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RESEARCH

A Counterexample Regarding a Two-phase Problem for Harmonic Measure in VMO

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Abstract
Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a vanishing Reifenberg flat domain such that Ω^+ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ have joint big pieces of chord-arc subdomains and such that the outer unit normal to $\partial\Omega^+$ belongs to $\mathrm{VMO}(\omega^+)$, where $\omega^\pm$ is the harmonic measure in $\Omega^\pm$. Up to now it was an open question if these conditions imply that $\log \dfrac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(\omega^+)$. In this paper we answer this question in the negative by constructing an appropriate counterexample in $\mathbb{R}^2$, with the additional property that the outer unit normal to $\partial\Omega^+$ is constant ω^+-a.e. in $\partial\Omega^+$.

Keywords Harmonic measure · Two-phase problem · Reifenberg flat domains

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1 Introduction

This paper deals with a two-phase problem for harmonic measure. In such problems one considers two disjoint domains $\Omega^+, \Omega^- \subset \mathbb{R}^{n+1}$ whose boundaries have non-empty intersection, and whose respective harmonic measures ω^+, ω^- are usually mutually absolutely continuous in some subset of $\partial\Omega^+ \cap \partial\Omega^-$. Then one has to relate the analytic properties of the density $\frac{d\omega^-}{d\omega^+}|_{\partial\Omega^+\cap\partial\Omega^-}$ to the geometric properties of $\partial\Omega^+ \cap \partial\Omega^-$. For example, in [3] and [5] it has been proved that if ω^+ and ω^- are mutually absolutely continuous in a subset $E \subset \partial\Omega$, then $\omega^\pm|_E$ is concentrated in an n-rectifiable set. For a previous related result see [10], and for a more recent work involving elliptic measure, see [2]. For other works of more quantitative nature where one assumes Ω^+, Ω^- to be complementary NTA domains and either that $\omega^- \in A_\infty(\omega^+)$ or stronger conditions, see [4, 8, 13, 15], and [16], for instance. See also [6] and [7] for other recent results dealing with the structure of the singular set of the boundary.

In connection with the precise question studied in this paper, by combining works of Prats, Tolsa, and Toro, the following is known:

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Theorem A *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a bounded NTA domain and let $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ be an NTA domain as well. Denote by ω^+ and ω^- the respective harmonic measures with poles $p^+ \in \Omega^+$ and $p^- \in \Omega^-$. Suppose that Ω^+ is a δ-Reifenberg flat domain, with $\delta > 0$ small enough. Then the following conditions are equivalent:*

(a) *ω^+ and ω^- are mutually absolutely continuous and* $\log \dfrac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(\omega^+)$.

(b) *Ω^+ is vanishing Reifenberg flat, Ω^+ and Ω^- have joint big pieces of chord-arc subdomains, and*

$$\lim_{\rho\to 0} \sup_{\substack{x\in\partial\Omega^+\\ 0<r\leq\rho}} \fint_{B(x,r)} |N - N_{B(x,r)}|\, d\omega^+ = 0, \tag{1.1}$$

where N is the outer unit normal to $\partial\Omega^+$ and $N_{B(x,r)}$ is the unit normal (pointing to Ω^-) to the n-plane $L_{B(x,r)}$ minimizing

$$\max\Big\{ \sup_{y\in\partial\Omega^+\cap B(x,r)} \mathrm{dist}(y, L_{B(x,r)}),\ \sup_{y\in L_{B(x,r)}\cap B(x,r)} \mathrm{dist}(y, \partial\Omega^+)\Big\}. \tag{1.2}$$

(c) *Ω^+ and Ω^- have joint big pieces of chord-arc subdomains and Eq. 1.1 holds.*

(d) *Ω^+ is a chord-arc domain and the outer unit normal N to $\partial\Omega^+$ belongs to* $\mathrm{VMO}(\mathcal{H}^n|_{\partial\Omega^+})$.

The equivalence (a) $\Leftrightarrow$ (b) was proven in [15], while (c) $\Rightarrow$ (d) $\Rightarrow$ (a) was shown more recently in [16]. For the definitions of NTA, Reifenberg flat, and chord-arc domains, and VMO, see Section 2. Remark that, for a Reifenberg flat domain (and more generally, for a two-sided NTA domain) the property of having joint big pieces of chord-arc subdomains is equivalent to the fact that $\omega^- \in A_\infty(\omega^+)$, as shown in [4]. This property implies that both ω^+ and ω^- are concentrated in an n-rectifiable set, by [3], and so the outer unit normal N is defined $\omega^\pm$-a.e. A key point in the theorem is that a priori it is not assumed that $\partial\Omega^+$ has locally finite surface measure in (a), (b), and (c). Nevertheless, (c) $\Rightarrow$ (d) ensures that $\mathcal{H}^n(\partial\Omega^+) < \infty$ (recall that a chord-arc domain is an n-Ahlfors regular NTA domain).

It is worth comparing the previous theorem with the following result of Kenig and Toro about the one-phase problem for harmonic measure:

Theorem B [11, 12, 14] *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded δ-Reifenberg flat chord-arc domain, with $\delta > 0$ small enough. Denote by ω the harmonic measure in Ω with pole $p \in \Omega$ and write $\sigma = \mathcal{H}^n|_{\partial\Omega}$. Then the following conditions are equivalent:*

(i) $\log \dfrac{d\omega}{d\sigma} \in \mathrm{VMO}(\sigma)$.

(ii) *Ω is vanishing Reifenberg flat and the outer unit normal N to $\partial\Omega$ belongs to* $\mathrm{VMO}(\sigma)$.

(iii) *The outer unit normal N to $\partial\Omega$ exists σ-a.e. and it belongs to* $\mathrm{VMO}(\sigma)$.

In view of the similarities between the statements (b) in Theorem A and (ii) in Theorem B, a natural question (which is left open in the works [15] and [16]) is if under the assumptions in Theorem A, the statement (b) is equivalent to the following:

(b') Ω^+ is vanishing Reifenberg flat, Ω^+ and Ω^- have joint big pieces of chord-arc subdomains, and the outer unit normal N to $\partial\Omega^+$ belongs to $\mathrm{VMO}(\omega^+)$.

Notice that (b') is the same as (b), with $N_{B(x,r)}$ in Eq. 1.1 replaced by $\fint_{B(x,r)} N\, d\omega^+$. In this paper we provide a negative answer by constructing a suitable counterexample in $\mathbb{R}^2$. The precise result is the following:

Theorem 1.1 *There exists a bounded vanishing Reifenberg flat domain $\Omega^+ \subset \mathbb{R}^2$ such that Ω^+ and $\Omega^- := \mathbb{R}^2 \setminus \overline{\Omega^+}$ have joint big pieces of chord-arc subdomains, the outer unit normal N is constant ω^+-a.e. in $\partial\Omega^+$, and such that Eq. 1.1 does not hold, and so* $\log \dfrac{d\omega^-}{d\omega^+} \notin \mathrm{VMO}(\omega^+)$.

Observe that in the theorem, although $\log \dfrac{d\omega^-}{d\omega^+} \notin \mathrm{VMO}(\omega^+)$, we still have $\omega^- \in A_\infty(\omega^+)$, because Ω^+ and Ω^- have joint big pieces of chord-arc subdomains. Of course, the theorem also implies that in the statement (c) in Theorem A one cannot replace the assumption Eq. 1.1 by the fact that $N \in \mathrm{VMO}(\omega^+)$.

