ON FAKE ES-IRREDUCIBILE COMPONENTS OF CERTAIN STRATA OF SMOOTH PLANE SEXTICS

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Abstract. We construct the first examples of what we call *fake ES-irreducible components*; Definition 2.8. In our way to do so, we classify the automorphism groups of smooth plane sextics that only have automorphisms of order ≤ 3; Theorems 2.1, 2.4 and 2.5, Corollaries 2.9 and 2.11.

1. Introduction

Let $\mathcal{M}_g^{\text{Pl}}$ be the set of K-isomorphism classes of smooth plane curves C of a fixed degree $d \geq 4$. Here K is an algebraically closed field of characterisitc $p = 0$ or $p > 2g + 1$, where $g = (d-1)(d-2)/2 \geq 3$ is the geometric genus of C.

We can associate to any $[C] \in \mathcal{M}_g^{\text{Pl}}$ infinitely many non-singular plane models, each of them is given by a homogeneous polynomial equation $C : F(X, Y, Z) = 0$ of degree d in $\mathbb{P}^2(K)$. Moreover, two such plane models for C are K-isomorphic and their automorphism groups are $\text{PGL}_3(K)$-conjugated via a projective change of variables $\phi \in \text{PGL}_3(K)$.

Now, suppose that G is a finite non-trivial group that can be embedded into $\text{PGL}_3(K)$. We write $[C] \in \mathcal{M}_g^{\text{Pl}}(G)$ when there exists an injective representation $\varrho : G \hookrightarrow \text{PGL}_3(K)$ such that $\varrho(G)$ is a subgroup of $\text{Aut}(C)$; the automorphism group of $C : F(X, Y, Z) = 0$ inside $\text{PGL}_3(K)$. More precisely, we say that $[C]$ belongs to the component $\mathcal{M}_g^{\text{Pl}}(\varrho(G))$ of $\mathcal{M}_g^{\text{Pl}}(G)$. Similarly, we write $[C] \in \widetilde{\mathcal{M}_g^{\text{Pl}}}(G)$ when $\varrho(G) = \text{Aut}(C)$ for some ϱ, and again we say that $[C]$ belongs to the component $\widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho(G))$ of $\widetilde{\mathcal{M}_g^{\text{Pl}}}(G)$.

Clearly, if $\varrho_i : G \hookrightarrow \text{PGL}_3(K)$, for $i = 1, 2$, are $\text{PGL}_3(K)$-conjugated, then $\mathcal{M}_g^{\text{Pl}}(\varrho_1(G)) = \mathcal{M}_g^{\text{Pl}}(\varrho_2(G))$ and $\widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho_1(G)) = \widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho_2(G))$. Accordingly,

$$\mathcal{M}_g^{\text{Pl}}(G) = \bigcup_{[\varrho] \in R_G} \mathcal{M}_g^{\text{Pl}}(\varrho(G)) \text{ and } \widetilde{\mathcal{M}_g^{\text{Pl}}}(G) = \bigsqcup_{[\varrho] \in R_G} \widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho(G)).$$

Here $R_G := \{\varrho : G \hookrightarrow \text{PGL}_3(K)\} / \sim$, where $\varrho_1 \sim \varrho_2$ if and only if $\varrho_1(G)$ and $\varrho_2(G)$ are $\text{PGL}_3(K)$-conjugated.

Definition 1.1 (ES-irreducibility [3])**.** Each $[\varrho] \in R_G$ such that $\widetilde{\mathcal{M}_g^{\text{Pl}}}(\varrho(G)) \neq \emptyset$ is called an *ES-irreducible component* for $\widetilde{\mathcal{M}_g^{\text{Pl}}}(G)$. We call $\widetilde{\mathcal{M}_g^{\text{Pl}}}(G)$ *ES-irreducible* if it has exactly one ES-irreducible component.

Clearly, if a non-empty $\widetilde{\mathcal{M}_g^{\text{Pl}}}(G)$ is not ES-irreducible, then it is not irreducible and the number of its ES-irreducible components is a lower bound for the number of its irreducible components inside the coarse moduli space $\mathcal{M}_g$ of K-isomorphism classes of smooth curves of genus g.

Now, in the language of ES-irreducibility, one can interpret the results of Henn [10] and Komiya-Kuribayashi [11] for smooth plane quartic curves, which are genus

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$g = 3$ curves, as follows: the strata $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)$ are either empty or ES-irreducible. Thus each non-empty $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)$ is described by a single *normal form*; a homogenous polynomial equation $F(X, Y, Z) = 0$ in $\mathbb{P}^2(K)$ equipped with parameters as its coefficients such that any $[C] \in \widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)$ can be described by a smooth plane model through a specialization of those parameters.

Notation. Throughout the paper, $L_{i,B}$ denotes the generic homogeneous polynomial of degree i in the variables $\{X, Y, Z\} - \{B\}$.

By ζ_n we mean a fixed primitive nth root of unity in K.

A projective linear transformation $A = (a_{i,j}) \in \mathrm{PGL}_3(K)$ is sometimes written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z].$$

For example, $[X : Z : Y]$ represents the projective change of variables $X \mapsto X$, $Y \mapsto Z$, $Z \mapsto Y$, and $\mathrm{diag}(1, a, b)$ represents $X \mapsto X$, $Y \mapsto aY$, $Z \mapsto bZ$ with $a, b \in K^*$.

We use the formal GAP library notations "GAP(n, m)" to refer the finite group of order n that appears in the m-th position of the atlas for small finite groups [8]. See also GroupNames.

Fix the following subgroups in $\mathrm{PGL}_3(K)$:

- $\varrho_1(\mathbb{Z}/2\mathbb{Z}) := \langle \mathrm{diag}(1, 1, -1)\rangle$ and $\varrho_1((\mathbb{Z}/2\mathbb{Z})^2) := \langle \varrho_1(\mathbb{Z}/2\mathbb{Z}), \mathrm{diag}(1, -1, 1)\rangle$,
- $\varrho_1(\mathbb{Z}/3\mathbb{Z}) := \langle \mathrm{diag}(1, 1, \zeta_3)\rangle$ and $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2) := \langle \varrho_1(\mathbb{Z}/3\mathbb{Z}), \mathrm{diag}(1, \zeta_3, 1)\rangle$,
- $\varrho_2(\mathbb{Z}/3\mathbb{Z}) := \langle \mathrm{diag}(1, \zeta_3, \zeta_3^{-1})\rangle$ and $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2) := \langle \varrho_2(\mathbb{Z}/3\mathbb{Z}), [Y : Z : X]\rangle$,
- $\varrho_1(\mathrm{S}_3) := \langle [Y : Z : X], [X : Z : Y]\rangle$ and $\varrho_2(\mathrm{S}_3) := \langle \varrho_2(\mathbb{Z}/3\mathbb{Z}), [X : Z : Y]\rangle$,
- $\varrho_1(\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3) := \langle \varrho_1(\mathrm{S}_3), \varrho_2(\mathbb{Z}/3\mathbb{Z})\rangle$,
- $\varrho_1(\mathrm{A}_4) := \langle \varrho_1((\mathbb{Z}/2\mathbb{Z})^2), [Y : Z : X]\rangle$ and $\varrho_2(\mathrm{A}_4) := \langle \varrho_1((\mathbb{Z}/2\mathbb{Z})^2), [\zeta_6^{-1}Y : Z : X]\rangle$.

Remark 1.2. *P. Henn observed that $\mathcal{M}_3^{\mathrm{Pl}}(\mathbb{Z}/3\mathbb{Z})$ admits two ES-components. One component corresponds to $\varrho_1(\mathbb{Z}/3\mathbb{Z})$ where any $[C] \in \mathcal{M}_3^{\mathrm{Pl}}(\varrho_1(\mathbb{Z}/3\mathbb{Z}))$ is given by an equation of the form $Z^3Y + L_{4,Z} = 0$. The second component corresponds to $\varrho_2(\mathbb{Z}/3\mathbb{Z})$ such that any $[C'] \in \mathcal{M}_3^{\mathrm{Pl}}(\varrho_2(\mathbb{Z}/3\mathbb{Z}))$ is given by an equation of the form $X^4 + X(Y^3 + Z^3) + \alpha_{2,1}X^2YZ + \alpha_{1,1}X(YZ)^2 = 0$ for some $\alpha_{2,1}, \alpha_{1,1} \in K$. In particular, C' has $[X : Z : Y]$ as an extra involution, thus C' always has the symmetry group S_3 as a subgroup of automorphisms. Therefore, $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(\varrho_2(\mathbb{Z}/3\mathbb{Z})) = \emptyset$ and $\mathcal{M}_3^{\mathrm{Pl}}(\varrho_2(\mathbb{Z}/3\mathbb{Z})) \subseteq \mathcal{M}_3^{\mathrm{Pl}}(\mathrm{S}_3)$.*

Concerning smooth plane quintic curves, which are genus $g = 6$ curves, Badr-Bars [1] showed that all the strata $\widetilde{\mathcal{M}_6^{\mathrm{Pl}}}(G)$ are either empty or ES-irreducible except when $G = \mathbb{Z}/4\mathbb{Z}$. In this case, $\widetilde{\mathcal{M}_6^{\mathrm{Pl}}}(\mathbb{Z}/4\mathbb{Z})$ has exactly two ES-irreducible components. Moreover, we generalized this result in [3] for any odd degree $d \geq 5$. More precisely, we proved that $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\mathbb{Z}/(d-1)\mathbb{Z})$ has at least two ES-irreducible components for any $g = (d-1)(d-2)/2$ with $d \geq 5$ odd. However, each of the strata $\widetilde{\mathcal{M}_6^{\mathrm{Pl}}}(\varrho(G))$ is described again by a single normal form.

Accordingly, we were wondering if this is the situation in general. That is to say, there always exists a single normal form describing the elements of $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G))$ for each $\varrho \in R_G$. In this article, we will show that this impression is not true at least for smooth plane sextic curves, which are genus $g = 10$ curves. We establish three counter examples corresponding to $G = \mathbb{Z}/3\mathbb{Z}$ and A_4 respectively.

On the other hand, classifying automorphism groups of smooth curves is a long standing problem that receives interest by many people. In the case of hyperelliptic curve, the structure of the automorphism group is quite explicit, see [6, 7, 16, 17]. For non-hyperelliptic curves, we still have a lack of knowledge about the structure, except for some special cases. For example, the cases of low genus and also Hurwitz curves, see [5, 10, 12, 13, 14]. This lack motivates us to do more investigation in this direction, especially for the case of smooth plane curves of degree $d \geq 4$. In this paper, we classify the automorphism groups of smooth plane curves C of degree 6 such that 2 and 3 are the only divisors of $|\operatorname{Aut}(C)|$. A more detailed treatment of automorphisms of non-singular plane sextic curves is intended in [4].

