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ORIGINAL PAPER

Deforming reducible representations of surface and 2-orbifold groups

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Abstract

For a compact 2-orbifold with negative Euler characteristic $\mathcal{O}^2$, the variety of characters of $\pi_1(\mathcal{O}^2)$ in $\mathrm{SL}_n(\mathbb{R})$ is a non-singular manifold at $\mathbb{C}$-irreducible representations. In this paper we prove that when a $\mathbb{C}$-irreducible representation of $\pi_1(\mathcal{O}^2)$ in $\mathrm{SL}_n(\mathbb{R})$ is viewed in $\mathrm{SL}_{n+1}(\mathbb{R})$, then the variety of characters is singular, and we describe the singularity.

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Contents

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1 Introduction

Let $\mathcal{O}^2$ be a compact 2-orbifold with $\chi(\mathcal{O}^2) < 0$, possibly with boundary. The group $\mathrm{PSL}_n(\mathbb{R})$ acts by conjugation algebraically on the variety of representations of $\pi_1(\mathcal{O}^2)$ in $\mathrm{SL}_n(\mathbb{R})$, and its quotient in real invariant theory is called the variety of characters

$$X(\mathcal{O}^2, \mathrm{SL}_n(\mathbb{R})) = \hom(\pi_1(\mathcal{O}^2), \mathrm{SL}_n(\mathbb{R}))//\mathrm{PSL}_n(\mathbb{R}).$$

Along the paper we shall assume $n \geq 2$.

A representation $\pi_1(\mathcal{O}^2) \to \mathrm{SL}_n(\mathbb{R})$ is called $\mathbb{C}$-irreducible if it has no proper invariant subspace in $\mathbb{C}^n$. By [10] the character of a $\mathbb{C}$-irreducible representation has a smooth (non-singular) neighborhood in the variety of characters $X(\mathcal{O}^2, \mathrm{SL}_n(\mathbb{R}))$. In this paper we show that its composition with the standard inclusion in $\mathrm{SL}_{n+1}(\mathbb{R})$ has a character topologically singular in $X(\mathcal{O}^2, \mathrm{SL}_{n+1}(\mathbb{R}))$, and we describe the singularity.

For a $\mathbb{C}$-irreducible representation $\rho\colon \pi_1(\mathcal{O}^2) \to \mathrm{SL}_n(\mathbb{R})$, let $\mathbb{R}^n_\rho$ denote $\mathbb{R}^n$ as $\pi_1(\mathcal{O}^2)$-module via ρ and set $d = \dim(H^1(\mathcal{O}^2, \mathbb{R}^n_\rho))$. View also ρ as a representation in $\mathrm{SL}_{n+1}(\mathbb{R})$, by composing it with the standard inclusion $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{SL}_{n+1}(\mathbb{R})$ and so that it is no longer $\mathbb{C}$-irreducible.

Theorem 1.1 *Let $\mathcal{O}^2$ be a compact and orientable 2-orbifold, satisfying $\chi(\mathcal{O}^2) < 0$. Let $\chi_\rho \in X(\mathcal{O}^2, SL_n(\mathbb{R}))$ be the character of a $\mathbb{C}$-irreducible representation ρ, $d = \dim(H^1(\mathcal{O}^2, \mathbb{R}^n_\rho))$, and $b = \dim(H^1(\mathcal{O}^2, \mathbb{R}))$ the first Betti number. A neighborhood of χ_ρ in $X(\mathcal{O}^2, SL_{n+1}(\mathbb{R}))$ is homeomorphic to*

$$\mathbb{R}^p \times \mathbb{R}^b \times Cone(X),$$

where X is as in Table 1 and $\mathbb{R}^p \times \{0\} \times \{0\}$ corresponds to a smooth neighborhood in the variety of characters $X(\mathcal{O}^2, SL_n(\mathbb{R}))$.

Here $S^{d-1} \subset \mathbb{R}^d$ denotes the $(d-1)$-dimensional unit sphere; $\mathrm{UT}(S^{d-1})$, its unit tangent bundle

$$\mathrm{UT}(S^{d-1}) = \{(u, v) \in S^{d-1} \times S^{d-1} \subset \mathbb{R}^{2d} \mid u \cdot v = 0\};$$

$\mathrm{UT}(\mathbb{RP}^{d-1})$, the unit tangent bundle of projective space; and $\sim$, the involution of $S^{d-1} \times S^{d-1}$ acting as the antipodal on each factor. The cone on a topological space X is denoted by $\mathrm{Cone}(X)$. When $d = 0$, we use the convention that $S^{-1} = \emptyset$ and $\mathrm{Cone}(\emptyset)$ is a point. The factor $\mathbb{R}^b$ is realized by representations in the stabilizer of ρ in $\mathrm{SL}_{n+1}(\mathbb{R})$, namely $\mathrm{diag}(\theta, \ldots, \theta, \theta^{-n})$ for a homomorphism $\theta\colon \Gamma \to \mathbb{R}^*$. The difference according to the parity of $n+1$ may be explained by the fact that we take into account the action of stabilizer of ρ: when $n+1$ is odd, the matrix $\mathrm{diag}(-1, \ldots, -1, 1)$ stabilizes ρ but it is not central, though when $n+1$ is even, $\mathrm{diag}(-1, \ldots, -1)$ is central.

As motivation, when $n = 3$ we consider projective structures on $\mathcal{O}^2$ modeled in $\mathbb{P}^2$, and we view them as projective structures on $\mathcal{O}^2 \times \mathbb{R}$ modeled in $\mathbb{P}^3$. When $\mathcal{O}^2$ is closed and

Table 1 The link X of the singularity in Theorem 1.1

X	$n+1$ even	$n+1$ odd
$\mathcal{O}^2$ is closed	$\mathrm{UT}(S^{d-1})$	$\mathrm{UT}(\mathbb{RP}^{d-1})$
$\mathcal{O}^2$ has boundary	$S^{d-1} \times S^{d-1}$	$(S^{d-1} \times S^{d-1})/\sim$

orientable, the space of convex projective structures on $\mathcal{O}^2$ is a component of $X(\mathcal{O}^2, \mathrm{SL}_3(\mathbb{R}))$, homeomorphic to a cell, by Choi-Goldman [6]. When embedding $\mathrm{SL}_3(\mathbb{R})$ in $\mathrm{SL}_4(\mathbb{R})$ by Theorem 1.1 we get a singularity (with few exceptions for small orbifolds). More precisely (Corollary 4.3) a neighborhood of the character of its projective holonomy in $X(\mathcal{O}^2, \mathrm{SL}_4(\mathbb{R}))$ is homeomorphic to

$$\begin{cases} \mathbb{R}^{p+b} \times \mathrm{Cone}(\mathrm{UT}(S^{t-1})) & \text{if } \mathcal{O}^2 \text{ is closed} \\ \mathbb{R}^{p+b} \times \mathrm{Cone}(S^{t-1} \times S^{t-1}) & \text{if } \mathcal{O}^2 \text{ has boundary} \end{cases}$$

where t is the dimension of its Teichmüller space and p is the dimension of its Choi-Goldman's space of convex projective structures. In particular (Corollary 4.4) for a turnover $t = b = 0$, so every deformation in $\mathrm{SL}_4(\mathbb{R})$ of its projective holonomy is conjugate to a deformation in $\mathrm{SL}_3(\mathbb{R})$, as the cone of the empty set is a point.

The theorem applies also to the Barbot component [3]. Given a discrete and faithful representation of the fundamental group of a surface F_g of genus $g \geq 2$ in $\mathrm{SL}_2(\mathbb{R})$, Barbot proves in [3] that the composition with the standard embedding in $\mathrm{SL}_3(\mathbb{R})$ yields Anosov representations that do not lie in the Hitchin component. The corresponding component in the character variety $X(F_g, \mathrm{SL}_3(\mathbb{R}))$ is singular by Theorem 1.1, and a neighborhood of singular points is homeomorphic to $\mathbb{R}^{8g-6} \times \mathrm{Cone}(\mathrm{UT}(\mathbb{RP}^{4g-5}))$.

At the end of the paper we also discuss the deformation space of projective structures on a hyperbolic 3-orbifold $\mathcal{O}^3$ of finite type. For this purpose, the holonomy representation $\rho\colon \pi_1(\mathcal{O}^3) \to \mathrm{SO}(3,1)$ of the hyperbolic structure is composed with the standard inclusion $\mathrm{SO}(3,1) \to \mathrm{SL}_4(\mathbb{R})$, yielding the holonomy of the corresponding projective structure. Thus a neighborhood of the character of ρ in the variety of characters $X(\mathcal{O}^3, \mathrm{SL}_4(\mathbb{R}))$ yields the deformation space of this projective structure.

To put our result in context, we recall a theorem of Ballas, Danciger, and Lee [2, Theorem 3.2]. According to their theorem, if a finite volume orbifold $\mathcal{O}^3$ is infinitesimally rigid with respect to the boundary, then the character of its hyperbolic holonomy is a smooth point of $X(\mathcal{O}^3, \mathrm{SL}_4(\mathbb{R}))$. Here "infinitesimally rigid with respect to the boundary" is a technical hypothesis on cohomology on the Lie algebra (Definition 5.1), it essentially means that all infinitesimal deformations of $\mathcal{O}^3$ come from its ends. We prove the following theorem:

Theorem 1.2 *Let $\mathcal{O}^3$ be a compact orientable orbifold, with $\partial\mathcal{O}^3 \neq \emptyset$ and hyperbolic interior int$(\mathcal{O}^3)$ so that it is not elementary, nor Fuchsian. Assume that $\mathcal{O}^3$ is infinitesimally projectively rigid with respect to the boundary. Then the character of the hyperbolic holonomy is a smooth point of $X(\mathcal{O}^3, SL_4(\mathbb{R}))$ if, and only if, all ends of int$(\mathcal{O}^3)$ are either non-Fuchsian or turnovers. Furthermore, if the character is a singular point, then the singularity is quadratic.*

We say that one end is *Fuchsian* if its holonomy is a Fuchsian group: namely the end corresponding to a totally geodesic boundary component.

Let us comment on the proof of Theorem 1.1. Under the hypothesis of this theorem, when $\partial\mathcal{O}^2 \neq \emptyset$ the variety of representations (not of characters) $\hom(\pi_1(\mathcal{O}^2), \mathrm{SL}(n+1,\mathbb{R}))$ is smooth at ρ. The singularity occurs in the variety of characters, when considering the real GIT quotient. This singularity appears because ρ is not irreducible, and there is a one-parameter subgroup of $\mathrm{SL}_{n+1}(\mathbb{R})$ that commutes with $\mathrm{SL}_n(\mathbb{R})$. To understand this singularity when $\partial\mathcal{O}^2 \neq \emptyset$, notice that $\mathbb{R}^n_\rho$ and $\mathbb{R}^n_{\rho^*}$ appear in the complement of $\mathfrak{sl}(n,\mathbb{R})$ in $\mathfrak{sl}(n+1,\mathbb{R})$, where $\rho^*(\gamma) = \rho(\gamma^{-1})^t$ is the contragredient representation. When $n+1$ is even the action of the commutator of $\mathrm{SL}_n(\mathbb{R})$ on $H^1(\mathcal{O}^2, \mathbb{R}^n_{\rho^*}) \oplus H^1(\mathcal{O}^2, \mathbb{R}^n_\rho)$ is equivalent to the action of $\mathbb{R} - \{0\}$ on $\mathbb{R}^d \times \mathbb{R}^d$ defined by $t \cdot (x, y) \mapsto (t^{n+1}x, t^{-n-1}y)$, for $t \in \mathbb{R} - \{0\}$ and $x, y \in \mathbb{R}^d$.

The quotient $(\mathbb{R}^d \times \mathbb{R}^d)//\mathbb{R}_{>0}$ is homeomorphic to the cone $|x|^2 = |y|^2$, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. When $n+1$ is odd, in addition we take into account of the antipodal on $\mathbb{R}^n_{\rho^*} \oplus \mathbb{R}^n_{\rho}$.

When $\mathcal{O}^2$ is closed, the variety of representations of $\hom(\pi_1(\mathcal{O}^2), \mathrm{SL}(n+1, \mathbb{R}))$ has a quadratic singularity, by a theorem of Goldman [9] (see also Goldman and Millson [11] and Simpson [25]). Then both singularities have to be combined, the quadratic singularity and the singularity from passing to the quotient. Thus for $x, y \in \mathbb{R}^d$, we combine the equality $|x|^2 = |y|^2$ with $x \cdot y = 0$, that yields the cone in the unit tangent bundle of the sphere when $n+1$ is even, or its quotient by the antipodal when $n+1$ is odd.

The paper also discusses non-orientable orbifolds, so we consider the group

$$\mathrm{SL}_n^{\pm}(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = \pm 1\}.$$

In this case there are two natural extensions of $\mathrm{SL}_n^{\pm}(\mathbb{R})$. One extension is in $\mathrm{SL}_{n+1}(\mathbb{R})$, so that every transformation of $\mathbb{P}^{n-1}$ extends to an orientation preserving transformation of $\mathbb{P}^n$. Namely, every matrix $A \in \mathrm{SL}_n^{\pm}(\mathbb{R})$ is mapped to a matrix $\mathrm{SL}_{n+1}(\mathbb{R})$ by adding a last raw and last column of zeros, except for the $(n+1, n+1)$ entry that equals $\det(A)$. The other extension $\mathrm{SL}_n^{\pm}(\mathbb{R}) \to \mathrm{SL}_{n+1}^{\pm}(\mathbb{R})$ is just the trivial extension, so that the $(n+1, n+1)$ entry equals 1. Theorem 1.1 is easily adapted to the non-orientable case, just taking into account these different extensions, see Theorem 3.6 for details.