To prove Theorem 1.1 we will construct a snowflake-type domain satisfying the required properties. Its construction is vaguely inspired by the example in [4, Section 8]. We will show that in this domain harmonic measure is concentrated in a countable collection of vertical segments by a probabilistic argument which uses the central limit theorem (see Lemma 3.3) and which has its own interest.

2 Preliminaries

We denote by C or c some constants that may depend on the dimension and perhaps other fixed parameters. Their values may change at different occurrences. On the contrary, constants with subscripts, like C_0, retain their values. For $a, b \geq 0$, we write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$. We write $a \approx b$ to mean $a \lesssim b \lesssim a$. If we want to emphasize that the implicit constant depends on some parameter $\eta \in \mathbb{R}$, we write $a \approx_\eta b$.

2.1 Ahlfors Regularity and Uniform Rectifiability

All measures in this paper are assumed to be Borel measures. A measure μ in $\mathbb{R}^d$ is called *doubling* if there is some constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\,\mu(B(x, r)) \quad \text{for all } x \in \operatorname{supp}\mu \text{ and } r > 0.$$

A measure μ in $\mathbb{R}^d$ is called *Ahlfors regular* (or n-Ahlfors regular) if

$$C^{-1} r^n \leq \mu(B(x, r)) \leq C r^n \quad \text{for all } x \in \operatorname{supp}\mu \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}\mu). \tag{2.1}$$

A set $E \subset \mathbb{R}^d$ is Ahlfors regular (or n-Ahlfors regular) if $\mathcal{H}^n|_E$ is Ahlfors regular.

A measure μ in $\mathbb{R}^d$ is called *uniformly n-rectifiable* if it is n-Ahlfors regular and there exist constants $\theta, M > 0$ such that for all $x \in \operatorname{supp}\mu$ and all $0 < r \leq \operatorname{diam}(\operatorname{supp}\mu)$ there is a Lipschitz mapping g from the ball $B_n(0, r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\mathrm{Lip}(g) \leq M$ such that

$$\mu(B(x, r) \cap g(B_n(0, r))) \geq \theta r^n.$$

A set $E \subset \mathbb{R}^d$ is called uniformly n-rectifiable if the measure $\mathcal{H}^n|_E$ is uniformly n-rectifiable. Recall that E is called n-rectifiable if there are Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\Big(E \setminus \textstyle\bigcup_i g_i(\mathbb{R}^n)\Big) = 0.$$

It is easy to check that if E is uniformly n-rectifiable, then it is n-rectifiable.

2.2 NTA and Reifenberg Flat Domains

Given $\Omega \subset \mathbb{R}^{n+1}$ and $C \geq 2$, we say that Ω satisfies the *C-Harnack chain condition* if for every $\rho > 0, k \geq 1$, and every pair of points $x, y \in \Omega$ with $\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega) \geq \rho$ and $|x-y| < 2^k \rho$, there is a chain of open balls $B_1, \ldots, B_m \subset \Omega, m \leq Ck$, with $x \in B_1, y \in B_m$, $B_k \cap B_{k+1} \neq \varnothing$ and $C^{-1}\operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C \operatorname{diam}(B_k)$. The chain of balls is called a *Harnack chain*.

For $C \geq 2$, Ω is a *C-corkscrew set* if for all $\xi \in \partial\Omega$ and $r \in (0, R)$ there is a ball of radius r/C contained in $B(\xi, r) \cap \Omega$. Finally, we say that Ω is *C-non-tangentially accessible (or C-NTA, or just NTA)* if it satisfies the Harnack chain condition and both Ω and $(\overline{\Omega})^c$ are C-corkscrew sets. Also, Ω is *two-sided C-NTA* if both Ω and $(\overline{\Omega})^c$ are C-NTA. A chord-arc domain is an NTA domain $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is n-Ahlfors regular. NTA domains were introduced by Jerison and Kenig in [9]. In this type of domains, the authors showed that harmonic measure is doubling and it satisfies other remarkable properties, such as the following change of pole formula.

Theorem 2.1 ([9, Lemma 4.11]) *Let $n \geq 1$, Ω be a C-NTA open set in $\mathbb{R}^{n+1}$ and let B be a ball centered in $\partial\Omega$. Let $p_1, p_2 \in \Omega$ be such that* $\operatorname{dist}(p_i, B \cap \partial\Omega) \geq c_0^{-1} r(B)$ *for $i = 1, 2$. Then, for any Borel set $E \subset B \cap \partial\Omega$,*

$$\frac{\omega^{p_1}(E)}{\omega^{p_1}(B)} \approx \frac{\omega^{p_2}(E)}{\omega^{p_2}(B)}, \tag{2.2}$$

with the implicit constant depending only on n, c_0 and C.

Given a set $E \subset \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$, $r > 0$, and a hyperplane P, we set

$$D_E(x, r, P) = r^{-1} \max\left\{ \sup_{y \in E \cap B(x,r)} \operatorname{dist}(y, P), \sup_{y \in P \cap B(x,r)} \operatorname{dist}(y, E) \right\}. \tag{2.3}$$

We also define

$$D_E(x, r) = \inf_P D_E(x, r, P) \tag{2.4}$$

where the infimum is taken over all hyperplanes P. Given $\delta, R > 0$, a set $E \subset \mathbb{R}^{n+1}$ is (δ, R)-*Reifenberg flat* (or just δ-Reifenberg flat) if $D_E(x, r) < \delta$ for all $x \in E$ and $0 < r \leq R$, and it is *vanishing Reifenberg flat* if

$$\lim_{r \to 0} \sup_{x \in E} D_E(x, r) = 0.$$

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and let $0 < \delta < 1/2$. We say that Ω is a (δ, R)-Reifenberg flat domain (or just δ-Reifenberg flat) if it satisfies the following conditions:

(a) $\partial\Omega$ is (δ, R)-Reifenberg flat.
(b) For every $x \in \partial\Omega$ and $0 < r \leq R$, denote by $P(x, r)$ an n-plane that minimizes $D_E(x, r)$. Then one of the connected components of

$$B(x, r) \cap \left\{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, P(x, r)) \geq 2\delta\, r\right\}$$

is contained in Ω and the other is contained in $\mathbb{R}^{n+1} \setminus \Omega$.

If, additionally, $\partial\Omega$ is vanishing Reifenberg flat, then Ω is said to be vanishing Reifenberg flat, too. It is well known that if Ω is a δ-Reifenberg flat domain, with δ small enough, then it is also an NTA domain (see [11]).

Given two NTA domains $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$, we say that Ω^+ and Ω^- have joint big pieces of chord-arc subdomains if for any ball B centered in $\partial\Omega^+$ with radius at most $\operatorname{diam}(\partial\Omega^+)$ there are two chord-arc domains $\Omega_B^s \subset \Omega^s$, with $s = +, -$, such that $\mathcal{H}^n(\partial\Omega_B^+ \cap \partial\Omega_B^- \cap B) \gtrsim r(B)^n$.

2.3 The Space VMO

Given a Radon measure μ in $\mathbb{R}^{n+1}$, $f \in L^1_{loc}(\mu)$, and $A \subset \mathbb{R}^{n+1}$, we write

$$m_{\mu,A}(f) = \fint_A f\, d\mu = \frac{1}{\mu(A)} \int_A f\, d\mu.$$

Assume μ to be doubling. We say that $f \in \mathrm{VMO}(\mu)$ if

$$\lim_{r\to 0} \sup_{x\in \operatorname{supp}\mu} \fint_{B(x,r)} \left|f - m_{\mu,B(x,r)}(f)\right|^2 d\mu = 0. \tag{2.5}$$

It is well known that the space $\mathrm{VMO}(\mu)$ coincides with the closure of the set of bounded uniformly continuous functions on $\operatorname{supp}\mu$ in the BMO norm.