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2. Main Results

Theorem 2.1. *Let C be a smooth plane sextic curve that admits an automorphism of maximal order 2. Up to K-isomorphism, C is defined by an equation of the form:*

$$C : Z^6 + Z^4 L_{2,Z} + Z^2 L_{4,Z} + L_{6,Z} = 0$$

such that $L_{6,Z}$ is of degree ≥ 5 in both X and Y, and at least one of the binary forms $L_{2,Z}$ and $L_{4,Z}$ is non-zero. Moreover, $\operatorname{Aut}(C) = \varrho_1(\mathbb{Z}/2\mathbb{Z})$ unless $L_{2,Z}$, $L_{4,Z}$ and $L_{6,Z}$ belong to the ring $K[X^2, Y^2]$. In the latter case, $\operatorname{Aut}(C) = \varrho_1((\mathbb{Z}/2\mathbb{Z})^2)$.

Corollary 2.2. *The strata $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/2\mathbb{Z})$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}((\mathbb{Z}/2\mathbb{Z})^2)$ are ES-irreducible.*

Definition 2.3 ([15])**.** An homology of period n is a projective linear transformation of the plane $\mathbb{P}^2(K)$, which is $\mathrm{PGL}_3(K)$-conjugate to $\operatorname{diag}(1, 1, \zeta_n)$. Such a transformation fixes pointwise a line $\mathcal{L}$ (its axis) and a point P off this line (its center). In its canonical form, $\mathcal{L} : Z = 0$ and center $P = (0 : 0 : 1)$.

Otherwise, it is called a *non-homology*.

Theorem 2.4. *Let C be a smooth plane sextic curve that admits an homology of period 3 as an automorphism of maximal order. Up to K-isomorphism, C is defined by an equation of the form $Z^6 + Z^3 L_{3,Z} + L_{6,Z} = 0$ where neither $L_{3,Z}$ nor $L_{6,Z}$ equals 0. Moreover, $\operatorname{Aut}(C)$ is always $\varrho_1(\mathbb{Z}/3\mathbb{Z})\rangle$ except when C is K-isomorphic to C' of the form $C' : X^6 + Y^6 + Z^6 + Z^3\left(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3\right) + \alpha_{3,3}X^3Y^3 = 0$, such that $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ are pair-wise distinct modulo $\{\pm 1\}$. In this case, $\operatorname{Aut}(C') = \varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.*

Theorem 2.5. *Let C be a smooth plane sextic curve that admits a non-homology of period 3 as an automorphism of maximal order. Up to K-isomorphism, C is a member of one of the following families:*

$$\begin{aligned}
\mathcal{C}_1 \quad &: \quad X^6 + Y^6 + Z^6 + XYZ\left(\alpha_{4,1}X^3 + \alpha_{1,4}Y^3 + \alpha_{1,1}Z^3\right) + \alpha_{2,2}X^2Y^2Z^2 \\
&+ \quad \alpha_{3,3}X^3Y^3 + \alpha_{3,0}X^3Z^3 + \alpha_{0,3}Y^3Z^3 = 0 \\
\mathcal{C}_2 \quad &: \quad X^5Y + Y^5Z + XZ^5 + XYZ\left(\alpha_{3,2}X^2Y + \alpha_{1,3}Y^2Z + \alpha_{2,1}XZ^2\right) \\
&+ \quad \alpha_{2,4}X^2Y^4 + \alpha_{0,2}Y^2Z^4 + \alpha_{4,0}X^4Z^2 = 0.
\end{aligned}$$

In either way, $\sigma = \operatorname{diag}(1, \zeta_3, \zeta_3^{-1})$ is an automorphism of maximal order 3.

(1) The automorphism group $\operatorname{Aut}(\mathcal{C}_1) = \varrho_2(\mathbb{Z}/3\mathbb{Z})$ except when one of the following conditions holds.

(i) *If* $\alpha_{4,1}=\alpha_{1,4}=\alpha_{1,1}=\alpha_{2,2}=0$ *such that* $\alpha_{3,3}\neq\alpha_{3,0}$, *then* $\mathcal{C}_1$ *reduces to*

$$X^6+Y^6+Z^6+X^3\left(\alpha_{3,3}Y^3+\alpha_{3,0}Z^3\right)+\alpha_{0,3}Y^3Z^3=0,$$

where $\mathrm{Aut}(\mathcal{C}_1)=\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$.

(ii) *If* ***(a)*** $\alpha_{4,1}=\pm\alpha_{1,4}$ *and* $\alpha_{3,0}=\pm\alpha_{0,3}$, ***(b)*** $\alpha_{1,4}=\pm\alpha_{1,1}$ *and* $\alpha_{3,3}=\pm\alpha_{3,0}$, *or* ***(c)*** $\alpha_{4,1}=\pm\alpha_{1,1}$ *and* $\alpha_{3,3}=\pm\alpha_{0,3}$, *then* $\mathcal{C}_1$ *is* K*-isomorphic to*

$$\begin{aligned}\mathcal{C}_1' \quad:\quad & X^6+Y^6+Z^6+\alpha'_{4,1}X^4YZ+\alpha'_{3,3}X^3(Y^3+Z^3)+\alpha'_{2,2}X^2Y^2Z^2\\ + \quad & \alpha'_{1,2}XYZ(Y^3+Z^3)+\alpha'_{0,3}Y^3Z^3=0,\end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}_1')=\varrho_2(\mathrm{S}_3)$ *if* $\alpha'_{4,1}\neq\alpha'_{1,2}$ *or* $\alpha'_{3,3}\neq\alpha'_{0,3}$, *and* $\mathrm{Aut}(\mathcal{C}_1')=\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$ *otherwise.*

Remark 2.6. $\left(\alpha'_{3,3},\alpha'_{1,2}\right)\neq(0,0)$ *or* $\mathrm{diag}(1,\zeta_6,\zeta_6^{-1})$ *will be an automorphism of order* $6>3$.

(iii) *If* $\alpha_{4,1}=\zeta_6^{\ell}\alpha_{1,1},\alpha_{1,4}=\pm\zeta_6^{-\ell}\alpha_{1,1},\ \alpha_{3,3}=\pm(-1)^{\ell}\alpha_{3,0},\ \alpha_{0,3}=\pm\alpha_{3,0}$ *for some* $\ell\neq 0$ *or* $3 \bmod 6$, *then* $\mathcal{C}_1$ *is* K*-isomorphic to*

$$\begin{aligned}\mathcal{C}_1'' \quad:\quad & X^6+\zeta_6^{2\ell}Y^6+\zeta_6^{-2\ell}Z^6+\alpha'_{1,1}XYZ(X^3+\zeta_6^{2\ell}Y^3+\zeta_6^{-2\ell}Z^3)+\\ + \quad & \alpha'_{3,0}(X^3Y^3+\zeta_6^{-2\ell}X^3Z^3+\zeta_6^{2\ell}Y^3Z^3)=0.\end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}_1'')=\varrho_2((\mathbb{Z}/3\mathbb{Z})^2)$.

(iv) *If* ***(a)*** $(\alpha_{4,1},\alpha_{1,1},\alpha_{1,4})$, $(\alpha_{1,4},\alpha_{4,1},\alpha_{1,1})$ *or* $(\alpha_{1,1},\alpha_{1,4},\alpha_{4,1})$ *equals*

$$\left(\frac{2\left(29-54\lambda^6-54\mu^6\right)}{27\lambda\mu},\frac{2\left(27\mu^6-54\lambda^6-52\right)}{27\lambda\mu^4},\frac{2\left(27\lambda^6-54\mu^6-52\right)}{27\lambda^4\mu}\right),$$

(b) $(\alpha_{3,0},\alpha_{3,3},\alpha_{0,3})$, $(\alpha_{3,3},\alpha_{0,3},\alpha_{3,0})$ *or* $(\alpha_{0,3},\alpha_{3,0},\alpha_{3,3})$ *equals*

$$\left(\frac{2\left(81\lambda^6-27\mu^6-26\right)}{27\mu^3},\frac{2\left(81\mu^6-27\lambda^6-26\right)}{27\lambda^3},\frac{2\left(82-27\lambda^6-27\mu^6\right)}{27\lambda^3\mu^3}\right),$$

and ***(c)*** $\alpha_{2,2}=\dfrac{9\lambda^6+9\mu^6+10}{3\lambda^2\mu^2}$ *for some* $\lambda,\mu\in K^*$, *then* $\mathcal{C}_1$ *is* K*-isomorphic to*

$$\begin{aligned}\mathcal{C}_{1,\lambda,\mu}: X^6+Y^6+Z^6 \quad + \quad & f_1(\lambda,\mu)X^2Y^2Z^2+f_2(\lambda,\mu)(X^4Y^2+X^2Z^4+Y^4Z^2)\\ + \quad & f_2(\mu,\lambda)(X^4Z^2+X^2Y^4+Y^2Z^4)=0,\end{aligned}$$

where

$$\begin{aligned}f_1(\lambda,\mu) &:= 3(80+81\lambda^6+81\mu^6),\\ f_2(\lambda,\mu) &:= 81\left(1+\zeta_3\lambda^6+\zeta_3^{-1}\mu^6\right).\end{aligned}$$

In this case, $\mathrm{Aut}(\mathcal{C}_{1,\lambda,\mu})=\varrho_1(\mathrm{A}_4)$.

(2) *The automorphism group* $\mathrm{Aut}(\mathcal{C}_2)=\langle\sigma\rangle=\varrho_2(\mathbb{Z}/3\mathbb{Z})$ *except when one of the following conditions holds.*

(i) *If* $\alpha_{0,2}=\zeta_{21}^{-12r}\alpha_{4,0},\ \alpha_{2,4}=\zeta_{21}^{3r}\alpha_{4,0},\ \alpha_{1,3}=\zeta_{21}^{-6r}\alpha_{3,2},\ \alpha_{2,1}=\zeta_{21}^{3r}\alpha_{3,2}$, *then* $\mathcal{C}_2$ *is* K*-isomorphic to*

$$\begin{aligned}\mathcal{C}_2' \quad:\quad & X^5Y+Y^5Z+XZ^5+\alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2+X^2Y^4+Y^2Z^4\right)\\ + \quad & \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y+XZ^2+Y^2Z\right)=0,\end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}_2')=\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.

Remark 2.7. $(\alpha_{2,4},\alpha_{1,3})\neq(0,0)$ *or* $\mathrm{diag}(1,\zeta_{21},\zeta_{21}^{-4})$ *will be an automorphism of order* $21>3$.

