3 + dxy^2, \tag{2}$$

with parameters $a, b, c, d \in \mathbb{R}$. Notice that the differential system (2) is invariant under the discrete symmetry $(x, y, t) \to (x, -y, -t)$, hence it has a local center at the origin. The global centers of systems (2) are classified in the next result.

Theorem 3. *System* (2) *has a global center at the origin of coordinates if and only if* $a^2 + 4c < 0$ *and* $d < 0$.

The paper is structured as follows. In section 2 we provide an sketch of the monodromy algorithm for singularities developed in [1,2]. Finally in section 3 we prove Theorems 2 and 3.

2. A monodromy algorithm for singularities

Let $\mathbb{N}$ be the set of non-negative integers. Given an analytic vector field $\mathcal{Z} = A(x, y)\partial_x + B(x, y)\partial_y$ with $A(x, y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij} x^i y^{j-1}$, $B(x, y) = \sum_{(i,j)\in\mathbb{N}^2} b_{ij} x^{i-1} y^j$, where clearly $a_{i0} = b_{0j} = 0$. We take the support $\text{supp}(\mathcal{Z}) = \{(i, j) \in \mathbb{N}^2 : (a_{ij}, b_{ij}) \neq (0, 0)\}$. The Newton diagram $\mathbf{N}(\mathcal{Z})$ of $\mathcal{Z}$ is composed by edges and vertices of the boundary of the convex hull of the set $\bigcup_{(i,j)\in\text{supp}(\mathcal{X})}\{(i, j) + \mathbb{R}^2_+\}$ joining both positive semi-axis. Each edge of $\mathbf{N}(\mathcal{Z})$ has endpoints in $\mathbb{N}^2$ and we associate to it the weights $(p, q) \in \mathbb{N}^2$, with p and q coprimes, given by the tangent q/p of the angle between that edge and the y-axis. From now on we denote by $W(\mathbf{N}(\mathcal{Z})) \subset \mathbb{N}^2$ the set of all weights. A vertex in $\mathbf{N}(\mathcal{Z})$ is called exterior vertex if it lies on the axis, otherwise it is called an interior vertex. We also define (a_{ij}, b_{ij}) as the vector coefficient of the vertex $(i, j) \in \text{supp}(\mathcal{Z})$.

From now on we will describe some elements associated to $\mathbf{N}(\mathcal{Z})$, explained in [1, 2] and needed for the forthcoming monodromy algorithm. Given $(p, q) \in W(\mathbf{N}(\mathcal{Z}))$, we take the (p, q)-quasihomogeneous expansion $\mathcal{Z} = \sum_{i\geq r} \mathcal{Z}_i$, where $\mathcal{Z}_i$ are (p, q)-quasihomogeneous vector fields of degree i, that is $\mathcal{Z}_i = A_{p+i}(x, y)\partial_x + B_{q+i}(x, y)\partial_y$ where $A_{p+i}(\lambda^p x, \lambda^q y) = \lambda^i A_{p+i}(x, y)$ and $B_{q+i}(\lambda^p x, \lambda^q y) = \lambda^i B_{q+i}(x, y)$ for any $\lambda \in \mathbb{R}$. Now we perform the conservative-dissipative decomposition of $\mathcal{Z}_r \not\equiv 0$ given by $\mathcal{Z}_r = \mathcal{Z}_{h_{r+p+q}} + \mu_r \mathcal{D}_0^{(p,q)}$, where $\mathcal{Z}_{h_{r+p+q}}$ is the Hamiltonian vector field with Hamiltonian h_{r+p+q} and $\mathcal{D}_0^{(p,q)} = px\partial_x + qy\partial_y$ is the Euler field. A factor of h_{r+p+q} of the form x, y or $y^p - \alpha x^q$ with $\alpha \neq 0$ is called a strong factor if one of the following conditions holds: (i) its multiplicity is odd; (ii) its multiplicity is $2m$ and, either it is not a factor of μ_r when $\mu_r \not\equiv 0$, or is a factor of μ_r with multiplicity $2n$ with $0 < n < m$.

