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Limit cycles and critical periods with non-hyperbolic slow-fast systems

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Abstract

By considering planar slow-fast systems with a curve of double singular points, we obtain lower bounds on the number of limit cycles of polynomial systems surrounding a single singular point, as well as on the number of critical periods in one annulus of periodic orbits. In some circumstances, orbits of such slow-fast systems do not exhibit the typical slow-fast behavior but instead follow a hit-and-run pattern: they quickly move toward the critical curve, pause briefly there, and then continue their path.
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1. Introduction

Planar centers are extensively studied from various viewpoints. Beyond their qualitative properties, which have applications in multiple scientific fields and in the reduction of partial differential equations, these systems also hold theoretical interest. The challenges arising in these

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theoretical branches reveal a complexity that might not initially seem connected to the qualitative study of planar systems. See, for example, [9] for more background.

We will briefly outline two such research questions and at the same time state our main related results. One is related to limit cycles. If

$$X_\lambda : \begin{cases} \dot{x} = -y + f(x, y, \lambda), \\ \dot{y} = x + g(x, y, \lambda), \end{cases}$$

is a family of vector fields for which the origin is a center of X_0, then the annulus of periodic orbits may be perturbed for $\lambda \neq 0$, and interestingly one or more limit cycles may appear. Techniques to deal with this include (among others) Melnikov analysis, Bautin ideals, singular perturbations, symmetries, ...; instead of citing a per definition incomplete list of references, it might be better to refer to the Hilbert 16th problem related literature from the last 30–40 years. Though the majority of interest is towards obtaining an upper bound on the number of limit cycles that could appear within certain classes of systems (polynomial, quadratic, cubic, homogeneous nonlinearities, Liénard, reversible, Hamiltonian, integrable, ...), the quest has also naturally led to the question of what such an upper bound might look like, and in that sense providing good *lower* bounds on the number of such cycles is of interest. In order to find examples with many limit cycles, one typically starts with a center X_0 and one considers the presence of any of the coefficients in the perturbation as opportunities to break the center into a cycle. In the class of polynomial systems of at most degree N for example, the number of such coefficients is of the order $\mathcal{O}(N^2)$, but the best known lower bounds on limit cycles present in a single period annulus around a singular point is up to now $\mathcal{O}(3N^2/16)$ (see Theorem B in [14]). (For specific low degree cases, one can find better bounds, see for example [12,16].)

It was therefore a rather surprising result that up to $\mathcal{O}(\frac{1}{2}N^2 \log N)$ limit cycles could exist in degree N if one considered multiple period annuli; see [1,3,13,17]. Examples with sharper bounds exist for specific low-degree cases; see [15]. Restricting to the study of one nest, the expectation to find at most a number of limit cycles given by the number of free coefficients in the class of systems has kept up quite well. In this light, we improve the known lower bound but do not meet the expected $\mathcal{O}(N^2)$.

Theorem 1. *There exist polynomial systems of odd degree N with up to at least $(N-1)^2/2 = (N^2+1)/2 - N$ isolated periodic orbits, all of which are present in a single nest surrounding a unique singular point.*

We will prove this theorem by considering slow-fast perturbations of a system equivalent to the harmonic oscillator. Our proof is based on counting sign changes of a Melnikov function. As we do not go as far as proving the simplicity of the roots that are deduced from Rolle's Theorem, we have not included conclusions concerning the hyperbolicity of the limit cycles in the statement, but we expect this property to hold true in the provided examples.

In a second research question, the focus is on the period function associated to an annulus of periodic orbits, as previously studied, for example, in [5,10,11]. Given a section Σ transverse to the flow in such an annulus, the time for orbits to return after one loop is denoted $\tau : \Sigma \to \mathbb{R}^+$. This function naturally depends on the parametrization but the periodic orbits at which a critical point of τ is located is an intrinsic notion; they are called critical periods (one could agree the terminology does not remove confusion between the location of the critical points and their values; here we mean with critical period the associated periodic orbit). The multiplicity and whether the period function is locally extremal are also intrinsic properties.

Similar to the study of limit cycles, attention in the literature goes both to upper bounds and lower bounds. The most recent and best result on critical periods is obtained using Hamiltonian systems: Following Cen's result [2], where $\mathcal{O}(N^2/2)$ critical periods are found in degree N, we refined Cen's idea to obtain $\mathcal{O}(N^2)$ critical periods in [8]. In fact, this paper can also be seen as an offspring of Cen's result, this time using slow-fast techniques. We have again managed to improve Cen's lower bound, but remain slightly below the bound in the Hamiltonian class from [8].

Theorem 2. *There exist polynomial systems of odd degree $N \geq 3$ with up to $N^2 - 2N$ critical periods, all of which are present in a single annulus of periodic orbits that contains a unique singular point and all of which correspond to simple local minima and maxima of the period function.*

Interestingly, both results are proven using a vector field in a joint 4-parameter family

$$\begin{cases} \dot{x} = -y(F(x,y)^2 + \varepsilon(G(x,y)^2 + \delta)) + \alpha x(G(x,y)^2 - \beta), \\ \dot{y} = x(F(x,y)^2 + \varepsilon(G(x,y)^2 + \delta)) + \alpha y(G(x,y)^2 - \beta), \end{cases} \tag{1}$$

where F and G are polynomials up to degree $(N-1)/2$ and where $(\varepsilon, \alpha, \beta, \delta) \sim (0,0,0,0)$ (being $\varepsilon > 0$, $\delta > 0$, $\beta > 0$). We will assume that $F(0,0) \neq 0$, so the unperturbed system (e.g. for $\alpha = 0$) is a simple center near the origin. (Notice the quite unusual squaring of the function F, completely ruling out normal hyperbolicity of the fast vector field — we will go into that matter in the next section.) In fact, for proving Theorem 2, we will keep $\alpha = 0$ to maintain the center property.

Unlike in the setting of limit cycles, relieving the restriction of having only one nest, and thus allowing multiple period annuli, does not lead (yet) to better lower bounds, we are currently far from the equivalent lower bound of $\mathcal{O}(N^2 \log N)$ that we have for limit cycles.

2. Approach of the problem

When $\alpha = 0$, system (1) is orbitally equivalent to the simple harmonic oscillator

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases} \tag{2}$$

so it is natural to study the system in polar coordinates. Writing $f(r,\theta) = F(r\cos\theta, r\sin\theta)$ and $g(r,\theta) = G(r\cos\theta, r\sin\theta)$, then we study the system

$$\begin{cases} \dot{r} = \alpha r(g(r,\theta)^2 - \beta), \\ \dot{\theta} = f(r,\theta)^2 + \varepsilon(g(r,\theta)^2 + \delta), \end{cases} \tag{3}$$

which reduces in case $\alpha = 0$ to the r-family of scalar differential equations

$$\frac{d\theta}{dt} = f(r,\theta)^2 + \varepsilon(g(r,\theta)^2 + \delta). \tag{4}$$

Parameterizing the periodic orbits γ_r by the radius, we hence have an elementary expression for the period function

$$T_{\varepsilon,\delta}(r) = \int_0^{2\pi} \frac{d\theta}{f(r,\theta)^2 + \varepsilon(g(r,\theta)^2 + \delta)}. \tag{5}$$

Lemma 3. *When $r_0 > 0$ is such that $f(r_0,\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, then $(r, \varepsilon, \delta) \mapsto T_{\varepsilon,\delta}(r)$ is smoothly defined for all $\varepsilon \geq 0$, $\delta \geq 0$, and r sufficiently close to r_0.*

Under the conditions of the lemma, the period function is smooth but we have a little control over the critical periods. In this case, we observe that the period function

$$T_{0,\delta}(r) = \int_0^{2\pi} \frac{d\theta}{f(r,\theta)^2}$$

becomes unbounded as the orbit γ_r approaches the zero set of $f(r,\theta)$. This naturally leads us to a slow-fast analysis. In essence, the period function will be dominantly determined by the time spent near $f = 0$. But unlike in most papers on slow-fast systems, orbits will not slide along slow curves, but instead will briefly be halted, yet the slowdown is sufficiently long for the passage point to be dominant; this will be the basis for proving Theorem 2.

In proving Theorem 1 we have to perturb the center and take $\alpha \neq 0$. The main tool to study the effect of such a perturbation is the Melnikov-integral

$$\mathcal{M}_{\varepsilon,\beta,\delta}(r) = r^2 \int_0^{T_{\varepsilon,\delta}(r)} \left(g(r,\theta)^2 - \beta\right)\Big|_{\theta=\theta_{\varepsilon,\delta,r}(t)} dt. \tag{6}$$

It is therefore convenient to keep track of the integrand as dynamic variable, i.e. we extend the vector field for $\alpha = 0$ with the equation

$$\dot{M} = g(r,\theta)^2 - \beta.$$

We will of course rely on the well-known property that sign changes of the Melnikov integral translate to limit cycles of the perturbed system for α sufficiently close to 0. So Theorem 1 is a direct consequence of the following result.

