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THE MATCHING OF TWO MARKUS-YAMABE PIECEWISE SMOOTH SYSTEMS IN THE PLANE

DENIS DE CARVALHO BRAGA[†], FABIO SCALCO DIAS[†], JAUME LLIBRE[‡]
AND LUIS FERNANDO MELLO[†♯]

Abstract. A Markus-Yamabe vector field is a smooth vector field in $\mathbb{R}^n$ having only one equilibrium point and such that the spectrum of its Jacobian matrix at any point of $\mathbb{R}^n$ is on the left of the imaginary axis in the complex plane. A vector field is globally asymptotically stable if it has an equilibrium point p and all the other orbits tend to p in forward time. One of the great results of the Qualitative Theory of Differential Equations establishes that a planar Markus-Yamabe vector field is globally asymptotically stable, but a Markus-Yamabe vector field defined in $\mathbb{R}^n$, $n \geqslant 3$, does not have in general this property. We prove that planar crossing piecewise smooth vector fields defined in two zones formed by two Markus-Yamabe vector fields sharing the same equilibrium point located on the separation straight line are not necessarily globally asymptotically stable.

1. Introduction and statement of the main result

For $n \geqslant 2$ let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 vector field. As usual we will identify the vector field F with the ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = F(x), \quad x = (x_1, \ldots, x_n), \tag{1}$$

where the dot denotes the derivative with respect to the independent variable t, called here the time.

Assume that $x^* \in \mathbb{R}^n$ is an equilibrium point of system (1). Assume further that there exists an open neighborhood U of x^* such that the orbits of (1) starting from U tend to x^* in forward time. The basin of attraction of x^* is the largest open set whose elements satisfy the above condition. The equilibrium point x^* of the vector field F is *globally asymptotically stable* if its basin of attraction is the whole $\mathbb{R}^n$. Thus, if F is globally asymptotically stable, no matter how far from the equilibrium point the initial condition is, the positive trajectory through it will converge to the equilibrium point.

The problem of determining the basin of attraction of an equilibrium point of a vector field is of great importance for applications of the stability theory of ordinary differential equations. Historically the global asymptotic stability problem is closely related to the Markus-Yamabe Conjecture [11].

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Conjecture 1 (Markus-Yamabe). *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 vector field having the origin as its unique equilibrium point. If the eigenvalues of the Jacobian matrix of F for all $x \in \mathbb{R}^n$ have negative real parts, then F is globally asymptotically stable.*

A *Markus-Yamabe vector field* is a vector field satisfying the hypotheses of Conjecture 1.

Independently, Feßler [7], Glutsyuk [9], and Gutiérrez [10] proved that the Markus-Yamabe Conjecture is true in dimension two. On the other hand, if n is greater than or equal to three, the conjecture has been proven to be false. See the articles Bernat and Llibre [2] and Cima, van den Essen, Gasull, Hubbers and Mañosas [6].

For a planar Markus-Yamabe vector field F the condition on the eigenvalues of $DF(x, y)$ is equivalent to $\operatorname{tr} DF(x, y) < 0$ and $\det DF(x, y) > 0$, for all $(x, y) \in \mathbb{R}^2$, and, certainly, it is not necessary for the global asymptotic stability. In fact, consider the vector field

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F(x, y) = (-x + xy, -y). \tag{2}$$

The origin is the only equilibrium point which is locally asymptotically stable. The eigenvalues of $DF(x, y)$ are $\lambda_1 = -1$ and $\lambda_2 = -1 + y$, for all $(x, y) \in \mathbb{R}^2$. If $y \geqslant 1$ then $\lambda_2 \geqslant 0$, that is, the vector field (2) is not a Markus-Yamabe vector field. Nevertheless, the origin is globally asymptotically stable since

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad L(x, y) = \ln(1 + x^2) + y^2,$$

is a proper strict Lyapunov function.

In short, the class of planar Markus-Yamabe vector fields is properly contained in the class of planar globally asymptotically stable vector fields.

Here we are interested in the study of the global asymptotic stability of Markus-Yamabe piecewise smooth differential systems in the plane.