For each interior vertex $V \in \mathbf{N}(\mathcal{Z})$ we denote by (p_1, q_1) and (p_2, q_2) the weights of its upper and lower adjacent edges, and we assume that its associated Hamiltonians $h_{r_1+p_1+q_1}(x, y)h_{r_2+p_2+q_2}(x, y) \not\equiv 0$. Then we define the constant associated to V as $\beta_V = \tilde{c}_{j_0} c_{i_0}$, where $i_0 = \min\{i \geq 0 : c_i \neq 0\}$, and $j_0 = \min\{j \geq 0 : \tilde{c}_j \neq 0\}$, being c_i and $\tilde{c}_j$ the coefficients of the polynomials $h_{r_1+p_1+q_1}$ and $h_{r_2+p_2+q_2}$ ordered from the highest to the lowest exponent in x and y, respectively.

The following results correspond to Theorems 3 and 4 in [1] and provide the necessary and sufficient conditions for the monodromy of a singular point.

Necessary monodromic conditions: If the origin of $\mathcal{Z}$ is a monodromic singularity then its Newton diagram $\mathbf{N}(\mathcal{Z})$ satisfies the following restrictions:

(I) All its vertices have even coordinates;
(II) it has two exterior vertices and if $(a, 0)$ and $(0, b)$ are the vector coefficients of these vertices, then $ab < 0$;
(III) all its interior vertices V satisfy $\beta_V > 0$; and
(IV) for each bounded edge, its associated Hamiltonian is non-null and does not have any strong factor.

Sufficient monodromic conditions: If $\mathbf{N}(\mathcal{Z})$ satisfies: (I), (II), (III) and (V) for each $(p, q) \in W(\mathbf{N}(\mathcal{Z}))$, its associated Hamiltonian $h_{r+p+q} \not\equiv 0$ and does not have any factor of the form $y^p - \alpha x^q$ with $\alpha \neq 0$ real, then the origin of $\mathcal{Z}$ is monodromic.

3. Proofs

Let $p \in \mathbb{R}^2$ be a center. The period annulus $\mathcal{P}$ of the center p is the union of all the periodic orbits surrounding p.

3.1. **Proof of Theorem 1.** The Bendixson compactification can be found in Chapt. 13 of [4] or Chapt. 5 of [10]. Then the monomials in the components of $\tilde{\mathcal{X}}$ have minimum degree $n + 2$, hence $(0, 0)$ is a singular point of $\tilde{\mathcal{X}}$. Notice that Bendixson compactification blows-down the polycycle Γ at infinity of $\mathcal{X}$ into the singularity at the origin of $\tilde{\mathcal{X}}$ preserving their monodromic nature.

First we assume that $\mathcal{X}$ has a global center at the origin. Then $\tilde{\mathcal{X}}$ has a center at the origin and consequently it is monodromic.

To prove the converse we suppose that the origin of $\tilde{\mathcal{X}}$ is monodromic. Consequently, the polycycle Γ is monodromic. Now we shall prove that the local center at the origin of $\mathcal{X}$ is global, i.e. $\mathcal{P} = \mathbb{R}^2 \backslash \{(0,0)\}$ or equivalently the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ is $\partial\mathcal{P} = \{(0,0)\} \cup \Gamma$. Indeed, since the unique equilibrium point of $\mathcal{X}$ is the origin, if $\Gamma \cap \partial\mathcal{P} = \emptyset$ then $\partial\mathcal{P} = \{(0,0)\} \cup \gamma$, being γ a periodic orbit of $\mathcal{X}$. We shall prove that this cannot occur. Consider a local transversal section Σ to γ. Since the Poincaré return map defined on Σ is an analytic function in one variable and it is the identity map in $\Sigma \cap \mathcal{P}$ it follows that it must be also the identity map in the whole Σ, in contradiction with the fact that $\gamma \subset \partial\mathcal{P}$. In summary $\Gamma \subset \mathcal{P}$ and this completes the proof.