Theorem 4. *With $f(r,\theta) = F(r\cos\theta, r\sin\theta)$ and $g(r,\theta) = G(r\cos\theta, r\sin\theta)$, there exist polynomials F and G of odd degree $N \geq 3$, and constants ε, β, and δ for which the Melnikov integral $\mathcal{M}_{\varepsilon,\delta}(r)$ has $(N-1)^2/2$ sign changes.*

Like before, the Melnikov integral will be dominantly determined by its behavior near the points of the double curve $f^2 = 0$. We will therefore continue the analysis after a digression on slow-fast systems with curves of double points in general.

Remark 1. A straightforward but technical approach to integrals (5) and (6) is certainly possible and would lead to a somewhat shorter proof; we prefer to adopt the geometric singular perturbation point of view, at the same time taking advantage of working out a framework that is more general than the application here. Of course, we do realize that appearance of curves of double points in real-world applications is probably very limited, but we envisage possible other theoretical applications in the future.

Remark 2. The unperturbed system for $\alpha = 0$ is typically neither Hamiltonian nor reversible. For this reason, the Melnikov integral is not referred to as an Abelian integral in this context; however, one could call it a pseudo-Abelian integral, since the system becomes Hamiltonian after dividing out an integrating factor. In any case, Theorem 1 provides a new lower bound for the infinitesimal Hilbert 16th problem.

3. Non-hyperbolic slow-fast systems

3.1. Slow-fast systems in non-standard form

Standard presentations of slow-fast systems would start, for example, by considering

$$X_\varepsilon\colon \begin{cases} \dot x = f(x,y,\varepsilon), \\ \dot y = \varepsilon g(x,y,\varepsilon), \end{cases} \tag{7}$$

and would refer to X_0 as the fast system and $\{f(x,y,0)=0,\, y'=g(x,y,0)\}$ as the slow system. The critical curve $\mathcal{S}=\{f(x,y,0)=0\}$ is divided into normally hyperbolic points (where $\frac{\partial f}{\partial x}\neq 0$) and non-normally hyperbolic points. It is well-known that near such normally hyperbolic points, $\mathcal{S}$ perturbs to locally invariant curves called slow curves; they are used to explain the dynamics of slow-fast systems: up to a possible time reversal, orbits are fastly attracted towards the slow curves and then follow the slow curve(s) in a slow fashion. This qualitative description remains valid when we change X_ε by an $\mathcal{O}(\varepsilon^2)$ amount, i.e. well-known notions such as normal hyperbolicity, jump points or turning points, slow and fast vector fields all do not change. It leads one to consider the principal part of a slow-fast system: it is a triplet $(F, Z, Q|_{F=0})$ with

$$F = f(x,y,0), \qquad Z\colon \begin{cases} \dot x = 1, \\ \dot y = 0, \end{cases} \qquad Q\colon \begin{cases} \dot x = \frac{\partial f}{\partial \varepsilon}(x,y,0), \\ \dot y = g(x,y,0). \end{cases}$$

We will write, in this case, $X_\varepsilon = F\cdot Z + \varepsilon Q + \mathcal{O}(\varepsilon^2)$. In this formulation one might be tempted to call Q the slow system, but Q is not exactly that: to obtain the slow system from the (F, Z, Q)-triplet one needs to compute the unique vector field $Q+\lambda Z$ that is tangent to the critical curve $\{F=0\}$. (It is given by $(\dot x, \dot y) = (-g\frac{\partial F}{\partial y}/\frac{\partial F}{\partial x}, g)$. One can easily check that it is tangent to $F=0$. In most papers, the critical curve is a graph and then the slow system can be defined by one variable, specifying either $\dot x$ or $\dot y$.) But why bother decomposing a slow-fast system like (7) this way?

The main benefit of presenting slow-fast systems in the form $X_\varepsilon = F\cdot Z + \varepsilon Q + \mathcal{O}(\varepsilon^2)$, like is done in [6], is that this presentation is much more general than (7) and it does not rely on a trivial splitting of the slow and fast variables like in (7). It is then easier to identify and describe coordinate-free notions.

In the next subsection we use the same terminology for introducing fully non-normally hyperbolic slow-fast systems.

3.2. Double critical curves

Following the notations in [6], we consider families of vector fields $X_{\varepsilon,\lambda} = F_\lambda^2 Z_\lambda + \varepsilon Q_\lambda + \mathcal{O}(\varepsilon^2)$, but in contrast to that reference, the function F_λ, whose zero set forms the critical curve S_λ, is squared here. We repeat that this is a very degenerate situation, probably with limited applicability, and for that reason we do not seek a discussion in the utmost general setting, but it does reveal convenient to define the notions in a coordinate-free way. (A quick reader might want to jump directly to the presentation of the normal form (8) which we will rely upon in the next subsections.) We assume following conditions:

- (C1) The family of vector fields is smoothly defined near some point $p \in \mathbb{R}^2$, for $\varepsilon \in [0, \varepsilon_0[$ and for $\lambda \in \mathbb{R}^m$ in a neighborhood of some λ_0.
- (C2) The fast system, $X_{0,\lambda}$, reduces after division by F_λ^2, to a vector field Z_λ without singular points.
- (C3) The curves $S_\lambda = \{F_\lambda = 0\}$ form a family of regular curves, i.e. there are no points where $\{\frac{\partial F_\lambda}{\partial x} = 0, \frac{\partial F_\lambda}{\partial y} = 0\}$.

We denote the principal part of the slow-fast vector field with the triplet $\left(F_\lambda, Z_\lambda, Q_\lambda|_{S_\lambda}\right)_{\text{dbl}}$, marking with the subscript $(\dots)_{\text{dbl}}$ the difference from the similar triplet in [6].

Remark 3. We remark that there is some ambiguity involved in this principal part: $\left(F_\lambda, Z_\lambda, Q_\lambda|_{S_\lambda}\right)_{\text{dbl}}$ and $\left(c_\lambda F_\lambda, c_\lambda^{-2} Z_\lambda, Q_\lambda|_{S_\lambda}\right)_{\text{dbl}}$ stand for the same slow-fast system, for any function $c_\lambda \neq 0$, but besides this ambiguity the principal part is intrinsically defined. (Unlike in [6], the action of a family of diffeomorphisms $\Psi_{\varepsilon,\lambda}$ on $X_{\varepsilon,\lambda}$ simply induces the action of the diffeomorphism $\Psi_{0,\lambda}$ on Z_λ and on $Q_\lambda|_{S_\lambda}$, i.e. there is no extra source of ambiguity on Q_λ here, thanks to the double nature of the critical curve, see the Appendix.)

The triplet can be used to identify the most common kind of points on the double curve S_λ: *regular transverse points*, i.e. points p where we can locally write the slow-fast vector field in the form

$$\begin{cases} \dot{x} &= x^2 + q(y,\lambda)x^3 + \varepsilon c(x,y,\varepsilon,\lambda), \\ \dot{y} &= \varepsilon d(x,y,\varepsilon,\lambda), \end{cases} \tag{8}$$

with $c(0,0,0,\lambda) > 0$ and the regular transverse point p being located at the origin (Fig. 1). Note that in this form

$$(F_\lambda, Z_\lambda, Q_\lambda)_{\text{dbl}} = \left(x, (1+qx)\frac{\partial}{\partial x}, c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\right)_{\text{dbl}},$$

and that $Z_\lambda(F_\lambda) = 1 + qx$, and $Q_\lambda(F_\lambda) = c$. Inspired by the normal form we give a definition in intrinsic way:

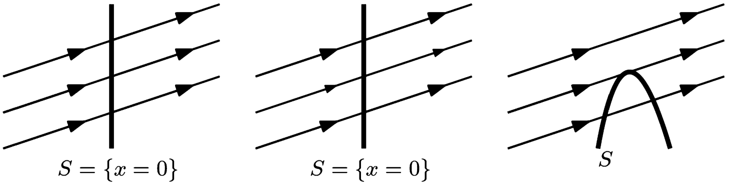

Fig. 1. Regular transverse point (left), Singular transverse point (middle), non-transverse point (right).

Definition 5. Let $X_{\varepsilon,\lambda}$ be a family of slow-fast systems with double critical curve, and let $(F_\lambda, Z_\lambda, Q_\lambda)_{\mathrm{dbl}}$ be the principal part as introduced above. A point p is called a *transverse point* of the critical curve $S_\lambda = F_\lambda^{-1}(\{0\})$ when

$$Z_\lambda(F_\lambda)(p) \neq 0, \qquad Q_\lambda(F_\lambda)(p) \cdot Z_\lambda(F_\lambda)(p) \geq 0.$$

It is called a *regular transverse point* when $Q_\lambda(F_\lambda)(p) \cdot Z_\lambda(F_\lambda)(p) > 0$, and singular otherwise.

We will prove the existence of a local normal form (8) in the next subsection. From the definition it is clear that, when p_0 is a regular transverse point at $\lambda = \lambda_0$, then there is a neighborhood of (p_0, λ_0) where all (p, λ) with $F_\lambda(p) = 0$ are regular transverse points. On the boundary of the regular transverse points we distinguish the singular transverse points, i.e. those where

$$Z_\lambda(F_\lambda)(p) \neq 0, \qquad Q_\lambda(F_\lambda)(p) = 0,$$

and the non-transverse points, i.e. those where

$$Z_\lambda(F_\lambda)(p) = 0.$$

The singular transverse points are easily understood from the normal form: it is a point where $c = 0$; at non-transverse points the normal form is invalid because of a tangency between the fast fibers (i.e. the orbits of the reduced fast vector field Z_λ) and the critical curve S_λ. Singular transverse points have appeared before in the literature, see [7].