Piecewise smooth differential systems with two zones in the plane separated by the line $\mathcal{H}(x, y) = 0$ are defined by

$$(\dot{x}, \dot{y}) = \begin{cases} F^+(x, y) & \text{if } \mathcal{H}(x, y) \geqslant 0, \\ F^-(x, y) & \text{if } \mathcal{H}(x, y) \leqslant 0, \end{cases} \tag{3}$$

where $F^\pm$ are smooth vector fields defined in the plane. The function $\mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is smooth having the zero as a regular value and the set $\Sigma = \mathcal{H}^{-1}(\{0\})$, called the *separation line* (*boundary*), divides the plane into two unbounded components (zones) Σ^+ and Σ^- where $\mathcal{H}$ is positive and negative, respectively. Thus $\mathbb{R}^2 = \Sigma^+ \cup \Sigma \cup \Sigma^-$. We will denote such a system by $(F^+, F^-, \mathcal{H})$ or (F^+, F^-, Σ).

Usually the points on Σ are classified as crossing, sliding, escaping or tangency points, for more details see [12]. Recall that a point $(x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(\{0\})$ is a *crossing* point if

$$T(x_0, y_0) = \left(F^-(x_0, y_0) \cdot \nabla\mathcal{H}(x_0, y_0)\right) \left(F^+(x_0, y_0) \cdot \nabla\mathcal{H}(x_0, y_0)\right) > 0. \tag{4}$$

What somehow gives simplicity to a crossing point on Σ is that the orbit of the piecewise system (3) through such a point is the concatenation of the orbits of the vector fields $F^+|_{\Sigma^+}$

and $F^-|_{\Sigma^-}$ through this point. A crossing point $(x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(\{0\})$ is a *continuity* point if $F^+(x_0, y_0) = F^-(x_0, y_0)$.

Since the seminal work of Andronov et al. [1] on piecewise differential systems formed by linear differential systems, a lot of articles on such systems have been published about questions like the existence, number, stability and distribution of limit cycles mainly for the case where $F^\pm$ are linear.

Nevertheless, to the best of our knowledge, there are almost no articles studying the problem of the global asymptotic stability of an equilibrium point in the class of piecewise systems (3). The landmark of such a study is the article [8] in which the authors analyzed the following case:

- H1. $F^\pm$ are Hurwitz homogeneous linear vector fields (the real parts of all its eigenvalues are negative);
- H2. The separation boundary Σ is a straight line that contains the unique equilibrium point of the piecewise system at the origin;
- H3. The points on $\Sigma \setminus \{(0,0)\}$ are continuity points.

With such hypotheses it was proved in [8] that the unique equilibrium point is globally asymptotically stable. Similar result was obtained in [4] weakening the assumption of the continuity on $\Sigma \setminus \{(0,0)\}$ in [8] for crossing, that is, the hypothesis H3 was changed by:

- H3†. The points on $\Sigma \setminus \{(0,0)\}$ are crossing points.

Consider the function $\mathcal{H}_\rho$ defined by $(x, y) \mapsto \mathcal{H}_\rho(x, y) = y - \psi_\rho(x)$,

$$x \mapsto \psi_\rho(x) = \begin{cases} 0, & x \leqslant 0, \\ \rho x, & x \geqslant 0, \end{cases} \qquad \rho \geqslant 0. \tag{5}$$

In this case the separation line

$$\Sigma = \Sigma_\rho = \big\{(x, y) \in \mathbb{R}^2 : \mathcal{H}_\rho(x, y) = 0\big\} \tag{6}$$

is a broken line for each $\rho > 0$. Now consider the hypothesis:

- H2†. The separation boundary Σ is a broken line that contains the unique equilibrium point of the piecewise system at the origin.

The following theorem was proved in [8].

Theorem 2. *Consider the piecewise smooth system* (3) *with the hypotheses* H1*,* H2†*, and* H3†*. Then there exist Hurwitz homogeneous linear vector fields F^+ and F^- such that the unique equilibrium point at the origin is: either a stable focus, or a center, or an unstable focus.*

Therefore, under the hypotheses of Theorem 2, the equilibrium point of the piecewise system is not necessarily globally asymptotically stable.

Since the hypothesis H1 deals only with linear vector fields change it by the following one:

- H1†. $F^\pm$ are globally asymptotically stable vector fields.

The following question is very natural: *Are piecewise smooth systems with two zones in the plane satisfying the hypotheses* $\mathrm{H1}^\dagger$, H2, *and* $\mathrm{H3}^\dagger$ *globally asymptotically stable?*

In the article [3] the authors proved the following theorem.

