# Analyzing the Multiplicity of Solutions in Filters With Parallel-Connected Structures

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***Abstract*—Determining the solution for a specific topology in parallel-connected filters might be challenging, especially due to the nonuniqueness of the solutions. This work proposes a method to address the multiplicity of solutions in high-order filters by focusing on the reconfiguration of the coupling matrix (CM). Unlike conventional techniques, which can be complicated by analyzing the network as a whole, the proposed approach examines each branch of the filter independently, where each existing solution depends on the grouping of the eigenvalues in each branch. It is further demonstrated that the analysis can be performed for an inner transversal subnetwork by previously reconfiguring the $N+2$ transversal CM to a transversal CM of lower order than $N+2$. To validate this analysis, the results are compared with all existing solutions with the help of a verification software based on the numerical interval Newton algorithm (NINA), which guarantees the global convergence of all solutions. To validate the concept, an example of a 7th-order topology is presented.**

***Index Terms*—Eigenvalues, numerical interval Newton algorithm (NINA), parallel-connected filter, transversal coupling matrix (CM).**

## I. Introduction

OBTAINING multiple solutions for the same topology may help designers in selecting the one that best fits the design, taking into consideration its physical limitations. Usually, the starting point is the transversal, folded, or wheel topology, and then the target topology is obtained through similarity transformations on the coupling matrix (CM) without altering the response [1], [2], [3].

Finding the sequence of transformations might be challenging when working with high-order filters that may also have irregular couplings. Therefore, alternative methods such as general optimization [4], [5] are used but they depend on initial variables and are susceptible to fail in local minima. To address this drawback, new global optimization methods have been developed such as the one described in [6], which is capable of obtaining a single solution, unless the variables are randomly initialized many times. To overcome this, the use of interval arithmetic [7], [8] was introduced. This method does not suffer from initialization problems and it is capable of finding all the solutions in a single run, as well as delimiting the range of solutions if desired. Due to these advantages, numerical interval Newton algorithm (NINA) is used as verification software.

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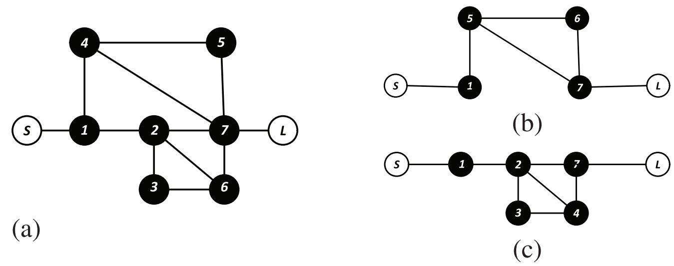



Fig. 1. (a) Target topology of 7th-order filter branching into two subnetworks. (b) 2nd branch, $k_2 = 2$. (c) 1st branch, $k_1 = 3$.

Since a parallel-connected network can be split into subnetworks or parallel branches [9], [10] as shown in Fig. 1, this characteristic is taken advantage of in this work to present an analysis of the reason for the nonuniqueness of the solutions by grouping the eigenvalues of the transversal network that will be transformed into the parallel-connected network.

## II. Proposed Analysis Method

The proposed analysis takes advantage of the properties of parallel-connected networks, where each branch can be reconfigured by grouping the eigenvalues. When constructing parallel-connected networks, grouping and assigning eigenvalues among each branch is a key step in obtaining the desired topology. The number of groupings depends on all the possible combinations according to the number of existing eigenvalues and the number of parallel subnetworks that the target topology has. The expression for determining the number of possible combinations is

$$C = \frac{N!}{\prod_{i=1}^{m} k_i! \cdot \prod_{j=1}^{p} n_j!} \tag{1}$$

where $N$ is the number of eigenvalues in the transversal network, which is split into $m$ parallel subnetworks, each with $k_i$ eigenvalues ($i = 1, 2, \ldots, m$). The values of $n_j$ ($j = 1, 2, \ldots, p$) represent the numbers of topologically identical branches. There may be $p$ different cases, so the noninclusion of combinations that represent the same network solution is ensured here.

A filter with a parallel subnetwork between nodes other than the source ($S$) and the load ($L$) can be analyzed by previously reconfiguring the $N+2$ transversal matrix into an internal subtransversal of order $N$ as shown in Fig. 2(b), reducing the

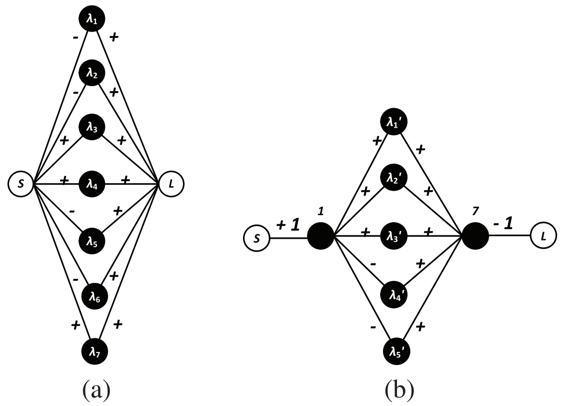



Fig. 2. (a) $N+2$ transversal CM. (b) $N$ transversal CM.