Both kinds of points reveal important in the application we have in mind but we will see that we can derive all conclusions from studying regular transverse points only.

3.3. Normal form at transverse points

Lemma 6. *Near a transverse point p_0 (at $\lambda = \lambda_0$) as in Definition 5, the normal form (8) can be obtained after a smooth family of coordinate changes. The coefficient $c(0, 0, 0, \lambda_0)$ is an invariant of the transverse point and is given by the formula*

$$c(0, 0, 0, \lambda_0) = Q_{\lambda_0}(F_{\lambda_0}) Z_{\lambda_0}(F_{\lambda_0})\big|_{p_0}.$$

Proof. Since Z_λ is nonzero we can use flow box coordinates for Z_λ so assume it is simply $\frac{\partial}{\partial x}$. Then the transversality condition $Z_\lambda(F_\lambda) \neq 0$ means that the Implicit Function Theorem can be used to solve $F_\lambda = 0$ as a curve $x = \phi_\lambda(y)$. An additional coordinate change rectifies the critical curve locally to $x = 0$ (and does not alter Z_λ). This puts $X_{0,\lambda} = F_\lambda^2 Z_\lambda$ already in the form

$$\begin{cases} \dot{x} = \mu(x, y, \lambda)^2 x^2, \\ \dot{y} = 0, \end{cases}$$

for some nonzero $\mu(x, y, \lambda)$. In this shape, it is a (y, λ)-family of saddle-node type vector fields for which it is well-known there exists an additional near-identity change of coordinates bringing the vector field $X_{0,\lambda}$ in the form

$$\begin{cases} \dot{x} = (1 + q(y, \lambda)x)x^2, \\ \dot{y} = 0, \end{cases}$$

hence we obtain the normal form (8) for $X_{\varepsilon,\lambda}$. In these coordinates, Z_λ is not necessarily $\frac{\partial}{\partial x}$, since it might have been affected by the final coordinate change. However, the triplet $\left(\tilde{F}_\lambda, \tilde{Z}_\lambda, Q_\lambda|_{S_\lambda}\right)_{\text{dbl}}$ with $\tilde{Z}_\lambda := (1 + q(y, \lambda)x)\frac{\partial}{\partial x}$ and $F_\lambda = x$, is an equivalent triplet. Since the formula in the statement of the lemma does not change under the ambiguity of the triplet mentioned in Remark 3 (see the Appendix), we can use the equivalent version to compute it, and we already did this computation (see the lines before Definition 5), it only remains to evaluate the expression $Q_\lambda(F)Z_\lambda(F)$ for $\lambda = \lambda_0$ and at the point p_0, which lies at $x = 0$. □

Remark 4. If $\tilde{X}_{\varepsilon,\lambda} = \rho_{\varepsilon,\lambda} X_{\varepsilon,\lambda}$ is the slow-fast system obtained after a change of time by multiplying the vector field with a nonzero factor $\rho_{\varepsilon,\lambda}$, then regular transverse points of $X_{\varepsilon,\lambda}$ remain regular transverse points of $\tilde{X}_{\varepsilon,\lambda}$, since the triplet transforms as

$$\left(\tilde{F}_\lambda, \tilde{Z}_\lambda, \tilde{Q}_\lambda\right)_{\text{dbl}} = \left(F_\lambda, \rho_{0,\lambda} Z_\lambda, \rho_{0,\lambda} Q_\lambda\right)_{\text{dbl}},$$

and thus

$$\tilde{Q}_\lambda(\tilde{F}_\lambda)\tilde{Z}_\lambda(\tilde{F}_\lambda)\Big|_p = \rho_{0,\lambda}^2 \; Q_\lambda(F_\lambda)Z_\lambda(F_\lambda)|_p .$$

(The $\mathcal{O}(\varepsilon)$-terms in ρ have no effect on $\tilde{Q}_\lambda$ because its domain is restricted to $F_\lambda = 0$.)

3.4. Passage near transverse points

Blow-up of transverse points. We add $\dot{\varepsilon} = 0$ to the system in normal form (8) and consider a cylindrical blow-up of

$$\begin{cases} \dot{x} = x^2 + q(y, \lambda)x^3 + \varepsilon c(x, y, \varepsilon, \lambda), \\ \dot{y} = \varepsilon d(x, y, \varepsilon, \lambda), \\ \dot{\varepsilon} = 0, \end{cases}$$

by writing

$$(x, \varepsilon) = (u\bar{x}, u^2\bar{\varepsilon}), \qquad u \geq 0, (\bar{x}, \bar{\varepsilon}) \in \mathbb{S}^1.$$

In this case, the blow-up is actually simply a weighted form of polar coordinates (Fig. 2). We study the circle in different charts. In the family chart $\bar{\varepsilon} = 1$, we write $(x, \varepsilon) = (u\bar{x}, u^2)$. This set of coordinates is used to study the part of the orbits where

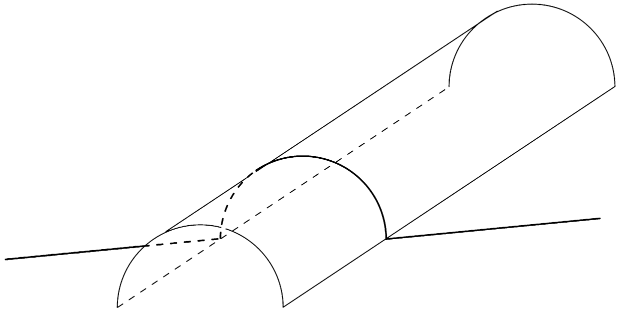

Fig. 2. Blow-up.

$$|x| \leq K\sqrt{\varepsilon}$$

for some large $K > 0$ chosen later (hereby restricting the chart to $|\bar{x}| \leq K$). We find the blow-up system after dividing by the positive common factor u:

$$\begin{cases} \dot{\bar{x}} = \bar{x}^2 + c(0, y, 0, \lambda) + \mathcal{O}(u), \\ \dot{y} = \mathcal{O}(u), \\ \dot{u} = 0. \end{cases} \tag{9}$$

When $c > 0$ there are no singular points here.

In the matching charts $\bar{x} = \pm 1$ we write $(x, \varepsilon) = (\pm u, u^2\bar{\varepsilon})$. This set of coordinates is used to study the part of the orbits where

$$K^{-1} \geq \pm x \geq K\sqrt{\varepsilon}$$

(hence restricting the chart to $0 \leq u \leq K^{-1}$, $0 \leq \bar{\varepsilon} \leq K^{-2}$ which implies assuming $\varepsilon \leq K^{-4}$). We obtain after dividing by the positive common factor u:

$$\begin{cases} \pm\dot{u} = u\left(1 + \bar{\varepsilon}c(0, y, 0, \lambda) + \mathcal{O}(u)\right), \\ \pm\dot{\bar{\varepsilon}} = -2\bar{\varepsilon}\left(1 + \bar{\varepsilon}c(0, y, 0, \lambda) + \mathcal{O}(u)\right), \\ \dot{y} = \mathcal{O}(u\bar{\varepsilon}). \end{cases}$$

By taking K sufficiently large the factor $(1 + \bar{\varepsilon}c + \mathcal{O}(u))$ is strictly positive and then we can divide out this factor after which the first two equations are in the form of a linear saddle:

$$\begin{cases} \pm\dot{u} = u, \\ \pm\dot{\bar{\varepsilon}} = -2\bar{\varepsilon}, \\ \dot{y} = \mathcal{O}(u\bar{\varepsilon}). \end{cases}$$

In a next step we can apply normal form theory: there exists an additional local $\mathcal{C}^\infty$-change of coordinates $y = \mathcal{Y}(Y, u, \bar{\varepsilon}, \lambda) = Y + \mathcal{O}(u\bar{\varepsilon})$ putting the system in form

$$\begin{cases} \pm\dot{u} = u, \\ \pm\dot{\bar{\varepsilon}} = -2\bar{\varepsilon}, \\ \dot{Y} = \varepsilon D_\pm(Y, \varepsilon, \lambda), \end{cases} \tag{10}$$

for some smooth function $D_\pm$. Observe that we conveniently use ε here as a shortcut for $u^2\bar{\varepsilon}$; it is the strength of the normal form that $\bar{\varepsilon}$ only appears paired with u^2 in the $\dot{Y}$-equation. The term $\varepsilon = u^2\bar{\varepsilon}$ is namely the only resonant monomial of the normal form. (For a proof of the smooth normal form see, for example, [6].)