Theorem 3. *Consider piecewise smooth differential systems in the plane separated by a straight line and formed by two differential systems satisfying the hypotheses* $\mathrm{H1}^\dagger$, H2, *and* $\mathrm{H3}^\dagger$. *Then there exist piecewise polynomial differential systems in this class which are not globally asymptotically stable.*

Since the class of planar Markus-Yamabe vector fields is more restrictive than the class of planar globally asymptotically stable vector fields, it seems natural to change the hypothesis $\mathrm{H1}^\dagger$ by the following one:

$\mathrm{H1}^\ddagger$. $F^\pm$ are Markus-Yamabe vector fields.

The main result of this article is the following.

Theorem 4. *Consider piecewise smooth differential systems in the plane separated by a straight line and formed by two differential systems satisfying the hypotheses* $\mathrm{H1}^\ddagger$, H2, *and* $\mathrm{H3}^\dagger$. *Then there exist piecewise polynomial differential systems in this class which are not globally asymptotically stable.*

The proof of Theorems 4 is presented in Section 2. We found instructive to offer two proofs of Theorem 4 and also to highlight the ideas underlying these proofs.

2. Proof of Theorem 4

The proof of Theorem 4 is given in this section. The ideas underlying the proof are the following:

(1) Start by considering a Markus-Yamabe vector field F with the property: there is an initial condition $(x_0, 0)$, with $x_0 > 0$, such that the positive orbit of F by $(x_0, 0)$ has the first intersection with the x-axis at the point $(x_1, 0)$, with $x_1 < 0$ and $|x_1| > x_0$. See Figure 1 (Left). See Lemma 5.

(2) Consider the piecewise system (F^+, A^-, Σ), where $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, $F^+ = F|_{\Sigma^+}$, $A^-(x, y) = (-y, x)$ is the homogeneous linear vector field with a symmetric center at $(0, 0) \in \Sigma$, and such that the points on $\Sigma \setminus \{(0, 0)\}$ are crossing points. The piecewise system (F^+, A^-, Σ) has an orbit that does not tend to the origin. See Figure 1 (Right). See Lemma 6.

(3) Slightly perturb the homogeneous linear vector field A^- to obtain a new one denoted by F^- with the same above properties but with a stable focus at $(0, 0)$. See Lemma 7.

(4) By construction, the vector fields F^+ and F^- are Markus-Yamabe and the piecewise system (F^+, F^-, Σ) has an orbit that does not tend to the equilibrium point $(0, 0)$.

Having presented the ideas underlying the proof of Theorem 4, we actually move on to its proof.

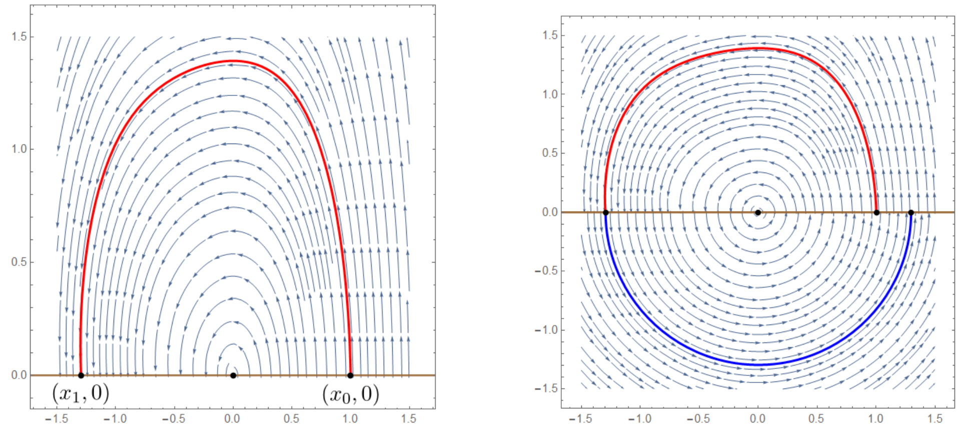

FIGURE 1. Left: The initial condition $(x_0, 0)$, $x_0 > 0$ and the point $(x_1, 0)$, $x_1 < 0$ such that $|x_1| > x_0$. Right: An orbit of the piecewise system (F^+, A^-, Σ) that does not tend to the origin.