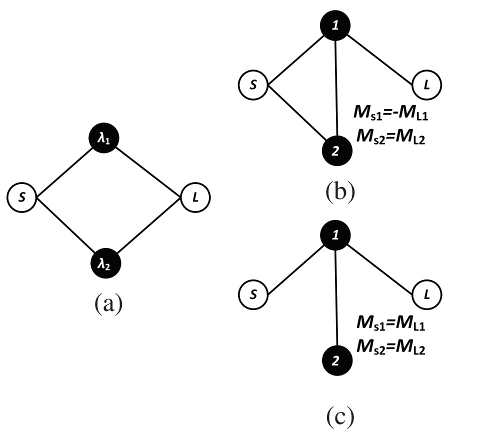



Fig. 3. (a) Transversal representation of a filter with two resonators that has one TZ. (b) Folded topology obtained by grouping two eigenvalues of different modes $N_e = N_o$. (c) Folded topology obtained by grouping two eigenvalues with the same mode (even) $N_e \neq N_o$.

number of eigenvalues from $N$ to $N-2$ and altering the original values, thus decreasing the number of possible combinations as shown in (1).

For classical topologies such as trisection and quartet form, a correspondence between even and odd modes must exist. If $N$ is even, the number of even ($N_e$) and odd ($N_o$) modes must be equal ($N_e = N_o$), and if $N$ is odd, the difference between the number of even and odd modes must be $|N_e - N_o| = 1$. This rule only holds in symmetric transversal networks where $S11 = S22$ and can help find particular topologies which may be difficult to obtain due to the existence of infinitely many solutions. For example, consider a filter of order $N = 2$ with one transmission zero (TZ), if ($N_e = N_o$) a trisection is formed as shown in Fig. 3(b), otherwise the dangling of Fig. 3(c) is formed.

To transform an initial topology $M_0$ to a target topology $M_T$, a transformation matrix $Q$ is needed. The matrix $Q$ contains the set of all the rotations necessary in the transformation sequence. To obtain this matrix, any of the currently known methods can be used. In this work, the NINA software described in [7] and [8] is used for this purpose.

The reconfiguration of each subnetwork is defined with a particular $Q$ matrix. The $m$ parallel branches are analyzed considering $C$ possible combinations of these branches and can be described as a system of nonlinear equations defined as

$$\begin{aligned} M_T^{(r,s)} &= Q^{(r,s)T} M_0^{(r,s)} Q^{(r,s)} \\ \left(M_T^{(r,s)}\right)_{ij} &= 0, \quad \text{when } (i,j) \in \mathrm{TM}^{(r,s)} \\ Q^{(r,s)T} Q^{(r,s)} &= I \\ &\text{where } r = 1, 2, \ldots, C \\ &\text{and } s = 1, 2, \ldots, m. \end{aligned} \tag{2}$$

Here, $r$ represents the index of the $C$ possible branch combinations, while $s$ represents the index of each of the $m$ parallel branches. The transversal CM for the $s$th branch in the $r$th combination is $M_0^{(r,s)}$, the target CM is $M_T^{(r,s)}$, and the transformation matrix is $Q^{(r,s)}$. $\mathrm{TM}^{(r,s)}$ is the set of indices for the elements of the matrix $M_T^{(r,s)}$ that are zero for the $s$th branch in the $r$th combination.

All the solutions in parallel-connected networks are related to different solutions that provide different possible combinations resulting from (2).

It should be noted that for cases where there are identical eigenvalues in the transversal network containing this section of the parallel-connected network, infinite solutions for the transversal network are obtained [11]. The analysis is valid for these situations because the initial point of the analysis is a transversal matrix and the nonuniqueness of the transversal does not imply the nonuniqueness of the reconfigurations.

The results obtained from the analysis are compared with the results provided by NINA. NINA works with the entire network and guarantees finding all the existing solutions.

Therefore, the step-by-step process is as follows.

1) Identification of the outer nodes that support the parallel-connected network.
   a) If the outer nodes are the source and load (nodes S and L), the analysis is performed directly with the original $N+2$ transversal matrix.
   b) If the outer nodes are internal nodes, the analysis is performed with the inner transversal topology of order lower than the original $N+2$.
2) Determination of the different possible combinations within the network according to (1).
3) Identify the symmetry of the transversal matrix.
   a) If $S_{11} = S_{22}$, it is possible to obtain information according to the nature of the grouped eigenvalues (even, odd) for each combination and use the information to find particular solutions that may be difficult to obtain.
   b) If $S_{11} \neq S_{22}$, the transversal matrix will be asymmetric, and the eigenvalues' identification cannot be applied.
4) Reconfiguration of all the branches based on NINA that is used to reconfigure each branch for each combination according to (2).
5) Verification of the solutions obtained in the analysis using NINA as verification software.