Dynamics near the blow-up locus. For describing the dynamics we divide the orbit in several parts

$$\{x=-x_0\} \xrightarrow{O_1} \{x=-1/K\} \xrightarrow{O_2} \{x=-K\sqrt{\varepsilon}\} \xrightarrow{O_3}$$
$$\{x=+K\sqrt{\varepsilon}\} \xrightarrow{O_4} \{x=1/K\} \xrightarrow{O_5} \{x=+x_0\}.$$

Under condition that $\varepsilon \le K^{-4}$, this sequence of passages is ordered from left to right. Orbit parts O_1 and O_5 are studied using (8); by choosing ε sufficiently small and x_0 sufficiently small and K sufficiently large, there are no singular points in the expression, so it is flow box behavior. The parts O_2 and O_4 are studied in the matching charts of the blow-up. In fact, for O_2 the orbit goes

$$\text{from} \quad (u,\bar{\varepsilon}) = (-1/K, \varepsilon K^2) \quad \text{to} \quad (u,\bar{\varepsilon}) = (-\sqrt{\varepsilon}K, 1/K^2)$$

and similarly along O_4. (To see this, from how O_2 and O_4 are defined, we know the start point and end point of x, hence also of u in the respective matching charts, from that and from the invariance of $\varepsilon = u^2\bar{\varepsilon}$ we deduce the start and end point of $\bar{\varepsilon}$.) We will use expression (10) to study the y-dependence of the orbit and hence will assume K is large enough for the expression to be valid along O_2 and O_4. The middle part O_3 is dealt with in the family chart, there we go

$$\text{from} \quad (\bar{x},u) = (-K, \sqrt{\varepsilon}) \quad \text{to} \quad (\bar{x},u) = (+K, \sqrt{\varepsilon}).$$

We will use expression (9) to study O_3.

Lemma 7. *Near regular transverse points in normal form (8), the orbits between the sections $\{x=-x_0\}$ and $\{x=+x_0\}$ are given by graphs of the form $y=\tilde{y}(x,y_0,\varepsilon,\lambda)$ where*

$$\tilde{y} = y_0 + \mathcal{O}(\varepsilon^{1/2}).$$

Proof. We distinguish the different parts of the orbits O_i, $i=1,\dots,5$, as explained above. Along O_1 we have a simple flow box behavior where orbits are smooth graphs of x. Furthermore since $\dot{y}=\varepsilon$, the y-coordinate has only changed from y_0 with at most an $\mathcal{O}(\varepsilon)$-change when the orbit arrives at $x=-1/K$.

Along O_2, we use the normal form (10). Note that at the onset of O_2, $Y = y + \mathcal{O}(\bar{\varepsilon}) = y_0 + \mathcal{O}(\sqrt{\varepsilon})$. To integrate up to $\bar{\varepsilon} = 1/K^2$, we work with the equivalent system

$$\begin{cases} \dot{u} &= -u/\varepsilon, \\ \dot{\bar{\varepsilon}} &= 2\bar{\varepsilon}/\varepsilon, \\ \dot{Y} &= -D_\pm(Y,\varepsilon,\lambda). \end{cases}$$

We note that the equations are decoupled. Clearly $Y=\tilde{Y}(t,Y_0,\varepsilon,\lambda)$ for some locally defined and smooth $\tilde{Y}$. The benefit of this equivalent system is that the transition time is $o(1)$ as $\varepsilon \to 0$:

$$u = \tilde{u}(t) = -K^{-1}\,\mathrm{e}^{-t/\varepsilon}, \qquad \bar{\varepsilon} = \tilde{\varepsilon}(t) = \varepsilon K^2\,\mathrm{e}^{2t/\varepsilon}, \qquad t \in \left[0, \frac{\varepsilon}{2}\log\frac{1}{\varepsilon K^4}\right].$$

Since the integration time is bounded by $\mathcal{O}(\varepsilon\log\varepsilon)$ and since $D_\pm$ is bounded as a smooth function, it follows that $\tilde{Y} = Y_0 + \mathcal{O}(\varepsilon\log\varepsilon)$ uniformly along O_2. Returning to the original co-ordinates, we obtain $y = Y_0 + \mathcal{O}(u\bar{\varepsilon}) + \mathcal{O}(\varepsilon\log\varepsilon)$, noting that $y = y_0 + \mathcal{O}(\sqrt{\varepsilon})$ (just notice that $u\bar{\varepsilon} = \sqrt{u^2\bar{\varepsilon}}\cdot\sqrt{\bar{\varepsilon}} = \sqrt{\varepsilon}\cdot\sqrt{\bar{\varepsilon}}$).

Continue with the orbit O_3 in the family chart. There, the orbit is clearly a graph of $\bar{x}$, a smooth solution of the regular ordinary differential equation

$$\frac{dy}{d\bar{x}} = \frac{\mathcal{O}(u)}{\bar{x}^2 + c(0, y, 0, \lambda) + \mathcal{O}(u)},$$

whose right-hand side is uniformly $\mathcal{O}(u) = \mathcal{O}(\sqrt{\varepsilon})$ when $c > 0$ (which is the case for regular transverse points). Orbits O_4 and O_5 are treated analogously. □

Remark 5. In the proofs of Theorems 1 and 2, this property is trivially satisfied, since $\dot{y} = 0$ there.

Transit time.

Lemma 8. *At regular transverse points in normal form (8), the orbits between the sections $\{x = -x_0\}$ and $\{x = +x_0\}$ have a transit time given by $t = T(y_0, \varepsilon, \lambda)$ with*

$$T(y_0, \varepsilon, \lambda) = \varepsilon^{-1/2}\left(\frac{\pi}{\sqrt{c(0, y_0, 0, \lambda)}} + \mathcal{O}(\varepsilon^{1/2}\log\varepsilon)\right), \qquad \varepsilon \to 0,$$

uniformly in (y_0, λ).

Proof. We again split up the orbits in O_i, $i = 1, \dots, 5$. Parts O_1 and O_5 have uniformly bounded transit time, so we may ignore these parts. We will in fact restrict to orbit O_4 and the part of orbit O_3 to the right of $\{x = 0\}$ and multiply the total time by 2 based on symmetry.

We first compute the time from $\bar{x} = 0$ to $\bar{x} = K$ in O_3 using the family chart. Recall that the vector field in that chart was given after a time rescaling, so, taking this into account, we find

$$\Delta T = \frac{1}{\sqrt{\varepsilon}}\int_0^K \frac{d\bar{x}}{\bar{x}^2 + c(0, y_0, 0, \lambda) + \mathcal{O}(\sqrt{\varepsilon})},$$

hereby using the fact that $\tilde{y} = y_0 + \mathcal{O}(\sqrt{\varepsilon})$, as established in the previous lemma. Since $c > 0$ for regular transverse points we obtain

$$\Delta T = \frac{1}{\sqrt{\varepsilon}}\int_0^K \frac{d\bar{x}}{\bar{x}^2 + c(0, y_0, 0, \lambda)} + \mathcal{O}(1). \tag{11}$$

Next we consider O_4. We integrate from $\bar{\varepsilon} = K^{-2}$ to $\bar{\varepsilon} = \varepsilon K^2$:

$$\Delta T = \int_{K^{-2}}^{\varepsilon K^2} \frac{d\bar{\varepsilon}}{-2u\bar{\varepsilon}\rho(u,\bar{\varepsilon},\tilde{y},\lambda)}, \tag{12}$$

with $\rho(u,\bar{\varepsilon},y,\lambda) = 1 + \bar{\varepsilon}c(0,y,0,\lambda) + \mathcal{O}(u)$ and of course bearing in mind that $\tilde{y}$ remains close to y_0 but is not a constant. Using the Mean Value Theorem twice we obtain uniformly along O_4:

$$\frac{1}{\rho(u,\bar{\varepsilon},y,\lambda)} = \frac{1}{\rho(0,\bar{\varepsilon},y,\lambda)} + \mathcal{O}(u) = \frac{1}{\rho(0,\bar{\varepsilon},y_0,\lambda)} + \mathcal{O}(u) + \mathcal{O}(y-y_0). \tag{13}$$

Since $\tilde{y}$ stays $\mathcal{O}(\sqrt{\varepsilon})$-close to y_0, both $\mathcal{O}$-terms are uniformly $\mathcal{O}(u)$, and these terms lead to a contribution in ΔT in (12) given by

$$\mathcal{O}\left(\int_{K^{-2}}^{\varepsilon K^2} \frac{d\bar{\varepsilon}}{-2\bar{\varepsilon}}\right) \leq \mathcal{O}(\log\varepsilon).$$

The other part of (13) leads to a contribution in ΔT in (12) given by

$$\frac{1}{\sqrt{\varepsilon}}\int_{K^{-2}}^{\varepsilon K^2} \frac{d\bar{\varepsilon}}{-2\sqrt{\bar{\varepsilon}}(1+\bar{\varepsilon}c(0,y,0,\lambda))} = \frac{1}{\sqrt{\varepsilon}}\int_{K^{-2}}^{0} \frac{d\bar{\varepsilon}}{-2\sqrt{\bar{\varepsilon}}(1+\bar{\varepsilon}c(0,y_0,0,\lambda))} + \mathcal{O}(1).$$

After a change of coordinates $\bar{\varepsilon} = \bar{x}^{-2}$ the integral smoothly collides with (11) computed in the family chart, together contributing as

$$\Delta T = \frac{1}{\sqrt{\varepsilon}}\int_{0}^{\infty} \frac{d\bar{x}}{\bar{x}^2 + c(0,y_0,0,\lambda)} + \mathcal{O}(1).$$

After multiplication by 2 due to symmetry, basic calculus can be used to evaluate the integral and to conclude the proof. □

By combining the above two lemmas with Lemma 6, we can now present a coordinate-free version of them:

Lemma 9. *Consider a slow-fast vector field with dominant part given by $(F_\lambda, Z_\lambda, Q_\lambda)_{dbl}$. Suppose that Σ and Σ' are two sections transverse to the flow of Z_λ and for which a transition map*

$$z \mapsto e_\lambda(z)$$

is well-defined, and that the orbits intersect $S_\lambda = \{F_\lambda = 0\}$ a single time, at regular transverse points $\omega_\lambda(z)$ (with z any regular coordinate on Σ). Then for sufficiently small $\varepsilon > 0$, the transition map $\Sigma \to \Sigma'$ is well-defined for the slow-fast vector field and is given by

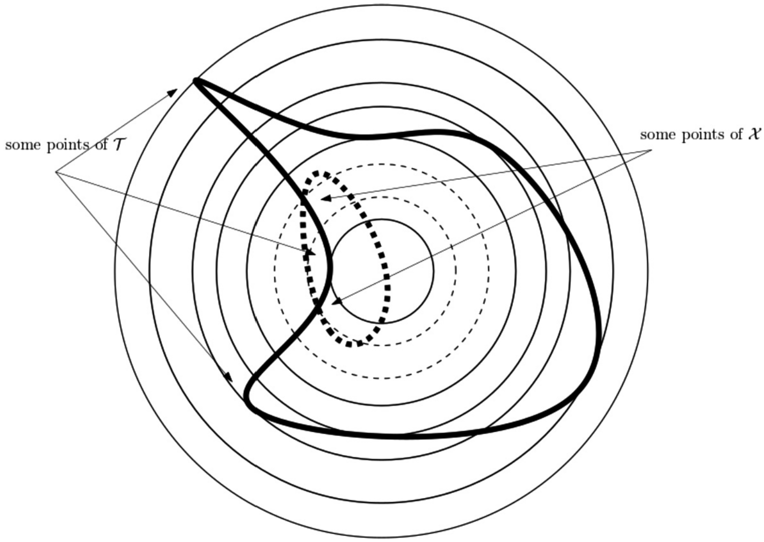

Fig. 3. Zero-set of F (fat curve), zero-set of G (dashed fat curve), their intersection points $\mathcal{X}$ and points of tangencies $\mathcal{T}$.

$$z \mapsto e_\lambda(z) + \mathcal{O}(\sqrt{\varepsilon}).$$

The transition time is given by

$$T(z,\varepsilon,\lambda) = \varepsilon^{-1/2}\left(\frac{\pi}{\sqrt{Q_\lambda(F_\lambda)|_{\omega_\lambda(z)} Z_\lambda(F_\lambda)|_{\omega_\lambda(z)}}} + \mathcal{O}(\varepsilon^{1/2}\log\varepsilon)\right) \text{ when } \varepsilon \to 0.$$

Proof. Outside S_λ, the deviation from the fast flow is at most $\mathcal{O}(\varepsilon)$ and the transition time is bounded, so we can reduce the problem to a study in the neighborhood of a transverse point, where a normal form can be used and the above lemmas are valid. Lemma 6 gives the link between the intrinsic formula for T and the specific one in Lemma 8. □

Of course, this lemma can and will be applied iteratively, when fast orbits intersect the critical curve more than once; it leads to a discrete sum of contributions in the leading order.

4. Proof of Theorem 2

Recall (3) for $\alpha = 0$:

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = f(r,\theta)^2 + \varepsilon(g(r,\theta)^2 + \delta). \end{cases} \tag{14}$$

We denote the zero set of f by $\mathcal{S}$ and the zero set of g by $\mathcal{G}$ (Fig. 3). *We assume $\mathcal{X} := \mathcal{S} \cap \mathcal{G}$ consists of a finite number of isolated points, all of them located on different concentric circles around the origin.* The set of points where $\mathcal{S}$ is tangent to the circles will be written as

$$\mathcal{T} := \mathcal{S} \cap \left\{(r,\theta) : \frac{\partial f}{\partial \theta} = 0\right\}.$$

We assume $\mathcal{T}$ consists of a finite number of isolated points, all of them located on different concentric circles around the origin and not on the concentric circles on which points of $\mathcal{X}$ are located. We finally assume $\mathcal{S}$ does not contain the origin.

In terms of the original functions in (x, y) variables we find

$$\mathcal{S} = \{(x, y) : F(x, y) = 0\},$$
$$\mathcal{X} = \{(x, y) : F(x, y) = G(x, y) = 0\},$$
$$\mathcal{T} = \left\{(x, y) : F(x, y) = y\frac{\partial F}{\partial x}(x, y) - x\frac{\partial F}{\partial y}(x, y) = 0\right\}.$$

Lemma 10. *Let $N \geq 3$ be odd. There are choices of polynomials of degree $(N-1)/2$ for which the above assumptions are satisfied and for which*

$$\#\mathcal{X} = \frac{(N-1)^2}{4}, \qquad \#\mathcal{T} = \frac{(N-1)^2}{4}.$$

Proof. We treat the case $N = 3$ separately, since we will have to assume $N \geq 5$ later on. In fact the case $N = 3$ is trivial: it suffices to take $F(x, y) = x - 1$ and $G(x, y) = y - 1$ for example; then $\mathcal{T} = \{(1, 0)\}$ and $\mathcal{X} = \{(1, 1)\}$, and both points have a distinct distance to the origin.

So let us continue with $N \geq 5$. Consider

$$L_k(x, y) = y + 1 + k^2 - 2kx$$

the zero set of L_k is tangent to a concentric circle around the origin at the point $p_k = (2k\frac{k^2+1}{4k^2+1}, -\frac{k^2+1}{4k^2+1})$, i.e. at the point of smallest distance to the origin. Let $K \subset \mathbb{R}$ be a finite set with at least 2 elements. Then besides the points p_k, $k \in K$, the function

$$F^0 = \prod_{k \in K} L_k$$

has a zero set (see for example Fig. 4) with double points $m_{k,\ell} = (\frac{k+\ell}{2}, k\ell - 1)$ at the intersections of the zero sets of each pair (L_k, L_ℓ). We impose on K the condition that all double points $m_{k,\ell}$ and all tangency points p_k lie on distinct concentric circles around the origin. There certainly exist sets K that satisfy this condition: one could remove all possible violations of the conditions one by one if necessary, by considering increasingly small perturbations of elements of K, one at a time. To make the set K in Fig. 4 conform to the condition, one would for example replace $K = \{\pm 1, \pm 2, \pm 3\}$ with $K = \{1, -\frac{11}{10}, 3, -\frac{21}{10}, 3, -\frac{31}{10}\}$. Note that this condition also implies that the tangent to the circle around the origin passing through any of the $m_{k,\ell}$ is different from the tangents of the lines L_k and L_ℓ.

We cannot choose $F^0 = 0$ as critical curve because at the double points the curve is not regular. So we perturb the curve by writing

$$F = F^0 + \mu P, \qquad \text{with } P \text{ some polynomial of degree } \#K - 2 \geq 0,$$

where $\mu > 0$ is sufficiently small. Interpolation theory (see [4] for example) tells us that there exists such a polynomial so that $P(m_{k,\ell}) = \pm 1$, for any choice of sign at any double point. In other words, in local coordinates near any of the points $m_{k,\ell}$ we would see

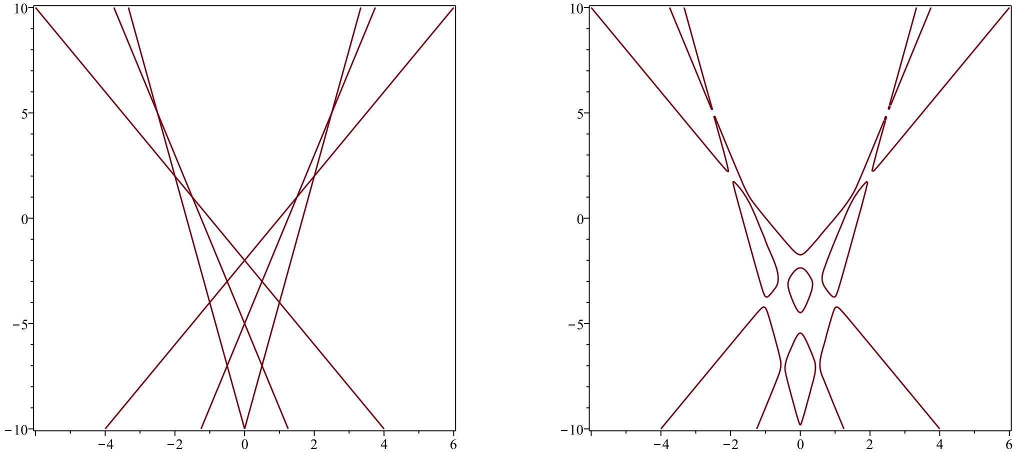

Fig. 4. Left: The zero set of F^0, for $K = \{\pm 1, \pm 2, \pm 3\}$. Right: some perturbation F of F^0.

