



# Limit cycles of discontinuous piecewise differential systems formed by two rigid systems separated by a straight line

Jaume Llibre, Angela C. T. Sánchez and Durval J. Tonon

**Abstract.** In this work we study the maximum number of limit cycles in some classes of discontinuous piecewise differential systems formed by two rigid systems separated by a straight line. These rigid systems are formed by a linear center plus a homogeneous polynomial of degree 2, 3, 4, 5, 6. For these classes of piecewise differential systems we solve the extended 16th Hilbert problem.

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## 1. Introduction and statements of the main results

Following [4, 12] a *center* is a singularity  $p \in \mathbb{R}$  of a planar analytical differential system such that there exists an open neighborhood  $U$  of  $p$  such that all solutions in  $U \setminus \{p\}$  are periodic. Denote by  $T_q$  the period of the periodic orbit through the point  $q \in U \setminus \{p\}$ . We say that  $p$  is an *isochronous center* if  $T_q$  is constant for  $q \in U \setminus \{p\}$ . Furthermore, an isochronous center is *uniform* or *rigid* if the angular velocity of the vector field is the same for all periodic orbits in  $U \setminus \{p\}$ . In other words, an isochronous center is rigid if in polar coordinates  $(r, \theta)$  defined by  $x = r \cos \theta$ ,  $y = r \sin \theta$ , it can be written as  $dr/dt = G(r, \theta)$ ,  $d\theta/dt = k$ , where  $k \neq 0$  is constant. Besides, according to [3] a rigid differential system in the plane  $\mathbb{R}^2$  having a focus or a center at the origin of coordinates can be written in the form

$$\dot{x} = \frac{dx}{dt} = -y + xF(x, y), \quad \dot{y} = \frac{dy}{dt} = x + yF(x, y), \quad (1)$$

where  $F(x, y)$  is a smooth real function. This system has only one singular point at the origin of coordinates.

For system (1), the center-focus problem, which consists of determining conditions that differentiate the cases in which the origin is a focus from the cases in which it is a center, is equivalent to the isochronicity problem [5]. The isochronicity problem has been studied by several authors due to its applications in physics, see [10, 15]. Furthermore, nowadays, isochronicity has also been explored for discontinuous piecewise differential systems, considering the coupling of two or more rigid systems and investigating their dynamics, see [8, 9].

In this paper we study the limit cycles of the piecewise differential systems separated by a straight line, that without loss of generality we take  $y = 0$ , and formed by the two rigid systems

$$\begin{aligned}\dot{x} &= -y + x(\lambda + P(x, y)) = X_m^1(x, y), \\ \dot{y} &= x + y(\lambda + P(x, y)) = X_m^2(x, y),\end{aligned}\tag{2}$$

where  $P(x, y)$  is a homogeneous polynomial of degree  $m - 1$ , and

$$\begin{aligned}\dot{x} &= -y + x(\lambda + Q(x, y)) = Y_n^1(x, y), \\ \dot{y} &= x + y(\lambda + Q(x, y)) = Y_n^2(x, y),\end{aligned}\tag{3}$$

where  $Q(x, y)$  is a homogeneous polynomial of degree  $n - 1$ . As usual the dot denotes derivative with respect to the time  $t$ . The interest in studying the limit cycles of discontinuous piecewise differential systems has increased in the last years due to a big number of applications of these systems, see for instance the recent papers [2, 6, 11, 13, 14].

The famous 16th Hilbert problem asks for an upper bound for the maximum number of limit cycles that the polynomial vector fields in the plane of a given degree can exhibit, see [7]. For the moment this problem remains open. Here we solve the extension of the 16th Hilbert problem for some classes of piecewise differential systems separated by the straight line  $y = 0$  and formed by the two rigid polynomial differential systems (2) and (3) of degree  $m$  and  $n$ , respectively.

Our main result is the following one.

**Theorem 1.** *Consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two rigid polynomial differential systems (2) and (3) of degree  $m$  and  $n$  respectively, with  $m \leq n$ , where  $P$  and  $Q$  are homogeneous polynomials of degree  $m - 1$  and  $n - 1$ , respectively. Then the piecewise differential system has*

- (a) at most 1 limit cycle if  $m = n = 2$  and the bound is reached;
- (b) at most 2 limit cycles if  $m = 2$  and  $n = 3$  and the bound is reached;
- (c) at most 3 limit cycles if  $m = 2$  and  $n = 4$  and the bound is reached;
- (d) at most 4 limit cycles if  $m = 2$  and  $n = 5$ ;
- (e) at most 5 limit cycles if  $m = 2$ , and  $n = 6$ ;
- (f) at most 1 limit cycles if  $m = n = 3$  and the bound is reached;
- (g) at most 4 limit cycles if  $m = 3$  and  $n = 4$ ;
- (h) at most 2 limit cycles if  $m = 3$  and  $n = 5$  and the bound is reached;
- (i) at most 5 limit cycles if  $m = 3$  and  $n = 6$ ;

- (j) at most 1 limit cycles if  $m = 4$  and  $n = 4$  and the bound is reached;
- (k) at most 6 limit cycles if  $m = 4$  and  $n = 5$ ;
- (l) at most 6 limit cycles if  $m = 4$  and  $n = 6$ ;
- (m) at most 1 limit cycles if  $m = n = 5$  and the bound is reached;
- (n) at most 8 limit cycles if  $m = 5$  and  $n = 6$ .

The rest of the paper is dedicated to prove Theorem 1.

## 2. Proof of theorem 1

If the degrees of the polynomials  $P(x, y)$  and  $Q(x, y)$  are  $\geq 2$ , then the eigenvalues of the linear part of the differential systems (2) and (3) at the origin of coordinates are  $\lambda \pm i$ . So the origin is a hyperbolic focus if  $\lambda \neq 0$ , and it is a weak focus or a center if  $\lambda = 0$ .

**Case  $(\mathbf{m}, \mathbf{n}) = (\mathbf{2}, \mathbf{2})$ .** Then the homogeneous polynomials  $P(x, y)$  and  $Q(x, y)$  have degree one, and we have the piecewise differential system separated by  $y = 0$  and formed by the two rigid systems defined by the vector fields

$$X_2(x, y) = \begin{cases} X_2^1(x, y) = -y + x(\lambda + a_1x + a_2y), \\ X_2^2(x, y) = x + y(\lambda + a_1x + a_2y), \end{cases} \quad (4)$$

and

$$Y_2(x, y) = \begin{cases} Y_2^1(x, y) = -y + x(\lambda + b_1x + b_2y), \\ Y_2^2(x, y) = x + y(\lambda + b_1x + b_2y). \end{cases} \quad (5)$$

We write these two differential systems in polar coordinates  $(r, \theta)$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then system (4) becomes

$$\dot{r} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta), \quad \dot{\theta} = 1.$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta).$$

Thus, the solution  $r_1(\theta)$  of the differential equation  $r'(\theta)$  such  $r_1(0) = r_0$  is

$$r_1(\theta) = \frac{e^{\theta\lambda}r_0(1 + \lambda^2)}{1 - a_2r_0 + a_1r_0\lambda + \lambda^2 + e^{(\theta\lambda)}r_0(a_2 - a_1\lambda) \cos \theta - e^{(\theta\lambda)}r_0(a_1 + a_2\lambda) \sin \theta}. \quad (6)$$

Similarly system (5) in polar coordinates taken  $\theta$  as the new independent variable writes

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(b_1 \cos \theta + b_2 \sin \theta).$$

and its solution  $r_2(\theta)$  such  $r_2(\pi) = r_1$  is

$$r_2(\theta) = \frac{e^{\theta\lambda}r_1(1 + \lambda^2)}{e^{\pi\lambda}(1 + b_2r_1 - b_1r_1\lambda + \lambda^2) - e^{(\theta\lambda)}r_1((-b_2 + b_1\lambda) \cos \theta + (b_1 + b_2\lambda) \sin \theta)}. \quad (7)$$

The trajectory  $r_1(\theta)$  of the vector field  $X_2$  such that  $r_1(0) = r_0$  intersects the straight line  $y = 0$  at the point  $(r_1(\pi), 0)$ . Then since we want to concatenate the solution  $r_1(\theta)$  with a trajectory of vector field  $Y_2$ , we consider the solution  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Now we want that after a time  $\pi$  the solution  $r_2(\theta)$  intersects the straight line  $y = 0$  at  $r_0$ , in this way the piecewise differential system will have a periodic orbit. If this periodic orbit is isolated in the set of all periodic orbits of the piecewise differential system it will be a limit cycle.