*(ii) If **(a)*** $(\alpha_{2,4},\alpha_{4,0},\alpha_{0,2})$, $(\alpha_{0,2},\alpha_{2,4},\alpha_{4,0})$ *or* $(\alpha_{4,0},\alpha_{0,2},\alpha_{2,4})$ *equals*

$$\left(\frac{\lambda^5\mu+4\mu^5}{2\lambda^4},\ \frac{\lambda+4\lambda^5\mu}{2\mu^2},\ \frac{4\lambda+\mu^5}{2\lambda^2\mu^4}\right)$$

*and **(b)*** $(\alpha_{1,3},\alpha_{3,2},\alpha_{2,1})$, $(\alpha_{2,1},\alpha_{1,3},\alpha_{3,2})$ *or* $(\alpha_{3,2},\alpha_{2,1},\alpha_{1,3})$ *equals*

$$\left(\frac{2\left(2\lambda^5\mu+2\lambda+\mu^5\right)}{\lambda^3\mu^2},\ \frac{2\lambda^5\mu+4\lambda+4\mu^5}{\lambda^2\mu},\ \frac{2\left(2\lambda^5\mu+\lambda+2\mu^5\right)}{\lambda\mu^3}\right),$$

then $\mathcal{C}_2$ *is* K*-isomorphic to*

$$\begin{aligned}\mathcal{C}_{2,\lambda,\mu}: X^6+Y^6+Z^6 \quad &+\quad g_1(\lambda,\mu)(\zeta_3^{-1}X^4Y^2+X^2Z^4+Y^4Z^2)\\ &+\quad g_2(\lambda,\mu)(X^4Z^2+\zeta_3X^2Y^4+Y^2Z^4)=0,\end{aligned}$$

where

$$\begin{aligned}g_1(\lambda,\mu) &:= \frac{\sqrt{3}\zeta_9\left(\zeta_4\lambda^5\mu+\zeta_{12}\lambda+\zeta_{12}^5\mu^5\right)}{\lambda^5\mu+\lambda+\mu^5},\\ g_2(\lambda,\mu) &:= \frac{\sqrt{3}\zeta_{18}\left(\zeta_{12}^5\lambda^5\mu+\zeta_{12}\lambda+\zeta_4\mu^5\right)}{\lambda^5\mu+\lambda+\mu^5}.\end{aligned}$$

In this case, $\mathrm{Aut}(\mathcal{C}_{2,\lambda,\mu})=\varrho_2(\mathrm{A}_4)$.

We now introduce the notion of *fake ES-irreducible components.*

Definition 2.8. An ES-irreducible component $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\varrho(G))$ is *fake* if it is not defined by a single normal form.

As a consequence of Theorems 2.4 and 2.5:

Corollary 2.9. *The strata* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z})$ *and* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}((\mathbb{Z}/3\mathbb{Z})^2)$ *are not ES-irreducible and each of them has exactly two ES-irreducible components namely,* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_i(\mathbb{Z}/3\mathbb{Z}))$ *and* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_i\left((\mathbb{Z}/3\mathbb{Z})^2\right))$ *respectively with* $i=1$ *and* 2.

On the other hand, the components $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2(\mathbb{Z}/3\mathbb{Z}))$ *and* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right))$ *are the first examples of fake ES-irreducible components. Any* $[C]$ *in the family* $\mathcal{C}_2$ *that belongs to* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2(\mathbb{Z}/3\mathbb{Z}))$ *or* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right))$ *has the property that its automorphism group fixes the triangle* $\triangle$ *whose vertices* $P_1=(1:0:0)$, $P_2=(0:1:0)$ *and* $P_2=(0:0:1)$ *lie on* C*. This does not hold if* $[C]$ *is in the family* $\mathcal{C}_1$*, in the sense that it is not necessarily true that* $\mathrm{Aut}(C)=\varrho_2(\mathbb{Z}/3\mathbb{Z})$ *or* $\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ *fixes a triangle whose vertices lie on* C*. For example, take* $[C]$ *as in* $\mathcal{C}_1''$ *with* $1+\alpha_{1,1}'+\alpha_{3,0}'\neq 0$.

Corollary 2.10. *The strata* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{S}_3)$ *and* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$ *are ES-irreducible. More precisely,* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{S}_3)=\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_2(\mathrm{S}_3))$ *and* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)=\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3))$.

Corollary 2.11. *The stratum* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{A}_4)$ *is ES-irreducible determined by* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathrm{A}_4))$*. It represents the second example of fake ES-irreducible components. Indeed,* $\mathcal{C}_{2,\lambda,\mu}$ *is* K*-isomorphic, via a change of variables* $\phi=\mathrm{diag}(1,s,t)$ *such that* $s=t^2$ *and* $t^3=\zeta_6$*, to* ${}^{\phi}\mathcal{C}_{2,\lambda,\mu}: X^6+\zeta_3^{-1}Y^6+\zeta_3Z^6+$ *lower order terms, where* $\mathrm{Aut}({}^{\phi}\mathcal{C}_{2,\lambda,\mu})=\varrho_1(\mathrm{A}_4)$*. Moreover, any* $[C]\in\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_1(\mathrm{A}_4))$ *in the family* $\mathcal{C}_{1,\lambda,\mu}$ *is a descendant of the Fermat curve* $\mathcal{F}_6$ *in the sense of Theorem 3.1 via a change of variables in the normalizer of* $\varrho_1(\mathrm{A}_4)$ *in* $\mathrm{PGL}_3(K)$*. This does not hold if* $[C]$ *is in the family* ${}^{\phi}\mathcal{C}_{2,\lambda,\mu}$.

3. Preliminaries about automorphism groups

Based entirely on geometrical methods, H. Mitchell [15, §1-10] proved that if G is a finite subgroups of $\mathrm{PGL}_3(K)$, then it fixes a point, a line or a triangle unless it is primitive and conjugate to some group in a specific list. However, as a consequence of Maschke's theorem in group representation theory, the first two cases are equivalent, in the sense that if G fixes a point (respectively a line), then it also fixes a line not passing through the point (respectively a point not lying the line).

Notation. For a non-zero monomial $cX^{i_1}Y^{i_2}Z^{i_3}$ with $c \in K^*$, its exponent is defined to be $\max\{i_1, i_2, i_3\}$. For a homogenous polynomial $F(X, Y, Z)$, the core of it is defined to be the sum of all terms of F with the greatest exponent. Now, let C_0 be a non-singular plane curve over K, a pair (C, G) with $G \leq \mathrm{Aut}(C)$ is said to be a descendant of C_0 if C is defined by a homogenous polynomial whose core is a defining polynomial of C_0 and G acts on C_0 under a suitable change of the coordinates system, i.e. G is $\mathrm{PGL}_3(K)$-conjugate to a subgroup of $\mathrm{Aut}(C_0)$.

An element of $\mathrm{PGL}_3(K)$ is called *intransitive* if it has the matrix shape

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The subgroup of $\mathrm{PGL}_3(K)$ of all intransitive elements is denoted by $\mathrm{PBD}(2, 1)$. Obviously, there is a natural map $\Lambda : \mathrm{PBD}(2, 1) \to \mathrm{PGL}_2(K)$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PBD}(2, 1) \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \mathrm{PGL}_2(K).$$

Theorem 3.1 below is very helpful for determining the full automorphism groups of smooth plane curves. For more details, we refer to the work of T. Harui [9, Theorem 2.1].

Theorem 3.1. *Let C be a non-singular plane curve of degree $d \geq 4$ defined over an algebraically closed field K of characteristic 0. Then, one of the following situations holds:*

1. $\mathrm{Aut}(C)$ *fixes a point on C and then it is cyclic.*
2. $\mathrm{Aut}(C)$ *fixes a point not lying on C where we can think about $\mathrm{Aut}(C)$ in the following commutative diagram, with exact rows and vertical injective morphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^* & \longrightarrow & \mathrm{PBD}(2,1) & \xrightarrow{\Lambda} & \mathrm{PGL}_2(K) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & N & \longrightarrow & \mathrm{Aut}(C) & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

Here, N is a cyclic group of order dividing the degree d and G' is a subgroup of $\mathrm{PGL}_2(K)$, which is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m with $m \leq d-1$, a Dihedral group D_{2m} of order $2m$ with $|N| = 1$ or $m|(d-2)$, one of the alternating groups A_4, A_5, or the symmetry group S_4.

Remark 3.2. *We note that N is viewed as the part of $\mathrm{Aut}(C)$ acting on the variable $B \in \{X, Y, Z\}$ and fixing the other two variables, while G' is the part acting on $\{X, Y, Z\} - \{B\}$ and fixing B. For example, if $B = X$, then every automorphism in N has the shape $\mathrm{diag}(\zeta_n, 1, 1)$ for some nth root of unity ζ_n.*

3. *$\mathrm{Aut}(C)$ is conjugate to a subgroup G of $\mathrm{Aut}(\mathcal{F}_d)$, where $\mathcal{F}_d$ is the Fermat curve $X^d+Y^d+Z^d=0$. In particular, $|G|$ divides $|\mathrm{Aut}(\mathcal{F}_d)|=6d^2$, and (C,G) is a descendant of $\mathcal{F}_d$.*
4. *$\mathrm{Aut}(C)$ is conjugate to a subgroup G of $\mathrm{Aut}(\mathcal{K}_d)$, where $\mathcal{K}_d$ is the Klein curve curve $X^{d-1}Y+Y^{d-1}Z+XZ^{d-1}=0$. In this case, $|\mathrm{Aut}(C)|$ divides $|\mathrm{Aut}(\mathcal{K}_d)|=3(d^2-3d+3)$, and (C,G) is a descendant of $\mathcal{K}_d$.*
5. *$\mathrm{Aut}(C)$ is conjugate to one of the finite primitive subgroup of $\mathrm{PGL}_3(K)$ namely, the Klein group $\mathrm{PSL}(2,7)$, the icosahedral group A_5, the alternating group A_6, or to one of the Hessian groups Hess_* with $*\in\{36,\ 72,\ 216\}$.*

Finally, we have:

Proposition 3.3. *The automorphism groups of the Fermat sextic curve $\mathcal{F}_6$ generated by $[X:Z:Y]$, $[Y:Z:X]$, $\mathrm{diag}(\zeta_6,1,1)$ and $\mathrm{diag}(1,\zeta_6,1)$ of orders $2,3,6$ and 6 respectively is isomorphic to $\mathrm{GAP}(216,92)=(\mathbb{Z}/6\mathbb{Z})^2\rtimes\mathrm{S}_3$. On the other hand, the automorphism group of the Klein sextic curve $\mathcal{K}_6$ generated by $\mathrm{diag}(1,\zeta_{21},\zeta_{21}^{-4})$ and $[Y:Z:X]$ of orders 21 and 3 respectively is isomorphic to $\mathrm{GAP}(63,3)=\mathbb{Z}/21\mathbb{Z}\rtimes\mathbb{Z}/3\mathbb{Z}$.*

Proof. Regarding the generators of $\mathrm{Aut}(\mathcal{F}_6)$ and $\mathrm{Aut}(\mathcal{K}_6)$, we refer the reader to [9, Propositions 3.3, 3.5]. Now, for the Fermat curve $\mathcal{F}_6$, take $a=[X:Z:Y]$, $b=[Y:Z:X]$, $c=\mathrm{diag}(\zeta_6,1,1)$ and $d=\mathrm{diag}(1,\zeta_6,1)$. One verifies that

$$(ab)^2=(ac)(ca)^{-1}=(cd)(dc)^{-1}=ada(cd)^{-5}=bcb^{-1}(cd)^{-5}=1.$$

These relations give us the 4th semidirect product of $(\mathbb{Z}/6\mathbb{Z})^2$ and S_3 acting faithfully, see semidirect products of $(\mathbb{Z}/6\mathbb{Z})^2$ and S_3 for more details.