3 Construction of the Domain

We will construct a domain $\Omega = \Omega^+ \subset \mathbb{R}^2$ by a limiting procedure. First we let Ω_0 be the interior of a regular polygon with 100 sides, say, with side length equal to 1. Given Ω_n, to construct Ω_{n+1}, we assume that Ω_n is a Jordan domain such that its boundary is a piecewise linear curve made up of m_n closed segments $L_1^n, \dots, L_{m_n}^n$ with equal length ℓ_n. We suppose that they are ordered clockwise, so that there are points $z_1^n, \dots, z_{m_n}^n \in \partial\Omega_n$ such that $L_j^n = [z_j^n, z_{j+1}^n]$ for each $j = 1, \dots, m_n$, with $z_{m_n+1}^n = z_1^n$. We assume also that, if B_j^n is a closed ball centered in the mid-point of L_j^n with radius $\ell_n/2$, then one component of $B_j^n \setminus L_j^n$ is contained in Ω_n and the other in $\mathbb{R}^2 \setminus \Omega_n$.

The first step to construct Ω_{n+1} consists in replacing each segment L_j^n by a piecewise linear Jordan arc Γ_j^n with the same end-points as L_j^n.

3.1 Construction of Γ_j^n

To shorten notation, denote $L = L_j^n$, and let $N(L)$ be the outer unit normal of Ω_n at L. Denote by γ_L the angle between $N(L)$ and the horizontal vector $(1, 0)$. We adopt the convention that $\gamma_L \in (-\pi, \pi]$. We will define Γ_j^n by a snowflake type construction. First we denote

$$\alpha = \alpha_L = \frac{|\gamma_L|}{M_n},$$

where M_n is a big integer (say $M_n \geq 20$) which will be chosen inductively later. For the moment, let us say that $M_1 = 20$ and that $\{M_n\}_n$ will be a monotone sequence of natural numbers tending to ∞. Now we start the usual construction of an arc of the Koch snowflake with angle α instead of $\pi/3$. To this end, we denote $\ell = \mathcal{H}^1(L)$ (so $\ell = \ell_n$) and we replace L by four segments $S_1, \dots, S_4$ with equal length

$$s := \frac{1}{2(1+\cos\alpha)}\,\ell,$$

as in Fig. 1. That is, in the case when $L = [(0,0),(\ell,0)]$ is a horizontal segment and $N(L) = (0,1)$, we consider the points (in complex notation)

$$y_0 = 0, \quad y_1 = s, \quad y_2 = \frac{\ell}{2} + i\,s\,\sin\alpha, \quad y_3 = \ell - s, \quad y_4 = \ell,$$

and we let $S_i = [y_{i-1}, y_i]$.

For an arbitrary segment L, we define $S_1, \ldots, S_4$ so that after a suitable translation and rotation we are in the preceding situation. We denote by $F(L)$ the curve generated in this way. That is,

$$F(L) = S_1 \cup S_2 \cup S_3 \cup S_4.$$

We also denote by $N(S_i)$ the unit normal to each vector S_i, so that the angle between $N(L)$ and $N(S_i)$ is at most α. In other words, $N(S_i)$ is the outer unit normal at S_i of the domain enclosed by the curve obtained by replacing L by $F(L)$ in $\partial\Omega_n$.

In the first iteration, we let $\Gamma_1(L) = F(L)$. To construct, $\Gamma_2(L)$ we iterate the construction in the usual way: we let

$$\Gamma_2(L) = F(S_1) \cup F(S_2) \cup F(S_3) \cup F(S_4),$$

with $F(S_i)$ defined in the same way as $F(L)$, with L replaced by S_i. We denote the four segments which compose $F(S_i)$ by $S_{i,1}, \ldots, S_{i,4}$. To construct $\Gamma_3(L)$ we proceed similarly.

However, we introduce a special rule in the iteration of the next curves $\Gamma_{k+1}(L)$: if one of the segments $S_{i_1,\ldots,i_k}$ of $\Gamma_k(L)$ is vertical and the associated outer normal $N(S_{i_1,\ldots,i_k})$ is the vector $(1,0)$, then in the construction of $\Gamma_{k+1}(L)$ we keep $S_{i_1,\ldots,i_k}$ unchanged. That is, for a segment $S_{i_1,\ldots,i_k}$ we let

$$\widetilde{F}(S_{i_1,\ldots,i_k}) = \begin{cases} F(S_{i_1,\ldots,i_k}) & \text{if } N(S_{i_1,\ldots,i_k}) \neq (1,0), \\ S_{i_1,\ldots,i_k} & \text{if } N(S_{i_1,\ldots,i_k}) = (1,0). \end{cases} \tag{3.1}$$

In the latter case, we let $S_{i_1,\ldots,i_k,1}, \cdots, S_{i_1,\ldots,i_k,4}$ be the four closed segments obtained by splitting $S_{i_1,\ldots,i_k}$ into four segments with disjoint interiors and the same length, and we let $N(S_{i_1,\ldots,i_j,h}) = (1,0)$ for $h = 1,2,3,4$. Then we let

$$\Gamma_{k+1}(L) = \bigcup_{1 \le i_1,\ldots,i_k \le 4} \widetilde{F}(S_{i_1,\ldots,i_k}).$$

Notice that, by the definition of α, the first appearance of a vertical segment $S_{i_1,\ldots,i_k}$ such that $N(S_{i_1,\ldots,i_k}) = (1,0)$ when $\alpha \neq 0$ occurs for $k = M_n$, and so the last definition is coherent with the construction described for the first curves $\Gamma_1(L), \Gamma_2(L), \Gamma_3(L)$.

We iterate this construction $M_n^2 \equiv (M_n)^2$ times, and we let

$$\Gamma_j^n = \Gamma_{M_n^2}(L).$$

Observe that Γ_j^n is made up of $4^{M_n^2}$ segments $S_{i_1,\ldots,i_{M_n^2}}$. See Fig. 2 (where we took $M_n = 2$ for an easy viewing of the vertical segments with associated normal $(1,0)$).

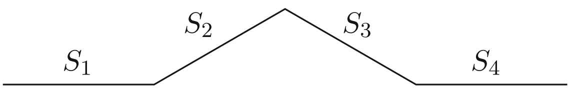

Fig. 1 The segments S_1, S_2, S_3, S_4, with $\alpha = \pi/6$

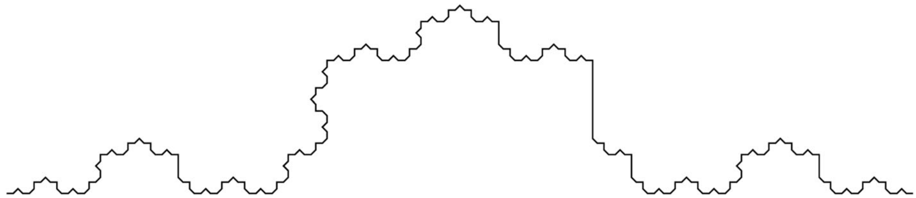

Fig. 2 The curve Γ_j^n generated by a horizontal segment L_j^n with parameters $N(L_j^n) = (0, 1)$, $\alpha = \pi/4$, and $M_n = 2$