3.2. **Proof of Theorem 2.**

Proof. We consider system (1) with n even. Using the inverse Bendixson map $\phi^{-1}(u,v) = (x,y) = (u/(u^2+v^2), v/(u^2+v^2))$, we get the explicit expression of the associated vector field $\tilde{\mathcal{X}} = \tilde{P}(u,v)\partial_u + \tilde{Q}(u,v)\partial_v$ defined in Theorem 1 that is

$$\begin{aligned}
\tilde{P}(u,v) &= (u^2+v^2)^n \left((v^2-u^2)\,(P\circ\phi^{-1})(u,v) - 2uv\,(Q\circ\phi^{-1})(u,v)\right),\\
\tilde{Q}(u,v) &= (u^2+v^2)^n \left((v^2-u^2)\,(Q\circ\phi^{-1})(u,v) - 2uv\,(P\circ\phi^{-1})(u,v)\right).
\end{aligned}$$

We write $P(x,y) = \sum_{1\le i\le n} P_i(x,y)$ and $Q(x,y) = \sum_{1\le i\le n} Q_i(x,y)$ with P_i and Q_i homogeneous polynomials of degree i. Then we have

$$\begin{aligned}
\tilde{P}(u,v) &= (v^2-u^2)\sum_{1\le i\le n}(u^2+v^2)^{n-i}P_i(u,v) - 2uv\sum_{1\le i\le n}(u^2+v^2)^{n-i}Q_i(u,v),\\
\tilde{Q}(u,v) &= (v^2-u^2)\sum_{1\le i\le n}(u^2+v^2)^{n-i}Q_i(u,v) - 2uv\sum_{1\le i\le n}(u^2+v^2)^{n-i}P_i(u,v).
\end{aligned}$$

We therefore have the expansion $\tilde{P}(u,v) = \tilde{P}_{n+2}(u,v) + \cdots$, $\tilde{Q}(u,v) = \tilde{Q}_{n+2}(u,v) + \cdots$ where $\tilde{P}_{n+2}$ and $\tilde{Q}_{n+2}$ are homogeneous polynomials of degree $n+2$ and the dots denote higher order terms. More specifically

$$\begin{aligned}
\tilde{P}_{n+2}(u,v) &= (v^2-u^2)P_n(u,v) - 2uvQ_n(u,v),\\
\tilde{Q}_{n+2}(u,v) &= (v^2-u^2)Q_n(u,v) - 2uvP_n(u,v).
\end{aligned}$$

We remark that, when n is even, the origin of $\tilde{\mathcal{X}}$ always possesses characteristic directions because $u\tilde{Q}_{n+2}(u,v) - v\tilde{P}_{n+2}(u,v)$ is a homogeneous polynomial of odd degree. Since $\mathcal{X}$ has degree n, that is, $P_n^2(x,y) + Q_n^2(x,y) \not\equiv 0$, it follows that $\tilde{P}_{n+2}^2(u,v) + \tilde{Q}_{n+2}^2(u,v) \not\equiv 0$ too. In order that the origin of $\tilde{\mathcal{X}}$ be monodromic it is necessary that $\mathbf{N}(\tilde{\mathcal{X}})$ has two exterior vertices. Thus the monomials v^{n+2} and u^{n+2} must be present in $\tilde{P}_{n+2}$ and $\tilde{Q}_{n+2}$, respectively. But these monomials are associated with the vertices $(0, n+3)$ and $(n+3, 0)$ having odd coordinates in contradiction with the monodromy at the origin of $\tilde{\mathcal{X}}$. $\square$

3.3. **Proof of Theorem 3.** The following result characterizes systems (2) having the origin of coordinates as the unique finite real singular point.

Proposition 4. *The unique finite real singular point of system* (2) *is the origin of coordinates if and only if* $a^2 + 4c < 0$.

Proof. It is straight after showing that the eventual finite singular points of (2) different of $(0,0)$ are $\left(0, \frac{-a \pm \sqrt{a^2+4c}}{2c}\right)$. $\square$

Proof of Theorem 3. We will use Theorem 1 to the vector field $\mathcal{X}$ of degree $n = 3$ associated to system (2). The outcome is that $\tilde{\mathcal{X}} = \tilde{P}(u,v)\partial_u + \tilde{Q}(u,v)\partial_v$ where

$$\begin{aligned}\tilde{P} =& u^5(-2a+u)v + u^2(-2d + u(-2(a+b)+3u))v^3 + u(-2b+3u)v^5 + v^7,\\ \tilde{Q} =& (a-u)u^6 + u^3(d+(b-3u)u)v^2 - u(d+u(a+3u))v^4 - (b+u)v^6.\end{aligned}$$

The Newton diagram of $\tilde{\mathcal{X}}$ has two edges with exterior vertices $(0,8)$ and $(6,0)$ and interior vertex $(2,4)$, hence the set of weights is $W(\mathbf{N}(\tilde{\mathcal{X}})) = \{(2,1),(1,1)\}$ whose elements are ordered from the upper to the lower in $\mathbf{N}(\tilde{\mathcal{X}})$. Notice that all the vertices have even coordinates, a necessary monodromic condition.