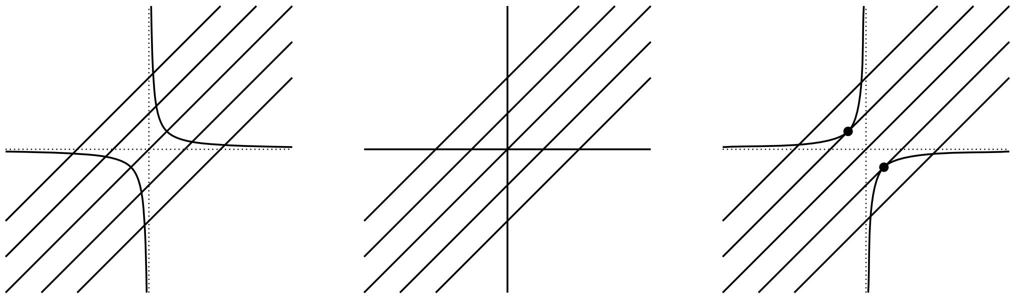

Fig. 5. Middle: unperturbed $L_k \cup L_\ell$ in local normal coordinates. Left: perturbation without tangency points, Right: perturbation with two tangency points. The diagonal lines are level sets of circles around the origin (almost straight lines in zoomed-in local coordinates around $m_{k,\ell}$).

$$F = XY(1 + \mathcal{O}(X, Y)) + \mu(\pm 1 + \mathcal{O}(X, Y)).$$

The points p_k smoothly perturb to tangency points of F, i.e. points of $\mathcal{T}$. We will now prove that near each double point $m_{k,\ell}$, we find two additional points of $\mathcal{T}$. Indeed, near $m_{k,\ell}$ the zero set of F^0 is a transcritical intersection of two lines that will perturb locally to a hyperbola in one or the other direction depending on the chosen sign in the above local expression. Taken into account that the nearby circles have different tangents (as noted above) it implies that for the well-chosen sign there is on each sheet of the hyperbola a tangency point, see Fig. 5.

If $n = \#K$, then there are n points p_k and $\binom{n}{2}$ points $m_{k,\ell}$ so after perturbation we have

$$n + 2\binom{n}{2} = n + n(n-1) = n^2$$

tangency points. With $n = (N-1)/2$ this proves the claim on $\mathcal{T}$. For the claim on $\mathcal{X}$ it suffices to choose $G = \prod_{k \in \tilde{K}} L_k$, the set $\tilde{K}$ being a perturbation of the set K; generic choices of these perturbations will cause the intersection points to lie on distinct concentric circles. (Actually, the order of perturbations is: first start with a general K, perturb a bit so that the genericity condition on F^0 is satisfied, then define $\tilde{K}$ as above, this way fixing G, and finally perturb F^0 to F by choosing a sufficiently small μ.) $\square$

We now distinguish three kinds of circles $C_{r_0} := \{r = r_0\}$:

- $C_{r_0} \cap \mathcal{S} = \emptyset$. Then the period function near C_r is smooth and remains bounded as $\varepsilon \to 0$ uniformly around $r = r_0$. We have no control on the location of any critical periods in those cases and do not try to count any possible occurrences. In particular, we have

$$\sqrt{\varepsilon} T_{\varepsilon,\delta}(r_0) \to 0, \text{ as } \varepsilon \to 0,$$

uniformly for all δ close enough to 0, if we keep r_0 in a compact set of values for which the intersection C_{r_0} with $\mathcal{S}$ is empty.

- $C_{r_0} \cap \mathcal{S}$ contains a finite number of points, but none of them belongs to $\mathcal{T}$ or $\mathcal{X}$. The intersection points are in that case regular transverse points. In order to express the period function, we note that at such a point p we have

$$(F, Z, Q)_{\text{dbl}} = \left(f, \frac{\partial}{\partial \theta}, (g^2 + \delta) \frac{\partial}{\partial \theta} \right)_{\text{dbl}}$$

so the invariant in Lemma 6 is given by $c = (g^2 + \delta) \frac{\partial f}{\partial \theta}(p)^2$ and using Lemma 9 we find

$$T_{\varepsilon,\delta}(r_0) = \frac{1}{\sqrt{\varepsilon}} T_\delta^0(r_0) + \mathcal{O}(\log \varepsilon), \qquad T_\delta^0(r_0) = \sum_{p \in C_{r_0} \cap \mathcal{S}} \frac{\pi}{\sqrt{g^2 + \delta} \left| \frac{\partial f}{\partial \theta}(p) \right|}. \tag{15}$$

We will not need asymptotics for $T'_{\varepsilon,\delta}(r_0)$ despite the fact that we are looking for critical points of $T_{\varepsilon,\delta}$. Only note as before that the convergence

$$\sqrt{\varepsilon} T_{\varepsilon,\delta}(r_0) \to T_\delta^0(r_0), \text{ as } \varepsilon \to 0, \tag{16}$$

is uniform as long as we keep $|\delta|$ sufficiently small and restrict r_0 to arbitrary large compact sets for which C_{r_0} avoids $\mathcal{X}$ and $\mathcal{T}$.

- $C_{r_0} \cap \mathcal{S}$ contains a finite number of points and exactly one of them, p_*, belongs to $\mathcal{T}$ or $\mathcal{X}$. Recall that by construction all points of $\mathcal{X} \cup \mathcal{T}$ lie on different circles. We will not need to control the asymptotics here. We only observe that in case a point p of $\mathcal{X}$ is encountered, $g(p) = 0$. We have not really proven an analogue to (16) but only need that

$$T_\delta^0(r) \to \infty, \text{ as } (r, \delta) \to (r_0, 0).$$

Similarly, when a point p of $\mathcal{T}$ is encountered then we know $\frac{\partial f}{\partial \theta}(p) = 0$, causing

$$T_\delta^0(r) \to \infty, \text{ as } r \to r_0,$$

uniformly for sufficiently small $|\delta|$.

Let us now finish the proof of the theorem. Let

$$0 < r_1 < r_2 < \cdots < r_M < \infty$$

be all the radii that belong to the last category, i.e. the corresponding circles have a point on $\mathcal{X}$ or $\mathcal{T}$. Choose radii in between in an arbitrary way:

$$r_1' \in]0, r_1[, \qquad \ldots, \qquad r_j' \in \big]r_{j-1}, r_j\big[, \qquad \ldots, \qquad r_{M+1}' \in]r_M, \infty[. \tag{17}$$

Then for each r_j', the corresponding circle is in one of the first two categories, which implies that $\sqrt{\varepsilon}T_\delta^0(r_j')$ is uniformly bounded by some T^{00} for all δ sufficiently close to 0 and all j.

Now for each of the radii r_i, there will be at least a $\delta > 0$ sufficiently small and a $\tilde{r}_i$ sufficiently close to r_i so that $T_\delta^0(\tilde{r}_i)$ is well-defined and is larger than T^{00}. This is possible since we know that $T_\delta^0(r) \to \infty$ as $(r, \delta) \to (r_i, 0)$. We can refine (17) to

$$r_1' \in \big]0, \tilde{r}_1\big[, \qquad \ldots, \qquad r_j' \in \big]\tilde{r}_{j-1}, \tilde{r}_j\big[, \qquad \ldots, \qquad r_{M+1}' \in \big]\tilde{r}_M, \infty\big[.$$

As $\{r_j', \tilde{r}_i\}_{i,j}$ is a set of radii for which the circles have no points of $\mathcal{X}$ or $\mathcal{T}$, formula (16) is valid, so it means $\sqrt{\varepsilon}T_{\varepsilon,\delta}$ must oscillate between values above and below T^{00} near each crossing of $\mathcal{X}$ or $\mathcal{T}$: for each of the r_j', $\sqrt{\varepsilon}T_{\varepsilon,\delta}$ tends towards a value bounded by T^{00} and for the $\tilde{r}_i$ radii, the limit is strictly larger than T^{00}. Since $T_{\varepsilon,\delta}$ is smooth for a fixed choice of (ε, δ) with $\varepsilon > 0$ and $\delta \neq 0$, this will imply the presence of the critical points.

Associated to each peak, we must have at least one local maximum of the period function, and between two peaks at least one local minimum, so the number of critical periods is at least

$$2 \times (\#\mathcal{X} + \#\mathcal{T}) - 1.$$

Keeping Lemma 10 in mind, it proves Theorem 2.

Remark 6. If one wanted to obtain upper bounds for the number of critical periods, then the approach with point-wise limits would not be sufficient and a detailed slow-fast analysis of singular transverse points and non-transverse (contact) points would be required, as well as information on the partial derivatives of T in Lemma 8.

5. Numerical example

Before we tackle the proof of Theorem 1, we discuss in this section a numerical example of degree $N = 5$, with 4 points in $\mathcal{X}$ and 4 points in $\mathcal{T}$. The proof of Theorem 2 does not rely on this section, we merely included it for the sake of a better comprehension of the proof. The computations in this section are partly verifiable by hand, and the numerical simulation is the result of an elementary numerical integration of (5), using mainstream mathematical software. We deviated a bit from the construction proposed in Lemma 10 (though we will follow the same spirit), in order to be able to take $|\mu|$ only moderately close to 0, since the smaller $|\mu|$, the closer the radii of the tangency points that unfold from the double points are to each other, implying the need to take extremely small values of ε if one wants to expose a critical period in between the radii.