In this paper we describe the topological singularities. The variety of characters is homeomorphic to a real semi-algebraic set (cf. Theorem 2.1), in particular every character has a neighborhood homeomorphic to a semi-analytic set. We shall not discuss this semi-analytic structure.

Organization of the paper Section 2 is devoted to preliminaries on varieties of (real) representations and characters, as well as cohomology and orbifolds. In particular we review Goldman's results. In Section 3 we prove the main theorem, as well as its generalization to non-orientable orbifolds. Section 4 applies the main theorem to deformation spaces of projective 2-orbifolds viewed in the space of three-dimensional projective structures, including some examples. Finally we apply these results in Sect. 5 to determine when the space of projective structures on hyperbolic 3-orbifolds is singular (assuming that all infinitesimal projective deformations come from the ends of the hyperbolic three-orbifold).

2 Preliminaries on varieties of characters

This first section is devoted to preliminaries on varieties of (real) representations and characters.

2.1 Variety of characters for real reductive groups

We recall the basic results on varieties of representations and characters in algebraic real reductive groups. Let Γ be a finitely generated discrete group (eg the fundamental group of a compact two-orbifold) and let G be either

$$G = \mathrm{SL}_n(\mathbb{R}) \qquad \text{or} \qquad G = \mathrm{SL}_n^{\pm}(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = \pm 1\},$$

for $n \geq 2$. The results here apply to more general semi-simple real algebraic groups (and some definitions need to be adapted), but we consider only those groups.

The *variety of representations* $\hom(\Gamma, G)$ is a real algebraic set. The group G acts by conjugation on $\hom(\Gamma, G)$ and to define the real GIT quotient we follow the results of Luna

[18], Richardson [21], and Richardson and Slodowy [22]. See also a modern treatment by Böhm and Lafuente [4], as well as an interesting approach using symmetric spaces by Parreau [19].

Theorem 2.1 *([18, 22]) The closure of each orbit by conjugation of G on* $\hom(\Gamma, G)$ *has precisely one closed orbit. Furthermore, the space of closed orbits is homeomorphic to a real semi-algebraic set.*

Here the space of closed orbits is equipped with the quotient topology of a subset of $\hom(\Gamma, G)$. One of the ingredients of Theorem 2.1 is Kempf-Ness theorem for real coefficients, proved first in [22] (see also [4]). Kempf-Ness provides an algebraic subset $\mathcal{M} \subset \hom(\Gamma, G)$ that intersects all closed orbits and only closed orbits. Furthermore the space of closed orbits is homeomorphic to $\mathcal{M}/K$ for $K \subset G$ compact (and therefore it is homeomorphic to a semi-algebraic set).

Using Theorem 2.1 one defines the *variety of characters* $X(\Gamma, G)$ as the space of closed orbits. This construction is also called the *real or* $\mathbb{R} -$ GIT *quotient*:

$$X(\Gamma, G) = \hom(\Gamma, G)/\!/G.$$

It is the "Hausdorff quotient" of the topological quotient $\hom(\Gamma, G)/G$: every continuous map from $\hom(\Gamma, G)/G$ to a Hausdorff space Y factors through $\hom(\Gamma, G)/\!/G$ in a unique way.

Next we also mention a theorem that identifies the orbits that are closed.

Definition 2.2 A representation $\rho \in \hom(\Gamma, G)$ is called:

- $\mathbb{R}$-*irreducible* or $\mathbb{R}$-*simple* if it has no proper invariant subspace in $\mathbb{R}^n$;
- $\mathbb{C}$-*irreducible* or $\mathbb{C}$-*simple* if it has no proper invariant subspace in $\mathbb{C}^n$;
- *semi-simple* if $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_k$ so that each E_i is ρ-invariant and the restriction of ρ to E_i is simple.

Notice that $\mathbb{C}$-irreducibility implies $\mathbb{R}$-irreducibility. However, the converse is not true, consider representations in SO(2). Notice also that semi-simplicity does not depend on considering $\mathbb{R}^n$ or $\mathbb{C}^n$ (an $\mathbb{R}$-irreducible space decomposes as a possibly trivial direct sum of $\mathbb{C}$-irreducible spaces). The following lemma is elementary:

Lemma 2.3 *A representation $\rho \in$* $\hom(\Gamma, G)$ *is semi-simple iff every invariant subspace $V \subset \mathbb{R}^n$ has an invariant complement, eg a ρ-invariant subspace $W \subset \mathbb{R}^n$ satisfying $V \oplus W = \mathbb{R}^n$.*

Theorem 2.4 *([21]) Let $\rho \in$* $\hom(\Gamma, G)$. *The orbit $G \cdot \rho$ is closed if and only if ρ is semi-simple.*

From this theorem, we can think of elements in $X(\Gamma, G)$ as conjugacy classes of semi-simple representations, that we may call characters by abuse of notation. Furthermore we may talk about irreducible or simple characters.

The center of G is $\mathcal{Z}(G) = \{h \in G \mid hg = gh \text{ for every } g \in G\}$.

Definition 2.5 An *analytic slice* at ρ is analytic subvariety $S \subset \hom(\Gamma, G)$ such that

(i) $S \cap G \cdot \rho = \{\rho\}$,
(ii) There is an bi-analytic map $G/\mathcal{Z}(G) \times S \cong G \cdot S$ and $G \cdot S$ is a neighborhood of $G \cdot \rho$.

The following is [15, Theorem 1.2] by Johnson and Millson:

Theorem 2.6 *([15]) Let $\rho \in \hom(\Gamma, G)$ be a $\mathbb{C}$-irreducible representation. Then the action by conjugation admits an analytic slice at ρ.*

We discuss a more general version of this theorem in Theorem 3.1. In general the slice takes into account the stabilizer of ρ, when the representation is irreducible, the stabilizer is just the center of G:

Lemma 2.7 *Let $\rho \in \hom(\Gamma, G)$ be a $\mathbb{C}$-irreducible representation. Then the stabilizer by conjugation of ρ is the center of G: $Stab_G(\rho) = \mathcal{Z}(G)$.*

Proof The center is always contained in the stabilizer and we prove the other inclusion. Let $g \in G$ be an element of the stabilizer of ρ. The relation $g\rho = \rho g$ implies that ρ preserves the eigenspaces of g, thus g has no proper eigenspaces. Equivalently, g is an scalar multiple of the identity, so an element of the center. □

Remark 2.8 It follows from Theorem 2.6 that there exists a well defined analytic structure in a neighborhood of a $\mathbb{C}$-irreducible character, as in [15]. Without assuming irreducibility, one should consider semi-analytic structures on $X(\Gamma, G)$.

2.2 Group cohomology

We follow [5] for basics on group cohomology. Fix a finitely generated group Γ, a $\mathbb{R}$-vector space V, and a representation $\Gamma \to \mathrm{GL}(V)$, so that V is called a Γ-*module*. We are mainly interested in the case where there is a representation $\rho\colon \Gamma \to G$ and $V = \mathfrak{g}$, that is a Γ-module via the adjoint of ρ, but we shall also consider other Γ-modules.

The i-chains of the *bar resolution* are defined as maps from $\Gamma \times \overset{(i)}{\cdots} \times \Gamma$ to V:

$$C^i(\Gamma, V) = \{\theta\colon \Gamma \times \overset{(i)}{\cdots} \times \Gamma \to V\}, \text{ for } i > 0 \qquad \text{and} \qquad C^0(\Gamma, V) = V.$$

The coboundary $\delta^i : C^i(\Gamma, V) \to C^{i+1}(\Gamma, V)$ is defined by

$$\begin{aligned} &\delta^i(\theta)(\gamma_0, \ldots, \gamma_i) = \gamma_0\theta(\gamma_1, \ldots, \gamma_i) \\ &+\sum_{j=0}^{i-1}(-1)^{j+1}\theta(\gamma_0, \ldots, \gamma_j\gamma_{j+1}, \ldots, \gamma_i) \\ &+(-1)^{i+1}\theta(\gamma_0, \ldots, \gamma_{i-1}). \end{aligned}$$

The space of cycles and coboundaries are denoted respectively by

$$Z^i(\Gamma, V) = \ker \delta^i \qquad \text{and} \qquad B^i(\Gamma, V) = \operatorname{Im} \delta^{i-1}$$

so that the cohomology is

$$H^i(\Gamma, V) = Z^i(\Gamma, V)/B^i(\Gamma, V).$$

The zero-th cohomology group is naturally isomorphic to the subspace of invariants

$$H^0(\Gamma, V) \cong Z^0(\Gamma, V) \cong V^\Gamma, \tag{1}$$

because $\delta^0(v)(\gamma) = \gamma v - v$ for $\gamma \in \Gamma$ and $v \in V$.

2.3 Products in cohomology

Let V_1, V_2 and V_3 be Γ-modules and let

$$\varphi\colon V_1 \times V_2 \to V_3$$

be a bilinear map that is Γ-equivariant. Combined with the cup product it induces a pairing in cohomology, that we denote by $\varphi(\cdot \cup \cdot)$.

$$\varphi(\cdot \cup \cdot)\colon H^i(\Gamma, V_1) \times H^j(\Gamma, V_2) \xrightarrow{\cup} H^{i+j}(\Gamma, V_1 \otimes V_2) \xrightarrow{\varphi} H^{i+j}(\Gamma, V_3).$$

We are interested in the explicit description for 1-cocycles. The space of 1-cocycles is

$$Z^1(\Gamma, V) = \{\theta\colon \Gamma \to V \mid \theta(\gamma_1\gamma_2) = \theta(\gamma_1) + \gamma_1\theta(\gamma_2), \forall \gamma_1, \gamma_2 \in \Gamma\}.$$

The space of 1-coboundaries is

$$B^1(\Gamma, V) = \{\theta_v \in Z^1(\Gamma, V) \mid \text{ there is a } v \in V \text{ s.t. } \theta_v(\gamma) = \gamma v - v, \forall \gamma \in \Gamma\}.$$

For 1-cocycles, we describe the cup product following [5]: if $z_i \in Z^1(\Gamma, V_i)$, then $\varphi(z_1 \cup z_2) \in Z^2(\Gamma, V_3)$ is the 2-cocycle

$$\varphi(z_1 \cup z_2)(\alpha, \beta) = \varphi(z_1(\alpha), \alpha z_2(\beta)), \qquad \forall \alpha, \beta \in \Gamma. \tag{2}$$

In addition we have:

Lemma 2.9 *Assume $V_2 = V_1$, then*

$$\varphi(\cdot \cup \cdot)\colon H^1(\Gamma, V_1) \times H^1(\Gamma, V_1) \to H^2(\Gamma, V_3)$$

is skew-symmetric when φ is symmetric, and symmetric when φ is skew-symmetric.

Proof Consider the 1-chain $c\colon \Gamma \to V_3$ defined by $c(\gamma) = \varphi(z_1(\gamma), z_2(\gamma))$. A direct application of the definition of the coboundary δ^1 yields

$$\begin{aligned}\delta^1(c)(\alpha, \beta) &= \varphi(z_1(\beta), z_2(\beta)) - \varphi(z_1(\alpha\beta), z_2(\alpha\beta)) + \varphi(z_1(\alpha), z_2(\alpha)) \\ &= -\varphi(z_1(\alpha), \alpha z_2(\beta)) - \varphi(\alpha z_1(\beta), z_2(\alpha)), \qquad \forall \alpha, \beta \in \Gamma.\end{aligned}$$

Hence by (2) we deduce:

$$-\delta^1(c) = \varphi(z_1 \cup z_2) \pm \varphi(z_2 \cup z_1)$$

where the sign in $\pm$ is $+$ for φ symmetric, and $-$ for φ skew-symmetric. □

Below we describe the role of 1-cocycles in tangent spaces and infinitesimal deformations.

2.4 Tangent space to the variety of representations

Let $\rho\colon \Gamma \to G$ be a representation. We view $\mathfrak{g}$ as a Γ-module by composing ρ with the the adjoint: $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$.

We describe Weil's construction [17, 26]. Given a 1-cocycle $d \in Z^1(\Gamma, \mathfrak{g})$, the assignment for each $\gamma \in \Gamma$:

$$\gamma \to (1 + t\, d(\gamma))\rho(\gamma)$$

is an infinitesimal path in $\hom(\Gamma, G)$; namely a path of representations up to a factor t^2, or a 1-jet from $(\mathbb{R}, 0)$ to $(\hom(\Gamma, G), \rho)$, eg an element in $J^1_{0,\rho}(\mathbb{R}, \hom(\Gamma, G))$.

Proposition 2.10 *[17, 26] Weil's construction yields an isomorphism*

$$Z^1(\Gamma, \mathfrak{g}) \cong T_\rho^{Zar} \hom(\Gamma, G)$$

that maps $B^1(\Gamma, \mathfrak{g})$ to the tangent space to the orbit by conjugation.