3.2 The Chord-arc Property of Γ_j^n

Notice that for $1 \leq k \leq M_n^2$,

$$\frac{1}{4}\mathcal{H}^1(S_{i_1,\ldots,i_{k-1}}) \leq \mathcal{H}^1(S_{i_1,\ldots,i_k}) \leq \frac{1}{2(1+\cos\alpha)}\mathcal{H}^1(S_{i_1,\ldots,i_{k-1}}). \tag{3.2}$$

Thus,

$$4^{-k}\ell \leq \mathcal{H}^1(S_{i_1,\ldots,i_k}) \leq \left(\frac{1}{2(1+\cos\alpha)}\right)^k \ell \quad \text{for } 1 \leq k \leq M_n^2. \tag{3.3}$$

We can also write the second inequality as follows:

$$\mathcal{H}^1(S_{i_1,\ldots,i_k}) \leq 4^{-k}\left(\frac{2}{1+\cos\alpha}\right)^k \ell = 4^{-k}\left(1+\frac{1-\cos\alpha}{1+\cos\alpha}\right)^k \ell.$$

Next we use the fact that, by the definition of α,

$$1+\frac{1-\cos\alpha}{1+\cos\alpha} \leq 1+(1-\cos\alpha) \leq 1+C\alpha^2 \leq 1+\frac{C}{M_n^2}.$$

Thus, using also that $k \leq M_n^2$,

$$\mathcal{H}^1(S_{i_1,\ldots,i_k}) \leq 4^{-k}\left(1+\frac{1-\cos\alpha}{1+\cos\alpha}\right)^{M_n^2} \ell \leq 4^{-k}\left(1+\frac{C}{M_n^2}\right)^{M_n^2} \ell \lesssim 4^{-k}\ell. \tag{3.4}$$

Consequently, since Γ_j^n consists of $4^{M_n^2}$ segments of the form $S_{i_1,\ldots,i_{M_n^2}}$, we derive

$$\mathcal{H}^1(\Gamma_j^n) \leq 4^{M_n^2} \sup_{i_1,\ldots,i_{M_n^2}} \mathcal{H}^1(S_{i_1,\ldots,i_{M_n^2}}) \lesssim 4^{M_n^2}\, 4^{-M_n^2}\, \ell = \ell. \tag{3.5}$$

The above calculation suggests that Γ_j^n is an Ahlfors regular curve. In the next lemma we show that this is the case. In fact, we will prove that this is a bilipschitz image of the segment L, thus a chord-arc curve, and in particular an Ahlfors regular curve.

Lemma 3.1 *Suppose that α is small enough. For each $k = 1, \ldots, M_n^2$, the curve $\Gamma_k(L)$ is a bilipschitz image of the segment L. That is, there exist a bijective map $\varphi_k : L \to \Gamma_k(L)$ and a constant C_k such that for all $x, y \in L$,*

$$C_k^{-1}|x-y| \leq |\varphi_k(x)-\varphi_k(y)| \leq C_k\,|x-y|. \tag{3.6}$$

Further the constant C_k is bounded above by some absolute constant.

Of course, in particular the lemma implies that Γ_j^n is an Ahlfors regular curve, with the Ahlfors regularity constant bounded by an absolute constant.

Proof We argue by induction. The case $k = 1$ is clear. Fix now $k \in [2, M_n^2]$, and suppose that $\Gamma_j(L)$ is a chord-arc curve and that Eq. 3.6 holds for $1 \le j \le k-1$. Denote $L = \Gamma_0(L)$. For $0 \le j \le k-1$, we define a map $\Pi_j : \Gamma_j(L) \to \Gamma_{j+1}(L)$ as follows. In the case $j = 0$, we let Π_0 be equal to the projection from L to $\Gamma_1(L)$ orthogonal to L. In the case $j \ge 1$, recall that

$$\Gamma_j(L) = \bigcup_{1 \le i_1,\dots,i_j \le 4} S_{i_1,\dots,i_j}.$$

Then, on each segment $S_{i_1,\dots,i_j}$ we let $\Pi_j|_{S_{i_1,\dots,i_j}}$ be equal to the projection from $S_{i_1,\dots,i_j}$ to $\widetilde{F}(S_{i_1,\dots,i_j})$ orthogonal to $S_{i_1,\dots,i_j}$ (recall that $\widetilde{F}(S_{i_1,\dots,i_j})$ is defined in Eq. 3.1). Notice that Γ_j does not have self-intersections, since it is a chord-arc curve, by assumption. So the definition of Π_j is correct because each $z \in \Gamma_j(L)$ belongs at most to one segment $S_{i_1,\dots,i_j}$, except in the case when z is a common end point of two adjacent segments of the form $S_{i_1,\dots,i_j}$ (in which case $\Pi_j(z) = z$).

We define

$$\varphi_k = \Pi_{k-1} \circ \cdots \Pi_1 \circ \Pi_0.$$

By construction it is clear that $\varphi_k : L \to \Gamma_k(L)$ is continuous, piecewise affine, and surjective. To check that Eq. 3.6 holds for all $x, y \in L$, notice first that, by the definition of φ_k and the construction of the $\Gamma_k(L)$, we can split L into 4^k segments $I_{i_1,\dots,i_k}$, with $1 \le i_1, \dots, i_k \le 4$, with disjoint interiors, with $\mathcal{H}^1(I_{i_1,\dots,i_k}) = 4^{-k}\ell$, such that

$$\varphi_k(I_{i_1,\dots,i_k}) = S_{i_1,\dots,i_k} \quad \text{for } 1 \le i_1, \dots, i_k \le 4,$$

and with $\varphi_k|_{I_{i_1,\dots,i_k}}$ being affine in $I_{i_1,\dots,i_k}$. Then, recalling that, by Eqs. 3.3 and 3.4,

$$\mathcal{H}^1(S_{i_1,\dots,i_k}) \approx \mathcal{H}^1(I_{i_1,\dots,i_k}) = 4^{-k}\ell,$$

we deduce $|\nabla\varphi_k| \approx 1$ in $I_{i_1,\dots,i_k}$. By connectivity, this implies that φ_k is Lipschitz on L, with $\mathrm{Lip}(\varphi_k) \lesssim 1$. Remark that almost the same argument shows that $\Pi_m \circ \cdots \Pi_1 \circ \Pi_0$ is affine in $I_{i_1,\dots,i_m}$, for $0 \le m \le k-1$ and

$$|\nabla(\Pi_m \circ \cdots \Pi_1 \circ \Pi_0)| \approx 1. \tag{3.7}$$

It remains to check that

$$|\varphi_k(x) - \varphi_k(y)| \gtrsim |x - y| \quad \text{for all } x, y \in L. \tag{3.8}$$

To this end, for $1 \le j \le k$, we denote

$$x_j = \Pi_{j-1} \circ \cdots \Pi_1 \circ \Pi_0(x), \quad y_j = \Pi_{j-1} \circ \cdots \Pi_1 \circ \Pi_0(y),$$

so that, in particular, $\varphi_k(x) = x_k$ and $\varphi_k(y) = y_k$. Notice also that $x_j, y_j \in \Gamma_j(L)$. Observe also that, by the definition of Π_j and $F(S_{i_1,\dots,i_j})$ and Eq. 3.4 with $k = j$, one easily gets

$$|x_{j+1} - x_j| \le \mathcal{H}^1(S_{i_1,\dots,i_j}) \sin\alpha \le C\, 4^{-j}\, \ell\, \sin\alpha. \tag{3.9}$$