We consider (1) with $\alpha = 0$. We choose

$$F = F^0 + \mu, \qquad F^0 = (x + 2y - 1)(y - 2), \qquad G = (x - 3)(2y - x - 12).$$

(For the choice of G we certainly deviate from the construction in Lemma 10, again in order to avoid radii that are numerically too close to each other.) For $\mu = 0$, the curves $F = 0$ and $G = 0$ intersect at the points $(3, -1)$, $(3, 2)$, $(-\frac{11}{2}, \frac{13}{4})$ and $(-8, 2)$. The curve $F = 0$ has 1 double point at $(-3, 2)$ and two points tangent to concentric circles around the origin at $(\frac{1}{5}, \frac{2}{5})$ and $(0, 2)$. The perturbation for $\mu > 0$, has the benefit that the 4 intersection points do not bifurcate, nor the 2 tangent points and the 2 additional tangent points that arise from the unfolding of the double point, for as long as $\mu < \mu_{max}$ with $\mu_{max} \approx 0.6$. We want the radii of the circles on which all these points to be distinct, so strategically choose μ somewhere between $\mu = 0$ and $\mu = \mu_{max}$ (at $\mu_{\max}$ some saddle-node bifurcation takes place). So we continue with $\mu = 0.125$. Using a numerical solver, we find

$$\mathcal{X} \approx \{(3.0, -0.98),\ (3.0, 1.98),\ (-5.55, 3.22),\ (-7.95, 2.03)\},$$

$$\mathcal{T} \approx \{(0.22, 0.43),\ (-0.03, 1.96),\ (-1.98, 1.79),\ (-4.01, 2.22)\},$$

the last two points originating from the double point. We also find the respective distinct radii, approximately given by

$$3.2,\ 3.6,\ 6.4,\ 8.2 \qquad \text{and} \qquad 0.5,\ 2.0,\ 2.7,\ 4.6,$$

and near which the theory from the previous sections predicts the presence of local maxima of the period function.

Using a numerical integrator we have produced a simulation of the period function, see Fig. 6. (We have used Maple software and have instructed Maple to use a general purpose numerical integrator and well-documented algorithm from the NAG library, called d01ajc; other integration methods confirm the outcome, but appear to be somewhat slower.) There seem to be 15 critical points: 8 local maxima, at the locations very close to those predicted by our theory, and 7 local minima in between. Some peaks are sharp, near other local maxima the growth is more moderate. It might reveal a difference in the order at which $\sqrt{\varepsilon}T_{\varepsilon,\delta}(r)$ tends to infinity as $(\varepsilon, \delta) \to (0, 0)$ at the various radii. In fact, we have not established the speed of divergence, we have only shown the divergence towards infinity. The proof of Theorem 2 consisted of establishing this divergence in combination with proving the convergence of $\sqrt{\varepsilon}T_{\varepsilon,\delta}(r)$ towards bounded values as $(\varepsilon, \delta) \to (0, 0)$, for radii in between those of points of $\mathcal{X} \cup \mathcal{T}$. Note that to fully illustrate our proof, one would however need to take values of (ε, δ) even closer to $(0, 0)$ than the values used to produce Fig. 6: in the proof, we need the highest local minimum to be lower than the lowest local maximum (which is not the case in Fig. 6), because T^{00} should lie in between. Theoretically, it is clear because the local minima converge whereas the local maxima diverge in the singular limit; numerically it is more difficult since integration becomes increasingly cumbersome towards the singular limit, even for an elementary-looking integral as in (5).

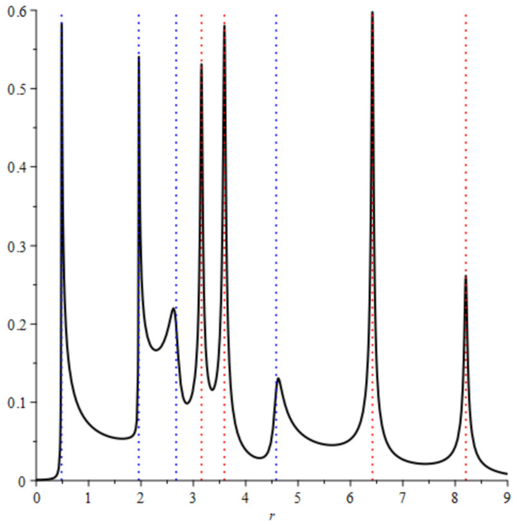

Fig. 6. Numerical simulation of the period function (multiplied by $\sqrt{\varepsilon}$). Integration parameters are $(\varepsilon, \delta) = (5 \times 10^{-7}, 0.1)$.

6. Proof of Theorem 1

Though the limit cycles are found for the perturbed system with $\alpha \neq 0$, the presence of the cycles is obtained by computing a Melnikov integral along the unperturbed system, i.e. with $\alpha = 0$. The integral (6) is equivalent to the integral

$$M_{\varepsilon,\delta,\beta}(r) = \int_0^{2\pi} \frac{g(r,\theta)^2 - \beta}{f(r,\theta)^2 + \varepsilon(g(r,\theta)^2 + \delta)} d\theta,$$

it is again an integral along a circle C_r. Then clearly

$$T_{\varepsilon,\delta}(r) \times \min_{C_r}(g(r,\theta)^2 - \beta) \le M_{\varepsilon,\delta,\beta}(r) \le T_{\varepsilon,\delta}(r) \times \max_{C_r}(g(r,\theta)^2 - \beta),$$

(it is the simplest lower/upper sum estimate) however such a crude estimate will not help to find zeros of $M_{\varepsilon,\delta,\beta}$. But if we focus on the parts of C_r that contribute most in the period, we will gain control.

Let us restrict our attention to Melnikov integrals computed at cycles that come close to points of $\mathcal{X}$, ignoring other parts of the annulus. We focus on one of such points $p_* \in \mathcal{X}$, with radius r_*.

Since the cycle through p_* contains no tangency point, i.e. no point of $\mathcal{T}$, the point p_* on $\mathcal{X}$ perturbs to a nearby point $p(r)$ on $C_r \cap \mathcal{S}$ for nearby radii r. Since points of $\mathcal{X}$ are isolated we may assume that $p(r) \notin \mathcal{X}$ for $r \neq r_*$.

Then there exists at least one cycle on each side of C_{r_*}, i.e. C_{r_-} and C_{r_+} with $0 < r_- < r_* < r_+$, that meets neither $\mathcal{X}$ nor $\mathcal{T}$ but that contains at least one transverse point namely $p(r_\pm)$, see Fig. 7.

For the rest of the proof we will fix choices $r_\pm$ near each of the finite number of points in $\mathcal{X}$.

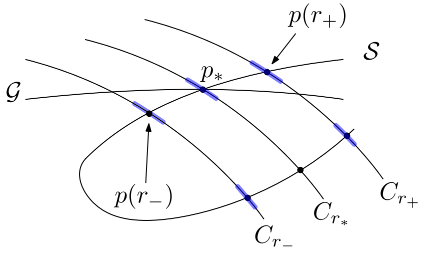

Fig. 7. A (part of a) circle C_{r_*} containing a singular transverse point p_* and nearby circles $C_{r_\pm}$ containing at most a finite number of regular transverse points. Each circle is divided in two parts, referenced in the text with index (1) and (2). The thick part is part (1), the thin part is part (2). On each of the three circles, passage through the part (1) is asymptotically much slower than through part (2).

Theorem 4 will be a direct consequence (and hence so will be Theorem 1) of the following proposition, applied near each point r_* of $\mathcal{X}$ (keeping in mind the existence of polynomials with $\#\mathcal{X} = \frac{(N-1)^2}{4}$: recall Lemma 10 to that end).

Proposition 11. *There exists a choice of $\beta > 0$ so that for each of the (r_*, r_+, r_-) and for (ε, δ) sufficiently close to $(0, 0)$ (both of them positive), we have*

$$M_{\varepsilon,\delta,\beta}(r_\pm) > 0 > M_{\varepsilon,\delta,\beta}(r_*).$$

In other words near r_* the function $M_{\varepsilon,\delta,\beta}$ has at least two sign changes.

Proof. Let $C_{r_\pm} = C^{(1)}_{r_\pm} \cup C^{(2)}_{r_\pm}$ where $C^{(1)}_{r_\pm}$ is a union of small parts of $C_{r_\pm}$ near all its points on $\mathcal{S}$, including at least a part near $p(r_\pm)$, and where $C^{(2)}_{r_\pm}$ is the rest of the cycle, away from the critical curve. We then infer

$$M_{\varepsilon,\delta,\beta}(r_\pm) \geq T^{(1)}_{\varepsilon,\delta}(r_\pm) \times \min_{C^{(1)}_{r_\pm}}(g(r,\theta)^2 - \beta) - T^{(2)}_{\varepsilon,\delta}(r_\pm) \times \max_{C^{(2)}_{r_\pm}} \left|\beta - g(r,\theta)^2\right|.$$

(It is again a simple lower sum estimate for the integral where we have used $\min X = -\max(-X) \geq -|\max(-X)|$.) Of course the second term is uniformly bounded (in absolute value) by some $K > 0$ as $(\varepsilon, \beta, \delta) \to (0, 0, 0)$ (since $C^{(2)}_{\pm}$ is kept away from $\mathcal{S}$). So the first term will dominate the right-hand side of the above expression because it involves the time spent near the *regular* transverse point(s). Formula (15) is valid for that part of the circle.