Here $T_\rho^{\mathrm{Zar}} \hom(\Gamma, G)$ means the Zariski tangent space as scheme, as the coordinate ring may be non-reduced, cf. [13]. Using Theorem 2.6 on the existence of a slice, we get:

Theorem 2.11 *Let $[\rho] \in X(\Gamma, G)$ be a $\mathbb{C}$-irreducible character. Weil's construction factors to an isomorphism:*

$$H^1(\Gamma, \mathfrak{g}) \cong T_{[\rho]}^{Zar} X(\Gamma, G).$$

We discuss the general case (without assuming irreducibility) in Sect. 3.1. We also need a result in the zero-th cohomology group in the $\mathbb{C}$-irreducible case.

Lemma 2.12 *If ρ is $\mathbb{C}$-irreducible, then the space of invariants of the Lie algebra is trivial: $\mathfrak{g}^{Ad\rho(\Gamma)} = 0$. In particular $H^0(\Gamma, \mathfrak{g}) = 0$.*

Proof Assume $H^0(\Gamma, \mathfrak{g}) \neq 0$, by (1) there exists $0 \neq a \in \mathfrak{g}^{\mathrm{Ad}\rho(\Gamma)}$. Then the one-parameter group $\{\exp(t\, a) \mid t \in \mathbb{R}\}$ commutes with the image of ρ. By considering the eigenspaces of $\exp(a)$, this commutativity contradicts that ρ is $\mathbb{C}$-irreducible by Lemma 2.7. □

2.5 Orbifolds

We relate the group cohomology with orbifold cohomology, that can be defined as the simplicial cohomology for a CW-complex structure on the orbifold [20, §3]. An orbifold is *very good* if it has a finite orbifold covering that is a manifold, and in this case the orbifold cohomology is the equivariant cohomology of the manifold covering. We are also interested in orbifolds that are *aspherical*: the universal covering is a contractible manifold.

In this paper we consider 2-dimensional orbifolds with negative Euler characteristic (eg hyperbolic) and hyperbolic 3-orbifolds, hence very good and aspherical.

Lemma 2.13 *If $\mathcal{O}^n$ is an aspherical and very good orbifold, then there is a natural isomorphism*

$$H^i(\mathcal{O}^n, V) \cong H^i(\pi_1(\mathcal{O}^n), V)$$

See for instance [20, Prop. 3.4] for a proof (for an aspherical manifold this is standard, and for a very good orbifold use a manifold covering and work equivariantly).

We shall need Poincaré duality for cohomology with coefficients. Let V_1 and V_2 be Γ-modules and let $\varphi\colon V_1 \times V_2 \to \mathbb{R}$ be a pairing that is Γ-equivariant. Combined with the cup product it induces a pairing in cohomology, that we denote by $\varphi(\cdot \cup \cdot)$.

$$\varphi(\cdot \cup \cdot)\colon H^i(\mathcal{O}^n, V_1) \times H^j(\mathcal{O}^n, \partial\mathcal{O}^n; V_2) \xrightarrow{\cup} H^{i+j}(\mathcal{O}^n, \partial\mathcal{O}^n; V_1 \otimes V_2)$$
$$\xrightarrow{\varphi} H^{i+j}(\mathcal{O}^n, \partial\mathcal{O}^n; \mathbb{R}).$$

Theorem 2.14 *(Poincaré duality with coefficients) Let $\mathcal{O}^n$ be a compact, connected, orientable, and very good n-orbifold. For V_1, V_2 and φ as above; if φ is a perfect pairing, then the product*

$$\varphi(\cdot \cup \cdot)\colon H^k(\mathcal{O}^n, V_1) \times H^{n-k}(\mathcal{O}^n, \partial\mathcal{O}^n; V_2) \to H^n(\mathcal{O}^n, \partial\mathcal{O}^n; \mathbb{R}) \cong \mathbb{R}$$

is a perfect pairing.

See for instance [20, Proposition 3.4] for a proof of this orbifold version of Poincaré duality.

2.6 Obstructions to integrability

Here we review the theory of obstructions to integrability. We start by reviewing the following theorem of Goldman:

Proposition 2.15 *([10]) Set $\Gamma = \pi_1(\mathcal{O}^2)$ and assume $\rho\colon \Gamma \to G$ is $\mathbb{C}$-irreducible. Then:*

(i) $\hom(\Gamma, G)$ *is smooth at ρ and $Z^1(\Gamma, \mathfrak{g})$ is isomorphic to $T_\rho \hom(\Gamma, G)$.*
(ii) $X(\Gamma, G)$ *is smooth at the character $[\rho]$ and $H^1(\Gamma, \mathfrak{g})$ is isomorphic to $T_{[\rho]}X(\Gamma, G)$.*

Furthermore, $X(\Gamma, G)$ is locally homeomorphic to the topological quotient $\hom(\Gamma, G)/G$.

The statement of this proposition deals with T instead of T^{Zar}, because in the smooth case, the Zariski tangent space equals the standard tangent space (here smooth means smooth not only as a variety but as scheme, in particular reduced).

A key tool for Proposition 2.15 is Goldman's obstruction to integrability, defined in [10]. The first obstruction to integrability uses the cup product combined with the Lie bracket, as in (2). At the level of 1-cocycles it is defined as follows: for $\sigma, \varsigma \in Z^1(\Gamma, \mathfrak{g})$,

$$\begin{aligned} [\sigma \cup \varsigma]\colon \Gamma \times \Gamma &\to \mathfrak{g} \\ (\gamma_1, \gamma_2) &\mapsto [\sigma(\gamma_1), \mathrm{Ad}_{\rho(\gamma_1)}\varsigma(\gamma_2)] \end{aligned}$$

As the Lie bracket is skew-symmetric, by Lemma 2.9 the induced product in cohomology is a symmetric bilinear form

$$[\cdot \cup \cdot]\colon H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \to H^2(\Gamma, \mathfrak{g})$$

For a tangent vector $z \in Z^1(\Gamma, \mathfrak{g})$, the first obstruction to integrability of z is the class of $[z \cup z]$ in $H^2(\Gamma, \mathfrak{g})$, see [10, 14] for details.

The element $z \in Z^1(\Gamma, \mathfrak{g})$ can be seen as a deformation of representations at first order, a 1-jet $J^1_{0,\rho}(\mathbb{R}, \hom(\Gamma, G))$. When $[z \cup z] \sim 0$ there is a deformation up to second order, equivalently it extends to a 2-jet in $J^2_{0,\rho}(\mathbb{R}, \hom(\Gamma, G))$. We state the existence of Goldman's obstructions as a lemma (see [10] or [14] for a proof):

Lemma 2.16 *(Goldman [10]) An n-jet in $J^n_{0,\rho}(\mathbb{R}, \hom(\Gamma, G))$ has a natural obstruction in $H^2(\Gamma, \mathfrak{g})$ that vanishes if and only if it extends to an $(n+1)$-jet.*

In the proof of Proposition 2.15, Goldman uses that $H^2(\Gamma, \mathfrak{g}) = 0$, by Lemma 2.12 and Poincaré duality, so all obstructions vanish.

When $H^2(\Gamma, \mathfrak{g})$ does not vanish, for certain classes of groups Γ including $\pi_1(\mathcal{O}^2)$, Goldman [9], Goldman and Millson [11], and Simpson [25] prove that the cup product is the only obstruction to integrability, hence the singularities in $\hom(\pi_1(\mathcal{O}^2), G)$ are at most quadratic. In terms of analytic geometry, the result is as follows:

Theorem 2.17 *([9, 11, 25]) Let $\mathcal{O}^2$ be closed and oriented. If the representation $\rho \in$ $\hom(\pi_1(\mathcal{O}^2), G)$ is semi-simple, then the analytic germ of $\hom(\pi_1(\mathcal{O}^2), G)$ at ρ is analytically equivalent to the quadratic cone*

$$\{z \in Z^1(\Gamma, G) \mid [z \cup z] \sim 0\}.$$

3 Deformation space

In this section we prove Theorem 1.1. It requires a slice theorem, in Subsection 3.1, and some discussion on the decomposition of the Lie algebra, Subsection 3.2. Theorem 1.1 is proved in Subsection 3.3. In Subsection 3.4 we discuss the non-orientable case.

3.1 An analytic slice at semisimple representations

There are several results in the literature dealing with slices of algebraic actions of reductive groups. For completeness, we have decided to include a slice theorem, that we did not find in the literature in this precise form.

We shall use some notation of real analytic geometry: analytic varieties are viewed as subsets of some Euclidean space $\mathbb{R}^n$. Furthermore, a *germ at a point* means an equivalence class of open sets centered at a point. For convenience, we use the term *germ* loosely to denote an open set in the equivalence class.

Let Γ be a finitely generated group, $G = \mathrm{SL}_{n+1}(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R})$, and let $\rho\colon \Gamma \to G$ be a semi-simple representation. Let $G_\rho = \mathrm{Stab}_G(\rho)$ denote the stabilizer of ρ in G by the action by conjugation. For an analytic variety $\mathcal{S}$ where G_ρ acts, we shall use the notation

$$G \times_{G_\rho} \mathcal{S} = (G \times \mathcal{S})/G_\rho,$$

where $g \in G_\rho$ maps $(h, s) \in G \times \mathcal{S}$ to (hg^{-1}, gs).

Theorem 3.1 *In the previous setting, there exists a G_ρ-invariant real analytic subvariety $\mathcal{S} \subset \hom(\Gamma, G)$ containing ρ such that:*

(i) $T_\rho^{Zar}\mathcal{S} \oplus B^1(\Gamma, \mathfrak{g}) = T_\rho^{Zar}\hom(\Gamma, G)$. *In particular* $T_\rho^{Zar}\mathcal{S} \cong H^1(\Gamma, \mathfrak{g})$.
(ii) If $\hom(\Gamma, G)$ *is smooth at ρ, then so is $\mathcal{S}$.*
(iii) There is a G-equivariant map $G \times_{G_\rho} \mathcal{S} \to \hom(\Gamma, G)$ that is a bi-analytic equivalence between $G \times_{G_\rho} \mathcal{S}$ and a G-saturated neighborhood of ρ.

Furthermore, the germ of $\mathcal{S}$ at ρ is G_ρ-equivariantly equivalent to a germ of a subvariety of $H^1(\Gamma, G)$ at the origin, that is unique up to G_ρ-invariant bi-analytic equivalence.

Proof Assume first that $\Gamma = F_k = \langle \gamma_1, \ldots, \gamma_k \mid \rangle$ is a free group of rank k. We consider the algebraic isomorphism

$$\begin{aligned} \phi\colon \hom(F_k, G) &\to G^k \\ \rho' &\mapsto (\rho'(\gamma_1)\rho(\gamma_1^{-1}), \ldots, \rho'(\gamma_k)\rho(\gamma_k^{-1})) \end{aligned}$$

and the map

$$\begin{aligned} f\colon G &\to \mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R}) \\ A &\mapsto A - \tfrac{\mathrm{trace}(A)}{n+1}\mathrm{Id} \end{aligned}$$

We are interested in the composition

$$\Phi = (f, \ldots, f) \circ \phi\colon \hom(F_k, G) \to \mathfrak{g}^k. \tag{3}$$

By construction Φ is G_ρ-equivariant, $\Phi(\rho) = 0$, and its tangent map at ρ is the isomorphism

$$\begin{aligned} \Phi_*\colon Z^1(F_k, \mathfrak{g}) &\to \mathfrak{g}^k \\ d &\mapsto (d(\gamma_1), \ldots, d(\gamma_k)). \end{aligned}$$

We shall consider an open neighborhood $U \subset \hom(F_k, G)$ of ρ such that Φ defines a bi-analytic map with $\Phi(U) \subset \mathfrak{g}^k$, a neighborhood of the origin.

The stabilizer G_ρ is easily determined, as ρ is direct sum of simple representations, and one checks that the adjoint representation of G_ρ in $\mathrm{End}(\mathfrak{g})$ is also is semi-simple. Hence there exists a G_ρ-invariant linear subspace $H \subset \mathfrak{g}^k$ such that $\Phi_*(B^1(F_k, \mathfrak{g})) \oplus H = \mathfrak{g}^k$. In particular $H \cong H^1(F_k, \mathfrak{g})$.

Define the slice as the union of G_ρ-translates:

$$\mathcal{S}_{F_k} = G_\rho \cdot (U \cap \Phi^{-1}(H)) = \bigcup_{g \in G_\rho} g \cdot (U \cup \Phi^{-1}(H))$$

As Φ is G_ρ-equivariant and H is G_ρ-invariant, we have the inclusion

$$\mathcal{S}_{F_k} \cap U = \big(G_\rho \cdot (U \cap \Phi^{-1}(H))\big) \cap U \subset \big(G_\rho \cdot \Phi^{-1}(H)\big) \cap U = \Phi^{-1}(H) \cap U.$$

Hence

$$\mathcal{S}_{F_k} \cap U = \Phi^{-1}(H) \cap U.$$

In particular $\mathcal{S}_{F_k}$ is an analytic subvariety that satisfies (i) and (ii) for a free group.