Lemma 5. *Consider the smooth vector field*

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F(x,y) = \left(-\varepsilon x - y, x + x^2 + x^3\right), \tag{7}$$

where ε is a non-negative real number. The following statements are true:

(a) *The vector field F is Markus-Yamabe for $\varepsilon > 0$.*
(b) *For $\varepsilon > 0$ small enough the positive orbit of F by the initial condition $(x_0, y_0) = (1, 0)$ has the first intersection with the x-axis at the point $(x_1, 0)$ with $x_1 < 0$ and $|x_1| > 1$. See Figure 1 (Left) for an illustration.*

Proof. The Jacobian matrix of the vector field F in (7) is given by

$$DF(x,y) = \begin{bmatrix} -\varepsilon & -1 \\ 1 + 2x + 3x^2 & 0 \end{bmatrix}.$$

It is immediate that

$$\operatorname{tr} DF(x,y) = -\varepsilon \quad \text{and} \quad \det DF(x,y) = 1 + 2x + 3x^2 > 0,$$

for all $(x, y) \in \mathbb{R}^2$. So the vector field F is Markus-Yamabe for $\varepsilon > 0$. The proof of statement (a) is complete.

When $\varepsilon = 0$ the vector field F is Hamiltonian with Hamiltonian function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$H(x,y) = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{y^2}{2}.$$

For each $c > 0$, the equation $H(x, y) = c$ implicitly defines a function $y = \phi_c(x)$ that is non-negative on its domain of definition $[X_1^c, X_0^c] \subset \mathbb{R}$ where $X_1^c < 0$ and $X_0^c > 0$. Furthermore, by implicitly differentiating, it is easy to see that $\phi_c'(x) > 0$ if $x \in (X_1^c, 0)$, $\phi_c'(0) = 0$, $\phi_c'(x) < 0$ if $x \in (0, X_0^c)$, and $\phi_c''(x) < 0$ for all $x \in (X_1^c, X_0^c)$.

It follows from these properties of derivatives that the function ϕ_c has only two real zeros corresponding to the endpoints of its domain. Therefore for each $c > 0$ we can build the map that associates each $X_0^c > 0$ to $X_1^c < 0$. In particular, for $\bar{c} = H(1,0) = 13/12$ we have $X_0^{\bar{c}} = 1$ and $X_1^{\bar{c}} < 0$ so that $|X_1^{\bar{c}}| > X_0^{\bar{c}}$. In fact, the real roots of the equation $\phi_{\bar{c}}(x) = 0$, or equivalently of $H(x,0) = \bar{c} = H(1,0)$ are also roots of the polynomial equation

$$p(x) = (x-1)(3x^3 + 7x^2 + 13x + 13) = 0.$$

In addition to the root $X_0^{\bar{c}} = 1$, the polynomial function p has two complex roots and the real root $X_1^{\bar{c}} \in [-2,-1]$, since $p(-2)p(-1) = -216 < 0$. In fact,

$$X_1^{\bar{c}} = \frac{1}{9}\left(-7 - \frac{34\sqrt[3]{4}}{\sqrt[3]{27\sqrt{273} - 347}} + \sqrt[3]{54\sqrt{273} - 694}\right) = -1.425746.... \tag{8}$$

It follows from the arguments presented about the function ϕ_c that the transition map $P^+ : L^+ \longrightarrow L^-$ associated with the vector field F is well defined for each $\varepsilon > 0$ small enough, where

$$L^- = \left\{(x,y) \in \mathbb{R}^2 : x < 0, y = 0\right\}, \quad L^+ = \left\{(x,y) \in \mathbb{R}^2 : x > 0, y = 0\right\}. \tag{9}$$

Furthermore, $P^+ = P^+(a,\varepsilon)$ is differentiable with respect to $a \equiv (a,0) \in L^+$ and ε. By the Mean Value Theorem, setting $a = 1$ and taking $\varepsilon > 0$ sufficiently small, there exists $\bar{\varepsilon} \in (0,\varepsilon)$ such that

$$P^+(1,\varepsilon) = P^+(1,0) + \frac{\partial P^+(1,\bar{\varepsilon})}{\partial \varepsilon}\varepsilon,$$

where $P^+(1,0) = X_1^{\bar{c}}$ as in (8). Therefore taking $x_0 = X_0^{\bar{c}} = 1$, there exists $x_1 = P^+(1,\varepsilon) < 0$ with $|x_1| > x_0$ for $\varepsilon > 0$ sufficiently small and statement (b) is proved. □