In summary, for studying the limit cycles of the piecewise differential system formed by the vector fields  $X_2$  and  $Y_2$  we must solve the equation  $r_2(2\pi) = r_0$  for  $r_0$ , i.e. the equation

$$\frac{e^{2\pi\lambda}(\lambda^2 + 1)r_0}{1 + \lambda^2 + r_0(1 + e^{\pi\lambda})(a_1\lambda - a_2 + e^{\pi\lambda}(b_2 - b_1\lambda))} = r_0. \quad (8)$$

The solutions of this equation are

$$r_{0,1} = 0 \quad \text{and} \quad r_{0,2} = \frac{(1 - e^{\pi\lambda})(\lambda^2 + 1)}{a_2 - a_1\lambda + e^{\pi\lambda}(b_1\lambda - b_2)}. \quad (9)$$

Since the origin is always an equilibrium point of the system, we only have the solution  $r_0^2$  providing a periodic orbit. More precisely, if  $\lambda = 0$ , then  $r_0^2 = 0$  and if  $(a_2 - a_1\lambda + e^{\pi\lambda}(b_1\lambda - b_2)) = 0$ , then  $r_0^2$  does not exist. So, in order that  $r_0^2$  be positive and provides a periodic orbit we must have  $(1 - e^{\pi\lambda})(a_2 - a_1\lambda + e^{\pi\lambda}(b_1\lambda - b_2)) > 0$ , and then since the piecewise differential system has only one periodic orbit, this is a limit cycle. Statement (a) of Theorem 1 is proved. **Case(m,n)=(2,3).** Now the piecewise differential system is formed by the vector field  $X_2$  defined in the previous case and the vector field (3) with  $n = 3$ , i.e.

$$Y_3(x, y) = \begin{cases} Y_3^1(x, y) = -y + x(\lambda + b_1x^2 + b_2xy + b_3y^2), \\ Y_3^2(x, y) = x + y(\lambda + b_1x^2 + b_2xy + b_3y^2). \end{cases} \quad (10)$$

This vector field in polar coordinates (10) provides the differential system

$$\dot{r} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)), \quad \dot{\theta} = 1, \quad (11)$$

or the equivalent differential equation

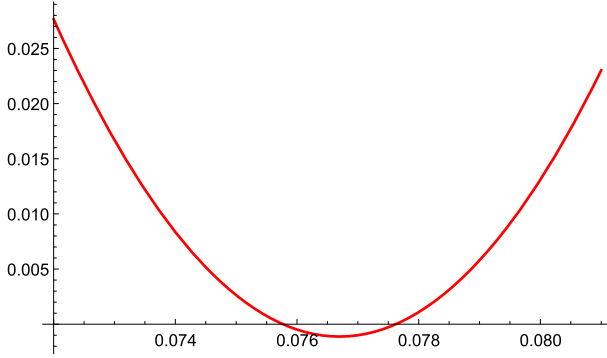
$$r' = \frac{dr}{d\theta} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)).$$

The solution  $r_2(\theta)$  of this differential equation such that  $r_2(\pi) = r_1$  is

$$r_2(\theta) = \frac{e^{\theta\lambda}\sqrt{2\lambda(1 + \lambda^2)}}{\sqrt{E_{11}/r_1^2 + E_{12}}} \quad (12)$$

where

$$\begin{aligned} E_{11} &= e^{2\pi\lambda}((b_1 + b_3)r_1^2 + (2 - b_2r_1^2)\lambda + 2b_1r_1^2\lambda^2 + 2\lambda^3), \\ E_{12} &= -e^{2\theta\lambda}((b_1 + b_3)(1 + \lambda^2) + \lambda(-b_2 + (b_1 - b_3)\lambda)\cos(2\theta) \\ &\quad + \lambda(b_1 - b_3 + b_2\lambda)\sin(2\theta)). \end{aligned}$$

FIGURE 1. Real positive roots for case  $(\mathbf{m}, \mathbf{n}) = (2, 3)$ 

In this way, the limit cycles satisfy the equation  $(r_2(2\pi))^2 - r_0^2 = 0$ , then we consider the numerator of the equation  $(r_2(2\pi))^2 - r_0^2 = 0$ , and we obtain  $(1 + e^{\pi\lambda}) r_0^2 p_1(r_0)$ , being  $p_1(r_0)$  the following polynomial of degree 2 in  $r_0$

$$p_1(r_0) = (C_{0,1} + C_{1,1}r_0 + C_{2,1}r_0^2), \quad (13)$$

where

$$\begin{aligned} C_{0,1} &= 2(e^{\pi\lambda} - e^{2\pi\lambda} + e^{3\pi\lambda} - 1)\lambda(\lambda^2 + 1)^2, \\ C_{1,1} &= 4\lambda(\lambda^2 + 1)(a_2 - a_1\lambda), \\ C_{2,1} &= -\lambda(2a_1^2(e^{\pi\lambda} + 1)\lambda^2 - 4a_1a_2(e^{\pi\lambda} + 1)\lambda + 2a_2^2(e^{\pi\lambda} + 1) \\ &\quad + b_2e^{2\pi\lambda}(e^{\pi\lambda} - 1)(\lambda^2 + 1)) \\ &\quad + b_1e^{2\pi\lambda}(e^{\pi\lambda} - 1)(2\lambda^4 + 3\lambda^2 + 1) + b_3e^{2\pi\lambda}(e^{\pi\lambda} - 1)(\lambda^2 + 1). \end{aligned}$$

The polynomial  $p_1(r_0)$  can have two real positive roots. For instance, for the values  $\lambda = -1$ ,  $a_1 = 10$ ,  $a_2 = 15$ ,  $b_1 = 14$ ,  $b_2 = 12$  and  $b_3 = -2$  it has two real positive roots given by  $r_0^1 = 0.0757703$  and  $r_0^2 = 0.0776237$ , see Figure 1. For these values of the variables and of the roots the denominator of the equation  $(r_2(2\pi))^2 - r_0^2$  is different from zero. So the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_2$  and  $Y_3$  can have two limit cycles. Statement (b) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (2, 4)$ .** Consider the piecewise differential system separated by the straight line  $y = 0$  and formed by vector field  $X_2$  defined in (4) and the vector field (3) with  $n = 4$ , i.e.

$$Y_4(x, y) = \begin{cases} Y_4^1(x, y) = -y + x(\lambda + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3), \\ Y_4^2(x, y) = x + y(\lambda + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3). \end{cases} \quad (14)$$

The vector field  $Y_4$  provides in polar coordinates the differential system

$$\dot{r}_2 = r\lambda + r^4(b_1\cos^3\theta + b_2\cos^2\theta\sin\theta + b_3\cos\theta\sin^2\theta + b_4\sin^3\theta), \quad \dot{\theta} = 1,$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(b_1 \cos^3 \theta + b_2 \cos^2 \theta \sin \theta + b_3 \cos \theta \sin^2 \theta + b_4 \sin^3 \theta).$$

Then the solution  $r_2(\theta)$  of this differential equation such that  $r_2(\pi) = r_1$  is

$$r_2(\theta) = -\frac{2^{\frac{2}{3}} e^{\theta\lambda} r_1 ((1 + \lambda^2)(1 + 9\lambda^2))^{\frac{1}{3}}}{E_2^{\frac{1}{3}}} \quad (15)$$

with

$$\begin{aligned} E_2 = & -4e^{3\pi\lambda}(1+\lambda^2)(1+9\lambda^2) - 4e^{3\pi\lambda}r_1^3(b_2+2b_4+3b_2\lambda^2 - \lambda(7b_1+2b_3+9b_1\lambda^2)) \\ & + e^{3\theta\lambda}r_1^3(-3(b_2+3b_4-3(3b_1+b_3)\lambda)(1+\lambda^2)\cos\theta + (-b_2+b_4+b_1\lambda-b_3\lambda) \\ & (1+9\lambda^2)\cos 3\theta + 3(3b_1+b_3+3b_2\lambda+9b_4\lambda)(1+\lambda^2)\sin\theta \\ & + (b_1-b_3+b_2\lambda-b_4\lambda)(1+9\lambda^2)\sin 3\theta). \end{aligned}$$

Thus, the limit cycles satisfy the equation  $(r_2(2\pi))^3 - r_0^3 = 0$ . Then we consider the numerator of equation  $(r_2(2\pi))^3 - r_0^3 = 0$  and we obtain the following polynomial of degree 3 in  $r_0$