To show that $\mathcal{S}_{F_k}$ satisfies (iii), notice that the natural map $G \times \mathcal{S}_{F_k} \to \hom(F_k, G)$ is a submersion (by the choice of H) and it factors to

$$\Psi \colon G \times_{G_\rho} \mathcal{S}_{F_k} \to \hom(F_k, G).$$

The group G_ρ acts freely and properly on $G \times \mathcal{S}_{F_k}$ and $G \times_{G_\rho} \mathcal{S}_{F_k}$ is an analytic manifold. By the submersion theorem, Ψ defines a bi-analytic map between $V \subset G \times_{G_\rho} \mathcal{S}_{F_k}$ a neighborhood of the class $G_\rho \times \{0\}$ and U, the neighborhood of ρ. The image of Ψ is the the union of G-iterates $G \cdot U = \bigcup_{g \in G} g \cdot U$, which is open and saturated, and Ψ is locally bi-analytic by equivariance, in particular it is open. It remains to check that Ψ is injective: let $x, y \in G \times_{G_\rho} \mathcal{S}_{F_k}$ satisfy $\Psi(x) = \Psi(y)$. By G-equivariance we may assume $\Psi(x) = \Psi(y) \in U$. From the construction of Ψ as a a submersion, U is contained in the union of orbits of $\mathcal{S}_{F_k} \cap U$ and acting by G we may even assume that $\Psi(x) = \Psi(y) \in \mathcal{S}_{F_k} \cap U$. As $\Psi(x) \in \mathcal{S}_{F_k} \cap U$ and Ψ is locally by-analytic, x is the class of $e \times \Psi(x) \in G \times \mathcal{S}_{F_k}$ modulo G_ρ. Since $\Psi(x) = \Psi(y)$, $x = y$. This proves (iii) for a free group.

For a group with a finite presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_k \mid (r_j)_{j \in J} \rangle$, let F_k denote the group freely generated by $\gamma_1, \ldots, \gamma_k$. The projection $F_k \to \Gamma$ induces an inclusion of varieties of representations

$$\hom(\Gamma, G) \hookrightarrow \hom(F_k, G),$$

which in its turn induces an inclusion of Zariski tangent spaces

$$Z^1(\Gamma, \mathfrak{g}) \subset Z^1(F_k, \mathfrak{g}).$$

Let $\mathcal{S}_{F_k} \subset \hom(F_k, G)$ denote the slice for a free group constructed above. We consider the intersection

$$\mathcal{S}_\Gamma = \mathcal{S}_{F_k} \cap \hom(\Gamma, G).$$

It satisfies properties (i), (ii) and (iii) of a slice, in particular

$$T_\rho^{\mathrm{Zar}} \mathcal{S}_\Gamma = T_\rho^{\mathrm{Zar}} \mathcal{S}_{F_k} \cap Z^1(\Gamma, \mathfrak{g}) \cong H^1(\Gamma, \mathfrak{g}),$$

because $B^1(\Gamma, \mathfrak{g}) = B^1(F_n, \mathfrak{g})$.

We next construct a G_ρ equivariant bi-analytic equivalence between $\mathcal{S}_\Gamma$ and a germ $\mathcal{S}$ in $T_\rho^{\mathrm{Zar}} \mathcal{S}_\Gamma$. At the beginning of the proof for F_k, we chose $H \subset \mathfrak{g}^k$ a G_ρ-invariant complement to the space of coboundaries; again by semi-simplicity of G_ρ, we find G_ρ-invariant complement L:

$$H = T_\rho^{\mathrm{Zar}} \mathcal{S}_\Gamma \oplus L \tag{4}$$

Then the projection in (4) of $\mathcal{S}_\Gamma$ to $T_\rho^{\mathrm{Zar}} \mathcal{S}_\Gamma \cong H \cap Z^1(\Gamma, \mathfrak{g})$ is the required subset $\mathcal{S}$. By construction, this map is G_ρ-equivariant, we claim that it is a bi-analytic equivalence between

$\mathcal{S}_\Gamma$ and $\mathcal{S}$. When $\mathcal{S}_\Gamma$ is non-singular the claim is clear from standard arguments, because it is a projection to the tangent space of a non-singular subvariety. When $\mathcal{S}_\Gamma$ is singular, we embed $\mathcal{S}_\Gamma$ in an analytic subvariety M that it is non-singular and has the same Zariski tangent space at ρ. Namely, we use the argument in [23, Theorem V.A.14] to find a real analytic germ $\mathcal{S}_\Gamma \subset M \subset \mathcal{S}_{F_n}$ with $T_\rho \mathcal{S}_\Gamma = T_\rho M$. We notice that the projection defines a bi-analytic equivalent between M and the germ of $T_\rho \mathcal{S}_\Gamma = T_\rho M$ at ρ.

Next we prove uniqueness. Two presentations of Γ yield two surjections $p_1 \colon F_{k_1} \to \Gamma$ and $p_2 \colon F_{k_2} \to \Gamma$. Let $f \colon F_{k_1} \twoheadrightarrow F_{k_2}$ and $g \colon F_{k_2} \twoheadrightarrow F_{k_1}$ be lifts of the identity of Γ:

$$\begin{array}{ccc} F_{k_1} & \overset{g}{\underset{f}{\leftrightarrows}} & F_{k_2} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & \Gamma & \end{array}$$

Following the previous construction, this yields two analytic germs of subvarieties

$$\mathcal{S}_i \subseteq M_i \subset V_i \subset \mathfrak{g}^{k_i}, \qquad \text{for } i = 1, 2,$$

such that $T_0^{\mathrm{Zar}} \mathcal{S}_i = T_0^{\mathrm{Zar}} M_i \cong H^1(\Gamma, \mathfrak{g})$, with M_i non-singular and V_i an open neighborhood of 0. The induced maps $f^* \colon V_2 \to V_1$ and $g^* \colon V_1 \to V_2$ are analytic and satisfy

$$f^* \circ g^*|_{S_1} = \mathrm{Id}_{\mathcal{S}_1} \qquad g^* \circ f^*|_{\mathcal{S}_2} = \mathrm{Id}_{\mathcal{S}_2}.$$

Therefore the tangent morphisms df^* and dg^* satisfy

$$df^* \circ dg^*|_{T_0^{\mathrm{Zar}} \mathcal{S}_1} = \mathrm{Id}_{T_0^{\mathrm{Zar}} \mathcal{S}_1} \qquad dg^* \circ df^*|_{T_0^{\mathrm{Zar}} \mathcal{S}_2} = \mathrm{Id}_{T_0^{\mathrm{Zar}} \mathcal{S}_2}.$$

In particular $f^*(M_1)$ is a non-singular analytic germ in $\mathfrak{g}^{k_2}$ and we may chose $M_2 = f^*(M_1)$. Thus the pairs of germs at the origin $(\mathcal{S}_1, M_1)$ and $(\mathcal{S}_2, M_2)$ are equivalent, and uniqueness follows. □

Corollary 3.2 *Under the hypothesis of the theorem, a neighborhood of* $[\rho]$ *in* $X(\Gamma, G)$ *is homeomorphic to* $S//G_\rho$.

When G_ρ is trivial, eg when ρ is $\mathbb{C}$-irreducible, this provides the natural analytic structure in a neighborhood of the character of ρ.

3.2 Decomposing the Lie algebra

Before proving Theorem 1.1, we discuss some results related to the decomposition of the Lie algebra.

Let $\mathcal{O}^2$ be a compact and orientable 2-orbifold, with fundamental group $\Gamma = \pi_1(\mathcal{O}^2)$. Set $G_0 = \mathrm{SL}_n(\mathbb{R})$, $G = \mathrm{SL}_{n+1}(\mathbb{R})$, and view G_0 as a subgroup of G:

$$\begin{aligned} G_0 &\hookrightarrow G \\ A &\mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix} \end{aligned}$$

Let $\mathfrak{g}$ and $\mathfrak{g}_0$ denote the corresponding Lie algebras. We have a direct sum of G_0-modules

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbf{m}_\mathsf{c} \oplus \mathbf{m}_\mathsf{r} \oplus \mathbf{d} \tag{5}$$

where $\mathbf{m}_\mathsf{c}$ is the subspace of $\mathfrak{g}$ with vanishing entries away from the last column, $\mathbf{m}_\mathsf{r}$ away from the last row, and $\mathbf{d} \cong \mathbb{R}$ is the subspace of diagonal matrices that commute with every element in $\mathfrak{g}_0$. As G_0-modules, $\mathbf{m}_\mathsf{c} \cong \mathbb{R}^n_{G_0}$ (eg $\mathbb{R}^n$ as module by the standard action of G_0), $\mathbf{m}_\mathsf{r} = (\mathbf{m}_\mathsf{c})^*$ (the contragredient of $\mathbf{m}_\mathsf{c}$, which is also its dual), and $\mathbf{d} \cong \mathbb{R}$ is the trivial module.

In terms of elementary matrices, if $e_{i,j}$ denotes the elementary matrix with 1 in the i-th row and j-th column, and 0 in the other entries, then as vector spaces we have:

$$\mathfrak{g}_0 = \langle e_{i,j} \mid 1 \leq i, j \leq n\rangle \cap \mathfrak{g} \qquad \mathbf{m}_\mathsf{c} = \langle e_{i,n+1} \mid 1 \leq i \leq n\rangle$$
$$\mathbf{m}_\mathsf{r} = \langle e_{n+1,j} \mid 1 \leq j \leq n\rangle \qquad \mathbf{d} = \langle e_{1,1} + \cdots + e_{n,n} - n\, e_{n+1,n+1}\rangle$$

The G_0-modules $\mathfrak{g}_0$ and $\mathbf{d}$ are both self-dual: the Killing form on $\mathfrak{g}$ restricts to a non-degenerate form in both $\mathfrak{g}_0$ and $\mathbf{d}$ (a multiple of the Killing form of $\mathfrak{g}_0$, and a multiple of the isomorphism $\mathbf{d} \otimes \mathbf{d} \cong \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$).

The modules $\mathbf{m}_\mathsf{c}$ and $\mathbf{m}_\mathsf{r}$ are dual from each other. The Killing form B on $\mathfrak{g}$ restricts to a perfect pairing $B\colon \mathbf{m}_\mathsf{c} \times \mathbf{m}_\mathsf{r} \to \mathbb{R}$ which is G_0-invariant. In terms of representations, the adjoint action on $\mathfrak{g}$ induces the Γ-module structures:

$$\mathbf{m}_\mathsf{c} \cong \mathbb{R}^n_\rho \qquad \text{and} \qquad \mathbf{m}_\mathsf{r} \cong \mathbb{R}^n_{\rho^*}$$

where ρ^* denotes the contragredient representation, defined by $\rho^*(\gamma) = (\rho(\gamma)^{-1})^t, \forall \gamma \in \Gamma$. The restriction of the Killing form is also Γ-invariant perfect pairing $B\colon \mathbf{m}_\mathsf{r} \times \mathbf{m}_\mathsf{c} \to \mathbb{R}$, that can be seen as a multiple of canonical pairing $\mathbb{R}^n_\rho \times \mathbb{R}^n_{\rho^*} \to \mathbb{R}$, after the identifications $\mathbf{m}_\mathsf{c} \cong \mathbb{R}_\rho$ and $\mathbf{m}_\mathsf{r} \cong \mathbb{R}_{\rho^*}$. It induces two pairings in cohomology

$$H^1(\Gamma, \mathbf{m}_\mathsf{r}) \times H^1(\Gamma, \mathbf{m}_\mathsf{c}) \to H^2(\Gamma, \mathbb{R}),$$
$$H^1(\Gamma, \mathbf{m}_\mathsf{c}) \times H^1(\Gamma, \mathbf{m}_\mathsf{r}) \to H^2(\Gamma, \mathbb{R})$$

respectively defined at the level of 1-cocycles by

$$B(z_\mathsf{r} \cup z_\mathsf{c})(\gamma_1, \gamma_2) = z_\mathsf{r}(\gamma_1)^t \rho(\gamma_1) z_\mathsf{c}(\gamma_2),$$
$$B(z_\mathsf{c} \cup z_\mathsf{r})(\gamma_1, \gamma_2) = z_\mathsf{c}(\gamma_1)^t \rho^*(\gamma_1) z_\mathsf{r}(\gamma_2),$$

following (2). We remark two further properties.

Remark 3.3 (a) (Non-degeneracy.) When Γ is the fundamental group of a closed, orientable, and aspherical 2-orbifold $\mathcal{O}^2$, these pairings in cohomology are non-degenerate by Poincaré duality, stated in Theorem 2.14.
(b) (Skew-symmetry.) For $\theta_\mathsf{r} \in H^1(\Gamma, \mathbf{m}_\mathsf{r})$ and $\theta_\mathsf{c} \in H^1(\Gamma, \mathbf{m}_\mathsf{c})$, $B(\theta_\mathsf{c} \cup \theta_\mathsf{r}) = -B(\theta_\mathsf{r} \cup \theta_\mathsf{c})$, by Lemma 2.9.

In the formulation of the next lemma we require

$$\pi_\mathsf{r} : \mathfrak{g} \to \mathbf{m}_\mathsf{r} \quad \text{and} \quad \pi_\mathsf{c} : \mathfrak{g} \to \mathbf{m}_\mathsf{c} \tag{6}$$

the projections from the direct sum (5).