With weights $(p_1, q_1) = (2,1)$, the vector field $\tilde{\mathcal{X}}$ has the $(2,1)$-quasihomogeneous expansion $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_{r_1} + \cdots$ with $r_1 = 5$ and $\tilde{\mathcal{X}}_5 = v^3(-2du^2 - 2buv^2 + v^4)\partial_u - v^4(du + bu^2)\partial_v$. The conservative-dissipative decomposition of $\tilde{\mathcal{X}}_5$ is $\tilde{\mathcal{X}}_5 = \tilde{\mathcal{X}}_{h_8} + \mu_5 \mathcal{D}_0^{(2,1)}$ where $\mathcal{D}_0^{(2,1)} = 2u\partial_u + v\partial_v$, $h_8(u,v) = -v^8/8$, and $\mu_5(u,v) = -v^3(du + bv^2)$.

Similarly, with weights $(p_2, q_2) = (1,1)$ one has the following $(1,1)$-quasihomogeneous expansion $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_{r_2} + \cdots$ with $r_2 = 4$ and $\tilde{\mathcal{X}}_4 = -2u^2v(cu^2 + dv^2)\partial_u + u(u-v)(u+v)(cu^2+dv^2)\partial_v$, with conservative-dissipative decomposition $\tilde{\mathcal{X}}_4 = \tilde{\mathcal{X}}_{h_6} + \mu_4 \mathcal{D}_0^{(1,1)}$ where $\mathcal{D}_0^{(1,1)} = u\partial_u + v\partial_v$, $h_6(u,v) = u^2(u^2+v^2)(cu^2+dv^2)/6$, and $\mu_4(u,v) = -uv(5cu^2 - du^2 + 4dv^2)/3$.

The constant β_V associated to the interior vertex $V = (2,4)$ is $\beta_V = -d/48$. The necessary monodromy condition $\beta_V > 0$ means $d < 0$. Moreover, $c < 0$ is another necessary monodromy condition because

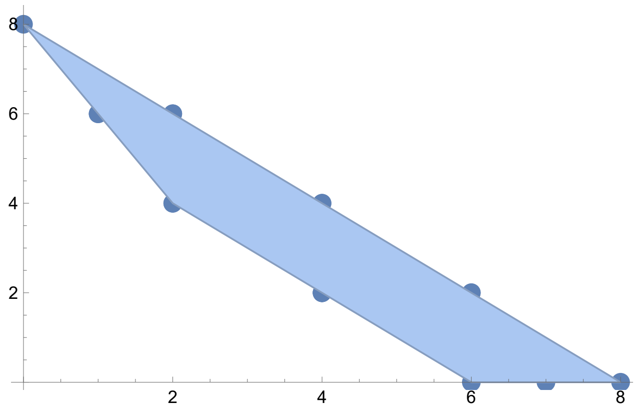

Figure 1. The Newton diagram of $\tilde{\mathcal{X}} = (\tilde{P}, \tilde{Q})$.

$(1,0)$ and $(0,c)$ are the vector coefficients of the exterior vertices. All the necessary monodromy conditions hold since there are no strong factors in both Hamiltonians $h_8(u,v)$ and $h_6(u,v)$.

The sufficient monodromy conditions also hold because the Hamiltonians $h_8(u,v) \not\equiv 0$, $h_6(u,v) \not\equiv 0$ and they have no factor of the form $v^2 - \alpha_1 u$ and $v - \alpha_2 u$ with $\alpha_i \in \mathbb{R}$, respectively.

In summary, the origin is a monodromic singularity of $\tilde{\mathcal{X}}$ if and only if $d < 0$ and $c < 0$.

The proof finishes applying Theorem 1 and taking into account Proposition 4. □

Data availability

Our manuscript has no associated data.

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