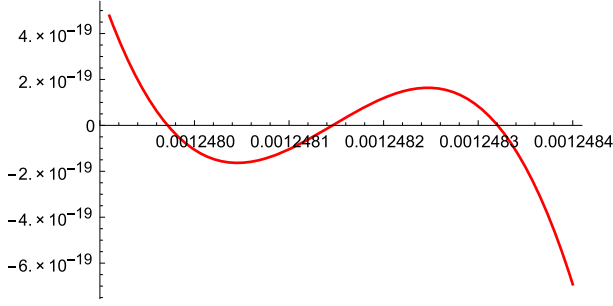
$$p_2(r_0) = r_0^3(C_{0,2} + C_{1,2}r_0 + C_{2,2}r_0^2 + C_{3,2}r_0^3), \quad (16)$$

where

$$\begin{aligned} C_{0,2} = & -(e^{6\pi\lambda} - 1)(1 + \lambda^2)^3(1 + 9\lambda^2), \\ C_{1,2} = & 3(1 + e^{\pi\lambda})(-a_2 + a_1\lambda)(1 + \lambda^2)^2(1 + 9\lambda^2), \\ C_{2,2} = & 3(1 + e^{\pi\lambda})^2(a_2 - a_1\lambda)^2(1 + \lambda^2)^2(1 + 9\lambda^2), \\ C_{3,2} = & e^{3\pi\lambda}(\lambda(\lambda^3(9a_1^3\lambda + b_2(3\lambda^2 + 7) + 2b_4) + \lambda(a_1^3\lambda + 5b_2 + 4b_4) \\ & - b_1(\lambda^2 + 1)^2(9\lambda^2 + 7) \\ & - 2b_3(\lambda^2 + 1)^2) - 3a_1^2a_2\lambda^2(9\lambda^2 + 1) + 3a_1a_2^2\lambda(9\lambda^2 + 1) \\ & - (a_2^3(9\lambda^2 + 1)) + b_2 + 2b_4) \\ & + 3e^{\pi\lambda}(9\lambda^2 + 1)(a_1\lambda - a_2)^3 + 3e^{2\pi\lambda}(9\lambda^2 + 1)(a_1\lambda - a_2)^3 \\ & + (9\lambda^2 + 1)(a_1\lambda - a_2)^3 \\ & + e^{6\pi\lambda}(\lambda^2 + 1)^2(-\lambda(9b_1\lambda^2 + 7b_1 + 2b_3) + 3b_2\lambda^2 + b_2 + 2b_4)). \end{aligned}$$

The polynomial  $p_2(r_0)$  can have three real positive roots. For instance, for the values  $\lambda = -1$ ,  $a_1 = -512$ ,  $a_2 = 2048$ ,  $b_1 = b_2 = b_3 = 0$  and  $b_4 = 414992$  it has three real positive roots given by  $r_0^1 = 0.00124798$ ,  $r_0^2 = 0.00124814$  and  $r_0^3 = 0.00124832$ , see Figure 2. With these values the denominator of the equation  $(r_2(2\pi))^3 - r_0^3$  is different from zero. So the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_2$  and  $Y_4$  can have three limit cycles. Statement (c) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (2, 5)$ .** We consider the piecewise differential system separated by straight line  $y = 0$  and formed by two vector fields  $X_2$  defined in (4) and

FIGURE 2. Real positive roots for case  $(\mathbf{m}, \mathbf{n}) = (2, 4)$ 

$Y_5$

$$Y_5(x, y) = \begin{cases} Y_5^1(x, y) = -y + x(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4), \\ Y_5^2(x, y) = x + y(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4). \end{cases} \quad (17)$$

The vector field (17) in polar coordinates becomes

$$\begin{aligned} \dot{r}_2 &= r\lambda + r^5 (b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta) \\ \dot{\theta} &= 1 \end{aligned}$$

or equivalently

$$\begin{aligned} r' &= \frac{dr}{d\theta} \\ &= r\lambda + r^5 (b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta). \end{aligned}$$

Thus, the solution  $r_2(\theta)$  of equation  $r'(\theta)$  such that  $r_2(\pi) = r_1$  is

$$r_2(\theta) = \frac{2^{3/4} e^{\theta\lambda} \lambda^{1/4} (1 + 5\lambda^2 + 4\lambda^4)^{1/4} r_1}{E_3^{1/4}},$$

where

$$\begin{aligned} E_3 &= 8e^{4\pi\lambda} (\lambda + 5\lambda^3 + 4\lambda^5) - r_1^4 (-e^{4\pi\lambda} (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 \\ &\quad + b_1(3 + 32\lambda^2 + 32\lambda^4)) + e^{4\theta\lambda} ((3b_1 + b_3 + 3b_5)(1 + 5\lambda^2 + 4\lambda^4) \\ &\quad - 4\lambda(b_2 + b_4 + 4(-b_1 + b_5)\lambda)(1 + \lambda^2) \cos(2\theta) + \lambda(-b_2 + b_4 \\ &\quad + (b_1 - b_3 + b_5)\lambda)(1 + 4\lambda^2) \cos(4\theta) + 8\lambda(b_1 - b_5 + (b_2 + b_4)\lambda)(1 + \lambda^2) \\ &\quad \times \sin(2\theta) + \lambda(b_1 - b_3 + b_5 + b_2\lambda - b_4\lambda)(1 + 4\lambda^2) \sin(4\theta))) \end{aligned}$$

Then assuming  $r_1 = r_1(\pi) = r_2(\pi)$ , where  $r_1(\theta)$  is the solution of (4) defined in (6). Now consider the numerator of the equation  $(r(2\pi))^4 - r_0^4 = 0$ , and we obtain  $(1 + e^{\pi\lambda})r_0^4 p_3(r_0)$ , being  $p_3(r_0)$  the following polynomial of degree 4 in  $r_0$

$$p_3(r_0) = C_{0,3} + C_{1,3}r_0 + C_{2,3}r_0^2 + C_{3,3}r_0^3 + C_{4,3}r_0^4,$$

where

$$\begin{aligned}
 C_{0,3} &= 8(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda})\lambda(1 + \lambda^2)^4(1 + 4\lambda^2), \\
 C_{1,3} &= -32\lambda(a_1\lambda - a_2)(1 + \lambda^2)^3(1 + 4\lambda^2); \\
 C_{2,3} &= -48(1 + e^{\pi\lambda})\lambda(a_2 - a_1\lambda)^2(1 + \lambda^2)^2(1 + 4\lambda^2), \\
 C_{3,3} &= 32(1 + e^{\pi\lambda})^2\lambda(a_2 - a_1\lambda)^3(1 + 5\lambda^2 + 4\lambda^4), \\
 C_{4,3} &= -8(1 + e^{\pi\lambda})^3\lambda(a_2 - a_1\lambda)^4(1 + 4\lambda^2) + e^{4\pi\lambda}(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + \lambda^2)^3 \\
 &\quad \times (3b_1 + b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + (32b_1 + 4b_3)\lambda^2 - 8b_2\lambda^3 + 32b_1\lambda^4).
 \end{aligned}$$

Now we shall use the Descartes' theorem (see [1]).

**Theorem 2.** (Descartes Theorem) *Consider the real polynomial  $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$  with  $0 \leq i_1 < i_2 < \dots < i_r$  and  $a_{i_j} \neq 0$  real constants for  $j \in \{1, 2, \dots, r\}$ . When  $a_{i_j}a_{i_{j+1}} < 0$ , we say that  $a_{i_j}$  and  $a_{i_{j+1}}$  have a variation of sign. If the number of variations of signs is  $m$ , then  $p(x)$  has at most  $m$  positive real roots. Moreover, if the coefficients of the polynomial  $p(x)$  are independent, then it is always possible to choose the coefficients of  $p(x)$  in such a way that  $p(x)$  has exactly  $m$  positive real roots.*

By Descartes Theorem the polyomial  $p_3(r_0)$  can have at most four real positive roots. For instance, with the conditions,  $a_1 = -0.1$ ,  $a_2 = 1$ ,  $b_1 = -2$ ,  $b_2 = 0$ ,  $b_3 = 10$ ,  $b_4 = -1$ ,  $b_5 = -\frac{4}{3}$  and  $\lambda = -1$  we have four variations of sign. Therefore the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_2$  and  $Y_5$  can have at most four limit cycles. Statament (d) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (2, 6)$ .** We consider the piecewise differential system separated by the straight line  $y = 0$  and formed by the vector field  $X_2$  defined in (4) and the vector field (3) for  $n = 6$

$$Y_6(x, y) = \begin{cases} Y_6^1(x, y) = -y + x(\lambda + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5), \\ Y_6^2(x, y) = x + y(\lambda + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5). \end{cases} \quad (18)$$