For the Klein curve $\mathcal{K}_6$, the two generators $a=\mathrm{diag}(1,\zeta_{21},\zeta_{21}^{-4})$ and $b=[Y:Z:X]$ of orders 21 and 3 respectively produce $\mathrm{GAP}(63,3)=\mathbb{Z}/21\mathbb{Z}\rtimes\mathbb{Z}/3\mathbb{Z}$ as $ba=(ab)^{-5}$. □

4. Proof of Theorem 2.4

In this case, $C:F(X,Y,Z)=0$ has an homology σ of period 3 in its automorphism group. The results in [2] allows us to assume that σ acts as

$$(X:Y:Z)\mapsto(X:Y:\zeta_3Z)$$

up to K-isomorphism, where ζ_3 is a fixed primitive 3rd root of unity in K. In particular, C is defined over K by a non-singular plane equation of the form:

$$C:Z^6+Z^3L_{3,Z}+L_{6,Z}=0,$$

where $\sigma=\mathrm{diag}(1,1,\zeta_3)$ is an automorphism of maximal order 3. By non-singularity, $L_{6,Z}$ should be of degree at least 5 in both variables X and Y. Also, $L_{3,Z}\neq0$ or $\mathrm{diag}(1,1,\zeta_6)$ would be an automorphism of order $6>3$.

In the sense of Theorem 3.1, we have the following:

- First, $\mathrm{Aut}(C)$ is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$ since each of them contains elements of order >3. Also, C is not a descendant of the Klein sextic curve $\mathcal{K}_6$ because $\mathrm{Aut}(\mathcal{K}_6)$ by Proposition 3.3 equals $\mathbb{Z}/21\mathbb{Z}\rtimes\mathbb{Z}/3\mathbb{Z}$ and it does not contains homologies of order 3 similar to σ.
- Secondly, suppose that C is a descendant of the Fermat curve $\mathcal{F}_6$. So there is a $\phi\in\mathrm{PGL}_3(K)$ such that $\phi^{-1}\,\mathrm{Aut}(C)\phi\leq\mathrm{Aut}(\mathcal{F}_6)$ and the transformed equation ${}^{\phi}C$ is $X^6+Y^6+Z^6+$ lower order terms in $X,Y,Z\ =0$. There is no loss of generality to impose $\phi^{-1}\langle\sigma\rangle\phi=\langle\sigma\rangle$ since homologies of period

3 inside $\mathrm{Aut}(\mathcal{F}_6)$ form two conjugacy classes represented by σ and σ^{-1}. Hence ${}^{\phi}C$ reduces to

$$ {}^{\phi}C : X^6 + Y^6 + Z^6 + Z^3 L_{3,Z} + \text{lower order terms in } X, Y = 0 $$

Furthermore, by assumption, the automorphisms of C have orders ≤ 3, then the group structure of $\mathrm{Aut}(\mathcal{F}_6) = (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$ assures that $\mathrm{Aut}({}^{\phi}C)$ would be one of the following groups inside $\mathrm{Aut}(\mathcal{F}_6)$:

$$ \mathbb{Z}/3\mathbb{Z},\ (\mathbb{Z}/3\mathbb{Z})^2,\ \mathrm{S}_3,\ \mathrm{A}_4,\ \mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3,\ \mathrm{He}_3\,. $$

For more details, check the subgroups lattice of $\mathrm{Aut}(\mathcal{F}_6)$.

Now we tackle each of the above situations.

- Any copy of S_3 (respectively A_4) inside $\mathrm{Aut}(\mathcal{F}_6)$ is $\mathrm{Aut}(\mathcal{F}_6)$-conjugate to either $\varrho_i(\mathrm{S}_3)$ (respectively $\varrho_i(\mathrm{A}_4)$) with $i = 1$ or 2. But none of these subgroups has homologies of period 3 similar to σ. So $\mathrm{Aut}({}^{\phi}C)$ can not be an S_3 or A_4 inside $\mathrm{Aut}(\mathcal{F}_6)$.

- If $\mathrm{Aut}({}^{\phi}C)$ equals a $(\mathbb{Z}/3\mathbb{Z})^2$, $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$ or He_3 in $\mathrm{Aut}(\mathcal{F}_6)$, then there must be $\sigma' \in \mathrm{Aut}(\mathcal{F}_6) \cap \mathrm{Aut}({}^{\phi}C)$ of order 3 that commutes with σ as in any of these groups $\mathbb{Z}/3\mathbb{Z}$ is always contained in a $(\mathbb{Z}/3\mathbb{Z})^2$. By Proposition 3.3, the elements of order 3 in $\mathrm{Aut}(\mathcal{F}_6)$ are $\mathrm{diag}(1, s, t)$ with $s^3 = t^3 = 1, [sY : tZ : X]$ and $[tZ : X : sY]$ with $s^6 = t^6 = 1$. One easily verifies that only the diagonal shapes satisfies the description, equivalently, $\sigma' \in \langle \sigma, \mathrm{diag}(1, \zeta_3, 1)\rangle$. In any case, we can reduce C up to K-isomorphism to

$$ {}^{\phi}C : X^6 + Y^6 + Z^6 + Z^3\left(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3\right) + \alpha_{3,3}X^3Y^3 = 0, $$

where $\varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right) \leq \mathrm{Aut}({}^{\phi}C)$.

Remark 4.1. *In this scenario, the parameters $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ must be pairwise distinct modulo $\{\pm 1\}$ or ${}^{\phi}C$ will admit automorphisms of order > 3. For example, $[\zeta_3 Y : X : Z] \in \mathrm{Aut}({}^{\phi}C)$ has order 6 if $\alpha_{3,0} = \alpha_{0,3}$ and $[\zeta_3 Y : X : -Z] \in \mathrm{Aut}({}^{\phi}C)$ has order 6 if $\alpha_{3,0} = -\alpha_{0,3}$.*

A similar discussion shows that any $\sigma'' \in \mathrm{Aut}(\mathcal{F}_6)$ that commutes with σ or σ' belongs to $\langle \sigma, \sigma' \rangle$. Therefore, $\mathrm{Aut}({}^{\phi}C)$ can not be the Heisenberg group He_3 because this requires another automorphism $\sigma'' \notin \langle \sigma, \sigma' \rangle$ that commutes with either σ or σ'.

Finally, for $\mathrm{Aut}({}^{\phi}C)$ to be $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$, it is necessary that $\mathrm{Aut}(\mathcal{F}_6) \cap \mathrm{Aut}({}^{\phi}C)$ has involutions in it. Proposition 3.3 tells us that the involutions of $\mathcal{F}_6$ are $\mathrm{diag}(-1, 1, 1)$, $\mathrm{diag}(1, -1, 1)$, $\mathrm{diag}(1, 1, -1)$, $[X : sZ : s^{-1}Y]$, $[s^{-1}Y : sX : Z]$ and $[sZ : Y : s^{-1}X]$ with $s^6 = 1$. If any of these involutions lies in $\mathrm{Aut}({}^{\phi}C)$, then two of the parameters are equal modulo $\{\pm 1\}$, which is absurd by Remark 4.1. For example, $\mathrm{diag}(-1, 1, 1) \in \mathrm{Aut}({}^{\phi}C)$ only if $\alpha_{3,0} = \alpha_{3,3} = 0, [sY : s^{-1}X : Z] \in \mathrm{Aut}({}^{\phi}C)$ only if $\alpha_{3,0} = \pm\alpha_{0,3}$, and so on.

- Third, if $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point P not lying on $\mathcal{L}$, then by Theorem 3.1 we can think about $\mathrm{Aut}(C)$ in a short exact sequence

$$ 1 \to N = \langle \sigma \rangle \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1, $$

where $\Lambda(\mathrm{Aut}(C)) \simeq \mathbb{Z}/3\mathbb{Z}$, D_4 or A_4.

- Any group of order 36 (respectively 12) that has a normal subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$ contains elements of order $6 > 3$, see Groups of order 12 and Groups of order 36 for more details. This allows us to exclude that $\Lambda(\mathrm{Aut}(C))$ equals A_4 or D_4.

- On the other hand, if $\Lambda(\mathrm{Aut})(C)$ equals $\mathbb{Z}/3\mathbb{Z}$ in $\mathrm{PGL}_2(K)$, then $\mathrm{Aut}(C)$ equals $(\mathbb{Z}/3\mathbb{Z})^2$ in $\mathrm{PBD}(2, 1)$. In particular, $C : Z^6 + Z^3L_{3,Z} +$

$L_{6,Z} = 0$ admits an automorphism $\sigma' \in \mathrm{PBD}(2,1) - \langle\sigma\rangle$ of order 3 that commutes with σ. Depending on whether σ' is an homology or a non-homology, it is conjugate via a change of variables $\phi \in \mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$, to $\mathrm{diag}(1,\zeta_3,1)$ or $\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ respectively. In either way, $\mathrm{Aut}(^{\phi}C) = \varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ which appeared earlier.

Summing up, we deduce that $\mathrm{Aut}(C)$ is always cyclic of order 3 generated by σ except when C is projectively equivalent to C' of the form

$$C' : X^6 + Y^6 + Z^6 + Z^3\left(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3\right) + \alpha_{3,3}X^3Y^3 = 0,$$

such that $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ are pair-wise distinct modulo $\{\pm 1\}$. In this case, $\mathrm{Aut}(C)$ is conjugate to $(\mathbb{Z}/3\mathbb{Z})^2$ generated by $\mathrm{diag}(1,\zeta_3,1)$ and $\mathrm{diag}(1,\zeta_3,1)$.

This proves Theorem 2.4.

5. Proof of Theorem 2.1

In this case, $C : F(X,Y,Z) = 0$ has an homology σ of period 2 in its automorphism group. By [2], there is no loss of generality to assume that σ acts as

$$(X : Y : Z) \mapsto (X : Y : -Z)$$

up to K-isomorphism. In particular, C is defined over K by a non-singular plane equation of the form:

$$C : Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z} = 0$$

where $\sigma = \mathrm{diag}(1,1,-1)$ is an automorphism of maximal order 2. Again $L_{6,Z}$ is of degree ≥ 5 in X and Y by non-singularity. Also, $L_{2,Z}$ or $L_{4,Z}$ does not vanish or $\mathrm{diag}(1,1,\zeta_6)$ will be an automorphism of order $6 > 3$ otherwise.