Lemma 3.4 *Let $\mathcal{O}^2$ be a compact orientable and hyperbolic 2-orbifold, and $\Gamma = \pi_1(\mathcal{O}^2)$. For a $\mathbb{C}$-irreducible representation $\rho\colon \Gamma \to G_0$:*

(i) If $\partial\mathcal{O}^2 \neq \emptyset$, then $H^2(\Gamma, \mathfrak{g}) = 0$.
(ii) If $\partial\mathcal{O}^2 = \emptyset$, then $H^2(\Gamma, \mathfrak{g}) = H^2(\Gamma, \boldsymbol{d}) \cong H^2(\Gamma, \mathbb{R}) = \mathbb{R}$.
In addition, for $\theta \in H^1(\Gamma, \mathfrak{g})$,

$$[\theta \cup \theta] = c_n\, B(\pi_r^*(\theta) \cup \pi_c^*(\theta)) = -c_n\, B(\pi_c^*(\theta) \cup \pi_r^*(\theta))$$

for some constant $c_n \neq 0$ depending only on the dimension n.

Proof (i) When $\partial\mathcal{O}^2 \neq \emptyset$, then $\mathcal{O}^2$ has virtually the homotopy type of a graph and therefore $H^2(\Gamma, \mathfrak{g}) \cong H^2(\mathcal{O}^2, \mathfrak{g}) = 0$.

(ii) When $\partial\mathcal{O}^2 = \emptyset$, since ρ is $\mathbb{C}$-irreducible in G_0 the invariant subspaces $(\mathfrak{g}_0)^{\mathrm{Ad}\rho(\Gamma)}$, $(\mathbf{m}_\mathsf{c})^{\mathrm{Ad}\rho(\Gamma)}$ and $(\mathbf{m}_\mathsf{v})^{\mathrm{Ad}\rho(\Gamma)}$ are trivial. In addition, using Poincaré duality and the isomorphism between the cohomology of $\mathcal{O}^2$ and $\Gamma = \pi_1(\mathcal{O}^2)$, we deduce $H^2(\Gamma, \mathfrak{g}_0) \cong H^0(\Gamma, \mathfrak{g}_0)^* \cong 0$, $H^2(\Gamma, \mathbf{m}_\mathsf{r}) \cong H^0(\Gamma, \mathbf{m}_\mathsf{r})^* \cong 0$, and $H^2(\Gamma, \mathbf{m}_\mathsf{c}) \cong H^0(\Gamma, \mathbf{m}_\mathsf{c})^* \cong 0$. Then the first assertion in (ii) follows from the direct sum (5).

From the vanishing of $H^2(\Gamma, \mathfrak{g}_0)$, $H^2(\Gamma, \mathbf{m}_\mathsf{r})$ and $H^2(\Gamma, \mathbf{m}_\mathsf{c})$, the only relevant term in $[\theta \cup \theta]$ comes from the product

$$[\cdot, \cdot] \colon (\mathbf{m}_\mathsf{r} \oplus \mathbf{m}_\mathsf{c}) \times (\mathbf{m}_\mathsf{r} \oplus \mathbf{m}_\mathsf{c}) \to \mathbf{d}.$$

This motivates the following computation. For $\gamma_1, \gamma_2 \in \Gamma$, $z_\mathsf{r} \in Z^1(\Gamma, \mathbf{m}_\mathsf{r})$, and $z_\mathsf{c} \in Z^1(\Gamma, \mathbf{m}_\mathsf{c})$,

$$\left[\begin{pmatrix} 0 & z_\mathsf{c}(\gamma_1) \\ z_\mathsf{r}^t(\gamma_1) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \rho(\gamma_1) z_\mathsf{c}(\gamma_2) \\ z_\mathsf{r}^t(\gamma_2)\rho(\gamma_1^{-1}) & 0 \end{pmatrix}\right] =$$
$$\begin{pmatrix} * & 0 \\ 0 & z_\mathsf{r}^t(\gamma_1)\rho(\gamma_1) z_\mathsf{c}(\gamma_2) - z_\mathsf{r}^t(\gamma_2)\rho(\gamma_1^{-1}) z_\mathsf{c}(\gamma_1) \end{pmatrix} =$$
$$\begin{pmatrix} * & 0 \\ 0 & z_\mathsf{r}^t(\gamma_1)\big(\rho(\gamma_1) z_\mathsf{c}(\gamma_2)\big) - z_\mathsf{c}^t(\gamma_1)\big(\rho^*(\gamma_1) z_\mathsf{r}(\gamma_2)\big) \end{pmatrix}.$$

The assertion follows from this formula, Remark 3.3, and Lemma 2.9. □

3.3 Proof of Theorem 1.1

The proof is based on the slice of Section 3.1 and the results on the Lie algebra and cup products in Section 3.2.

Proof of Theorem 1.1 We aim to apply Theorem 3.1, so we describe the stabilizer of ρ in G:

$$G_\rho = \{\mathrm{diag}(t, \ldots, t, t^{-n}) \mid t \in \mathbb{R} - \{0\}.\}$$

The identity component of G_ρ is $\exp(\mathbf{d}) = \{\mathrm{diag}(e^\lambda, \ldots, e^\lambda, e^{-n\lambda}) \mid \lambda \in \mathbb{R}\}$. The group G_ρ is generated by its identity component and the matrix

$$\begin{cases} \mathrm{diag}(-1, \ldots, -1, -1) & \text{if } n+1 \text{ is even,} \\ \mathrm{diag}(-1, \ldots, -1, +1) & \text{if } n+1 \text{ is odd.} \end{cases} \tag{7}$$

The stabilizer G_ρ preserves the decomposition (5) and acts trivially on $\mathfrak{g}_0$ and $\mathbf{d}$. A matrix $\mathrm{diag}(t, \ldots, t, t^{-n})$ acts by multiplication by t^{n+1} on $\mathbf{m}_\mathsf{c}$, and by $t^{-(n+1)}$ on $\mathbf{m}_\mathsf{r}$. In particular we must take into account the parity of $n+1$, as the matrix in (7) acts nontrivially on $\mathbf{m}_\mathsf{c}$ and $\mathbf{m}_\mathsf{r}$ iff $n+1$ is odd. It follows that G_ρ acts semi-simply on $\mathfrak{g}$.

The induced action in cohomology of G_ρ preserves the decomposition

$$H^1(\Gamma, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}_0) \oplus H^1(\Gamma, \mathbf{m}_\mathsf{c}) \oplus H^1(\Gamma, \mathbf{m}_\mathsf{r}) \oplus H^1(\Gamma, \mathbf{d})$$

and the action on each cohomology group is the same as in the corresponding coefficients. We set $p = \dim H^1(\Gamma, \mathfrak{g}_0)$, $d = \dim H^1(\Gamma, \mathbb{R}^n_\rho) = \dim H^1(\Gamma, \mathbf{m}_\mathsf{c})$, and $b = \dim H^1(\Gamma, \mathbf{d}) = \dim H^1(\Gamma, \mathbb{R})$.

Let $\mathcal{S}_\Gamma \subset \hom(\Gamma, G)$ be the slice of Theorem 3.1, so a neighborhood of $[\rho]$ is homeomorphic to the germ $\mathcal{S}_\Gamma//G_\rho$. To describe $\mathcal{S}_\Gamma$ we distinguish two cases. Assume first that $\partial\mathcal{O}^2 \neq \emptyset$, so $\hom(\Gamma, G)$ is smooth at ρ, because $H^2(\mathcal{O}^2, \mathfrak{g}) = 0$ (Proposition 2.15). So by Theorem 3.1 the germ of $\mathcal{S}_\Gamma$ at ρ is G_ρ-equivariantly by-analytic to the germ at the origin of $T_0^{\mathrm{Zar}}\mathcal{S}_\Gamma = H^1(\Gamma, \mathfrak{g})$. Thus, when $n+1$ is even, a neighborhood of ρ is homeomorphic to $U//\mathbb{R}$ where

$$U \subset \mathbb{R}^{p+b} \oplus (\mathbb{R}^d \oplus \mathbb{R}^d)$$

is a neighborhood of the origin and the action of $t \in \mathbb{R}$ on $(x, y, z) \in U \subset \mathbb{R}^{p+b} \oplus \mathbb{R}^d \oplus \mathbb{R}^d$ is given by $(x, y, z) \mapsto (x, e^t y, e^{-t} z)$. When $n+1$ is odd, we need to further quotient by the action of the antipodal on $\mathbb{R}^d \oplus \mathbb{R}^d$.

Define

$$\mathcal{M} = \{(x, y, z) \in \mathbb{R}^{p+b} \oplus \mathbb{R}^d \oplus \mathbb{R}^d \mid |y| = |z|\}. \tag{8}$$

Then:

$$U//\mathbb{R} \cong \mathcal{M} \cap U$$

because $\mathcal{M}$ intersects precisely once each closed orbit (at the point that minimizes the distance to $\mathbb{R}^{p+b} \times \{0, 0\}$, eg an elementary example of Kempf-Ness). Therefore a neighborhood of the character is homeomorphic to

$$\begin{cases} \mathcal{M} \cap U & \text{when } n+1 \text{ is even} \\ \mathcal{M} \cap U/\sim & \text{when } n+1 \text{ is odd,} \end{cases}$$

where $\sim$ denotes the antipodal relation on the last $2d$ coordinates.

Next we deal with the case $\partial\mathcal{O}^2 = \emptyset$. Here we use Theorem 2.17, the result of [9, 11, 25] on quadratic singularities. We describe a neighborhood using a power expansion. Fix a norm in $\mathfrak{g}$, say $|a| = \mathrm{trace}(a^t a)$, hence a norm in $\mathfrak{g}^k$ and in $Z^1(\Gamma, \mathfrak{g})$. By Theorem 2.17, a neighborhood $V \subset \hom(\Gamma, G)$ of ρ can be described as

$$V \cong \{\varepsilon z + O(\varepsilon^2) \mid \varepsilon \in [0, \varepsilon_0),\ z \in Z^1(\Gamma, \mathfrak{g}) \mid [z \cup z] \sim 0, |z| = 1\}. \tag{9}$$

Now we follow the construction of the slice $\mathcal{S}$ in the proof of Theorem 3.1, in particular we have a surjection of the free group of rank k, $F_k \twoheadrightarrow \Gamma$, an inclusion of varieties of representations $\hom(\Gamma, G) \subset \hom(F_k, G)$, and a map $\Phi\colon \hom(F_k, G) \to \mathfrak{g}^k \cong Z^1(F_k, \mathfrak{g}) \cong T_\rho^{\mathrm{Zar}} \hom(F_k, G)$ in (3) by-analytic in a neighborhood of ρ. The proof of Theorem 3.1 considers $H \subset Z^1(F_k, \mathfrak{g})$ a G_ρ-invariant complement to $B^1(F_k, \mathfrak{g})$ and define the slice in $\hom(F_k, G)$ as a neighborhood in $\Phi^{-1}(H)$. The slice in $\hom(\Gamma, G)$ is its trace in $\hom(\Gamma, G)$: $S_\Gamma = \Phi^{-1}(H) \cap V$, where V is as in (9). Thus

$$\mathcal{S}_\Gamma \cong \{\varepsilon z + O(\varepsilon^2) \mid \varepsilon \in [0, \varepsilon_0),\ z \in H \cap Z^1(\Gamma, \mathfrak{g}) \mid [z \cup z] \sim 0, |z| = 1\}.$$

Since $H \cap Z^1(\Gamma, \mathfrak{g}) \cong H^1(\Gamma, \mathfrak{g})$, and $\mathcal{S}$ is a projection of $\mathcal{S}_\Gamma$ to $H \cap Z^1(\Gamma, \mathfrak{g})$:

$$\mathcal{S} \cong \{\varepsilon\theta + O(\varepsilon^2) \mid \varepsilon \in [0, \varepsilon_0),\ \theta \in H^1(\Gamma, \mathfrak{g}) \mid [\theta \cup \theta] = 0, |\theta| = 1\}.$$

Therefore if $\mathcal{M} \subset H^1(\Gamma, \mathfrak{g})$ is a minimal set as in (8) and $(G_\rho)_0 \cong \mathbb{R}^*$ denotes the identity component of G_ρ:

$$\mathcal{S}//(G_\rho)_0 \cong \mathcal{S} \cap \mathcal{M} = \{\varepsilon\theta + O(\varepsilon^2) \mid \varepsilon \in [0, \varepsilon_0),\ \theta \in \mathcal{M} \mid [\theta \cup \theta] \sim 0, |\theta| = 1\}.$$

So $\mathcal{S} \cap \mathcal{M}$ is homeomorphic to the cone on the subset of the unit sphere in $H^1(\Gamma, \mathfrak{g})$ defined by $\mathcal{M}$ and the vanishing of the cup product. Notice that both the equation defining $\mathcal{M}$ and the cup product are trivial on the spaces $H^1(\Gamma, \mathfrak{g}_0)$ and $H^1(\Gamma, \mathbf{d})$. Hence, using the definition of $\mathcal{M}$ and Lemma 3.4, the neighborhood in $X(\Gamma, G)$ is homeomorphic to a neighborhood of the origin in

$$H^1(\Gamma, \mathfrak{g}_0) \oplus H^1(\Gamma, \mathbf{d}) \oplus \begin{cases} \mathcal{C} & \text{when } n+1 \text{ is even} \\ \mathcal{C}/\sim & \text{when } n+1 \text{ is odd,} \end{cases}$$

where $\mathcal{C}$ is the cone

$$\mathcal{C} = \{(\theta_{\mathsf{c}}, \theta_{\mathsf{r}}) \in H^1(\Gamma, \mathbf{m}_{\mathsf{c}}) \oplus H^1(\Gamma, \mathbf{m}_{\mathsf{r}}) \mid |\theta_{\mathsf{c}}|^2 = |\theta_{\mathsf{r}}|^2 \text{ and } \theta_{\mathsf{c}} \cdot \theta_{\mathsf{r}} = 0\}.$$

The cup product

$$H^1(\Gamma, \mathbf{m}_{\mathsf{c}}) \times H^1(\Gamma, \mathbf{m}_{\mathsf{r}}) \to H^2(\Gamma, \mathbf{d}) \cong \mathbb{R}$$

is a perfect pairing, by Theorem 2.14. Then using elementary linear algebra to combine the norms with the scalar product, we have that

$$\mathcal{C} \cong \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |x|^2 = |y|^2,\ x \cdot y = 0\} \cong \mathrm{Cone}(UT(S^{d-1})),$$

and the theorem follows. □

3.4 Non-orientable orbifolds

Next we discuss *non orientable* 2-orbifolds. Instead of working with $\mathrm{SL}_n(\mathbb{R})$, for non-orientable orbifolds it makes sense to work with representations of its fundamental group in

$$\mathrm{SL}_n^{\pm}(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = \pm 1\}.$$

Definition 3.5 A representation $\rho\colon \pi_1(\mathcal{O}^2) \to \mathrm{SL}_n^{\pm}(\mathbb{R})$ is called *type preserving* if it maps orientation preserving elements in $\pi_1(\mathcal{O}^2)$ to matrices of determinant 1 and orientation reversing elements to matrices of determinant -1.