## III. Analysis Example

To validate the proposed method, a 7th-order filter with three finite TZs at $\Omega_k = [-2 - 1.5\ 1.4]$ rad/s and return losses RL 23 dB is considered.

TABLE I

POSSIBLE GROUPING CONFIGURATIONS

| Case ($r$) | 1st Branch (s=1) | Mode Grouping | 2nd Branch (s=2) | Mode Grouping |
|---|---|---|---|---|
| 1 | $\lambda_4' \lambda_1' \lambda_2'$ | $O-E-E$ | $\lambda_5' \lambda_3'$ | $O-E$ |
| 2 | $\lambda_4' \lambda_1' \lambda_3'$ | $O-E-E$ | $\lambda_5' \lambda_2'$ | $O-E$ |
| 3 | $\lambda_4' \lambda_2' \lambda_3'$ | $O-E-E$ | $\lambda_5' \lambda_1'$ | $O-E$ |
| 4 | $\lambda_5' \lambda_1' \lambda_2'$ | $O-E-E$ | $\lambda_4' \lambda_3'$ | $O-E$ |
| 5 | $\lambda_5' \lambda_1' \lambda_3'$ | $O-E-E$ | $\lambda_4' \lambda_2'$ | $O-E$ |
| 6 | $\lambda_5' \lambda_2' \lambda_3'$ | $O-E-E$ | $\lambda_4' \lambda_1'$ | $O-E$ |
| 7 | $\lambda_4' \lambda_5' \lambda_1'$ | $O-O-E$ | $\lambda_2' \lambda_3'$ | $E-E$ |
| 8 | $\lambda_4' \lambda_5' \lambda_2'$ | $O-O-E$ | $\lambda_1' \lambda_3'$ | $E-E$ |
| 9 | $\lambda_4' \lambda_5' \lambda_3'$ | $O-O-E$ | $\lambda_1' \lambda_2'$ | $E-E$ |
| 10 | $\lambda_1' \lambda_2' \lambda_3'$ | $E-E-E$ | $\lambda_4' \lambda_5'$ | $O-O$ |

The target topology includes parallel branches between nodes 1 and 7. Therefore, to carry out the analysis, it is necessary to reconfigure the transversal topology from $N+2$ to $N$. The new $N$ transversal topology comprises five eigenvalues $\lambda_i'$ between nodes 1 and 7, consisting of three even and two odd eigenvalues. This new topology serves as the basis for eigenvalue grouping of the new $N$ transversal CM. To do this, the source and load are moved to nodes 1 and 7 and to have no modification in the magnitude response, and it is necessary to extract two unitary inverters of opposite signs. By doing this, the phase changes $-180°$ and the effect of the original inverter is absorbed by resonators 1 and 7 so that their values in the frequency variant CM $W$ will be different from unity. At the end of the reconfiguration procedure to obtain the response with the original phase, the extracted inverters are simply added and the source and load are returned to their original positions. If a unitary $W$ matrix is needed, a simple scaling must be performed at nodes 1 and 7 by altering the values of couplings in the matrix $M$ that connect to these nodes.

To start with the grouping, two subnetworks ($m = 2$) are defined. The 1st subnetwork ($s = 1$) is made up of three resonators ($k_1 = 3$) and the second subnetwork ($s = 2$) is made up of two resonators ($k_2 = 2$). The number of possible groupings is given by (1), resulting in ten possible combinations. Note that here $p = 0$ because the two branches are topologically different. The different possible combinations are depicted in Table I.

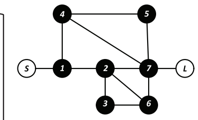
For the ten possible cases, NINA algorithm is applied to each subnetwork, taking the N transversal topology as the initial point. The equation system for the 1st and 2nd branch for the $r$th combination is defined by the equation expressed in (2).

In cases 1–6 the relation $N_e = N_o$ is fulfilled in the upper branch and $|N_e - N_o| = 1$ in the lower branch, resulting in precisely the desired solution with a quadruplet in the 1th branch and a trisection in the 2nd branch, as shown in Fig. 4(a).

In cases 7–9, the formation of a quadruplet in the 1st branch results from the fulfillment of the relation $|N_e - N_o| = 1$ as in the previous cases. While for the formation of the trisection of the 2nd branch, the relation $N_e = N_o$ is not fulfilled so the solution results in a dangling node. This scenario is illustrated in Fig. 4(b).