Let $h \ge 0$ be the integer such that

$$4^{-h-1}\ell \le |x - y| < 4^{-h}\ell. \tag{3.10}$$

Suppose first that $h \le k$. Then either exists a segment $I_{i_1,\dots,i_h} \subset L$ such that

$$x, y \in I_{i_1,\dots,i_h},$$

or there are two adjacent segments (i.e, with a common end-point) $I_{i_1,\dots,i_h}$, $I_{j_1,\dots,j_h} \subset L$ such that

$$x \in I_{i_1,\dots,i_h}, \qquad y \in I_{j_1,\dots,j_h}.$$

In the first case, since $\Pi_{h-1} \circ \cdots \Pi_1 \circ \Pi_0$ is affine in $I_{i_1,\dots,i_h}$, by Eq. 3.7 we have

$$|x-y| \approx |x_h - y_h|. \tag{3.11}$$

In the second case, this also holds. Indeed, let z be the common end-point of $I_{i_1,\dots,i_h}$ and $I_{j_1,\dots,j_h}$, and let $z_h = \Pi_{h-1} \circ \cdots \Pi_1 \circ \Pi_0(z)$. Then,

$$|x-y| = |x-z| + |z-y| \approx |x_h - z_h| + |z_h - y_h| \approx |x_h - y_h|, \tag{3.12}$$

where we applied Eq. 3.11 twice in the first estimate to show that $|x-z| \approx |x_h - z_h|$ and $|z-y| \approx |z_h - y_h|$. In the last estimate in Eq. 3.12 we used the fact that since the two segments $I_{i_1,\dots,i_h}$ and $I_{j_1,\dots,j_h}$ are adjacent in L, then $S_{i_1,\dots,i_h}$ and $S_{j_1,\dots,j_h}$ are adjacent segments in $\Gamma_h(L)$ and they form an angle either equal to $\pi - \alpha$ or π, with $0 < \alpha \ll 1$, by the construction above.

The estimate Eq. 3.11, together with Eq. 3.9, gives

$$\begin{aligned}|\varphi_k(x) - \varphi_k(y)| = |x_k - y_k| &\geq |x_h - y_h| - \sum_{m=h}^{k-1} |x_m - x_{m+1}| - \sum_{m=h}^{k-1} |y_m - y_{m+1}| \\ &\geq c\,|x-y| - C\,4^{-h}\,\ell\,\sin\alpha \gtrsim |x-y|,\end{aligned}$$

assuming α small enough in the last inequality.

In the case when the integer h in Eq. 3.10 is larger than k, then either exists a segment $I_{i_1,\dots,i_k}$ such that $x, y \in I_{i_1,\dots,i_k}$, or there are two adjacent segments $I_{i_1,\dots,i_k}$, $I_{j_1,\dots,j_k}$, such that $x \in I_{i_1,\dots,i_k}$ and $y \in I_{j_1,\dots,j_k}$. Then, we deduce that either Eqs. 3.11 or 3.12 hold replacing x_h, y_h, z_h by x_k, y_k, z_k, by the same arguments above. So in any case, Eq. 3.8 holds, which concludes the proof of the lemma. □

The curves Γ_j^n also satisfy the following:

Lemma 3.2 *For every $\xi \in \Gamma_j^n$ and $r > 0$, there exists a line $L_{\xi,r}$ such that*

$$\Gamma_j^n \cap B(\xi, r) \subset \mathcal{U}_{c\alpha r}(L_{\xi,r}) \cap B(\xi, r). \tag{3.13}$$

In the lemma, $\mathcal{U}_s(A)$ stands for the s-neighborhood of A.

Proof We will use the same notation as in the proof of Lemma 3.1.

We may assume that $r \leq \ell$. Let $k \geq 0$ be the integer such that $4^{-k-1}\ell \leq r < 4^{-k}\ell$. Suppose first that $k \leq M_n^2$. Let $x = (\varphi_{M_n^2})^{-1}(\xi)$ and let $x_k = \varphi_k(x)$. Let $S_{i_1,\dots,i_k}$ be such that $x_k \in S_{i_1,\dots,i_k}$ and let $L_{\xi,r}$ be the line supporting $S_{i_1,\dots,i_k}$. As in the proof of Lemma 3.1, denote $I_{i_1,\dots,i_k} = \varphi_k^{-1}(S_{i_1,\dots,i_k})$. For some constant $A_0 > 10$ to be fixed below, let $\mathcal{I}$ the family of segments $I_{j_1,\dots,j_k}$ such that

$$\operatorname{dist}(I_{i_1,\dots,i_k}, I_{j_1,\dots,j_k}) \leq A_0\,4^{-k}\ell,$$

and let $\mathcal{I}' = \{\varphi_k(J) : J \in \mathcal{I}\}$. Since the angle between $S_{i_1,\dots,i_k}$ and any segment $S \in \mathcal{I}'$ is between $\pi - C(A_0)\alpha$ and π, it follows that $S \subset \mathcal{U}_{C(A_0)\alpha r}(L_{\xi,r})$ for any $S \in \mathcal{I}'$. On the other hand, for $S \notin \mathcal{I}'$ and A_0 large enough, we have $S \cap B(x_k, 2r) = \varnothing$ by the chord-arc property of $\Gamma_k(L)$. Thus

$$\Gamma_k(L) \cap B(x_k, 2r) \subset \mathcal{U}_{C(A_0)\alpha r}(L_{\xi,r}). \tag{3.14}$$

From Eq. 3.9, it easily follows that $\Gamma_j^n = \Gamma_{M_n^2}(L) \subset \mathcal{U}_{c\alpha 4^{-k}}(\Gamma_k(L))$. The lemma is deduced from this fact and Eq. 3.14. The proof for the case $h > M_n^2$ is similar, and even simpler. □

For a point $z \in \mathbb{R}^2$, a line $\Lambda \subset \mathbb{R}^2$, and $\beta > 0$, we denote by $X(z, \Lambda, \beta)$ the cone

$$X(z, \Lambda, \beta) := \{y \in \mathbb{R}^2 : \operatorname{dist}(z, \Lambda) \leq \beta\, |y - z|\}.$$

Another useful property of Γ_j^n is the following: if $x_{L_j^n}, y_{L_j^n}$ are the end-points of L_j^n and Λ_j^n is the line supporting L_j^N, then

$$\Gamma_j^n \subset X(x_j^n, \Lambda_j^n, C\alpha) \cap X(y_j^n, \Lambda_j^n, C\alpha). \tag{3.15}$$

This can deduced by a quick inspection of the construction of Γ_j^n. We leave the details for the reader.