The vector field (18) in polar coordinates becomes

$$\begin{aligned}
 \dot{r}_2 &= r\lambda + r^6 (b_1 \cos^5 \theta + b_2 \cos^4 \theta \sin \theta + b_3 \cos^3 \theta \sin^2 \theta + b_4 \cos^2 \theta \sin^3 \theta \\
 &\quad + b_5 \cos \theta \sin^4 \theta + b_6 \sin^5 \theta) \\
 \dot{\theta} &= 1
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 r' = \frac{dr}{d\theta} &= r\lambda + r^6 (b_1 \cos^5 \theta + b_2 \cos^4 \theta \sin \theta + b_3 \cos^3 \theta \sin^2 \theta + b_4 \cos^2 \theta \sin^3 \theta \\
 &\quad + b_5 \cos \theta \sin^4 \theta + b_6 \sin^5 \theta)
 \end{aligned}$$

Thus, the solution  $r_2(\theta)$  of equation  $r'(\theta)$  such that  $r_2(\pi) = r_1$  is

$$r_2(\theta) = -\frac{2^{\frac{4}{5}} e^{\theta\lambda} ((1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2))^{\frac{1}{5}} r_1}{E_4^{\frac{1}{5}}}, \quad (19)$$



where

$$\begin{aligned}
 E_4 = & -16e^{5\pi\lambda} (9 + 3(3b_2 + 2b_4 + 8b_6)r_1^5 - (149b_1 + 26b_3 + 24b_5)r_1^5\lambda \\
 & + (259 + 10(11b_2 + 3b_4)r_1^5)\lambda^2 - 50(15b_1 + b_3)r_1^5\lambda^3 + 125(7 + b_2r_1^5)\lambda^4 \\
 & - 625b_1r_1^5\lambda^5 + 625\lambda^6) + e^{5\theta\lambda}r_1^5(-10(b_2 + b_4 + 5b_6 - 5(5b_1 + b_3 + b_5)\lambda) \\
 & \times (1 + \lambda^2)(9 + 25\lambda^2)\cos\theta - 5(1 + \lambda^2)(1 + 25\lambda^2)(9b_2 + 3b_4 \\
 & + 5((-5b_1 + b_3 + 3b_5)\lambda - 3b_6))\cos 3\theta + (b_4 - b_2 - b_6 + (b_1 - b_3 + b_5)\lambda) \\
 & \times (9 + 250\lambda^2 + 625\lambda^4)\cos 5\theta + 10(5b_1 + b_3 + b_5 + 5(b_2 + b_4 + 5b_6)\lambda) \\
 & \times (1 + \lambda^2)(9 + 25\lambda^2)\sin\theta + 5(15b_1 - 3b_3 - 9b_5 + 5(3b_2 + b_4 - 5b_6)\lambda) \\
 & \times (1 + \lambda^2)(1 + 25\lambda^2)\sin 3\theta + (b_1 - b_3 + b_5 + (b_2 - b_4 + b_6)\lambda) \\
 & \times (9 + 250\lambda^2 + 625\lambda^4)\sin 5\theta) .
 \end{aligned}$$

Now consider  $r_1 = r_1(\pi) = r_2(\pi)$  with  $r_1(\theta)$  defined in (6). Then consider the numerator the equation  $r_2(2\pi)^5 - r_0^5 = 0$  and we obtain  $(1 + e^{\pi\lambda})r_0^5 p_4(r_0)$ , being  $p_4(r_0)$  the following polynomial of degree 5 in  $r_0$

$$p_4(r_0) = C_{0,4} + C_{1,4}r_0 + C_{2,4}r_0^2 + C_{3,4}r_0^3 + C_{4,4}r_0^4 + C_{5,4}r_0^5,$$

where

$$\begin{aligned}
 C_{0,4} = & (1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})(1 - e^{5\pi\lambda})(1 + \lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4), \\
 C_{1,4} = & 5(a_1\lambda - a_2)(1 + \lambda^4)(9 + 250\lambda^2 + 625\lambda^4), \\
 C_{2,4} = & 10(1 + e^{\pi\lambda})(a_2 - a_1\lambda)^2(1 + \lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4), \\
 C_{3,4} = & 10(1 + e^{\pi\lambda})^2(a_1\lambda - a_2)^3(1 + \lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4), \\
 C_{4,4} = & 5(1 + e^{\pi\lambda})^3(a_2 - a_1\lambda)^4(1 + \lambda^2)(9 + 250\lambda^2 + 625\lambda^4), \\
 C_{5,4} = & (1 + e^{\pi\lambda})^4(a_1\lambda - a_2)^5(9 + 250\lambda^2 + 625\lambda^4) \\
 & + e^{5\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda}) \\
 & \times (1 + \lambda^2)^4(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) \\
 & - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)) .
 \end{aligned}$$

The polynomial  $p_4(r_0)$  can have at most 5 positive real roots. For example, consider  $\lambda = -1$ ,  $a_1 = 10$ ,  $a_2 = 15$ ,  $b_1 = b_2 = b_3 = b_4 = 0$ ,  $b_5 = -14$  and  $b_6 = -12$ , with these values there are 5 variations of sign. So By Descartes Theorem it can have at most 5 positive real roots, is that, the piecewise differential system formed by the two vector fields  $X_2$  and  $Y_6$  and separated by the straight line  $y = 0$  can have at most 5 limit cycles. Statement (e) of Theorem 1 is proved.

**Case (m, n) = (3, 3).** We consider the piecewise differential system separated by the straight line  $y = 0$  and formed by the vector field  $Y_3$  defined in (10) and the vector field

$$X_3(x, y) = \begin{cases} x_3^1(x, y) = -y + x(\lambda + a_1x^2 + a_2xy + a_3y^2), \\ x_4^2(x, y) = x + y(\lambda + a_1x^2 + a_2xy + a_3y^2). \end{cases} \quad (20)$$

Thus the solution  $r_1(\theta)$  of the vector field  $X_3$  in polar coordinates such that  $r_1(0) = r_0$  is

$$r_1(\theta) = \frac{e^{\theta\lambda} \sqrt{2\lambda(1+\lambda^2)}}{\sqrt{2(\lambda+\lambda^3)/r_0^2 + E_5}} \quad (21)$$

where

$$E_5 = e^{2\theta\lambda}(-(a_1 + a_3)(1 + \lambda^2) + \lambda(a_2 + (-a_1 + a_3)\lambda) \cos(2\theta) - \lambda(a_1 - a_3 + a_2\lambda) \sin(2\theta)) + \lambda(-a_2 + 2a_1\lambda) + a_1 + a_3.$$

The solution  $r_2(\theta)$  of the vector field  $Y_3$  such that  $r_2(\pi) = r_1$  is given in (12). Then solving the equation  $r_2(2\pi) = r_0$  for  $r_0$  we have the following solution

$$r_0 = \sqrt{\frac{-2\lambda(1 + e^{2\pi\lambda})(1 + \lambda^2)}{a_1 + a_3 - a_2\lambda + 2a_1\lambda^2 + e^{2\pi\lambda}(b_1 + b_3 - b_2\lambda + 2b_1\lambda^2)}}.$$

When  $r_0 > 0$  the piecewise differential system separated by the straight line  $y = 0$  and formed by the vector fields  $X_3$  and  $Y_3$  has at most one limit cycle. Statement (f) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{4})$ .** We consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_3$  defined in (20) and  $Y_4$  defined in (14). We know their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Then the numerator of the equation  $r_2(2\pi)^3 - r_0^3 = 0$  is  $A + B = 0$  where

$$\begin{aligned} A = & 2\sqrt{2}e^{3\pi\lambda}\lambda^{3/2}(\lambda^2 + 1)^{3/2}r_0^3(9b_1\lambda^3 + 7b_1\lambda - b_2(3\lambda^2 + 1) + 2b_3\lambda - 2b_4) \\ & + 2\sqrt{2}e^{6\pi\lambda}\lambda^{3/2}(\lambda^2 + 1)^{3/2}(\lambda r_0^3(7b_1 + 2b_3) + 9b_1\lambda^3r_0^3 \\ & - r_0^3(b_2 + 2b_4) + \lambda^2(10 - 3b_2r_0^3) + 9\lambda^4 + 1), \\ B = & (\lambda^2 + 1)(9\lambda^2 + 1)r_0\left((e^{2\pi\lambda} - 1)r_0^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3) - 2(\lambda^3 + \lambda)\right) \\ & \sqrt{a_1 + a_3 + \lambda(-a_2 + 2a_1\lambda) - e^{2\pi\lambda}(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) + 2(\lambda^3 + \lambda)/r_0^2}. \end{aligned}$$