- Obviously, $\mathrm{Aut}(C)$ is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$ as each of them contains elements of order > 2. Also, C can not be a descendant of the Klein sextic curve $\mathcal{K}_6$ since $2 \nmid |\,\mathrm{Aut}(\mathcal{K}_6)|$, recall that $|\,\mathrm{Aut}(\mathcal{K}_6)| = 63$ by Proposition 3.3.
- Secondly, if $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point P off $\mathcal{L}$, then, by Theorem 3.1, $\mathrm{Aut}(C)$ is inside $\mathrm{PBD}(2,1)$ and satisfies a short exact sequence

$$1 \to N = \langle\sigma\rangle \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1.$$

  Our assumptions that any automorphism of C has order ≤ 2 implies that $\Lambda(\mathrm{Aut}(C))$ is either $\mathbb{Z}/2\mathbb{Z}$ or D_4 inside $\mathrm{PGL}_2(K)$, so $\mathrm{Aut}(C)$ is conjugate to either $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^3$. In both situations $\mathrm{Aut}(C)$ has another involution σ' that commutes with σ. Up to projective equivalence via a change of variables $\phi \in \mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$ in $\mathrm{PGL}_3(K)$, we can assume that $\sigma' = \mathrm{diag}(1,-1,1)$. Consequently, C is K-isomorphic to $C' : Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z} = 0$ for some $L_{i,Z} \in K[X^2,Y^2]$. Moreover, $\mathrm{Aut}(C)$ equals $(\mathbb{Z}/2\mathbb{Z})^3$ only if there is an involution $\sigma'' \notin \mathrm{PBD}(2,1) - \langle\sigma,\sigma'\rangle$ that commutes with both σ and σ'. It is straightforward to check that such σ'' does not exist, hence $\mathrm{Aut}(C)$ is not $(\mathbb{Z}/2\mathbb{Z})^3$ in this case.
- If C is a descendant of the Fermat curve $\mathcal{F}_6$ via a change of variables $\phi \in \mathrm{PGL}_3(K)$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(^{\phi}C)$ is a copy of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\mathrm{Aut}(\mathcal{F}_6)$. Indeed any other subgroup of $\mathrm{Aut}(\mathcal{F}_6)$ has elements of order > 2, see subgroups lattice of $\mathrm{Aut}(\mathcal{F}_6)$.

  Up to $\mathrm{Aut}(\mathcal{F}_6)$-conjugation, there are two copies of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\mathrm{Aut}(\mathcal{F}_6)$ namely, $\langle\sigma,\sigma'\rangle$ and $\langle\sigma,\tau\rangle$ with $\sigma' = \mathrm{diag}(1,-1,1)$ and $\tau = [Y : X : Z]$. However, both groups are $\mathrm{PGL}_3(K)$-conjugated via a transformation in $\mathrm{PBD}(2,1)$, the normalizer of $\langle\sigma\rangle$ in $\mathrm{PGL}_3(K)$. Thus there is no loss of generality to assume that $\mathrm{Aut}(C)$ is conjugate to $\varrho_1\left((\mathbb{Z}/2\mathbb{Z})^2\right)$, which was treated earlier.

Summing up, we deduce that $\mathrm{Aut}(C)$ is always cyclic of order 2 generated by σ except when $L_{i,Z}\in K[X^2,Y^2]$ for $i=2,4,6$. In the latter case, $\mathrm{Aut}(C)$ equals $\varrho_1\left((\mathbb{Z}/2\mathbb{Z})^2\right)$, which shows Theorem 2.1.

6. Proof of Theorem 2.5

In this case, $C: F(X,Y,Z)=0$ has a non-homology σ of period 3 in its automorphism group. By [2], one can assume that σ acts as

$$(X:Y:Z)\mapsto(X:\zeta_3Y:\zeta_3^{-1}Z)$$

up to K-isomorphism, where ζ_3 is a fixed primitive 3rd root of unity in K. In particular, C is a K-isomorphic to a non-singular plane model in one of the following families:

$$\begin{aligned}\mathcal{C}_1\quad &: \quad X^6+Y^6+Z^6+XYZ\left(\alpha_{4,1}X^3+\alpha_{1,4}Y^3+\alpha_{1,1}Z^3\right)+\alpha_{2,2}X^2Y^2Z^2\\ &+\quad \alpha_{3,3}X^3Y^3+\alpha_{3,0}X^3Z^3+\alpha_{0,3}Y^3Z^3=0\\ \mathcal{C}_2\quad &: \quad X^5Y+Y^5Z+XZ^5+XYZ\left(\alpha_{3,2}X^2Y+\alpha_{1,3}Y^2Z+\alpha_{2,1}XZ^2\right)\\ &+\quad \alpha_{2,4}X^2Y^4+\alpha_{0,2}Y^2Z^4+\alpha_{4,0}X^4Z^2=0.\end{aligned}$$

where $\sigma:=\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ is an automorphism of maximal order 3.

- Again $\mathrm{Aut}(\mathcal{C}_i)$ for $i=1$ and 2 is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(K)$.
- Suppose that $\mathrm{Aut}(\mathcal{C}_i)$ fixes a line $\mathcal{L}$ and a point P not lying on this line. Since σ is a non-homology inside $\mathrm{Aut}(\mathcal{C}_i)$ in its canonical form, $\mathcal{L}$ must be one of the reference lines; $B=0$ with $B=X,Y$ or Z and P is the reference point $(1:0:0)$, $(0:1:0)$ or $(0:0:1)$ respectively.

  - For $\mathcal{C}_2$, the point P belongs to $C: F(X,Y,Z)=0$. Hence $\mathrm{Aut}(\mathcal{C}_2)$ is cyclic, generated by $\langle\sigma\rangle$.

  - For $\mathcal{C}_1$, we can further impose $\mathcal{L}: X=0$ and $P=(1:0:0)$ (in the worst case scenario, one just needs to permute two of the variables and to fix the third one, which preserves the property that σ remains an automorphism). In particular, by Theorem 3.1, $\mathrm{Aut}(\mathcal{C}_1)\subseteq\mathrm{PBD}(2,1)$ and lives in a short exact sequence: $1\to N\to\mathrm{Aut}(\mathcal{C}_1)\to\Lambda(\mathrm{Aut}(\mathcal{C}_1))\to1$, where $N=\langle\tau\rangle$ has order $1,2$ or 3 and $\Lambda(\mathrm{Aut}(C))$ is either $\mathbb{Z}/3\mathbb{Z}$, S_3 with $|N|=1$ or A_4 in $\mathrm{PGL}_2(K)$. First, we easily exclude the case when τ has order 2 because $\sigma\tau$ would be an automorphism of order $6>3$, a contradiction.

  Secondly, we handle each of the remaining cases:

  (i) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathbb{Z}/3\mathbb{Z}$ and $N=1$, then $\mathrm{Aut}(\mathcal{C}_1)=\mathbb{Z}/3\mathbb{Z}$ generated by σ.

  (ii) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathbb{Z}/3\mathbb{Z}$ and $N=\mathbb{Z}/3\mathbb{Z}$, then $\mathrm{Aut}(\mathcal{C}_1)=\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$ generated by σ and $\tau=\mathrm{diag}(\zeta_3,1,1)$. In particular, $\alpha_{4,1}=\alpha_{2,2}=\alpha_{1,1}=\alpha_{1,4}=0$, and $\mathcal{C}_1$ reduces to

  $$X^6+Y^6+Z^6+Z^3\left(\alpha_{3,0}X^3+\alpha_{0,3}Y^3\right)+\alpha_{3,3}X^3Y^3=0,$$

  which happened before in Theorem 2.4. We also remark that $\alpha_{3,0}\neq\alpha_{0,3}$ or $[Y:X:Z]$ will be an extra involution for $\mathcal{C}_1$.
  This clarifies part of Theorem 2.5, (1)-(i).

  (iii) If $\Lambda(\mathrm{Aut}(\mathcal{C}_1))=\mathrm{S}_3$ and $N=1$, then C should have an involution τ such that $\tau\sigma\tau=\sigma^{-1}$. So $\tau=[X:sZ:s^{-1}Y]$, $[sY:s^{-1}X:Z]$ or $[sZ:Y:s^{-1}X]$ with $s^6=1$. This holds if we are in one of the situations: $\alpha_{3,3}=\pm\alpha_{3,0}$ and $\alpha_{1,1}=\pm\alpha_{1,4},\alpha_{0,3}=\pm\alpha_{3,0}$ and

$\alpha_{4,1} = \pm\alpha_{1,4}$, or $\alpha_{3,3} = \pm\alpha_{0,3}$ and $\alpha_{1,1} = \pm\alpha_{4,1}$. Moreover, in all scenarios we can reduce to $\tau = [X : Z : Y]$ via a change of variables ϕ in the normalizer of $\langle\sigma\rangle$, more precisely, via $\phi = \mathrm{diag}(1, \lambda, s\lambda)$ modulo $\langle [X : Z : Y], [Y : Z : X]\rangle$ with $\lambda^6 = 1$. That is, $\mathcal{C}_1$ is K-isomorphic to

$$\begin{aligned}\mathcal{C}_1' \quad : \quad & X^6 + Y^6 + Z^6 + \alpha'_{4,1}X^4YZ + \alpha'_{3,3}X^3(Y^3 + Z^3) + \alpha'_{2,2}X^2Y^2Z^2 \\ + \quad & \alpha'_{1,2}XYZ(Y^3 + Z^3) + \alpha'_{0,3}Y^3Z^3 = 0.\end{aligned}$$

Here $\mathrm{Aut}(\mathcal{C}_1') = \langle\sigma, \tau\rangle = \varrho_1(\mathrm{S}_3)$. In particular, we should impose $\alpha'_{4,1} \neq \alpha'_{1,2}$ or $\alpha'_{3,3} \neq \alpha'_{0,3}$ to avoid having $[Y : Z : X]$ as an extra automorphism. Also, $(\alpha'_{3,3}, \alpha'_{1,2}) \neq (0, 0)$ to avoid having $\mathrm{diag}(1, \zeta_6, \zeta_6^{-1})$ as an extra automorphism of order $6 > 3$.
This shows part of Theorem 2.5, (1)-(ii).