From now on consider

$$\mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \mathcal{H}(x,y) = y, \quad \text{and} \quad \Sigma = \mathcal{H}^{-1}(\{0\}). \tag{10}$$

Lemma 6. *Consider $\mathcal{H}$ and Σ as in (10). Consider $F^+ = F|_{\Sigma^+}$ as in (7) with $\varepsilon > 0$ small enough, $A^- = A|_{\Sigma^-}$, where $A(x,y) = (-y,x)$. Let the piecewise system (F^+, A^-, Σ). Then:*

(i) *The origin $(0,0) \in \Sigma$ is the only equilibrium of both F^+ and A^-.*
(ii) *The points $(x,0) \in \Sigma$, $x \neq 0$, are crossing points.*
(iii) *The positive orbit by $(x_0,y_0) = (1,0)$ does not tend to $(0,0)$. See Figure 1 (Right).*

Proof. The only statement that needs to be proved is (ii), since (i) is immediate and (iii) follows from statement (b) of Lemma 5 and the definition of A^-. As the gradient of the function $\mathcal{H}$ is given by $\nabla\mathcal{H}(x,y) = (0,1)$, from (4) we obtain

$$T(x,0) = x^2\left(1 + x + x^2\right) > 0,$$

for all $x \neq 0$. This implies that the points $(x,0) \in \Sigma \setminus \{(0,0)\}$ are crossing points, proving statement (ii). □

Consider the following Markus-Yamabe vector field, obtained as a perturbation of the vector field A,

$$B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad B(x,y) = (-\mu\, x - y, x - \mu\, y), \tag{11}$$

where $\mu > 0$ is sufficiently small. The proof of the following lemma is immediate and is a consequence of the previous lemmas.

Lemma 7. *Consider $\mathcal{H}$ and Σ as in* (10). *Consider $F^+ = F|_{\Sigma^+}$ as in* (7) *with $\varepsilon > 0$ small enough, $F^- = B|_{\Sigma^-}$, where the linear homogeneous vector field B is given in* (11). *Let the piecewise system (F^+, F^-, Σ). The following statements hold for $\mu > 0$ small enough:*

(I) *The origin $(0,0) \in \Sigma$ is the only equilibrium of both F^+ and F^-.*
(II) *The points $(x,0) \in \Sigma$, $x \neq 0$, are crossing points.*
(III) *The positive orbit by $(x_0, y_0) = (1,0)$ does not tend to $(0,0)$.*

In short Theorem 4 is proved.

Remark. Without loss of generality, it is possible to take $\mu = \varepsilon$ in Lemma 7.

Now we will give a second proof of Theorem 4. Consider the piecewise system (F^+, F^-, Σ), where $\Sigma = \mathcal{H}^{-1}(\{0\})$ is as in (10) and the vector fields $F^\pm : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are given by

$$F^\pm(x,y) = \left(-y + \varepsilon g_1^\pm(x,y), x + \varepsilon g_2^\pm(x,y)\right), \tag{12}$$

with

$$g_1^-(x,y) = -x, \quad g_2^-(x,y) = -y, \quad g_1^+(x,y) = -c_1 x, \quad g_2^+(x,y) = c_2 x^2 + c_3 x^3, \tag{13}$$

c_1, c_2 and c_3 positive real numbers and $\varepsilon \geqslant 0$.

Note that $F^\pm$ are Markus-Yamabe vector fields since $F^- = B$ as in (11) and

$$\operatorname{tr} DF^+(x,y) = -\varepsilon c_1 < 0 \quad \text{and} \quad \det DF^+(x,y) = 1 + 2\varepsilon c_2 x + 3\varepsilon c_3 x^2 > 0$$

for $\varepsilon > 0$ small enough. In addition to this,

$$T(x,0) = x^2\left(1 + \varepsilon c_2 x + \varepsilon c_3 x^2\right) > 0$$

for $\varepsilon > 0$ sufficiently small, showing that all points on $\Sigma \setminus \{(0,0)\}$ are crossing points.