Now we consider the equation  $A^2 - B^2 = 0$  and we obtain the polynomial

$$p_5(r_0) = C_{0,5} + C_{25}r_0^2 + C_{3,5}r_0^3 + C_{4,5}r_0^4 + C_{6,5}r_0^6,$$

where

$$\begin{aligned} C_{0,5} = & 8(e^{12\pi\lambda} - 1)\lambda^3(\lambda^2 + 1)^5(9\lambda^2 + 1)^2, \\ C_{2,5} = & 12(e^{2\pi\lambda} - 1)(\lambda^2 + 1)^4(9\lambda^3 + \lambda)^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3), \\ C_{3,5} = & 16e^{9\pi\lambda}(e^{3\pi\lambda} + 1)\lambda^3(\lambda^2 + 1)^4(9\lambda^2 + 1)(9b_1\lambda^3 + 7b_1\lambda - b_2(3\lambda^2 + 1) \\ & + 2b_3\lambda - 2b_4), \\ C_{4,5} = & 6(1 - e^{2\pi\lambda})^2\lambda(\lambda^2 + 1)^3(9\lambda^2 + 1)^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3)^2, \\ C_{6,5} = & a_1^3(2\lambda^2 + 1)^3(9\lambda^4 + 10\lambda^2 + 1)^2(e^{2\pi\lambda} - 1)^3 \\ & + 3(18\lambda^6 + 29\lambda^4 + 12\lambda^2 + 1)^2(e^{2\pi\lambda} - 1)^3 \end{aligned}$$

$$\begin{aligned}
& (a_3 - a_2\lambda)a_1^2 + 3a_1(2\lambda^2 + 1)(9\lambda^4 + 10\lambda^2 + 1)^2(e^{2\pi\lambda} - 1)^3(a_3 - a_2\lambda)^2 \\
& + (\lambda^2 + 1)^2 \left( \lambda^3 \left( 8e^{6\pi\lambda} (e^{3\pi\lambda} + 1)^2 (\lambda^2 + 1) \right. \right. \\
& \left. \left. (-9b_1\lambda^3 - 7b_1\lambda + 3b_2\lambda^2 + b_2 - 2b_3\lambda + 2b_4)^2 \right. \right. \\
& \left. \left. - a_2^3 (e^{2\pi\lambda} - 1)^3 (9\lambda^2 + 1)^2 \right) + 3a_2^2 a_3 \lambda^2 (9\lambda^2 + 1)^2 \right. \\
& \left. (e^{2\pi\lambda} - 1)^3 - 3a_2\lambda (e^{2\pi\lambda} - 1)^3 \right. \\
& \left. (9a_3\lambda^2 + a_3)^2 + a_3^3 (9\lambda^2 + 1)^2 (e^{2\pi\lambda} - 1)^3 \right).
\end{aligned}$$

By Descartes Theorem the polynomial  $p_5(r_0)$  can have at most four positive real roots. So the piecewise differential system separated by the straight line  $y = 0$  and formed by the two vector fields  $X_3$  and  $Y_4$  can have at most four limit cycles. Statement (g) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (3, 5)$ .** We consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_3$  defined in (20) and  $Y_5$  defined in (17). We know their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Then we consider the numerator of the equation  $(r_2(2\pi))^4 - r_0^4 = 0$  and we get  $(-1 + e^{2\pi\lambda})r_0^4 p_6(r_0)$ , where  $p_6(r_0)$  is the following polynomial of degree 4 in  $r_0$

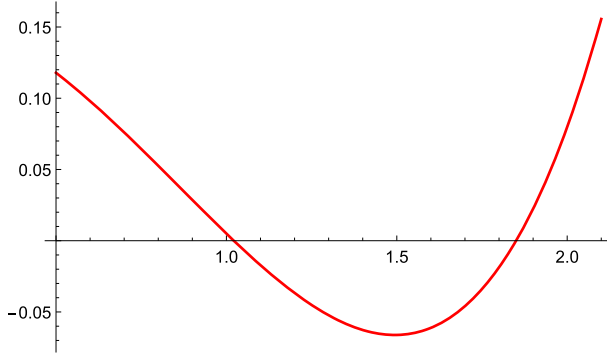
$$p_6(r_0) = C_{0,6} + C_{2,6}r_0^2 + C_{4,6}r_0^4,$$

with

$$\begin{aligned}
C_{0,6} &= 8(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda})\lambda^2(1 + \lambda^2)^2(1 + 4\lambda^2), \\
C_{2,6} &= 8\lambda(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(1 + 5\lambda^2 + 4\lambda^4), \\
C_{4,6} &= 2(1 - e^{2\pi\lambda})(1 + 4\lambda^2)(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^2 + e^{4\pi\lambda}(1 + e^{2\pi\lambda})(\lambda + \lambda^3) \\
&\quad \times (b_3 + 3b_5 - 5b_2\lambda - 3b_4\lambda + 4\lambda^2(b_3 - 2b_2\lambda) + b_1(3 + 32(\lambda^2 + \lambda^4))).
\end{aligned}$$

By Descartes Theorem the polynomial  $p_6(r_0)$  can have at most two positive real roots. For example, for the values  $\lambda = -0.1$ ,  $a_1 = -5.63853$ ,  $a_2 = 10$ ,  $a_3 = 5$ ,  $b_1 = b_2 = b_3 = 0$ ,  $b_4 = 250000$  and  $b_5 = -24999.9$  it has two real positive roots,  $r_0^1 = 1.02181$  and  $r_0^2 = 1.84989$ . See Figure 3. For these values the denominator of the equation  $(r_2(2\pi))^4 - r_0^4$  is different from zero. So the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_3$  and  $Y_5$  can have two limit cycles. Statement (h) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (3, 6)$ .** We consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_3$  defined in (20) and  $Y_6$  defined in (18). We know their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Then the numerator the equation  $r_2(2\pi)^5 - r_0^5 =$

FIGURE 3. Real positive roots for case  $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{5})$ 

0 is  $A + B = 0$  where

$$A = -r_0(1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2) \\ \sqrt{(1 - e^{2\pi\lambda})(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) + 2(\lambda + \lambda^3)/r_0^2}; \\ B = 4\sqrt{2}e^{5\pi\lambda}\lambda^{\frac{5}{2}}(1 + \lambda^2)^{\frac{5}{2}}(e^{5\pi\lambda}(9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) - (1 + e^{5\pi\lambda})r_0^5 \\ \times (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\ + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))) .$$

Now consider the equation  $A^2 - B^2 = 0$  and we obtain  $(1 + e^{\pi\lambda})(1 + \lambda^2)p_7(r_0)$ , where  $p_7(r_0)$  is a polynomial of degree 10 in  $r_0$

$$p_7(r_0) = C_{0,7} + C_{2,7}r_0^2 + C_{4,7}r_0^4 + C_{5,7}r_0^5 + C_{6,7}r_0^6 + C_{8,7}r_0^8 + C_{10,7}r_0^{10},$$

where

$$C_{0,7} = 32(1 - e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda} + e^{8\pi\lambda} + e^{12\pi\lambda} + e^{16\pi\lambda})\lambda^5 \\ \times (9 + 250\lambda^2 + 625\lambda^4)^2(1 + \lambda^2)^5, \\ C_{2,7} = 80(1 - e^{\pi\lambda})(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(\lambda + \lambda^3)^4(9 + 250\lambda^2 + 625\lambda^4)^2, \\ C_{4,7} = 80(-1 + e^{\pi\lambda})^2(1 + e^{\pi\lambda})\lambda^3(1 + \lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4)^2 \\ (a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^2, \\ C_{5,7} = 64e^{15\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})\lambda^5(1 + \lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4) \\ \times (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 \\ + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))) , \\ C_{6,7} = 40(1 - e^{2\pi\lambda})^3(1 + e^{\pi\lambda})^2\lambda^2(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^3(1 + \lambda^2)^2 \\ \times (9 + 250\lambda^2 + 625\lambda^4)^2, \\ C_{8,7} = 10(-1 + e^{\pi\lambda})^4(1 + e^{\pi\lambda})^3\lambda(1 + \lambda^2)(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^4 \\ \times (9 + 250\lambda^2 + 625\lambda^4)^2, \\ C_{10,7} = (-1 - e^{\pi\lambda})((-1 + e^{\pi\lambda})^5(1 + e^{\pi\lambda})^3(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^5$$

$$\begin{aligned}
& \times (9 + 250\lambda^2 + 625\lambda^4)^2 \\
& + 32e^{10\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})^2\lambda^5(1 + \lambda^2)^3 \\
& \times (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) \\
& - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^2).
\end{aligned}$$