(iv) If $\Lambda(\mathrm{Aut}(C)) = \mathrm{A}_4$, then the Group Structure of A_4 assures that $\Lambda(\mathrm{Aut}(C))$ contains $\Lambda(\tau)$ and $\Lambda(\tau')$ both of order 2 such that

$$\Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \Lambda(\tau'),\ \Lambda(\sigma)\Lambda(\tau')\Lambda(\sigma)^{-1} = \Lambda(\tau')\Lambda(\tau) = \Lambda(\tau)\Lambda(\tau').$$

We aim to show that such τ and τ' do not exist. Write $\Lambda(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then being of order 2 yields $(a + d)b = (a + d)c = 0$ and $a = \pm d$. So $\Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

- If $\Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, then

$$\Lambda(\tau') = \Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3 c & 0 \end{pmatrix} = \Lambda(\tau) \text{ in } \mathrm{PGL}_2(K),$$

which implies that $\Lambda(\tau')\Lambda(\tau) \neq \Lambda(\tau)\Lambda(\tau')$ a contradiction.

- If $\Lambda(\tau) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then $\Lambda(\tau') = \Lambda(\sigma)\Lambda(\tau)\Lambda(\sigma)^{-1} = \begin{pmatrix} a & \zeta_3^{-1}b \\ \zeta_3 c & -a \end{pmatrix}$ such that $\Lambda(\tau)\Lambda(\tau') = \Lambda(\tau')\Lambda(\tau)$. That is,

$$\begin{pmatrix} a^2 + \zeta_3 bc & (\zeta_3^{-1} - 1)ab \\ (1 - \zeta_3)ac & a^2 + \zeta_3^{-1}bc \end{pmatrix} = \begin{pmatrix} a^2 + \zeta_3^{-1}bc & -(\zeta_3^{-1} - 1)ab \\ -(1 - \zeta_3)ac & a^2 + \zeta_3 bc \end{pmatrix} \text{ in } \mathrm{PGL}_2(K).$$

For this to be true, either $ab = ac = 0$ or $a^2 + \zeta_3 bc = -(a^2 + \zeta_3^{-1}bc)$. Assuming $ab = ac = 0$ yields $\Lambda(\tau') = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3^{-1}c & 0 \end{pmatrix} = \Lambda(\tau)$ in $\mathrm{PGL}_2(K)$ or $\Lambda(\tau') = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = \Lambda(\tau)$ in $\mathrm{PGL}_2(K)$, which is again a contradiction. Assuming $a^2 + \zeta_3 bc = -(a^2 + \zeta_3^{-1}bc)$ yields $c = 2a^2/b$ with $ab \neq 0$. Moreover, $\Lambda(\sigma)\Lambda(\tau')\Lambda(\sigma)^{-1} = \Lambda(\tau)\Lambda(\tau')$, hence

$$\begin{pmatrix} a & \zeta_3 b \\ 2a^2/b & -a \end{pmatrix} = \begin{pmatrix} a(\zeta_3 - \zeta_3^{-1}) & (\zeta_3^{-1} - 1)b \\ 2a^2(1 - \zeta_3)/b & -a(\zeta_3 - \zeta_3^{-1}) \end{pmatrix} \text{ in } \mathrm{PGL}_2(K).$$

This is valid only if $(\zeta_3 - \zeta_3^{-1})\zeta_3 = (\zeta_3^{-1} - 1)$ and $(\zeta_3 - \zeta_3^{-1}) = (1 - \zeta_3)$, however, the second equation is never valid. This means that $\Lambda(\mathrm{Aut}(C)) \neq \mathrm{A}_4$.

- Thirdly, assume that $\mathcal{C}_i$ is a descendant of the Klein sextic curve $\mathcal{K}_6$.

Claim 1. *For $\mathcal{C}_1$ a descendant of $\mathcal{K}_6$,* $\mathrm{Aut}(\mathcal{C}_1) = \varrho_2(\mathbb{Z}/3\mathbb{Z})$.

Proof. (of Claim 1) If $\mathcal{C}_1$ is a descendant of $\mathcal{K}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then, from the Group Structure of $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and since the automorphisms of C have orders ≤ 3, $\mathrm{Aut}(\mathcal{C}_1)$ should be conjugate to a $(\mathbb{Z}/3\mathbb{Z})^2$ in $\mathrm{Aut}(\mathcal{K}_6)$. Thus $\mathcal{C}_1$ has another automorphism $\sigma' \notin \langle\sigma\rangle$ of order 3 that commutes with σ. Direct calculations show that we can take $\sigma' = \mathrm{diag}(1, s, t)$ with $s^3 = t^3 = 1$ or $[sY : tZ : X]$ with $s, t \in K^*$.

In the first case, σ' reduces to an homology as $\sigma' \notin \langle\sigma\rangle$. This is absurd because $\mathrm{Aut}(\mathcal{K}_6)$ does not contain any homologies of period 3. Regarding the second case, any descendant $\mathcal{C}'$ of the Klein curve $\mathcal{C}' : X^5Y + Y^5Z + Z^5X+$ lower terms in X, Y, Z satisfies the property that its automorphism group fixes the triangle Δ whose vertices are the three reference points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$, moreover, those points all lie on $\mathcal{C}'$. Because Δ is the only triangle fixed by $\langle\sigma, [sY : tZ : X]\rangle$ for any s, t and because none of its vertices lies on $\mathcal{C}_1$, we conclude that $\mathrm{Aut}(\mathcal{C}_1)$ can not equal $\langle\sigma, [sY : tZ : X]\rangle$. This proves the claim for $\mathcal{C}_1$. □

Claim 2. *For $\mathcal{C}_2$ a descendant of $\mathcal{K}_6$, $\mathrm{Aut}(\mathcal{C}_2)$ is either conjugate to $\varrho_2(\mathbb{Z}/3\mathbb{Z})$ or $\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$.*

Proof. (of Claim 2) If $\mathcal{C}_2$ is a descendant of $\mathcal{K}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(\mathcal{C}_2) = \langle\sigma, [sY : tZ : X]\rangle$ for some $s, t \in K^*$. For $\sigma' \in \mathrm{Aut}(\mathcal{C}_2)$, $s = \zeta_{21}^r$, $t = \zeta_{21}^{-4r}$, $\alpha_{0,2} = \zeta_{21}^{-12r}\alpha_{4,0}$, $\alpha_{2,4} = \zeta_{21}^{3r}\alpha_{4,0}$, $\alpha_{1,3} = \zeta_{21}^{-6r}\alpha_{3,2}$, $\alpha_{2,1} = \zeta_{21}^{3r}\alpha_{3,2}$, and $\mathcal{C}_2$ reduces to

$$\begin{aligned} X^5Y \quad &+ \quad Y^5Z + XZ^5 + \alpha_{4,0}\left(X^4Z^2 + \zeta_{21}^{3r}X^2Y^4 + \zeta_{21}^{-12r}Y^2Z^4\right) \\ &+ \quad \alpha_{3,2}XYZ\left(X^2Y + \zeta_{21}^{3r}XZ^2 + \zeta_{21}^{-6r}Y^2Z\right) = 0. \end{aligned}$$

In any situation, there exists a change of variables $\phi = \mathrm{diag}(1, \zeta_{21}^{r'}, \zeta_{21}^{17r'}) \in \mathrm{Aut}(\mathcal{K}_6)$ such that $21 \mid 18r' + r$, $12r' - 4r$ for some $r' \in \{0, 1, ..., 20\}$ that transforms $\mathcal{C}_2$ up to K-isomorphism to

$$\begin{aligned} \mathcal{C}_2' : X^5Y \quad &+ \quad Y^5Z + XZ^5 + \alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2 + X^2Y^4 + Y^2Z^4\right) \\ &+ \quad \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y + XZ^2 + Y^2Z\right) = 0, \end{aligned}$$

where $\mathrm{Aut}(\mathcal{C}_2') = \varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right) = \langle\sigma, [Y : Z : X]\rangle$. In particular, we must have $(\alpha_{2,4}, \alpha_{1,3}) \neq (0, 0)$ or $\mathrm{diag}(1, \zeta_{21}, \zeta_{21}^{-4}) \in \mathrm{Aut}(\mathcal{C}_2')$ of order $21 > 3$.

This completes the proof, which in turns shows Theorem 2.5, (2)-(i). □

- Now, assume that $\mathcal{C}_i$ is a descendant of the Fermat curve $\mathcal{F}_6$. From the Group structure of $\mathrm{Aut}(\mathcal{F}_6)$, one sees that if $\mathcal{C}_i$ is a descendant of $\mathcal{F}_6$ with bigger automorphism group than $\langle\sigma\rangle$, then $\mathrm{Aut}(\mathcal{C}_i)$ is conjugate to one of the following groups inside $\mathrm{Aut}(\mathcal{F}_6)$:

$$(\mathbb{Z}/3\mathbb{Z})^2, \ \mathrm{S}_3, \ \mathrm{A}_4, \ \mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3, \ \mathrm{He}_3\,.$$

In what follows, we treat each of these cases for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, more precisely, Claim 3 and Claim 4 below.

Claim 3. *For $\mathcal{C}_1$ a descendant of $\mathcal{F}_6$, $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to $\varrho_2(\mathbb{Z}/3\mathbb{Z})$, $\varrho_2\left(\mathrm{S}_3\right)$, $\varrho_1\left(\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3\right)$, $\varrho_1\left((\mathbb{Z}/3\mathbb{Z})^2\right)$, $\varrho_2\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ or $\varrho_1(\mathrm{A}_4)$.*

Claim 4. *For $\mathcal{C}_2$ a descendant of $\mathcal{F}_6$, $\mathrm{Aut}(\mathcal{C}_2)$ is conjugate to $\varrho_2(\mathbb{Z}/3\mathbb{Z})$, $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2)$ or $\varrho_2(\mathrm{A}_4)$.*

Proof. (of Claim 3) - If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to S_3 or $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$ inside $\mathrm{Aut}(\mathcal{F}_6)$, then $\mathcal{C}_1$ has an involution τ such that $\tau\sigma\tau = \sigma^{-1}$. Similarly as before, this holds only if $\alpha_{3,3} = \pm\alpha_{3,0}$ and $\alpha_{1,1} = \pm\alpha_{1,4}, \alpha_{0,3} = \pm\alpha_{3,0}$ and $\alpha_{4,1} = \pm\alpha_{1,4}$, or $\alpha_{3,3} = \pm\alpha_{0,3}$ and $\alpha_{1,1} = \pm\alpha_{4,1}$. In this scenario, $\mathcal{C}_1$ is K-isomorphic to

$$\begin{aligned} \mathcal{C}_1' \quad &: \quad X^6 + Y^6 + Z^6 + \alpha_{4,1}'X^4YZ + \alpha_{3,3}'X^3(Y^3 + Z^3) + \alpha_{2,2}'X^2Y^2Z^2 \\ &+ \quad \alpha_{1,2}'XYZ(Y^3 + Z^3) + \alpha_{0,3}'Y^3Z^3 = 0, \end{aligned}$$

where $\varrho_2(\mathrm{S}_3)$ generated by $\sigma=\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ and $\tau=[X:Z:Y]$ is a subgroup of $\mathrm{Aut}(\mathcal{C}_1')$. Furthermore, if $\mathrm{Aut}(\mathcal{C}_1')$ equals $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$, then it must contain another automorphism $\sigma'\notin\langle\sigma,\tau\rangle$ of order 3 that commutes with σ and satisfies $\tau\sigma'\tau=\sigma'^{-1}$. Thus $\sigma'=[s'Y:s'^{-1}Z:X]$ and the invariance of the defining equation for $\mathcal{C}_1'$ under the action of σ' yields $s'^3=1,\alpha'_{4,1}=\alpha'_{1,2}$ and $\alpha'_{3,3}=\alpha'_{0,3}$. Hence $\mathcal{C}_1'$ becomes

$$\begin{aligned}X^6 &+ Y^6+Z^6+\alpha'_{1,2}XYZ(X^3+Y^3+Z^3)+\alpha'_{3,3}(X^3Y^3+Y^3Z^3+Z^3X^3)\\ &+ \alpha'_{2,2}X^2Y^2Z^2=0\end{aligned}$$

with $\mathrm{Aut}(\mathcal{C}_1')=\varrho_1(\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3)$. This shows the rest of Theorem 2.5, (1)-(ii).