We consider two natural embeddings of $\mathrm{SL}_n^{\pm}(\mathbb{R})$:

- *Orientable embedding.* We view every projective transformation of $\mathbb{P}^{n-1}$ as an orientation preserving transformation of $\mathbb{P}^n$:

$$\begin{aligned} \mathrm{SL}_n^{\pm}(\mathbb{R}) &\hookrightarrow \mathrm{SL}_{n+1}(\mathbb{R}) \\ A &\mapsto \begin{pmatrix} A & \\ & \det(A) \end{pmatrix} \end{aligned}$$

- *Type preserving.* An element preserves the orientation on $\mathbb{P}^{n-1}$ if, and only if, it preserves the orientation on $\mathbb{P}^n$:

$$\begin{aligned} \mathrm{SL}_n^{\pm}(\mathbb{R}) &\hookrightarrow \mathrm{SL}_{n+1}^{\pm}(\mathbb{R}) \\ A &\mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix} \end{aligned}$$

To state the theorem, we introduce the following notation:

- $d_{\mathrm{tp}} = \dim H^1(\mathcal{O}^2, \mathbb{R}^n_\rho)$, and
- $d_{\mathrm{oe}} = \dim H^1(\mathcal{O}^2, \mathbb{R}^n_{\rho\otimes\alpha})$, where $\alpha(\gamma) = \det \rho(\gamma) \in \{\pm 1\}$ for $\alpha \in \Gamma$.

In the type preserving case, we work with d_{tp} and in the orientable embedding, with d_{oe}.

Theorem 3.6 *Let $\mathcal{O}^2$ be a compact, non-orientable 2-orbifold with $\chi(\mathcal{O}^2) < 0$, let $\rho\colon \pi_1(\mathcal{O}^2) \to SL_n^{\pm}(\mathbb{R})$ be a type preserving representation, and let $d_{oe} \geq 0$ and $d_{tp} \geq 0$ be as above. Assume that ρ restricted to the orientation covering of $\mathcal{O}^2$ is $\mathbb{C}$-irreducible.*

(i) For the orientable embedding, a neighborhood of the character of ρ *in* $X(\mathcal{O}^2, SL_{n+1}(\mathbb{R}))$ *is homeomorphic to*

$$\mathbb{R}^p \times \mathbb{R}^b \times \begin{cases} Cone(S^{d_{oe}-1} \times S^{d_{oe}-1}) & \text{if } n+1 \text{ is even} \\ Cone(S^{d_{oe}-1} \times S^{d_{oe}-1})/\sim & \text{if } n+1 \text{ is odd} \end{cases}$$

(ii) For the type preserving embedding, a neighborhood of the character of ρ *in* $X(\mathcal{O}^2, SL^{\pm}_{n+1}(\mathbb{R}))$ *is homeomorphic to*

$$\mathbb{R}^p \times \mathbb{R}^b \times Cone(S^{d_{tp}-1} \times S^{d_{tp}-1})/\sim$$

In both cases $X(\mathcal{O}^2, SL^{\pm}_n(\mathbb{R}))$ *corresponds to* $\mathbb{R}^p \times \{0\} \times \{0\}$*, and* $\sim$ *denotes the action of the antipodal map in* $S^{d-1} \times S^{d-1} \subset S^{2d-1} \subset \mathbb{R}^{2d}$.

The theorem has the same proof as Theorem 1.1 in the orientable case with some minor changes. Notice that by Lemma 3.7 below, $H^2(\mathcal{O}^2, \mathbb{R}) = 0$, so there is no obstruction to integrability and we are in the same situation as when the orientable orbifold has boundary. The minor changes in the proof depend on the kind of embedding:

- For the orientable embedding, in the $\pi_1(\mathcal{O}^2)$-module $\mathbb{R}^n$, instead of the action of ρ, we consider the action of $\rho \otimes \alpha$.
- In the orientable embedding we have to care about the parity of $n+1$, but in the type preserving embedding not, because the group $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ always contains the matrix $\mathrm{diag}(-1, \ldots, -1, 1)$.

Lemma 3.7 *Let* $\mathcal{O}^2$ *and* ρ *be as in the theorem. Then* $H^2(\mathcal{O}^2, \mathfrak{g}) = 0$.

Proof of the lemma Let $\widetilde{\mathcal{O}^2}$ denote the orientation covering of $\mathcal{O}^2$, the cohomology of $\mathcal{O}^2$ is isomorphic to the invariant subspace of the cohomology of $\widetilde{\mathcal{O}^2}$ invariant by the action of $\mathbb{Z}/2\mathbb{Z}$, the group of deck transformations: $H^2(\mathcal{O}^2, \mathfrak{g}) \cong H^2(\widetilde{\mathcal{O}^2}, \mathfrak{g})^{\mathbb{Z}/2\mathbb{Z}}$. By Lemma 3.4, $H^2(\widetilde{\mathcal{O}^2}, \mathfrak{g}) = H^2(\widetilde{\mathcal{O}^2}, \mathbf{d}) \cong H^2(\widetilde{\mathcal{O}^2}, \mathbb{R}) \cong \mathbb{R}$, and $\mathbb{Z}/2\mathbb{Z}$ acts by change of sign on $\mathbb{R}$, so the invariant subspace is trivial (by the same reason why $H^2(\mathcal{O}^2, \mathbb{R}) = 0$). □

4 Convex projective 2-orbifolds

Let $\mathcal{O}^2$ be a compact 2-orbifold with negative Euler characteristic, hence hyperbolic. The holonomy representation of its convex projective structure lies in $\mathrm{PGL}_3(\mathbb{R})$, and by Choi-Goldman [6] in lies in the same component in the variety of representations as the holonomy of its hyperbolic structure. The holonomy of a hyperbolic structure in $\mathrm{PO}(2,1)$ lifts to $\mathrm{O}(2,1)$, thus:

Remark 4.1 The projective holonomy of a convex projective structure on a 2-orbifold lifts to $\mathrm{SL}^{\pm}_3(\mathbb{R})$, or $\mathrm{SL}_3(\mathbb{R})$ when it is orientable.

Irreducibility over $\mathbb{R}$ of the holonomy is well known. Since 3 is odd, we also have:

Remark 4.2 The holonomy of a convex projective structure of $\mathcal{O}^3$ in $\mathrm{SL}^{\pm}_3(\mathbb{R})$ is $\mathbb{C}$-irreducible.

4.1 Orientable projective 2-orbifolds

For $\mathcal{O}^2$ *orientable* and compact, we consider the following spaces:

- The Teichmüller space $\mathrm{Teich}(\mathcal{O}^2) \cong \mathbb{R}^t$, which is an open set in a component of the variety of characters, $X_0(\mathcal{O}^2, \mathrm{SO}(2,1))$ (the whole component when $\mathcal{O}^2$ is closed).
- The space of convex projective structures $\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}^2)$, which is homeomorphic to a cell $\mathbb{R}^p$, as proved by Choi-Goldman [6]. The projective structures are required to have principal geodesic boundary components, so $\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}^2)$ is not the whole component of characters $X_0(\mathcal{O}^2, \mathrm{SL}_3(\mathbb{R}))$, but an open subset (see Remark 4.1 regarding the lift from PSL to SL).

For $\mathcal{O}^2$ compact and orientable, since the standard representation of $\mathrm{SO}(2,1)$ on $\mathbb{R}^3$ is equivalent to the adjoint representation in $\mathfrak{so}(2,1)$:

$$t = \dim(\mathrm{Teich}(\mathcal{O}^2)) = \dim(H^1(\mathcal{O}^2, \mathbb{R}^3_\rho)) = -3\chi(|\mathcal{O}^2|) + 2c(\mathcal{O}^2), \tag{10}$$

where $|\mathcal{O}^2|$ denotes the underlying surface of the orbifold and $c(\mathcal{O}^2)$ is the number of cone points of $\mathcal{O}^2$. See [26] or [20]. Theorem 1.1 immediately yields:

Corollary 4.3 *Let $\mathcal{O}^2$ be a compact orientable 2-orbifold, with negative Euler characteristic and a convex projective structure. Then a neighborhood of the character of its projective holonomy in $X(\mathcal{O}^2, SL_4(\mathbb{R}))$ is homeomorphic to*

$$\begin{cases} \mathbb{R}^{p+b} \times Cone(UT(S^{t-1})) & \text{if } \mathcal{O}^2 \text{ is closed} \\ \mathbb{R}^{p+b} \times Cone(S^{t-1} \times S^{t-1}) & \text{if } \mathcal{O}^2 \text{ has boundary} \end{cases}$$

where $t = \dim(Teich(\mathcal{O}^2))$, $p = \dim(Proj_{cvx}(\mathcal{O}^2))$ and $t = \dim(H^1(\mathcal{O}^2, \mathbb{R}))$.

Notice that for $\mathcal{O}^2$ compact, orientable and hyperbolic, by (10) $t = \dim(\mathrm{Teich}(\mathcal{O}^2)) = 0$ if and only if $\mathcal{O}^2$ is a turnover (a 2-sphere with 3 cone points). In addition, for a turnover $H^1(\mathcal{O}^2, \mathbb{R}) = 0$. So in this case we have:

Corollary 4.4 *Let $\mathcal{O}^2$ be a turnover, every deformation in $SL_4(\mathbb{R})$ of its projective holonomy is conjugate to a deformation in $SL_3(\mathbb{R})$.*

By [6], the space $\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}^2)$ may be non-trivial for a turnover. In fact, if all cone points of $\mathcal{O}^2$ have order at least three, then $\dim(\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}^2)) = 2$. If one of the cone points has order two, then $\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}^2)$ is a point. By hyperbolicity, at most one of the cone points of the turnover has order 2.

4.2 Non-orientable projective 2-orbifolds

We follow the notation of Subsection 3.4. In particular let $\alpha\colon \pi_1(\mathcal{O}^2) \to \{\pm 1\}$ be the orientation representation: $\alpha(\gamma) = 1$ if γ preserves the orientation and $\alpha(\gamma) = -1$ if γ reverses it.

The group of isometries of hyperbolic plane is identified to $\mathrm{O}_0(2,1)$, the components of the group of Lorentz transformations of $\mathbb{R}^2_1$ that preserve each leaf of the hyperboloid.

Lemma 4.5 *Let $\rho\colon \pi_1(\mathcal{O}^2) \to O_0(2,1)$ be the holonomy of a hyperbolic structure. Then the representation $\rho \otimes \alpha$ is equivalent to the adjoint representation, acting on $\mathfrak{so}(2,1)$.*

Proof We have isomorphisms of Lie groups

$$\mathrm{O}_0(2,1) \cong \mathrm{PO}_0(2,1) \cong \mathrm{PO}(2,1) \cong \mathrm{PSO}(2,1) \cong \mathrm{SO}(2,1).$$

Then the lemma follows from the following two assertions:

a) The isomorphism $\mathrm{O}_0(2,1) \cong \mathrm{SO}(2,1)$ consists in multiplying each matrix by its determinant. Notice also that $\alpha(\gamma) = \det(\rho(\alpha))$.
b) The tautological representation of $\mathrm{SO}(2,1)$ is equivalent to its adjoint action on $\mathfrak{so}(2,1)$.