Note that the signs of the coefficients  $C_0, C_4$  and  $C_8$  are determined only by the sign of  $\lambda$ . Assume  $\lambda < 0$ , then  $C_0 < 0$ ,  $C_4 < 0$  and  $C_8 < 0$ . In this way, between  $C_0, C_2$  and  $C_4$  there are at most two variations of sign, between  $C_4, C_5, C_6$  and  $C_8$  there are at most two variations of sign and between  $C_8$  and  $C_{10}$  there is at most one variation of sign. Therefore, there are at most five variations of sign between the coefficient of  $p_7(r_0)$ . More specifically, the sign of the coefficient  $C_2$  is determined by the sign of  $A_1 = (a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)$ . We assume  $A_1 > 0$  in order to  $C_2 > 0$ . The sign of  $C_5$  is given by

$$\begin{aligned}
A_2 = & (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\
& + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)),
\end{aligned}$$

then assuming  $A_2 < 0$  we have  $C_5 > 0$ . Now as  $A_1 > 0$  then  $C_6 > 0$ . Finally as  $A_1 > 0$  we have  $C_{10} > 0$ . Thus, we obtain five variations of sign, i.e  $C_0 < 0$ ,  $C_2 > 0$ ,  $C_4 < 0$ ,  $C_5 > 0$ ,  $C_6 > 0$ ,  $C_8 < 0$  and  $C_{10} > 0$  for  $\lambda < 0$ . On the other hand, assuming  $\lambda > 0$  we have  $C_0 < 0$ ,  $C_4 > 0$  and  $C_8 > 0$ . Thus between  $C_0, C_2$  and  $C_4$  there is at most one variation of sign and between  $C_4, C_5, C_6$  and  $C_8$  there are at most two variations of sign and between  $C_8$  and  $C_{10}$  there is one variation. In this way, for  $\lambda > 0$  there are at most four variations of sign. For instance, consider  $\lambda < 0$ ,  $a_1 > 0$ ,  $a_2 \geq 0$ ,  $a_3 \geq -a_1$ ,  $b_1 = b_2 = 0$ ,  $b_3 < 0$ ,  $b_4 < 0$ ,  $b_5 = -\frac{13b_3}{12}$  and  $b_6 < 0$ , with these conditions we have five variations of sign. In conclusion by Descartes Theorem the polynomial  $p_7(r_0)$  can have at most five positive real roots. In this way, it follows that the piecewise differential system separated by the straight line  $y = 0$  and formed by the two vector fields  $X_3$  and  $Y_6$  can have at most five limit cycles. Statement (i) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (4, 4)$ .** Now consider the piecewise differential system separated by the straight line  $y = 0$  and formed by vector field  $Y_4$  defined in (14) and the vector field (2) for  $m = 4$ , i.e

$$X_4(x, y) = \begin{cases} X_4^1(x, y) = -y + x(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3), \\ X_4^2(x, y) = x + y(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3). \end{cases} \quad (22)$$

The vector field (22) in polar coordinates provides the differential system

$$\dot{r}_1(\theta) = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta), \quad \dot{\theta} = 1$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta).$$

Then the solution  $r_1(\theta)$  of the equation  $r'(\theta)$  such that  $r_1(0) = r_0$  is

$$r_1(\theta) = -\frac{2^{\frac{2}{3}} e^{\theta\lambda} r_0 ((1 + \lambda^2)(1 + 9\lambda^2))^{\frac{1}{3}}}{E_6^{\frac{1}{3}}}. \quad (23)$$

Where

$$\begin{aligned} E_6 = & e^{3\theta\lambda} r_0^3 \left( -3(a_2 + 3a_4 - 3(3a_1 + a_3)\lambda)(1 + \lambda^2) \cos \theta + (-a_2 + a_4 + a_1\lambda - a_3\lambda) \right. \\ & \times (1 + 9\lambda^2) \cos 3\theta + 3(3a_1 + a_3 + 3a_2\lambda + 9a_4\lambda)(1 + \lambda^2) \sin \theta \\ & + (a_1 - a_3 + a_2\lambda - a_4\lambda)(1 + 9\lambda^2) \sin 3\theta - 4(1 + \lambda^2)(1 + 9\lambda^2) \\ & \left. + 4r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)) \right). \end{aligned}$$

Then assuming  $r_1 = r_1(\pi) = r_2(\pi)$ , where  $r_2(\theta)$  is the solution of (14) defined in (15), we solve  $r_2(2\pi) = r_0$  for  $r_0$ . More specifically we consider the numerator of equation  $(r_2(2\pi))^3 - r_0^3 = 0$  and we get  $(1 + e^{3\pi\lambda})r_0^3 p_8(r_0)$ , with  $p_8(r_0)$  being the following polynomial of degree 3 in  $r_0$

$$p_8(r_0) = C_{0,8} + C_{3,8}r_0^3,$$

where

$$\begin{aligned} C_{0,8} = & (1 - e^{3\pi\lambda})(1 + 10\lambda^2 + 9\lambda^4), \\ C_{3,8} = & (7a_1 + 2a_3)\lambda + 9a_1\lambda^3 - a_2(1 + 3\lambda^2) + e^{3\pi\lambda}(b_2 + 2b_4 \\ & - (7b_1 + 2b_3)\lambda + 3b_2\lambda^2 - 9b_1\lambda^3) - 2a_4. \end{aligned}$$

The polynomial  $p_8(r_0)$  can have at most one real positive root  $r_0 = \left(\frac{-C_{0,8}}{C_{3,8}}\right)^{1/3}$ . Therefore the piecewise differential system separated by the straight line  $y = 0$  formed by the vector fields  $X_4$  and  $Y_4$  can have at most one limit cycle. Statement (j) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (4, 5)$ .** Now consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_4$  defined in (22) and  $Y_5$  defined in (17). We consider their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Then the numerator of the equation  $(r_2(2\pi))^4 - r_0^4 = 0$  is  $A - B = 0$  where

$$\begin{aligned} A = & -8\lambda(1 + 5\lambda^2 + 4\lambda^4) \left( (1 + \lambda^2)(1 + 9\lambda^2) - (1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})r_0^3 \right. \\ & \times (a_2 + 2a_4 - 7a_1\lambda - 2a_3\lambda + 3a_2\lambda^2 - 9a_1\lambda^3) \\ & \times ((-1 - \lambda^2)(1 + 9\lambda^2) + (e^{\pi\lambda} + 1) \\ & \left. \times (1 - e^{\pi\lambda} + e^{2\pi\lambda})r_0^3(a_2 + 2a_4 - 7a_1\lambda - 2a_3\lambda + 3a_2\lambda^2 - 9a_1\lambda^3) \right)^{1/3} \\ B = & e^{4\pi\lambda}((1 + \lambda^2)(1 + 9\lambda^2))^{4/3} (8e^{4\pi\lambda}(\lambda + 5\lambda^3 + 4\lambda^5) \\ & + (-1 + e^{\pi\lambda})(1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})r_0^4 \\ & \times (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4b_3\lambda^2 - 8b_2\lambda^3 + b_1(3 + 32\lambda^2 + 32\lambda^4))) . \end{aligned}$$

Now we consider the equation  $A^3 - B^3$  and we obtain  $(1 + e^{\pi\lambda})(1 + \lambda^2)^3 p_9(r_0)$ , being  $p_9(r_0)$  the following polynomial of degree 12 in  $r_0$

$$p_9(r_0) = C_{0,9} + C_{3,9}r_0^3 + C_{4,9}r_0^4 + C_{6,9}r_0^6 + C_{8,9}r_0^8 + C_{9,9}r_0^9 + C_{12,9}r_0^{12}.$$

Where

$$\begin{aligned} C_{0,9} &= -512(1 - e^{\pi\lambda} + e^{2\pi\lambda})(-1 + e^{3\pi\lambda})(1 + e^{6\pi\lambda})(1 + e^{12\pi\lambda})\lambda^3 \\ &\quad \times (1 + \lambda^2)^4(1 + 4\lambda^2)^3(1 + 9\lambda^2)^4, \\ C_{3,9} &= -2048(1 - e^{\pi\lambda} + e^{2\pi\lambda})(\lambda + 14\lambda^3 + 49\lambda^5 + 36\lambda^7)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)), \\ C_{4,9} &= -192e^{20\pi\lambda}(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})\lambda^2(1 + \lambda^2)^3(1 + 9\lambda^2)^4 \\ &\quad \times (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda \\ &\quad + 4(b_3 - 2b_2\lambda)\lambda^2 + b_1(3 + 32\lambda^2 + 32\lambda^4))(1 + 4\lambda^2)^2, \\ C_{6,9} &= 3072(1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})^2(\lambda + 4\lambda^3)^3(1 + 10\lambda^2 + 9\lambda^4)^2 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2, \\ C_{8,9} &= -24e^{16\pi\lambda}(1 + e^{\pi\lambda})(-1 + e^{\pi\lambda} - e^{2\pi\lambda} + e^{3\pi\lambda})^2\lambda(1 + \lambda^2)^2(1 + 4\lambda^2)(1 + 9\lambda^2)^4 \\ &\quad \times (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 + b_1(3 + 32\lambda^2 + 32\lambda^4))^2, \\ C_{9,9} &= -2048(1 + e^{\pi\lambda})^2(1 - e^{\pi\lambda} + e^{2\pi\lambda})^3\lambda^3(1 + 4\lambda^2)^3(1 + 10\lambda^2 + 9\lambda^4) \\ &\quad (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^3, \\ C_{12,9} &= -(1 + e^{\pi\lambda})^2(-512(1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})^4(\lambda + 4\lambda^3)^3(a_2 + 2a_4 \\ &\quad + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^4 + e^{12\pi\lambda}(-1 + e^{\pi\lambda})^3(1 + e^{2\pi\lambda})^3 \\ &\quad \times (1 + \lambda^2)(1 + 9\lambda^2)^4(b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 \\ &\quad + b_1(3 + 32\lambda^2 + 32\lambda^4))^3). \end{aligned}$$