- If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to $(\mathbb{Z}/3\mathbb{Z})^2$ or He_3 inside $\mathrm{Aut}(\mathcal{F}_6)$, then $\mathcal{C}_1$ would have an automorphism $\sigma'\notin\langle\sigma\rangle$ of order 3 that commutes with σ since every copy of $\mathbb{Z}/3\mathbb{Z}$ in any of these groups is contained in a $(\mathbb{Z}/3\mathbb{Z})^2$. Similarly as before, we can take $\sigma'=\mathrm{diag}(1,s',t')$ with $s'^3=t'^3=1$ or $[s'Y:t'Z:X]$ with $s',t'\in K^*$.

(i) Suppose that $\sigma'=\mathrm{diag}(1,s',t')\in\mathrm{Aut}(\mathcal{C}_1)$. Because $\sigma'\notin\langle\sigma\rangle$, we have $\sigma'=\mathrm{diag}(1,1,\zeta_3),\ \mathrm{diag}(\zeta_3,1,1)$ or $\mathrm{diag}(1,\zeta_3,1)$. Consequently, $\alpha_{4,1}=\alpha_{2,2}=\alpha_{1,1}=\alpha_{1,4}=0$ and $\mathcal{C}_1$ reduces to

$$X^6+Y^6+Z^6+\alpha_{3,3}X^3Y^3+\alpha_{3,0}X^3Z^3+\alpha_{0,3}Y^3Z^3=0,$$

with $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)\subseteq\mathrm{Aut}(\mathcal{C}_1)$. On the other hand, $\mathrm{Aut}(\mathcal{C}_1)$ equals He_3 only if it contains an extra automorphism $\sigma''\notin\langle\sigma,\sigma'\rangle$ of order 3 that commutes with σ and satisfies $\sigma''\sigma'\sigma''^{-1}=\sigma'\sigma^{-1}$. This gives us $\sigma''=[s''Y:t''Z:X]$ for some $s'',t''\in K^*$. Hence $s''^6=t''^6=1,\ \alpha_{3,3}=s''^3\alpha_{3,0},\ \alpha_{0,3}=t''^3\alpha_{3,0}$, and $\mathcal{C}_1$ becomes of the form:

$$X^6+Y^6+Z^6+\alpha_{3,0}\left(\pm X^3Y^3+X^3Z^3+t''^3Y^3Z^3\right)=0.$$

In particular, $[Y:X:t''Z]$ is an automorphism for $\mathcal{C}_1$ of order divisible by 2. This is a contradiction as $2\nmid|\,\mathrm{He}_3\,|(=27)$.

(ii) Suppose that $\sigma'_{s,t}=[s'Y:t'Z:X]\in\mathrm{Aut}(\mathcal{C}_1)$. For this to be true, we should have $s'^6=t'^6=1,\ \alpha_{4,1}=s't'\alpha_{1,1},\ \alpha_{1,4}=s'^5t'^2\alpha_{1,1},\ \alpha_{3,3}=s'^3\alpha_{3,0}$, $\alpha_{0,3}=t'^3\alpha_{3,0}=\pm\alpha_{3,0}$, and $\mathcal{C}_1$ is defined by

$$\begin{aligned}X^6 &+ Y^6+Z^6+\alpha_{1,1}XYZ(s't'X^3\pm\frac{1}{s't'}Y^3+Z^3)+\alpha_{2,2}X^2Y^2Z^2\\ &+ \alpha_{3,0}(s'^3X^3Y^3+X^3Z^3\pm Y^3Z^3)=0,\end{aligned}$$

such that $(s't')^2=1$ whenever $\alpha_{2,2}\neq0$. Consequently, it must be the case that $\alpha_{2,2}=0$ and $\alpha_{1,1}\neq0$ or $[t'Y:t'^{-1}X:Z]$ would be an extra involution, which violates the fact that $|\,\mathrm{Aut}(\mathcal{C}_1)|=9$ or 27. That is, $s't'=\zeta_6^\ell$ for some $\ell\neq0$ or 3 mod 6, and $\mathcal{C}_1$ becomes

$$\begin{aligned}X^6+Y^6+Z^6 &+ \alpha_{1,1}XYZ(\zeta_6^\ell X^3\pm\zeta_6^{-\ell}Y^3+Z^3)+\\ &+ \alpha_{3,0}(\pm(-1)^\ell X^3Y^3+X^3Z^3\pm Y^3Z^3)=0,\end{aligned}$$

for some $\alpha_{1,1},\alpha_{3,0}\in K^*$. Applying the projective change of variables $\phi=\mathrm{diag}(1,\frac{\sqrt[3]{s't'}}{s'},\frac{1}{\sqrt[3]{s't'}})$ we get

$$\begin{aligned}\mathcal{C}_1'':X^6+\zeta_6^{2\ell}Y^6+\zeta_6^{-2\ell}Z^6 &+ \alpha'_{1,1}XYZ(X^3+\zeta_6^{2\ell}Y^3+\zeta_6^{-2\ell}Z^3)+\\ &+ \alpha'_{3,0}(X^3Y^3+\zeta_6^{-2\ell}X^3Z^3+\zeta_6^{2\ell}Y^3Z^3)=0.\end{aligned}$$

Now with σ and $\sigma'=[Y:Z:X]$ as automorphisms for $\mathcal{C}_1''$, we have that $\langle\sigma,\sigma'\rangle=\varrho_2((\mathbb{Z}/3\mathbb{Z})^2)\subseteq\mathrm{Aut}(\mathcal{C}_1'')$. Again it is impossible that we can enlarge $\mathrm{Aut}(\mathcal{C}_1'')$ to He_3, since this requires $\mathrm{diag}(1,1,\zeta_3)$ to be in $\mathrm{Aut}(\mathcal{C}_1'')$. This cannot be as $\alpha'_{1,1}=\frac{\alpha_{1,1}\zeta_6^\ell}{s}\neq0$.

- If $\mathrm{Aut}(\mathcal{C}_1)$ is conjugate to an A_4 inside $\mathrm{Aut}(\mathcal{F}_6)$, then it should be $\varrho_i(\mathrm{A}_4)$ with $i = 1$ or 2.

(i) First, suppose that $\phi^{-1}\,\mathrm{Aut}(\mathcal{C}_1)\phi = \varrho_1(\mathrm{A}_4)$. As all subgroups of A_4 of order 3 are A_4-conjugated, there is no loss of generality to take $\phi^{-1}\sigma\phi = [Y : Z : X]$ or $[Z : X : Y]$. In particular, ϕ has one of the following shapes:

$$\phi_1 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \\ \mu & \zeta_3\mu & \zeta_3^{-1}\mu \end{pmatrix},\ \phi_2 := \begin{pmatrix} \mu & \zeta_3\mu & \zeta_3^{-1}\mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \end{pmatrix},\ \phi_3 := \begin{pmatrix} \lambda & \zeta_3^{-1}\lambda & \zeta_3\lambda \\ \mu & \zeta_3\mu & \zeta_3^{-1}\mu \\ 1 & 1 & 1 \end{pmatrix},$$

$$\phi_4 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \\ \mu & \zeta_3^{-1}\mu & \zeta_3\mu \end{pmatrix},\ \phi_5 := \begin{pmatrix} \mu & \zeta_3^{-1}\mu & \zeta_3\mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \end{pmatrix},\ \phi_6 := \begin{pmatrix} \lambda & \zeta_3\lambda & \zeta_3^{-1}\lambda \\ \mu & \zeta_3^{-1}\mu & \zeta_3\mu \\ 1 & 1 & 1 \end{pmatrix},$$

for some $\lambda, \mu \in K^*$.

Now, we handle each of these situations to determine the restrictions on the defining equation of $\mathcal{C}_1$ for which this holds.

- For $\phi_1\,\mathrm{diag}(1, 1, -1)\phi_1^{-1}$ (respectively $\phi_4\,\mathrm{diag}(1, 1, -1)\phi_4^{-1}$) to be in $\mathrm{Aut}(\mathcal{C}_1)$, we must eliminate the coefficients of X^5Z, X^5Y, Y^5Z, XZ^5, YZ^5, X^4Y^2, X^4Z^2 from the transformed equation ${}^{\phi_i\,\mathrm{diag}(1,1,-1)\phi_i^{-1}}\mathcal{C}_1 = \mathcal{C}_1$ with $i = 1$ and 4 respectively. In this way, we obtain:

$$\begin{aligned}\alpha_{4,1} &= \frac{2\left(29 - 54\lambda^6 - 54\mu^6\right)}{27\lambda\mu},\ \alpha_{3,3} = \frac{2\left(81\mu^6 - 27\lambda^6 - 26\right)}{27\lambda^3},\\ \alpha_{3,0} &= \frac{2\left(81\lambda^6 - 27\mu^6 - 26\right)}{27\mu^3},\ \alpha_{1,4} = \frac{2\left(27\lambda^6 - 54\mu^6 - 52\right)}{27\lambda^4\mu},\\ \alpha_{1,1} &= \frac{2\left(27\mu^6 - 54\lambda^6 - 52\right)}{27\lambda\mu^4},\ \alpha_{0,3} = \frac{2\left(82 - 27\lambda^6 - 27\mu^6\right)}{27\lambda^3\mu^3},\\ \alpha_{2,2} &= \frac{9\lambda^6 + 9\mu^6 + 10}{3\lambda^2\mu^2}.\end{aligned}$$

  In particular, $\mathcal{C}_1$ is K-isomorphic via ϕ_1 (respectively ϕ_4 followed by $Y \leftrightarrow Z$) to $\mathcal{C}_{1,\lambda,\mu}$ described in Theorem 2.5, (1)-(iii).