The piecewise system (F^+, F^-, Σ) can be seen as a perturbation of a linear center. Then taking L^+ in (9) as a Poincaré section, the first return map $\Pi : L^+ \longrightarrow L^+$ has the following Taylor series expansion around $\varepsilon = 0$,

$$\Pi^+(a,\varepsilon) = a + \varepsilon(\mathcal{M}(a) + O(\varepsilon)),$$

where $O(\varepsilon)$ represents the higher-order terms in the Taylor series expansion and

$$\mathcal{M}(a) = \mathcal{M}^+(a) + \mathcal{M}^-(a), \tag{14}$$

with

$$\mathcal{M}^+(a) = \int_0^\pi \left[g_1^+(a\cos(s), a\sin(s))\cos(s) + g_2^+(a\cos(s), a\sin(s))\sin(s)\right] \mathrm{d}s$$

and

$$\mathcal{M}^-(a) = \int_\pi^{2\pi} \left[g_1^-(a\cos(s), a\sin(s))\cos(s) + g_2^-(a\cos(s), a\sin(s))\sin(s)\right] \mathrm{d}s.$$

It turns out that if $a_0 > 0$ is a simple zero of the function $\mathcal{M}$ for $\varepsilon > 0$ sufficiently small, then there is a limit cycle in the phase portrait of (F^+, F^-, Σ) which is stable if $\mathcal{M}'(a_0) < 0$ and unstable if $\mathcal{M}'(a_0) > 0$. See Theorem 1 of [5] and the results therein for more details.

From (13) and (14) we obtain $a_0 = (3(c_1+2)\pi)/(4c_2)$ as the only positive zero of

$$\mathcal{M}(a) = \frac{1}{6}a(4c_2 a - 3(c_1+2)\pi).$$

Since $\mathcal{M}'(a_0) = ((c_1+2)\pi)/2 > 0$, it follows that there is an unstable limit cycle in the phase portrait of (F^+, F^-, Σ). Figure 2 shows the comparison between the graphs of the separation function $a \longmapsto \Delta(a, \varepsilon) = P(a, \varepsilon) - a$ associated with the system (F^+, F^-, Σ) and the function $a \longmapsto \varepsilon\mathcal{M}(a)$ for $c_1 = 1$, $c_2 = c_3 = 100$ and $\varepsilon = 1/100$.

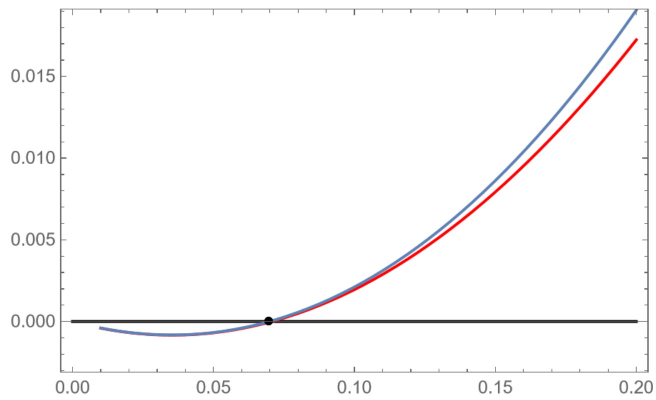

FIGURE 2. The graph of the separation function $a \longmapsto \Delta(a, \varepsilon) = P(a, \varepsilon) - a$ associated with the piecewise system (F^+, F^-, Σ) is represented by the blue line and the graph of the function $a \longmapsto \varepsilon\mathcal{M}(a)$ by the red line. The parameter values are $c_1 = 1$, $c_2 = c_3 = 100$ and $\varepsilon = 1/100$. The black dot corresponds to the only zero of the function $\mathcal{M}$ given by $a_0 = 9\pi/400$.

The second proof of Theorem 4 is complete.

Conflict of interest

The authors declare that they have no conflict of interest.

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† Instituto de Matemática e Computação, Universidade Federal de Itajubá, Avenida BPS 1303, Pinheirinho, CEP 37.500-903, Itajubá, MG, Brazil

Email address: braga@unifei.edu.br

Email address: scalco@unifei.edu.br

Email address, Corresponding author♯: lfmelo@unifei.edu.br

‡ Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jaume.llibre@uab.cat