Assertion (a) is easily checked by looking at the action on Lorentz space. Assertion (b) is "well known", and follows for instance from the classification of representations of $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(3,\mathbb{C})$. □

We view the holonomy of projective structures in $\mathrm{SL}_3^{\pm}(\mathbb{R})$, as deformations of the hyperbolic holonomy in $\mathrm{O}_0(2,1)$, in particular type preserving (Definition 3.5). Recall from Subsection 3.4 that

$$d_{\mathrm{tp}} = \dim H^1(\mathcal{O}^2, \mathbb{R}^n_{\rho}) \qquad \text{and} \qquad d_{\mathrm{oe}} = \dim H^1(\mathcal{O}^2, \mathbb{R}^n_{\rho\otimes\alpha}).$$

Lemma 4.6 *Let $\mathcal{O}^2$ be a compact non-orientable hyperbolic 2-orbifold. Let $\rho\colon \pi_1(\mathcal{O}^2) \to SL_3^{\pm}(\mathbb{R})$ be the holonomy of a convex projective structure. Then:*

(a) $H^i(\mathcal{O}^2, \mathbb{R}^n_{\rho}) = H^i(\mathcal{O}^2, \mathbb{R}^n_{\rho\otimes\alpha}) = 0$, *for* $i = 0, 2$.
(b) $d_{oe} = \dim(Teich(\mathcal{O}^2))$.
(c) $d_{tp} = \dim(Teich(\mathcal{O}^2)) - \mathsf{f}(\partial\mathcal{O}^2)$.

where $\mathsf{f}(\partial\mathcal{O}^2)$ denotes the number of components of the boundary that are full 1-orbifolds.

Definition 4.7 A *full* 1-orbifold is the quotient of the real line by the group generated by two different reflections.

A full 1-orbifold is closed and non-orientable. Its underlying space is an interval and the isotropy group of the endpoints of the interval is the cyclic group with two elements (generated by a reflection on the line). It is the quotient of the circle by a non-orientable involution.

Corollary 4.8 *Let $\mathcal{O}^2$ be a compact non-orientable 2-orbifold, with negative Euler characteristic and a convex projective structure.*

(i) For the orientable embedding, a neighborhood of the character of its projective holonomy in $X(\mathcal{O}^2, SL_4(\mathbb{R}))$ is homeomorphic to

$$\mathbb{R}^{p+b} \times Cone(S^{t-1} \times S^{t-1}).$$

(ii) For the type preserving embedding, a neighborhood of the character of its projective holonomy in $X(\mathcal{O}^2, SL_4^{\pm}(\mathbb{R}))$ is homeomorphic to

$$\mathbb{R}^{p+b} \times Cone(S^{t-\mathsf{f}-1} \times S^{t-\mathsf{f}-1})/\sim .$$

Here $t = \dim(Teich(\mathcal{O}^2))$, $p = \dim(Proj_{cvx}(\mathcal{O}^2))$, $b = \dim(H^1(\mathcal{O}^2, \mathbb{R}))$, and $\mathsf{f} = \mathsf{f}(\partial\mathcal{O}^2)$.

Proof of Lemma 4.6 Assertion (a) uses the fact that the cohomology of $\mathcal{O}^2$ is the invariant cohomology of its orientation covering, which is also a projective orbifold. Both ρ and $\rho\otimes\alpha$ restrict to the projective holonomy of the orientation covering, for which the 0th and 2nd cohomology group with coefficients in $\mathbb{R}^3_{\rho}$ vanish, as discussed in Section 3.2.

To prove the other statements, we use Proposition 3.2 of [20]. Namely, if K is a finite cell decomposition of $\mathcal{O}^2$, so that the isotropy group $\mathrm{Stab}(\tilde{e})$ is constant on each cell of K (here $\tilde{e}$ is a lift of e to the universal covering), then

$$\sum_i (-1)^i \dim H^i(\mathcal{O}^2, V) = \sum_{e \text{ cell of } K} (-1)^{\dim e} \dim V^{\mathrm{Stab}(\tilde{e})}. \tag{11}$$

Assertion (b) holds true for $\rho\colon \pi_1(\mathcal{O}^2) \to \mathrm{O}_0(2,1)$ the holonomy of a hyperbolic structure, because by Lemma 4.5 $\mathrm{Ad}\rho$ is equivalent to $\rho \otimes \alpha$. Then the equality of dimensions holds true in the whole connected component of the variety of representations by (a) and (11), because the conjugacy class of the image of a finite order element does not change along a connected component.

To prove (c), we shall compute the difference $d_{\mathrm{oe}} - d_{\mathrm{tp}}$. We aim to apply Assertion (a) and Equality (11), so we compute difference

$$\dim\left((\mathbb{R}^3_\rho)^{\mathrm{Stab}(\tilde{e})}\right) - \dim\left((\mathbb{R}^3_{\rho\otimes\alpha})^{\mathrm{Stab}(\tilde{e})}\right).$$

for each cell e of K. When the isotropy group of e preserves the orientation, then α is trivial on $\mathrm{Stab}(\tilde{e})$ and this difference vanishes.

When the stabilizer of a cell is non-orientable, it is a group generated by reflections, either $\mathbb{Z}/2\mathbb{Z}$ or a dihedral group D_{2n}. For $\mathbb{Z}/2\mathbb{Z}$ the images are generated by:

$$\rho(\mathbb{Z}/2\mathbb{Z}) = \langle \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \rangle \quad \text{and} \quad (\rho\otimes\alpha)(\mathbb{Z}/2\mathbb{Z}) = \langle \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \rangle.$$

Hence the dimension of the invariant spaces are

$$\dim(\mathbb{R}^3_\rho)^{\mathbb{Z}/2\mathbb{Z}} = 2 \quad \text{and} \quad \dim(\mathbb{R}^3_{\rho\otimes\alpha})^{\mathbb{Z}/2\mathbb{Z}} = 1.$$

For the dihedral group D_{2n}, the generators of the image are

$$\rho(D_{2n}) = \langle \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix} \rangle$$

$$(\rho\otimes\alpha)(D_{2n}) = \langle \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix} \rangle$$

and the dimension of the invariant spaces are

$$\dim(\mathbb{R}^3_\rho)^{D_{2n}} = 1 \quad \text{and} \quad \dim(\mathbb{R}^3_{\rho\otimes\alpha})^{D_{2n}} = 0.$$

Summarizing:

$$\dim(\mathbb{R}^3_\rho)^{\mathrm{Stab}(\tilde{e})} - \dim(\mathbb{R}^3_{\rho\otimes\alpha})^{\mathrm{Stab}(\tilde{e})} = \begin{cases} 0 & \text{if } \mathrm{Stab}(\tilde{e}) \text{ is or. pres.} \\ +1 & \text{if } \mathrm{Stab}(\tilde{e}) \text{ is not o. p.} \end{cases} \tag{12}$$

Let $\mathsf{nops}_i(K)$ denote the number of i-cells of K with non-orientation preserving stabilizer. Notice that all cells with non-orientation preserving stabilizer lie in the boundary of the underlying space of the orbifold $\partial|\mathcal{O}^2|$ (a subset of the boundary of the orbifold $\partial\mathcal{O}^2$), that is a union of circles. By looking at the contribution of the full orbifolds we may deduce:

$$\mathsf{f}(\partial\mathcal{O}^2) = \mathsf{nops}_0(K) - \mathsf{nops}_1(K). \tag{13}$$

Finally, using Equality (a) and Equalities (11), (12), and (13):

$$\begin{aligned} d_{\mathrm{oe}} - d_{\mathrm{tp}} &= \dim H^1(\mathcal{O}^2, \mathbb{R}^n_{\rho\otimes\alpha}) - \dim H^1(\mathcal{O}^2, \mathbb{R}^n_\rho) \\ &= \mathsf{nops}_0(K) - \mathsf{nops}_1(K) = \mathsf{f}(\partial\mathcal{O}^2). \end{aligned}$$

This finishes the proof of the lemma. □

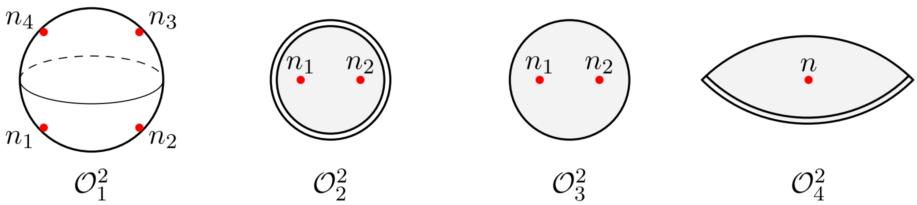

Fig. 1 The four examples in Section 4.3. The orbifold $\mathcal{O}_1^2$ is the orientation covering of $\mathcal{O}_2^2$, and $\mathcal{O}_3^2$, of $\mathcal{O}_4^2$ (for a suitable choice of labels)

Remark 4.9 The difference of dimensions also occurs when we consider both possible embedding of $\mathrm{Isom}(\mathbb{H}^2)$ in either $\mathrm{Isom}^+(\mathbb{H}^3)$ (orientation embedding) or $\mathrm{Isom}(\mathbb{H}^3)$ (type preserving), according to whether a non-orientable isometry of $\mathbb{H}^3$ extends to the unique orientable or non-orientable isometry of $\mathbb{H}^3$. For instance, consider a Fuchsian group $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ generated by a rotation of finite order and a reflection along a geodesic, $\mathcal{O}_4^2$ in Figure 1. Its deformation space as Fuchsian group has dimension one, the parameter being the distance between the fixed point of the rotation to the fixed line of the reflection, both in $\mathbb{H}^2$:

(i) The type preserving embedding Γ_{tp} is a group generated by a rotation and a reflection on a plane in $\mathbb{H}^3$. The deformation space of Γ_{tp} has again dimension one, as the relative position between a plane and a line in generic position is determined by their distance.
(ii) The orientation embedding Γ_{oe} is a group generated by two rotations along two axes, one of them of order two (the reflection in $\mathbb{H}^2$ extends to a rotation of order two in $\mathbb{H}^3$). The relative position between two lines in $\mathbb{H}^3$ is determined by two parameters, hence the deformation space as quasi-Fuchsian group has dimension 2.

Notice that in this case $\mathsf{f}(\partial\mathcal{O}^2) = 1$, it is Example 4.13 below ($\mathcal{O}_4^2$ in Figure 1).

4.3 Examples

In this section we describe the deformation spaces for four 2-orbifolds pictured in Figure 1. They have small deformation spaces and the topology of the singularities is easy to describe.

Example 4.10 Consider a sphere with 4 cone points $\mathcal{O}_1^2 = S^2(n_1, n_2, n_3, n_4)$. Since $\mathcal{O}_1^2$ is hyperbolic, at least one of the n_i is ≥ 2. The dimension of the space of convex projective structures is $p = \dim(\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}_1^2)) = 8 - 2k$, where $0 \leq k \leq 3$ is the number of n_i equal to 2. The dimension of the Teichmüller space is $d = \dim(\mathrm{Teich}(\mathcal{O}_1^2)) = 2$.

As $\mathcal{O}_1^2$ is orientable and closed, $X(\mathcal{O}_1^2, \mathrm{SL}_4(\mathbb{R}))$ has a neighborhood homeomorphic to

$$\mathbb{R}^{8-2k} \times \mathrm{Cone}(\mathrm{UT}(S^1)) = \mathbb{R}^{8-2k} \times \mathrm{Cone}(S^1 \sqcup S^1)$$

Namely, the neighborhood is homeomorphic to the union to 2 manifolds of dimension $10-2k$ that intersect along a manifold of codimension 2.

Example 4.11 Consider a disc with mirror boundary and two cone points of order n_1 and n_2, with at most one of the n_i equal to 2. This is a non-orientable closed 2-orbifold, that we denote by $\mathcal{O}_2^2 = D^2(n_1, n_2; \emptyset)$. Its orientation covering is $S^2(n_1, n_1, n_2, n_2)$. Now the dimension of the space of convex projective structures is $p = \dim(\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}_2^2)) = 4 - 2k$, where $0 \leq k \leq 1$ is the number of n_i equal to 2. The dimension of the Teichmüller space is $d = \dim(\mathrm{Teich}(\mathcal{O}_2^2)) = 1$.

As $\mathcal{O}_2^2$ is non-orientable, we distinguish the orientable embedding from the type preserving embedding:

(i) Consider the orientable embedding. A neighborhood in $X(\mathcal{O}_2^2, \mathrm{SL}_4(\mathbb{R}))$ is homeomorphic to

$$\mathbb{R}^{4-2k} \times \mathrm{Cone}(S^0 \times S^0).$$

Here $S^0 \times S^0$ is the set with cardinality 4. Namely, the neighborhood is homeomorphic to the union to 2 manifolds of dimension $5 - 2k$ that intersect along a manifold of codimension one.

(ii) Consider the type preserving embedding. A neighborhood of the character in $X(\mathcal{O}^2, \mathrm{SL}_\pm(4, \mathbb{R}))$ is homeomorphic to

$$\mathbb{R}^{4-2k} \times \mathrm{Cone}(S^0 \times S^0)/\!\sim\, = \mathbb{R}^{4-2k} \times \mathrm{Cone}(S^0) \cong \mathbb{R}^{5-2k}.$$

Thus the neighborhood is topologically non-singular.