3.3 The Abundance of Vertical Segments in the Curve Γ_j^n

Denote by $\mathcal{V}(\Gamma_j^n)$ the subfamily of the segments $S_{i_1,\ldots,i_{M_n^2}}$, with $1 \leq i_1, \ldots, i_{M_n^2} \leq 4$, which are vertical and such that $N(S_{i_1,\ldots,i_{M_n^2}}) = (1, 0)$. Before going on with the construction of Ω_{n+1}, we will prove the abundance of that type of segments. This will be the key property that we will use below to show that the outer unit normal to Ω equals $(1, 0)$ ω^+-a.e. The precise result we need is the following:

Lemma 3.3 *There exists an absolute constant $c_1 > 0$ such that, for any choice of M_n large enough,*

$$\sum_{S \in \mathcal{V}(\Gamma_j^n)} \mathcal{H}^1(S) \geq c_1\, \mathcal{H}^1(\Gamma_j^n). \tag{3.16}$$

Proof We will use a probabilistic argument. We can assume that $\alpha \neq 0$, because otherwise all the segments $S_{i_1,\ldots,i_{M_n^2}}$ are vertical and their associated unit normal is $(1, 0)$. Consider the set of codings of the segments $S_{i_1,\ldots,i_{M_n^2}}$, that is, $I_n := \{1, 2, 3, 4\}^{M_n^2}$. Let μ be the uniform probability measure on $\{1, 2, 3, 4\}$ (so that $\mu(\{1\}) = \cdots = \mu(\{4\}) = 1/4$), and let $\mu_{I_n} = \mu \times \cdots \times \mu$ be the product measure (M_n^2 times) of μ on I_n. Consider the function $g : \{1, 2, 3, 4\} \to \mathbb{R}$ defined by

$$g(1) = g(4) = 0, \quad g(2) = \alpha, \quad g(3) = -\alpha,$$

and consider the random variables $X_1, \ldots, X_{M_n^2}$ on I_n defined by

$$X_j\big((i_1, \ldots, i_{M_n^2})\big) = g(i_j).$$

It is immediate to check that the variables $X_1, \ldots, X_{M_n^2}$ are independent and identically distributed, and they have zero mean and variance $\sigma^2 = \alpha^2/2$.

Notice that, for $i \in I_n$ with $S_i \notin \mathcal{V}(\Gamma_j^n)$, we have that $\Sigma_{M_n^2}(i) := X_1(i) + \ldots + X_{M_n^2}(i)$ is the angle that the segment S_i has rotated with respect the initial segment[1] L_j^n. Suppose for simplicity that the angle γ_L between $N(L) = N(L_j^n)$ and the vector $(1, 0)$ is positive, i.e., $0 < \gamma_L \leq \pi$. Then, if for some given $i = (i_1, \ldots, i_{M_n^2}) \in I_n$, we have $\Sigma_{M_n^2}(i_1, \ldots, i_{M_n^2}) \leq$

[1] In fact, $\Sigma_{M_n^2}(i)$ coincides also with the angle that any segment from $\{S_i\}_{i \in I_n}$ would have rotated if we had not applied the special rule about the vertical segments in the construction of Γ_j^n.

$-M_n\alpha$, this implies that some segment $S_{i_1,\dots,i_k}$ has rotated clockwise by an angle equal to $M_N\alpha = \gamma_L$ with respect to L, so that it is vertical and $N(S_{i_1,\dots,i_k}) = (1,0)$. By construction this also ensures that $N(S_{i_1,\dots,i_{M_n^2}}) = (1,0)$. Consequently,

$$\#\mathcal{V}(\Gamma_j^n) \geq \#\big\{i \in I_n : \Sigma_{M_n^2}(i) \leq -M_n\alpha\big\}$$

and thus

$$\frac{\#\mathcal{V}(\Gamma_j^n)}{\#I_n} \geq \mu_{I_n}\big(\{i \in I_n : \Sigma_{M_n^2}(i) \leq -M_n\alpha\}\big). \tag{3.17}$$

Observe that

$$\frac{\Sigma_{M_n^2}}{M_n\,\sigma} = \frac{\sqrt{2}\,\Sigma_{M_n^2}}{M_n\,\alpha}.$$

By the central limit theorem, the random variable $\frac{\Sigma_{M_n^2}}{M_n\,\sigma}$ converges in law to the standard normal distribution as $M_n \to \infty$. Therefore, for some absolute constant $\eta > 0$ (determined by the standard normal distribution),

$$\mu_{I_n}\big(\{i \in I_n : \Sigma_{M_n^2}(i) \leq -M_n\alpha\}\big) = \mu_{I_n}\left(\left\{i \in I_n : \frac{\Sigma_{M_n^2}(i)}{M_n\,\sigma} \leq -\sqrt{2}\right\}\right) \to \eta \quad \text{as } M_n \to \infty.$$

Consequently, by Eq. 3.17,

$$\frac{\#\mathcal{V}(\Gamma_j^n)}{\#I_n} \geq \frac{\eta}{2},$$

for M_n big enough.

To finish the proof of Eq. 3.16, recall that any segment S_i, with $i \in I_n$, satisfies $\mathcal{H}^1(S_i) \geq 4^{-M_n^2}\,\mathcal{H}^1(L_j^n)$. Thus,

$$\sum_{S\in\mathcal{V}(\Gamma_j^n)} \mathcal{H}^1(S) \geq \#\mathcal{V}(\Gamma_j^n)\,4^{-M_n^2}\,\mathcal{H}^1(L_j^n) \geq \frac{\eta}{2}\,\#I_n\,4^{-M_n^2}\mathcal{H}^1(L_j^n) = \frac{\eta}{2}\,\mathcal{H}^1(L_j^n) \geq c\,\eta\,\mathcal{H}^1(\Gamma_j^n),$$

using Eq. 3.5 for the last inequality. □

3.4 The Domain Ω_{n+1}

Recall that $\partial\Omega_n = \bigcup_{j=1}^{m_n} L_j^n$ and that each segment L_j^n has the same end-points as the chord-arc curve Γ_j^n. We consider the curve

$$G_{n+1} = \bigcup_{j=1}^{m_n} \Gamma_j^n.$$

The fact that the curves Γ_j^n are chord-arc and the property Eq. 3.15 will ensure that G_{n+1} is a quasicircle.

Let V_{n+1} be the domain enclosed by G_{n+1}. This domain should be considered as a first approximation of Ω_{n+1}. In order to guaranty the vanishing Reifenberg flatness property of the final domain Ω, we have to modify V_{n+1}. Indeed, it is easy to check that the angle between the outer normals for V_{n+1} at any two neighboring segments contained in the same piecewise curve Γ_j^n is at most $\alpha_{L_j^n} \leq C/M_n$, and we will choose M_n so that $M_n \to \infty$. However, the angle between two consecutive segments of the curve G_{n+1} which are contained in different (but neighboring) curves $\Gamma_j^n, \Gamma_{j+1}^n$ is the same as the angle between L_j^n and L_{j+1}^n, which is not convenient, because we want this to become flatter.

To define a suitable smoothened version of G_{n+1} we need some additional notation. Let $T_1, \ldots, T_{t_{n+1}}$ the segments which compose the piecewise curve G_{n+1}. That is to say,

$$\{T_i\}_{1\le i\le t_{n+1}} = \{S_{i_1,\ldots,i_{M_n^2}}(\Gamma_j^n) : 1 \le j \le m_n,\ 1 \le i_1, \ldots, i_{M_n^2} \le 4\},$$

where $S_{i_1,\ldots,i_{M_n^2}}(\Gamma_j^n)$ stands for the segment $S_{i_1,\ldots,i_{M_n^2}}$ appearing in the construction of Γ_j^n. Suppose that $T_1, \ldots, T_{t_{n+1}}$ are ordered clockwise in the curve G_{n+1}. Let

$$a_{n+1} = \min_{1\le i\le t_{n+1}} \mathcal{H}^1(T_i), \tag{3.18}$$

and for each i let $\widetilde{T}_i$ be a closed segment such that $\widetilde{T}_i \subset T_i$ with the same mid-point as T_i and such that

$$\mathcal{H}^1(\widetilde{T}_i) = \mathcal{H}^1(T_i) - \frac{a_{n+1}}{200}. \tag{3.19}$$