By Descartes Theorem the polynomial  $p_9(r_0)$  can have at most six positive real roots. For example, for the values  $\lambda = 0.1$ ,  $a_1 = 1$ ,  $a_2 = 0$ ,  $a_3 = -2$ ,  $a_4 = -1$ ,  $b_1 = 1$ ,  $b_2 = b_3 = 0$ ,  $b_4 = -1$ ,  $b_5 = 1$  it has six variations of sign, that is,  $p_9(r_0)$  can have at most six positive real roots. So the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_4$  and  $Y_5$  can have at most six limit cycles. Statament (k) of Theorem (1) is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (4, 6)$ .** We consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_4$  defined in (22) and  $Y_6$  defined in (18). We know their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1(\pi)$ . Then the numerator of the equation

$r_2(2\pi)^5 - r_0^5 = 0$  is  $r_0^5(A - B) = 0$  where

$$\begin{aligned} A &= (1 + 10\lambda^2 + 9\lambda^4 - (1 + e^{3\pi\lambda})r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))) \\ &\quad \times (9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) (-1 - 10\lambda^2 - 9\lambda^4 + (1 + e^{3\pi\lambda})r_0^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)))^{2/3}; \\ B &= e^{5\pi\lambda}(1 + \lambda^2)^{2/3}(1 + 9\lambda^2)^{2/3}(1 + 10\lambda^2 + 9\lambda^4) \\ &\quad \times (e^{5\pi\lambda}(1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2) \\ &\quad + (r_0^5 + e^{5\pi\lambda}r_0^5)(-6(b_4 + 4b_6) + (149b_1 + 26b_3 + 24b_5)\lambda \\ &\quad - b_2(9 + 110\lambda^2 + 125\lambda^4) + 5\lambda^2(-6b_4 + 5\lambda(2b_3 + 5b_1(6 + 5\lambda^2))))). \end{aligned}$$

Then we consider the equation  $A^3 - B^3 = 0$  and we get the polynomial

$$\begin{aligned} p_{10}(r_0) &= C_{0,10} + C_{3,10}r_0^3 + C_{5,10}r_0^5 + C_{6,10}r_0^6 \\ &\quad + C_{9,10}r_0^9 + C_{10,10}r_0^{10} + C_{12,10}r_0^{12} + C_{15,10}r_0^{15}, \end{aligned}$$

where

$$\begin{aligned} C_{0,10} &= (1 - e^{30\pi\lambda})(1 + \lambda^2)^8(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^3, \\ C_{3,10} &= -5(1 + e^{3\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)), \\ C_{5,10} &= 3e^{25\pi\lambda}(1 + e^{5\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^2 \\ &\quad \times (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda \\ &\quad (149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))), \\ C_{6,10} &= 10(1 + e^{3\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2\lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2, \\ C_{9,10} &= -10(1 + e^{3\pi\lambda})^3(1 + \lambda^2)^5(1 + 9\lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2\lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^3; \\ C_{10,10} &= -3e^{20\pi\lambda}(1 + e^{5\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4) \\ &\quad \times (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) \\ &\quad - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)))^2, \\ C_{12,10} &= 5(1 + e^{3\pi\lambda})^4(1 + \lambda^2)^4(1 + 9\lambda^2)(9 + 250\lambda^2 + 625\lambda^4)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^4, \\ C_{15,10} &= -(1 + e^{\pi\lambda})(1 + \lambda^2)^3((1 + e^{3\pi\lambda})^2(1 - e^{\pi\lambda} + e^{2\pi\lambda})^5(9 + 250\lambda^2 + 625\lambda^4)^3 \\ &\quad \times (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^5 \\ &\quad + (1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})^3 \\ &\quad \times (- (24b_6 + 6b_4(1 + \lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 \\ &\quad + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))))^3 e^{15\pi\lambda}(1 + \lambda^2)^2(1 + 9\lambda^2)^5). \end{aligned}$$



Since we are assuming that  $C_i$  is not null for all  $i$ , then we have that  $C_6 > 0$ ,  $C_{10} < 0$  and  $C_{12} > 0$ . In this way, note that between  $C_0$ ,  $C_3$ ,  $C_5$  and  $C_6$  there are at most three variations of sign. Now between  $C_6$ ,  $C_9$ ,  $C_{10}$  and  $C_{12}$  there are at most two variations of sign and between  $C_{12}$  and  $C_{15}$  there is at most one variation. Therefore there are at most six variations of sign. For example, consider  $\lambda = 0.1$ ,  $a_1 = 0$ ,  $a_2 = -2$ ,  $a_3 = 1$ ,  $a_4 = 1$ ,  $b_1 = b_2 = 0$ ,  $b_3 = 1$ ,  $b_4 = -0.5$ ,  $b_5 = -13/12$  and  $b_6 = -1$ , with this conditions there are six variations of sign. So by the Descartes Theorem the polynomial  $p_{10}(r_0)$  can have at most six positive real roots. Thus, the piecewise differential system separated by the straight line  $y = 0$  and formed by the two vector fields  $X_4$  and  $Y_6$  can have at most 6 limit cycles. Statement (1) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (5, 5)$ .** Now consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $Y_5$  defined in (17) and

$$X_5(x, y) = \begin{cases} X_5^1(x, y) = -y + x(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4), \\ X_5^2(x, y) = x + y(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4). \end{cases} \quad (24)$$

Thus the solution  $r_1(\theta)$  of the vector field  $X_5$  in polar coordinates such that  $r_1(0) = r_0$  is

$$r_1(\theta) = \frac{2^{3/4}e^{\theta\lambda}\lambda^{1/4}(1 + 5\lambda^2 + 4\lambda^4)^{1/4}r_0}{E_7^{1/4}} \quad (25)$$

where

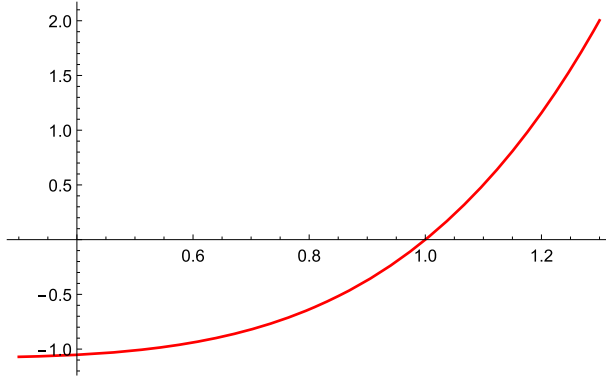
$$\begin{aligned} E_7 = & 8(\lambda + 5\lambda^3 + 4\lambda^5) - r_0^4 \left( e^{4\pi\lambda} (-3a_5 - a_3(1 + 4\lambda^2) - a_1(3 + 32\lambda^2 + 32\lambda^4) \right. \\ & \left. + \lambda(3a_4 + a_2(5 + 8\lambda^2))) + e^{4\theta\lambda} (3a_1 + a_3 + 3a_5 + (3a_1 + a_3 + 3a_5)\lambda^2(5 + 4\lambda^2) \right. \\ & \left. - 4\lambda(a_2 + a_4 + 4(-a_1 + a_5)\lambda)(1 + \lambda^2) \cos 2\theta + \lambda(-a_2 + a_4 + (a_1 - a_3 + a_5)\lambda) \right. \\ & \left. \times (1 + 4\lambda^2) \cos 4\theta + 8\lambda(a_1 - a_5 + (a_2 + a_4)\lambda)(1 + \lambda^2) \sin 2\theta + \lambda(a_1 - a_3 \right. \\ & \left. + a_5 + a_2\lambda - a_4\lambda)(1 + 4\lambda^2) \sin 4\theta \right). \end{aligned}$$