- For $\phi_2\,\mathrm{diag}(1, 1, -1)\phi_2^{-1}$ (respectively $\phi_5\,\mathrm{diag}(1, 1, -1)\phi_5^{-1}$) to be in $\mathrm{Aut}(\mathcal{C}_1)$, one notices that $\phi_2 = [Z : X : Y]\,\phi_1 = \phi_1 \circ [Z : X : Y]$ (respectively $\phi_5 = [Z : X : Y]\,\phi_4 = \phi_4 \circ [Z : X : Y]$). This means that we get the same conclusion as above up to a permutation of the parameters, more precisely, after

$$\begin{aligned}(\alpha_{4,1}, \alpha_{1,1}, \alpha_{1,4}) &\mapsto (\alpha_{1,1}, \alpha_{1,4}, \alpha_{4,1})\,,\\ (\alpha_{0,3}, \alpha_{3,3}, \alpha_{3,0}) &\mapsto (\alpha_{3,3}, \alpha_{3,0}, \alpha_{0,3})\,.\end{aligned}$$

  In other words, we have $\phi_i\,\mathrm{diag}(1, 1, -1)\phi_i^{-1}$ with $i = 2$ or 5 inside $\mathrm{Aut}(\mathcal{C}_1)$ only if

$$\begin{aligned}\alpha_{1,4} &= \frac{2\left(29 - 54\lambda^6 - 54\mu^6\right)}{27\lambda\mu},\ \alpha_{0,3} = \frac{2\left(81\mu^6 - 27\lambda^6 - 26\right)}{27\lambda^3},\\ \alpha_{3,3} &= \frac{2\left(81\lambda^6 - 27\mu^6 - 26\right)}{27\mu^3},\ \alpha_{1,1} = \frac{2\left(27\lambda^6 - 54\mu^6 - 52\right)}{27\lambda^4\mu},\\ \alpha_{4,1} &= \frac{2\left(27\mu^6 - 54\lambda^6 - 52\right)}{27\lambda\mu^4},\ \alpha_{3,0} = \frac{2\left(82 - 27\lambda^6 - 27\mu^6\right)}{27\lambda^3\mu^3},\\ \alpha_{2,2} &= \frac{9\lambda^6 + 9\mu^6 + 10}{3\lambda^2\mu^2}.\end{aligned}$$

  Once more $\mathcal{C}_1$ reduces to $\mathcal{C}_{1,\lambda,\mu}$ described in Theorem 2.5, (1)-(iii).

Similarly, $\phi_3 = \phi_1 \circ [Y : Z : X]$ and $\phi_6 = \phi_4 \circ [Y : Z : X]$. So $\phi_i \operatorname{diag}(1, 1, -1)\phi_i^{-1}$ with $i = 3$ or 6 is an automorphism for $\mathcal{C}_1$ only if

$$\begin{aligned}
\alpha_{1,1} &= \frac{2\left(29 - 54\lambda^6 - 54\mu^6\right)}{27\lambda\mu}, \ \alpha_{3,0} = \frac{2\left(81\mu^6 - 27\lambda^6 - 26\right)}{27\lambda^3},\\
\alpha_{0,3} &= \frac{2\left(81\lambda^6 - 27\mu^6 - 26\right)}{27\mu^3}, \ \alpha_{4,1} = \frac{2\left(27\lambda^6 - 54\mu^6 - 52\right)}{27\lambda^4\mu},\\
\alpha_{1,4} &= \frac{2\left(27\mu^6 - 54\lambda^6 - 52\right)}{27\lambda\mu^4}, \ \alpha_{3,3} = \frac{2\left(82 - 27\lambda^6 - 27\mu^6\right)}{27\lambda^3\mu^3},\\
\alpha_{2,2} &= \frac{9\lambda^6 + 9\mu^6 + 10}{3\lambda^2\mu^2},
\end{aligned}$$

where $\mathcal{C}_1$ becomes K-isomorphism to $C_{1,\lambda,\mu}$.

This shows Theorem 2.5, (1)-(iii).

(ii) Second, suppose that $\psi^{-1}\operatorname{Aut}(\mathcal{C}_1)\psi = \varrho_2(\mathrm{A}_4)$. Again, we can impose $\psi^{-1}\sigma\psi = [\zeta_6^{-1}Y : Z : X]$ or $[Z : \zeta_6 X : Y]$, in particular, ψ has the shape of ψ_i below.

$$\psi_1 := \begin{pmatrix} 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \\ \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \\ \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \end{pmatrix}, \ \psi_2 := \begin{pmatrix} \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \\ 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \\ \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \end{pmatrix}, \ \psi_3 := \begin{pmatrix} \lambda & \zeta_{18}^{-8}\lambda & \zeta_{18}^{5}\lambda \\ \mu & \zeta_{18}^{4}\mu & \zeta_{18}^{-7}\mu \\ 1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \end{pmatrix},$$

$$\psi_4 := \begin{pmatrix} 1 & \zeta_{18}^{2} & \zeta_{18} \\ \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \\ \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \end{pmatrix}, \ \psi_5 := \begin{pmatrix} \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \\ 1 & \zeta_{18}^{2} & \zeta_{18} \\ \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \end{pmatrix}, \ \psi_6 := \begin{pmatrix} \lambda & \zeta_{18}^{-4}\lambda & \zeta_{18}^{7}\lambda \\ \mu & \zeta_{18}^{8}\mu & \zeta_{18}^{-5}\mu \\ 1 & \zeta_{18}^{2} & \zeta_{18} \end{pmatrix},$$

for some $\lambda, \mu \in K^*$. However, it is straightforward to check that none of these transformation transforms $\mathcal{C}_1$ to $\mathcal{C}'$ whose core is $X^6 + Y^6 + Z^6$. Consequently, $\mathcal{C}_1$ is never a descendant of the Fermat curve $\mathcal{F}_6$ with $\operatorname{Aut}(\mathcal{C}_1)$ conjugate to $\varrho_2(\mathrm{A}_4)$.

This proves Claim 3. □

It remains to prove Claim 4 for $\mathcal{C}_2$ that is a descendant of the Fermat curve $\mathcal{F}_6$.

Proof. (of Claim 4) - We easily discard the cases when $\operatorname{Aut}(\mathcal{C}_2)$ equals an S_3 or $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$ inside $\operatorname{Aut}(\mathcal{F}_6)$ as none of the involutions $[X : sZ : s^{-1}Y]$, $[sY : s^{-1}X : Z]$ and $[sZ : Y : s^{-1}X]$ preserves the core $X^5Y + Y^5Z + Z^5X$ of $\mathcal{C}_2$.

- On the other hand, if $\operatorname{Aut}(\mathcal{C}_2)$ equals $(\mathbb{Z}/3\mathbb{Z})^2$ or He_3, then the discussion we had to show Claim 2 applies to conclude that $\mathcal{C}_2$ is K-isomorphic to

$$\begin{aligned}
\mathcal{C}' : X^5Y &+ Y^5Z + XZ^5 + \alpha_{4,0}\zeta_{21}^{4r}\left(X^4Z^2 + X^2Y^4 + Y^2Z^4\right)\\
&+ \alpha_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y + XZ^2 + Y^2Z\right) = 0,
\end{aligned}$$

where $\varrho_2((\mathbb{Z}/3\mathbb{Z})^2) \subseteq \operatorname{Aut}(\mathcal{C}')$. Next, if $\operatorname{Aut}(\mathcal{C}')$ is He_3, then there must be another automorphism $\sigma' \notin \varrho_2((\mathbb{Z}/3\mathbb{Z})^2)$ of order 3 that commutes with σ such that $\sigma'[Y : Z : X]\sigma'^{-1} = [Y : Z : X]\sigma^{-1}$. Straightforward calculations show that $\sigma' = [s'Y : t'Z : X]$ or $[s'Z : t'X : Y]$ with $s't' = \zeta_3$ and $s'^2t'^{-1} = \zeta_3^{-1}$. So σ' belongs to $\varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$ modulo $\langle[Y : Z : X]\rangle$. Obviously, none of these transformations leaves invariant the core of $\mathcal{C}'$. Therefore, $\operatorname{Aut}(\mathcal{C}_2)$ is never conjugate to He_3 inside $\mathcal{F}_6$.

- Thirdly, following the notations of Claim 3, a change of variables of the form $\phi = \phi_i$ for $i = 1, 2, ..., 6$ does not transform $\mathcal{C}_2$ to $\mathcal{C}_2' : X^6 + Y^6 + Z^6+$ lower order terms in X, Y, Z. Thus $\mathcal{C}_2$ is not a descendant of $\mathcal{F}_6$ such that $\phi^{-1}\operatorname{Aut}(\mathcal{C}_2)\phi =$

$\varrho_1(\mathrm{A}_4)$. On the other hand, $\psi_i\,\mathrm{diag}(1,1,-1)\psi_i^{-1}\in\mathrm{Aut}(\mathcal{C}_2)$ with $i=1$ or 4 only if

$$\begin{aligned}\alpha_{2,4} &= \frac{\lambda^5\mu+4\mu^5}{2\lambda^4},\ \alpha_{4,0}=\frac{\lambda+4\lambda^5\mu}{2\mu^2},\ \alpha_{0,2}=\frac{4\lambda+\mu^5}{2\lambda^2\mu^4}\\ \alpha_{1,3} &= \frac{2\left(2\lambda^5\mu+2\lambda+\mu^5\right)}{\lambda^3\mu^2},\ \alpha_{3,2}=\frac{2\lambda^5\mu+4\lambda+4\mu^5}{\lambda^2\mu},\ \alpha_{2,1}=\frac{2\left(2\lambda^5\mu+\lambda+2\mu^5\right)}{\lambda\mu^3}.\end{aligned}$$

The above restrictions are consequences of eliminating the coefficients of X^6, Y^6, Z^6, X^5Z, Y^4Z^2, X^4Y^2, X^4Z^2 from the transformed equation ${}^{\psi_i\,\mathrm{diag}(1,1,-1)\psi_i^{-1}}\mathcal{C}_2=\mathcal{C}_2$. Moreover, $\mathcal{C}_2$ is K-isomorphic via ψ_1 (respectively ψ_4 followed by $Y\leftrightarrow Z$) to $\mathcal{C}_{2,\lambda,\mu}$ described in Theorem 2.5, (2)-(ii). The rest is obvious by noticing that $\psi_2=\psi_1\circ[Z:X:Y]$, $\psi_5=\phi_4\circ[Z:X:Y]$, $\psi_3=\psi_1\circ[Y:Z:X]$ and $\psi_6=\psi_4\circ[Y:Z:X]$.

This proves Claim 4. □

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