Let C_i be a closed circular arc that joints $\widetilde{T}_i$ to $\widetilde{T}_{i+1}$, so that C_i is tangent both to T_i and T_{i+1} and the tangency points coincide with the two closest end-points of $\widetilde{T}_i$ and $\widetilde{T}_{i+1}$, as in Fig. 3. Of course, the arc C_i is determined once the end-points of each segment $\widetilde{T}_i$ are fixed by the condition Eq. 3.19. Also, by elementary geometry, it is clear that the angle subtended by C_i is equal to the supplementary of the angle formed by the two consecutive segments T_i and T_{i+1}. Then we let

$$\widetilde{G}_{n+1} = \bigcup_{1\le i\le t_{n+1}} (\widetilde{T}_i \cup C_i).$$

That is, roughly speaking, $\widetilde{G}_{n+1}$ is the curve obtained by erasing $T_i \setminus \widetilde{T}_i$, for $1 \le i \le t_{n+1}$ and replacing the erased parts by circular arcs, so that the resulting curve is of type C^1. We remark that the precise construction of $\widetilde{G}_{n+1}$ from G_{n+1} is not very relevant. What it is important is that $\widetilde{G}_{n+1}$ is a C^1 quasicircle (with a uniform character), close in Hausdorff distance to G_{n+1}, and it coincides with G_{n+1} in a big portion of each segment T_i.

The next step is to consider a family of points $z_1^{n+1}, \ldots, z_{m_{n+1}}^{n+1}$ from $\widetilde{G}_{n+1}$ ordered clockwise, with $m_{n+1} \ge t_{n+1}$, so that

$$|z_i^{n+1} - z_{i+1}^{n+1}| = |z_j^{n+1} - z_{j+1}^{n+1}| =: \ell_{n+1} \quad \text{for } 1 \le i, j \le m_{n+1},$$

understanding $z_{m_{n+1}+1}^{n+1} = z_1^{n+1}$. We denote $L_i^{n+1} = [z_i^{n+1}, z_{i+1}^{n+1}]$. Further, we take m_{n+1} large enough so that $\ell_{n+1} \le \frac{a_{n+1}}{200}$ and so that the supplementary of the angle between two neighboring segments L_i^{n+1}, L_{i+1}^{n+1} is at most $2^{-n-4}\pi$ (here we mean the angle equal to the smallest one of the two supplementary angles that form the two lines supporting L_i^{n+1} and

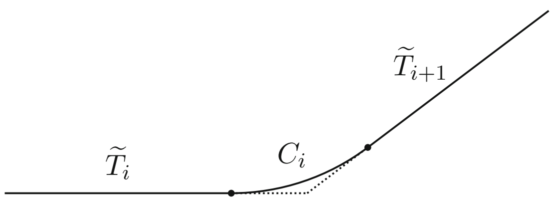

Fig. 3 The segments $\widetilde{T}_i, \widetilde{T}_{i+1}$, and the arc C_i that joins them. The length of each dotted segment is $a_{n+1}/100$

L_{i+1}^{n+1}). By a continuity argument and the C^1 character of $\widetilde{G}_{n+1}$ it is easy to prove the existence of the family of points $z_1^{n+1}, \dots, z_{m_{n+1}}^{n+1}$.

Finally, we let Ω_{n+1} be the domain enclosed by the curve formed by the union of the segments $L_1^{n+1}, \dots, L_{m_{n+1}}^{n+1}$, so that

$$\partial\Omega_{n+1} = \bigcup_{1 \le i \le m_{n+1}} L_i^{n+1}.$$

3.5 The Domain Ω

It is easy to check that the curves $\partial\Omega_n$ are quasi-circles which converge in Hausdorff distance to another quasi-circle Γ_∞ as $n \to \infty$. We let Ω be the domain bounded by the quasi-circle Γ_∞, and we also set $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$.

Recall that, for any $n \ge 1$, the curve $\partial\Omega_n$ is piecewise linear and that the supplementary angle between two adjacent segments L_i^{n+1}, L_{i+1}^{n+1} which compose $\partial\Omega_n$ is at most $2^{-n-4}\pi$. From this fact, and the construction above, using also Lemma 3.2, one can check that Ω is a vanishing Reifenberg-flat domain. Further, from Eq. 3.13 and the construction above, it follows easily that

$$\partial\Omega \subset \mathcal{U}_{c\ell_n/M_n}(\partial\Omega_n), \tag{3.20}$$

for some fixed $c > 0$.

Recall also that $T_1, \dots, T_{t_{n+1}}$ are the segments which compose the piecewise curve G_{n+1}. Denote by $\mathcal{V}_{n+1}$ the subfamily of segments $T \in \{T_1, \dots, T_{t_{n+1}}\}$ such that $N(T) = (1, 0)$, and let

$$F = \bigcup_{n \ge 1} \bigcup_{T \in \mathcal{V}_{n+1}} \tfrac{1}{2} T. \tag{3.21}$$

Lemma 3.4 *The set F is contained in $\partial\Omega$.*

Remark that we cannot ensure that the larger set

$$F' = \bigcup_{n \ge 1} \bigcup_{T \in \mathcal{V}_{n+1}} T$$

is contained in $\partial\Omega$ because in the smoothening process, i.e., the construction of $\widetilde{G}_{n+1}$ from G_{n+1}, some portion of the vertical segments T_i is deleted in order to obtain $\widetilde{T}_i$. Also, for $m \ge n+1$, in the construction of $\widetilde{G}_{m+1}$ from G_{m+1}, more portions of the segments T_i may be deleted. However, the deleted portion of the segments T_i is pretty small for each $m \ge n$ (because of the choice of the parameter $a_{m+1}/200$ in Eq. 3.19), so that we can ensure that $F \subset \partial\Omega$. This what the lemma claims.

Proof Let $T \in \mathcal{V}_{n+1}$. Then $T \subset G_{n+1}$ and $\frac{99}{100} T \subset \widetilde{G}_{n+1}$. Since we chose $\ell_{n+1} \le \frac{a_{n+1}}{200}$, it follows easily that $\frac{9}{10} T \subset \partial\Omega_{n+1}$, and moreover $\frac{9}{10} T$ is covered by a subfamily of segments from L_i^{n+1} such that $N(L_i^{n+1}) = (1, 0)$. We denote this family by $\{L_i^{n+1}\}_{i \in I^{n+1}(T)}$. By inspection of the above construction, it can be seen that (assuming the sequence $\{m_n\}_n$ to growth fast enough if necessary) at most two segments from the family $\{L_i^{n+1}\}_{i \in I^{n+1}(T)}$ are not included in $\partial\Omega_{n+2}$. Iterating, we see for each $m > n$, there is a segment T^m concentric with T contained in $\partial\Omega_m$ with length

$$\mathcal{H}^1(T^m) \ge \frac{9}{10} \mathcal{H}^1(T) - 2 \sum_{j=n+1}^{m} \ell_j \ge \frac{1}{2} \mathcal{H}^1(T).$$

Letting $m \to \infty$, this implies that $\frac{1}{2}T \subset \partial\Omega$. □

Lemma 3.5 *For any ball B centered in $\partial\Omega$ such that* $\operatorname{diam}(B) \le \operatorname{diam}(\partial\Omega)$ *there are two chord-arc subdomains $\Omega_B^\pm \subset \Omega^\pm$ such that*

$$\mathcal{H}^1(F \cap \partial\Omega_B^+ \cap \partial\Omega_B^-) \ge c \operatorname{diam}(B), \tag{3.22}$$

where $c > 0$ is an absolute constant and the chord-arc character of $\Omega_B^\pm$ does not depend on B.