Let us now present a first condition on β: we impose

$$0 < \beta \leq B := \frac{1}{3} \min_{r_\pm} \min_{p \in C_{r_\pm} \cap \mathcal{S}} g(p)^2 \in \mathbb{R}^+.$$

(The exterior minimum is taken over the finite set of all pairs $r_\pm$.) Then by taking the $C^{(1)}_{r_\pm}$ part of the circle sufficiently small (at the expense of possibly enlarging K) we have

$$\min_{C^{(1)}_{r_\pm}} g(r,\theta)^2 \geq 2B \implies \min_{C^{(1)}_{r_\pm}}(g(r,\theta)^2 - \beta) \geq B$$

and

$$M_{\varepsilon,\delta,\beta}(r_\pm) \geq T^{(1)}_{\varepsilon,\delta} \times B - K$$

which is strictly positive for sufficiently small (ε, δ) keeping in mind the asymptotics in formula (15) and knowing that there is at least one regular transverse point in $C_{r_\pm}$. We find that the Melnikov integral is strictly positive at $r_\pm$.

Let us now turn our attention to r_*. Like before we split up the circle in two parts: $C_{r_*} = C^{(1)}_{r_*} \cup C^{(2)}_{r_*}$ where $C^{(1)}_{r_*}$ contains this time only a part near p_* and $C^{(2)}_{r_*}$ all the rest, possibly including some regular transverse passages.

There we consider the following simple upper sum estimate:

$$M_{\varepsilon,\delta,\beta}(r_*) \leq -T^{(1)}_{\varepsilon,\delta}(r_*) \times \min_{C^{(1)}_{r_*}}(\beta - g(r,\theta)^2) + T^{(2)}_{\varepsilon,\delta}(r_\pm) \times \max_{C^{(2)}_{r_*}} \left| g(r,\theta)^2 - \beta \right|.$$

To control the first term we impose a restriction on the size of the part $C^{(1)}_{r_*}$: it should be small enough so that

$$g(r,\theta)^2 \leq \frac{\beta}{2}, \qquad \text{for all } (r,\theta) \in C^{(1)}_{r_*}.$$

It is possible because at each point $p_* \in \mathcal{X}$ the function g is zero.

The second term is bounded by $L/\sqrt{\varepsilon}$ for some $L > 0$, because again formula (15) applies. Thus, we find:

$$M_{\varepsilon,\delta,\beta}(r_*) \leq -T^{(1)}_{\varepsilon,\delta}(r_*) \times \frac{\beta}{2} + \frac{L}{\sqrt{\varepsilon}}.$$

Since we have shown that $\sqrt{\varepsilon}T_{\varepsilon,\delta}(r_*) \to \infty$ (and likewise $\sqrt{\varepsilon}T^{(1)}_{\varepsilon,\delta}(r_*) \to \infty$) as $(\varepsilon,\delta) \to (0,0)$ the Melnikov function will become negative at r_* for (ε,δ) sufficiently close to $(0,0)$.

As a consequence, $M_{\varepsilon,\delta,\beta}$ has at least two sign changes near r_*. This proves the proposition, and by consequence also Theorem 4 and Theorem 1. □

Remark 7. The proof of Proposition 11 uses very crude estimates to find a condition on β ($\beta \leq B$). One may expect that the upper bound is much too stringent. Effectively searching for zeros of the Melnikov numerically would most probably start with looking at values of β that are not necessarily small, since the price to pay when using small β is that ε will have to be much smaller as well since we want

$$T^{(1)}_{\varepsilon,\delta} \geq \frac{2}{\beta}\frac{L}{\sqrt{\varepsilon}}.$$

Remark 8. For future research we aim at establishing an asymptotic formula, similar to the one in (16), for computing divergence integrals along periodic orbits. Such a formula would constitute a *discrete analog of the slow divergence integral* for slow-fast cycles that contain a number of transverse points. Seen the fact that the slow divergence integral has shown itself as a successful tool in controlling the number of limit cycles (both for upper and lower bounds), such a discrete analog might be worthwhile studying from the theoretical point of view. In this paper, we have not really used the finite sum, any possible critical periods or zeros of the Melnikov integral that arise from an in-depth study of this finite sum could lead to improvements of the lower bounds presented here.

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Appendix A

We deal with Remark 3. Let (the principal part of) $X_{\varepsilon,\lambda}$ be defined by $\left(F_\lambda, Z_\lambda, Q_\lambda|_{S_\lambda}\right)_{\mathrm{dbl}}$. We mean that

$$X_{\varepsilon,\lambda} = F_\lambda^2 Z_\lambda + \varepsilon(Q_\lambda + \mathcal{O}(F_\lambda)) + \mathcal{O}(\varepsilon^2).$$

Now consider an (ε, λ)-family of changes of coordinates $\Phi_{\varepsilon,\lambda}$, and write

$$Y_{\varepsilon,\lambda} = \Phi_{\varepsilon,\lambda}^*(X_{\varepsilon,\lambda}) = \left(D\Phi_{\varepsilon,\lambda}\right)^{-1} \cdot (X_{\varepsilon,\lambda} \circ \Phi_{\varepsilon,\lambda}).$$

Then (keep in mind that F_λ is a scalar function and Z_λ and Q_λ are vector fields)

$$\begin{aligned} Y_{\varepsilon,\lambda} &= \left(D\Phi_{\varepsilon,\lambda}\right)^{-1} \cdot \left((F_\lambda^2 Z_\lambda) \circ \Phi_{\varepsilon,\lambda}\right) + \varepsilon \left(D\Phi_{0,\lambda}\right)^{-1} \cdot \left((Q_\lambda + \mathcal{O}(F_\lambda)) \circ \Phi_{0,\lambda}\right) + \mathcal{O}(\varepsilon^2) \\ &= \left(F_\lambda \circ \Phi_{\varepsilon,\lambda}\right)^2 \left(D\Phi_{\varepsilon,\lambda}\right)^{-1} \cdot \left(Z_\lambda \circ \Phi_{\varepsilon,\lambda}\right) + \varepsilon \left(D\Phi_{0,\lambda}\right)^{-1} \cdot \left((Q_\lambda + \mathcal{O}(F_\lambda)) \circ \Phi_{0,\lambda}\right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

If we write $G_\lambda = F_\lambda \circ \Phi_{0,\lambda}$, then

$$Y_{\varepsilon,\lambda} = (G_\lambda)^2 \left(D\Phi_{\varepsilon,\lambda}\right)^{-1} \cdot \left(Z_\lambda \circ \Phi_{\varepsilon,\lambda}\right) + \varepsilon(\Phi_{0,\lambda}^* Q_\lambda + \mathcal{O}(G_\lambda)) + \mathcal{O}(\varepsilon^2).$$

Note that $F_\lambda \circ \Phi_{\varepsilon,\lambda} = G_\lambda + \mathcal{O}(\varepsilon)$. The squaring of F ensures that the ε-dependent terms in $F_\lambda \circ \Phi_{\varepsilon,\lambda}$ disappear into the $\varepsilon \cdot \mathcal{O}(G_\lambda)$-terms and the $\mathcal{O}(\varepsilon^2)$ terms. (In the normally hyperbolic situation where F_λ is not squared, it causes Q_λ to only be defined up to $\mathcal{O}(Z_\lambda)$, here we do not have that issue.) We can thus conclude

$$Y_{\varepsilon,\lambda} = (G_\lambda)^2 \Phi_{0,\lambda}^* Z_\lambda + \varepsilon(\Phi_{0,\lambda}^* Q_\lambda + \mathcal{O}(G_\lambda)) + \mathcal{O}(\varepsilon^2),$$

in other words we have shown:

Lemma 12. *Any family of diffeomorphisms $\Phi_{\varepsilon,\lambda}$ pulls back a slow-fast vector field whose principal part is given by $\left(F_\lambda, Z_\lambda, Q_\lambda|_{S_\lambda}\right)_{dbl}$ to a slow-fast vector field whose principal part is given by*

$$\left(F_\lambda \circ \Phi_{0,\lambda}, \Phi_{0,\lambda}^* Z_\lambda, \Phi_{0,\lambda}^* Q_\lambda|_{C_\lambda}\right)_{dbl},$$

with $C_\lambda = \Phi_{0,\lambda}^{-1}(S_\lambda)$. This makes the components of the principal part intrinsically defined up to the ambiguity explained in Remark 3.

As a corollary, at points $p \in S_\lambda$ with $Z_\lambda(p) \neq 0$, the expression

$$Q_\lambda(F_\lambda) Z_\lambda(F_\lambda)|_p,$$

(seen in Lemma 6) is coordinate free. Indeed, if $\tilde{F}_\lambda = c_\lambda F_\lambda$ and $\tilde{Z}_\lambda = c_\lambda^{-2} Z_\lambda$, then evaluated at p we have

$$Q_\lambda(\tilde{F}_\lambda)\tilde{Z}_\lambda(\tilde{F}_\lambda) = Q_\lambda(c_\lambda F_\lambda) c_\lambda^{-2} Z_\lambda(c_\lambda F_\lambda) = Q_\lambda(F_\lambda) Z_\lambda(F_\lambda) + \mathcal{O}(F_\lambda)$$

which proves the claim since $F_\lambda(p) = 0$.

Data availability

No data was used for the research described in the article.

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