We consider the solution  $r_2(\theta)$  of  $Y_5$  such that  $r_2(\pi) = r_1$  and (25), then we assume that  $r_1 = r_1(\pi) = r_2(\pi)$ . Thus we consider the numerator of the equation  $(r_2(2\pi))^4 - r_0^4 = 0$  and we obtain  $(1 - e^{4\pi\lambda})r_0^4 p_{11}(r_0)$ , being  $p_{11}(r_0)$  the following polynomial of degree 4 in  $r_0$

$$p_{11}(r_0) = C_{0,11} + C_{4,11}r_0^4,$$

where

$$\begin{aligned} C_{0,11} = & 8(1 + e^{4\pi\lambda})(\lambda + 5\lambda^3 + 4\lambda^5); \\ C_{4,11} = & 3a_1 + a_3 + 3a_5 - (5a_2 + 3a_4)\lambda \\ & + (32a_1 + 4a_3)\lambda^2 - 8a_2\lambda^3 + 32a_1\lambda^4 + e^{4\pi\lambda} (3b_1 + b_3 \\ & + 3b_5 - (5b_2 + 3b_4)\lambda + (32b_1 + 4b_3)\lambda^2 - 8b_2\lambda^3 + 32b_1\lambda^4). \end{aligned}$$

FIGURE 4. Real positive root for case  $(\mathbf{m}, \mathbf{n}) = (5, 5)$ 

The polynomial  $p_{11}(r_0)$  can have one real positive root. For instance, for the values  $\lambda = -0.1$ ,  $a_1 = 2.83501$ ,  $a_2 = 9$ ,  $a_3 = -9$ ,  $a_4 = 1$ ,  $a_5 = -1$ ,  $b_1 = b_2 = b_3 = b_4 = 0$  and  $b_5 = -1$  it has one real positive root  $r_0 = 1$ , see Figure 4. With these values the denominator of the equation  $(r_2(2\pi))^4 - r_0^4$  is different from zero. So the piecewise differential system by the straight line  $y = 0$  with the vector fields  $X_5$  and  $Y_5$  can have one limit cycle. Statement (m) of Theorem 1 is proved.

**Case  $(\mathbf{m}, \mathbf{n}) = (5, 6)$ .** Now consider the piecewise differential systems separated by the straight line  $y = 0$  and formed by the two vector fields  $X_5$  defined in (24) and  $Y_6$  defined in (18). We consider their solutions  $r_1(\theta)$  such that  $r_1(0) = r_0$  and  $r_2(\theta)$  such that  $r_2(\pi) = r_1$ . Then the numerator of the equation  $(r_2(2\pi))^5 - r_0^5 = 0$  is  $r_0^5(A - B) = 0$  where

$$\begin{aligned} A &= 8 \times 2^{3/4} e^{10\pi\lambda} \lambda^{5/4} (1 + \lambda^2)^2 (1 + 5\lambda^2 + 4\lambda^4)^{5/4} (9 + 286\lambda^2 + 1625\lambda^4 + 2500\lambda^6) \\ &\quad + 8 \times 2^{3/4} e^{5\pi\lambda} (1 + e^{5\pi\lambda}) r_0^5 \lambda^{5/4} (1 + 5\lambda^2 + 4\lambda^4)^{5/4} ((149b_1 + 26b_3 + 24b_5)\lambda \\ &\quad - 6(b_4 + 4b_6) - b_2(9 + 110\lambda^2 + 125\lambda^4) + 5\lambda^2(-6b_4 + 5\lambda(2b_3 + 5b_1(6 + 5\lambda^2))))), \\ B &= (9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) \left( 8\lambda + 40\lambda^3 + 32\lambda^5 + (1 - e^{4\pi\lambda})r_0^4 \right. \\ &\quad \left. (a_3 + 3a_5 - (5a_2 + 3a_4)\lambda \right. \\ &\quad \left. + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4)) \right)^{5/4}. \end{aligned}$$

Now consider the equation  $A^4 - B^4 = 0$  and we obtain the polynomial

$$\begin{aligned} p_{12}(r_0) &= C_{0,12} + C_{4,12}r_0^4 + C_{5,12}r_0^5 + C_{8,12}r_0^8 + C_{10,12}r_0^{10} \\ &\quad + C_{12,12}r_0^{12} + C_{15,12}r_0^{15} + C_{16,12}r_0^{16} + C_{20,12}r_0^{20}, \end{aligned}$$

where

$$\begin{aligned} C_{0,12} &= 32768(-1 + e^{40\pi\lambda})(1 + \lambda^2)^9(\lambda + 4\lambda^3)^5(9 + 250\lambda^2 + 625\lambda^4)^4, \\ C_{4,12} &= 20480(-1 + e^{4\pi\lambda})\lambda^4(1 + \lambda^2)^8(1 + 4\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^4 \\ &\quad (a_3 + 3a_5 - (5a_2 + 3a_4) \\ &\quad \lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4)), \end{aligned}$$

$$\begin{aligned}
C_{5,12} &= -131072e^{35\pi\lambda}(1+e^{5\pi\lambda})(1+\lambda^2)^8(9+250\lambda^2+625\lambda^4)^3(24b_6+6b_4(1+5\lambda^2) \\
&\quad +b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5 \\
&\quad +50(15b_1+b_3)\lambda^2+625b_1\lambda^4))(\lambda+4\lambda^3)^5, \\
C_{8,12} &= -5120(-1+e^{4\pi\lambda})^2(1+\lambda^2)^7(\lambda+4\lambda^3)^3(9+250\lambda^2+625\lambda^4)^4 \\
&\quad (a_3+3a_5-(5a_2+3a_4) \\
&\quad \lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4))^2, \\
C_{10,12} &= 196608e^{30\pi\lambda}(1+e^{5\pi\lambda})^2(1+\lambda^2)^7(\lambda+4\lambda^3)^5(9+250\lambda^2+625\lambda^4)^2 \\
&\quad (24b_6+6b_4(1+5\lambda^2) \\
&\quad +b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5 \\
&\quad +50(15b_1+b_3)\lambda^2+625b_1\lambda^4))^2, \\
C_{12,12} &= 640(-1+e^{4\pi\lambda})^3(1+\lambda^2)^6(\lambda+4\lambda^3)^2(9+250\lambda^2+625\lambda^4)^4 \\
&\quad \times (a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4))^3, \\
C_{15,12} &= -131072e^{25\pi\lambda}(1+e^{5\pi\lambda})^3(1+\lambda^2)^6(9+250\lambda^2+625\lambda^4)(24b_6+6b_4(1+5\lambda^2) \\
&\quad +b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5+50(15b_1+b_3)\lambda^2 \\
&\quad +625b_1\lambda^4))^3(\lambda+4\lambda^3)^5, \\
C_{16,12} &= -40(-1+e^{4\pi\lambda})^4\lambda(1+\lambda^2)^5(1+4\lambda^2)(9+250\lambda^2+625\lambda^4)^4 \\
&\quad (a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4))^4, \\
C_{20,12} &= (-1+e^{4\pi\lambda})^5(1+\lambda^2)^4(9+250\lambda^2+625\lambda^4)^4(a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2 \\
&\quad -8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4))^5+32768e^{20\pi\lambda}\lambda^5(1+e^{5\pi\lambda})^4 \\
&\quad (24b_6+6b_4(1+5\lambda^2)+b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5 \\
&\quad +50(15b_1+b_3)\lambda^2+625b_1\lambda^4))^4(1+5\lambda^2+4\lambda^4)^5.
\end{aligned}$$

By Descartes Theorem the polynomial  $p_{12}(r_0)$  can have at most eight positive real roots. For instance, for the values  $\lambda = 1$ ,  $a_1 = -1$ ,  $a_2 = 0$ ,  $a_3 = 0$ ,  $a_4 = 1$ ,  $a_5 = -1$ ,  $b_1 = b_3 = b_4 = 0$ ,  $b_2 = b_6 = -1$  and  $b_5 = 1$  there are eight variations of sign between the coefficients, that is,  $p_{12}(r_0)$  can have at most eight positive real roots. Therefore the piecewise differential system separated by the straight line  $y = 0$  with the vector fields  $X_5$  and  $Y_6$  can have at most eight limit cycles. Statement (n) of Theorem 1 is proved.

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Jaume Llibre  
Departament de Matemàtiques  
Universitat Autònoma de Barcelona  
08193 Bellaterra, Barcelona Catalonia  
Spain  
e-mail: jaume.llibre@uab.cat

Angela C. T. Sánchez and Durval J. Tonon  
Institute of Mathematics and Statistics  
University Federal of Goiás  
Goiania Goiás  
Brazil  
e-mail: angelacarolina@discente.ufg.br

Durval J. Tonon  
e-mail: djtonon@ufg.br

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