To prove this lemma we will use following result of Jonas Azzam [1, Theorem 6.4]:

Theorem 3.6 *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a two-sided NTA domain and let $\Omega^- = (\overline{\Omega})^c$. Let $E \subset \partial\Omega \cap Z$, where Z is a uniformly n-rectifiable set. Then there are two chord-arc domain $\Omega_E^\pm$ such that $\Omega_E^\pm \subset \Omega^\pm$, with* $\operatorname{diam}(\partial\Omega_E^\pm) \gtrsim \operatorname{diam}(\partial\Omega^+)$*, such that $E \subset \partial\Omega^+ \cap \partial\Omega_E^+ \cap \partial\Omega_E^-$. The chord-arc parameters of $\Omega_E^\pm$ depend only on the NTA parameters of $\Omega^\pm$ and on the uniform rectifiability parameters of Z.*

Proof of Lemma 3.5 Given a ball B as in the lemma, let n be such that $10\ell_n \ge r(B) > 10\ell_{n+1}$. As shown in Lemma 3.1, each curve Γ_j^n is C-Ahlfors regular, for some fixed absolute constant C. Since, for n fixed, only a finite number of the curves Γ_j^n intersect $2B$, it follows that $G_n \cap 2B$ is also Ahlfors regular (perhaps with a slightly worse constant C'). By construction, this implies that also $\widetilde{G}_n \cap 1.5B$ and $\partial\Omega_{n+1} \cap B$ are upper C''-Ahlfors regular.

Let H be the arc of $B \cap \partial\Omega_{n+1}$ that passes through the center of B. Then H is a C''-Ahlfors regular curve with $\operatorname{diam}(H) \approx \mathcal{H}^1(H) \ge 2\,r(B)$. Denote by $\{L_i^{n+1}\}_{i\in J_H}$ the family of segments L_i^{n+1}, $1 \le i \le m_{n+1}$, such that $L_i^{n+1} \subset H$. By the choice of n, it follows easily that

$$\sum_{i\in J_H} \mathcal{H}^1(L_i^{n+1}) \approx r(B). \tag{3.23}$$

Next, form the curve Γ_H by replacing each L_i^{n+1}, $i \in J_H$, in the arc H by the corresponding curve Γ_i^{n+1}. Since $\mathcal{H}^1(\Gamma_i^{n+1}) \approx \mathcal{H}^1(L_i^{n+1})$ and Γ_i^{n+1} is Ahlfors regular for each i, we infer that Γ_H is a C'''-Ahlfors regular curve, where C''' is another absolute constant.

By Lemmas 3.3 and Eq. 3.4, we deduce that

$$\mathcal{H}^1(\Gamma_i^{n+1} \cap F \cap \partial\Omega) \gtrsim \mathcal{H}^1(L_i^{n+1}) \quad \text{for all } i \in J_H.$$

Summing on $i \in J_H$ and using Eq. 3.23, we deduce that

$$\mathcal{H}^1(\Gamma_H \cap F \cap \partial\Omega) \gtrsim r(B).$$

Finally we apply Theorem 3.6 with $E = F \cap B$ and $Z = \Gamma_H$ to deduce the existence of the required chord-arc domains $\Omega_B^\pm \subset \Omega^\pm$ satisfying Eq. 3.22. □

4 The Set *F* has Full Harmonic Measure and Eq. 1.1 does not Hold

Lemma 4.1 *The set F defined in Eq. 3.21 has full harmonic measure in Ω^+.*

Notice that the outer unit normal of Ω^+ at every $\xi \in F$ equals $(1, 0)$. From the fact that F has full harmonic measure, it follows then that the outer unit normal of Ω^+ is constant ω^+-a.e. in $\partial\Omega^+$.

Proof of Lemma 4.1 By standard arguments, it suffices to prove that there is a constant $c > 0$ such that for any ball B centered in $\partial\Omega^+$ such that $\mathrm{diam}(B) \leq \mathrm{diam}(\partial\Omega^+)$, it holds

$$\omega^+(F \cap B) \geq c\,\omega^+(B).$$

Here we assume that the pole for harmonic measure is a point $p \in \Omega$ such that $\mathrm{dist}(p, \partial\Omega) \gtrsim \mathrm{diam}(\Omega^+)$.

Let $\Omega_B^\pm$ be as in Lemma 3.5 and let $p_B \in B \cap \Omega_B^+$ be such that

$$\mathrm{dist}(p_B, \partial\Omega_B^+) \gtrsim \mathrm{diam}(B),$$

so that p_B is also a corkscrew point for B with respect to Ω^+. By the change of pole formula in Theorem 2.1, it suffices to show that

$$\omega^{+,p_B}(F \cap B) \gtrsim 1.$$

By Eq. 3.22, we know that

$$\mathcal{H}^1(F \cap \partial\Omega_B^+) \geq c\,\mathrm{diam}(B) \approx \mathcal{H}^1(\partial\Omega_B^+ \cap B).$$

Since the harmonic measure $\omega_{\Omega_B^+}^{p_B}$ is an A_∞ weight with respect to $\mathcal{H}^1|_{\partial\Omega_B^+}$, we deduce that

$$\omega_{\Omega_B^+}^{p_B}(F \cap B) \gtrsim 1.$$

Then, by the maximum principle it holds

$$\omega^{+,p_B}(F \cap B) \geq \omega_{\Omega_B^+}^{p_B}(F \cap B) \gtrsim 1,$$

as wished. □

To complete the proof of Theorem 1.1 it only remains to show that the condition Eq. 1.1 does not hold. To this end, it suffices to check that there are arbitrarily small balls B centered in $\partial\Omega^+$ such that

$$|N_B - (1,0)| \geq c_2, \tag{4.1}$$

for some fixed $c_2 > 0$.

For simplicity, suppose that one of the initial segments L_j^0 from Ω_0 is horizontal and satisfies $N(L_j^0) = (0,1)$. By construction, the segment with coding $S_{1,\dots,1}$ (with M_0^2 1's) is also horizontal and its associated outer normal is $(0,1)$. So one of the segments T_i, $1 \leq i \leq t_1$, from the curve G_1 is also horizontal and $N(T_i) = (0,1)$. By the smallness of the parameter a_1 defined in Eq. 3.18 and the fact that $\ell_1 \leq a_1/100$, it follows easily that there exists at least one segment L_k^1 from $\partial\Omega_1$ contained in T_i, so that $N(L_k^1) = (0,1)$. Iterating, we deduce that for any $n \geq 1$, there is a horizontal segment $L_{k_n}^n$ from $\partial\Omega_n$ such that $N(L_{k_n}^n) = (0,1)$.

Let B_n be a ball centered in the mid-point of $L_{k_n}^n$ with radius $r_n := \mathcal{H}^1(L_{k_n}^n)/4$. From Eq. 3.20, we deduce that $\partial\Omega \cap \frac{1}{2}B_n \neq \varnothing$ and $\partial\Omega \cap B_n \subset \mathcal{U}_{r_n/100}(L_{k_n}^n)$ for n big enough. Now let $\widetilde{B}_n \subset B_n$ be a ball centered in $\partial\Omega$ such that $\mathrm{diam}(\widetilde{B}_n) \geq \frac{9}{10}\mathrm{diam}(B_n)$. Then the vector $N_{\widetilde{B}_n}$ is close to being vertical and thus $|N(\widetilde{B}_n) - (1,0)| \gtrsim 1$. Hence there are arbitrary small balls satisfying Eq. 4.1.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval Not applicable.

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