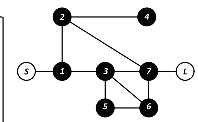
For case 10, an important observation arises: the system exhibits an infinite number of solutions because the

$$Mo_4 = \begin{bmatrix} 0 & 1.0491 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.0491 & -0.0038 & 0.5090 & 0 & -0.7095 & 0 & 0 & 0 & 0 \\ 0 & 0.5090 & 0.3947 & -0.6296 & 0 & 0 & -0.4398 & -0.1188 & 0 \\ 0 & 0 & -0.6296 & 0.1666 & 0 & 0 & -0.7986 & 0 & 0 \\ 0 & -0.7095 & 0 & 0 & -0.2098 & 0.3637 & 0 & -0.0852 & 0 \\ 0 & 0 & 0 & 0 & 0.3637 & -0.2977 & 0 & -0.7044 & 0 \\ 0 & 0 & -0.4398 & -0.7986 & 0 & 0 & 0.1837 & 0.4950 & 0 \\ 0 & 0 & -0.1188 & 0 & -0.0852 & -0.7044 & 0.4950 & -0.0038 & -1.0491 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0491 & 0 \end{bmatrix}$$



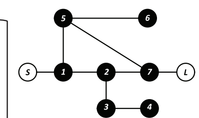

(a)

$$Mo_8 = \begin{bmatrix} 0 & 1.0491 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.0491 & -0.0038 & 0.5828 & -0.6503 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5828 & -0.081 & 0 & -0.4283 & 0 & 0 & 0.5828 & 0 \\ 0 & -0.6503 & 0 & 0.0572 & 0 & -0.6378 & 0.3224 & 0.5221 & 0 \\ 0 & 0 & -0.4283 & 0 & -0.8349 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6378 & 0 & 0.1709 & 0.211 & 0 & 0 \\ 0 & 0 & 0 & 0.3224 & 0 & 0.211 & 0.9253 & -0.3878 & 0 \\ 0 & 0 & 0.5828 & 0.5221 & 0 & 0 & -0.3878 & -0.0038 & -1.0491 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0491 & 0 \end{bmatrix}$$




(b)

$$Mo_{10} = \begin{bmatrix} 0 & 1.0491 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.0491 & -0.0038 & 0.6175 & 0 & 0 & 0.6175 & 0 & 0 & 0 \\ 0 & 0.6175 & 0.0408 & -0.5332 & 0 & 0 & 0 & 0.6175 & 0 \\ 0 & 0 & -0.5332 & -0.2318 & -0.8417 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.8417 & 0.3078 & 0 & 0 & 0 & 0 \\ 0 & 0.6175 & 0 & 0 & 0 & -0.0495 & -0.6718 & -0.6175 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.6718 & 0.1709 & 0 & 0 \\ 0 & 0 & 0.6175 & 0 & 0 & -0.6175 & 0 & -0.0038 & -1.0491 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0491 & 0 \end{bmatrix}$$




(c)

Fig. 4. Solutions of the target topology. (a) One of the six existing solutions from cases 1 to 6. (b) One of the three existing solutions from cases 7 to 9. (c) One of the infinitely many possible solutions of case 10.

equations for $(M_T^{(10,1)})_{1,3}$ and $(M_T^{(10,1)})_{3,7}$ are identical, resulting in an underdetermined system. To address this, an additional equation is introduced by setting $(M_T^{(10,1)})_{2,4}$ to any value that yields real solutions to increase the number of equations and form a well-determined system [8]. Specifically, a value of $(M_T^{(10,1)})_{2,4} = 0$ is set, which is part of the infinite set of solutions. Another way to approach this particular case in a simpler way is to realize that in the 1st branch the relation $|N_e - N_o| = 1$ is not fulfilled which results in an infinite number of solutions because the coupling $(M_T^{(10,1)})_{2,4}$ can take any value taking as pivot the nodes 3 and 4 and applying Givens rotations where one of them is the double dangling instead of the quartet forming a special topology called Cul-de-Sac [12], as shown in Fig. 4(c).

## IV. CONCLUSION

This work has described a method to analyze the multiplicity of solutions in filters connected in parallel. The inner N transversal network has been used as a starting point, grouping the new eigenvalues depending on the number of necessary branches and existing resonators. Once the branches have been split, each branch was reconfigured for each existing combination. It was demonstrated that the existence of multiple solutions is related to these different combinations of the eigenvalues in different subnetworks. The results have been compared with those obtained with NINA, which works with the entire network and offers the guarantee of providing all possible solutions. Furthermore, a specific case involving a Cul-de-Sac network was examined. Solutions were derived by constraining the solution range to mitigate the presence of infinite solutions in underdetermined systems.

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