Example 4.12 Next consider a disc with two cone points of order n_1 and n_2, with $(n_1, n_2) \neq (2, 2)$, $\mathcal{O}_3^2 = D^2(n_1, n_2)$. Even if the underlying space and the number of cone points is the same as the previous one, this orbifold is orientable and has boundary. The deformation space however looks like the previous example (orientable embedding): $\dim(\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}_3^2)) = 4-2k$, where $0 \leq k \leq 1$ is the number of n_i equal to 2 and $\dim(\mathrm{Teich}(\mathcal{O}_3^2)) = 1$. As $\partial\mathcal{O}_3^2 \neq \emptyset$, $X(\mathcal{O}_3^2, \mathrm{SL}_4(\mathbb{R}))$ is again homeomorphic to $\mathbb{R}^{4-2k} \times \mathrm{Cone}(S^0 \times S^0)$.

Example 4.13 Consider $\mathcal{O}_4^2$ a disc D^2 with one cone point of order $n \geq 3$ and ∂D^2 is the union of a full orbifold and a mirror interval. This is a non-orientable orbifold with boundary, and its orientation cover is $D^2(n, n)$, the previous example. Now $\dim(\mathrm{Proj}_{\mathrm{cvx}}(\mathcal{O}_4^2)) = 2$ and $\dim(\mathrm{Teich}(\mathcal{O}_4^2)) = 1$.

(i) For the orientable embedding, a neighborhood in $X(\mathcal{O}_4^2, \mathrm{SL}_4(\mathbb{R}))$ is homeomorphic to

$$\mathbb{R}^2 \times \mathrm{Cone}(S^0 \times S^0)$$

and the neighborhood is homeomorphic to the union of two 3-manifolds that intersect along a surface.

(ii) Consider the type preserving embedding. As $\mathsf{f}(\partial\mathcal{O}_4^2) = 1$,

$$\dim(\mathrm{Teich}(\mathcal{O}_4^2)) - \mathsf{f}(\partial\mathcal{O}_4^2) = 0.$$

Thus all deformations of representations of $\pi_1(\mathcal{O}_4^2)$ in $\mathrm{SL}_4^\pm(\mathbb{R})$ (after the type preserving embedding) remain in $\mathrm{SL}_3^\pm(\mathbb{R})$.

5 Projective deformations of hyperbolic 3-orbifolds

A hyperbolic 3-orbifold has a natural projective structure, by using the projective model of hyperbolic geometry, and there is no known general criterion to determine the deformation space of those projective structures, cf. [7, 8, 12]. Motivated by this question, the following definition was introduced in a joint paper with Heusener [12].

Definition 5.1 A compact hyperbolic 3-orbifold $\mathcal{O}^3$ is projectively infinitesimally rigid with respect to the boundary if the inclusion induces an injection

$$0 \to H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})).$$

For finite volume hyperbolic 3-manifolds, Ballas, Danciger, and Lee [2, Theorem 3.2] proved that infinitesimal rigidity with respect to the boundary implies that the space of $\mathrm{PSL}_4(\mathbb{R})$ characters is smooth. Here we consider compact 3-orbifolds with hyperbolic interior, possibly of infinite volume, namely orientable 3-orbifolds of finite type.

Example 5.2 A product $\mathcal{O}^3 = \mathcal{O}^2 \times [0, 1]$ it is infinitesimally projectively rigid for any hyperbolic structure.

Definition 5.3 We say that one end of a hyperbolic 3-orbifold is *Fuchsian* if its holonomy is a Fuchsian group: namely the end corresponding to a totally geodesic boundary component.

In this definition, finite volume ends (e.g. corresponding to a Euclidean boundary component) are not considered Fuchsian.

This section is devoted to the proof of Theorem 1.2 in the introduction, that we restate for convenience:

Theorem 5.4 *(Theorem 1.2) Let $\mathcal{O}^3$ be a compact orientable orbifold, with $\partial\mathcal{O}^3 \neq \emptyset$ and hyperbolic interior int($\mathcal{O}^3$), so that it is not elementary, nor Fuchsian. Assume that $\mathcal{O}^3$ is infinitesimally projectively rigid with respect to the boundary. Then the character of the hyperbolic holonomy is a smooth point of $X(\mathcal{O}^3, SL_4(\mathbb{R}))$ if, and only if, all ends of int($\mathcal{O}^3$) are either non-Fuchsian or turnovers. Furthermore, if the character is a singular point, then the singularity is quadratic.*

The variety of characters of $X(\mathcal{O}^3, \mathrm{SL}_4(\mathbb{R}))$ has a natural analytic structure at the character of the holonomy, but a priori it may be non-reduced, it may contain points or subvarieties with multiplicity higher than one. We follow the following convention:

Remark 5.5 A point with higher multiplicity is considered a singular point. We use the convention that a point is singular when the dimension of the Zariski tangent space is higher than the dimension of the (any) component containing it.

In particular, a point that lies in more than one irreducible component is singular.

Example 5.6 Consider again a product $\mathcal{O}^3 = \mathcal{O}^2 \times [0, 1]$. When $\mathcal{O}^2$ is Fuchsian, its holonomy is contained in $\mathrm{SO}(2, 1) \subset \mathrm{SL}_3(\mathbb{R})$, hence reducible as representation in $\mathrm{SL}_4(\mathbb{R})$. By Theorem 1.1, the space of characters in $\mathrm{SL}_4(\mathbb{R})$ is singular. But when $\mathcal{O}^2$ is non-Fuchsian it is smooth (by Lemma 5.7 below its holonomy in $\mathrm{SL}_4(\mathbb{R})$ is $\mathbb{C}$-irreducible and Proposition 2.15 applies).

The proof of Theorem 5.4 is divided into three statements: Propositions 5.12, 5.16, and 5.17.

Proposition 5.12 shows the smoothness of the variety of characters when there are no Fuchsian ends other than a turnover. It is proved in two subsections. In Subsection 5.1 we establish some preliminary results on the holonomy of a hyperbolic 3-orbifold, and in Subsection 5.2 we complete the proof, relying on naturality of Goldman's obstructions to integrability and smoothness of the character varieties of the boundary components.

In Proposition 5.16 we show that, when there is a Fuchsian end other than a turnover, the variety of characters is singular. This is also done in two subsections. The first step consists in deforming the holonomy representation to another representation in $\mathrm{Isom}^+(\mathbb{H}^3)$ with no Fuchsian ends. This is done in Subsection 5.3 and then in Subsection 5.4 we show that the dimension of the Zariski tangent space strictly decreases under this deformation, establishing the singularity.

Finally in Subsection 5.5 we establish that the singularity is quadratic, relying again on the boundary, where the fact that it is quadratic is due to Goldman and Millson [11].

5.1 The projective holonomy of a hyperbolic 3-orbifold

In this subsection we discuss some preliminary results on the hyperbolic holonomy representation of a three orbifold $\pi_1(\mathcal{O}^3) \to \mathrm{SO}_0(3,1)$ composed with the inclusion $\mathrm{SO}_0(3,1) \subset \mathrm{SL}_4(\mathbb{R})$.

Lemma 5.7 *Let $\mathcal{O}^3$ be a compact orientable orbifold, with hyperbolic interior. If it is not elementary nor Fuchsian, then its holonomy representation $\pi_1(\mathcal{O}^3) \to SO_0(3,1) \subset SL_4(\mathbb{R})$ is $\mathbb{C}$-irreducible.*

Proof The proof is by contradiction. Assume there is a proper subspace $V \subset \mathbb{C}^4$ invariant by the holonomy representation, then the image of this holonomy is contained in the closed subgroup $H = \{g \in \mathrm{SO}_0(3,1) \mid gV = V\}$, that has finitely many components. We may assume that the identity component H_0 of H is nontrivial, otherwise H would be finite and hence $\mathcal{O}^3$ elementary, yielding a contradiction. Since $\mathbb{C}^4$ is irreducible as $\mathrm{SO}_0(3,1)$-module, H_0 is a proper connected subgroup of $\mathrm{SO}_0(3,1)$. Consider the action of this subgroup H_0 in hyperbolic space: as it is a proper subgroup, H_0 preserves either a totally geodesic plane, a line, or a point in either hyperbolic space or the ideal boundary. The whole subgroup H also preserves the same totally geodesic plane, line, or (finite or ideal) point, because H_0 is a finite index normal subgroup of H. When H preserves a totally geodesic plane, $\mathcal{O}^3$ is Fuchsian, and in all other cases $\mathcal{O}^3$ is elementary. This contradiction concludes the proof. □

Lemma 5.7 applies not only to the orbifold $\mathcal{O}^3$ of Theorem 5.4 but also to its ends; namely to the orbifold $\mathbb{H}^3/\Gamma$, where $\Gamma < \pi_1(\mathcal{O}^3)$ denotes the peripheral subgroup corresponding to an end of $\mathcal{O}^3$.

Corollary 5.8 *Let $\mathcal{O}^3$ be a compact orientable orbifold, with hyperbolic interior. If it is not elementary nor Fuchsian, then the invariant subspace of the Lie algebra is trivial:*

$$\mathfrak{sl}_4(\mathbb{R})^{\pi_1(\mathcal{O}^3)} = 0.$$

In particular $H^0(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = 0$.

Proof By contradiction, assume that there exists $0 \neq v \in \mathfrak{sl}_4(\mathbb{R})^{\pi_1(\mathcal{O}^3)}$. Then the holonomy of $\pi_1(\mathcal{O}^3)$ commutes with the one parameter real group $\{\exp(tv) \mid t \in \mathbb{R}\}$. By Lie-Kolchin theorem, this real group has an invariant complex line in $\mathbb{C}^4$. This line is also preserved by $\pi_1(\mathcal{O}^3)$, contradicting Lemma 5.7. The last assertion of the corollary follows from the natural isomorphism $H^0(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \cong H^0(\pi_1(\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{R})) \cong \mathfrak{sl}_4(\mathbb{R})^{\pi_1(\mathcal{O}^3)}$. □

Remark 5.9 As $\mathbb{C}^{n+1}$ is $\mathrm{SO}(n,1)$-irreducible, the very same argument proves the analog statement in any dimension. In particular, for a non-elementary hyperbolic 2-orbifold $\mathcal{O}^2$, its (Fuchsian) holonomy representation $\pi_1(\mathcal{O}^2) \to \mathrm{SO}_0(3,1) \subset \mathrm{SL}_3(\mathbb{R})$ is $\mathbb{C}$-irreducible and $H^0(\mathcal{O}^2, \mathfrak{sl}_3(\mathbb{R})) = 0$.

Next we require some more preliminary results on cohomology with twisted coefficients in the Lie algebra $\mathfrak{sl}_4(\mathbb{R})$. For an orbifold $\mathcal{O}^3$ satisfying the hypothesis of Theorem 5.4, let

$$\partial\mathcal{O}^3 = \partial_1\mathcal{O}^3_1 \cup \cdots \cup \partial_k\mathcal{O}^3$$

denote the decomposition in connected components, so that

$$H^*(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = \bigoplus_{i=1}^{k} H^*(\partial_i\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \cong \bigoplus_{i=1}^{k} H^*(\pi_1(\partial_i\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{R})).$$

Being infinitesimally projectively rigid means that the restriction induces an inclusion

$$H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \hookrightarrow H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})). \tag{14}$$

We furthermore have:

Lemma 5.10 *Let $\mathcal{O}^3$ be a compact orientable orbifold, with $\partial\mathcal{O}^3 \neq \emptyset$ and with hyperbolic interior int($\mathcal{O}^3$), that is neither elementary and nor a product $\mathcal{O}^2 \times [0, 1]$. Assume that $\mathcal{O}^3$ is infinitesimally projectively rigid with respect to the boundary. Then:*

(i) $\dim H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = \frac{1}{2} \dim H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$.
(ii) The restriction induces an isomorphism

$$H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \cong H^2(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})). \tag{15}$$

Proof of Lemma 5.10 By (14), the long exact sequence in cohomology of the pair $(\mathcal{O}^3, \partial\mathcal{O}^3)$ writes as

$$0 \to H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \xrightarrow{i^*} H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \xrightarrow{\delta} H^2(\mathcal{O}^3, \partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to \cdots$$

We claim that the maps i^* and δ are compatible with the pairings used in Theorem 2.14 induced by the Killing form. Namely, let

$$B : \mathfrak{sl}_4(\mathbb{R}) \times \mathfrak{sl}_4(\mathbb{R}) \to \mathbb{R}$$

denote the Killing form. We have the non-degenerate pairings

$$B(\cdot \cup \cdot) : H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \times H^2(\mathcal{O}^3, \partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^3(\mathcal{O}^3, \partial\mathcal{O}^3, \mathbb{R}) \cong \mathbb{R},$$

and, for $j = 1, \ldots, k$,

$$(\cdot, \cdot)_j = B(\cdot \cup \cdot) : H^1(\partial_j\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \times H^1(\partial_j\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^2(\partial_j\mathcal{O}^3, \mathbb{R}) \cong \mathbb{R}.$$

We denote their (orthogonal) sum by

$$\Psi = \sum (\cdot, \cdot)_j : H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \times H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to \mathbb{R}.$$

By naturality, for $a \in H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$ and $b \in H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$,

$$\Psi(i^*(a), b) = B(a \cup \delta(b))). \tag{16}$$

Next we claim that the image of i^* is a maximal isotropic subspace of Ψ. This is a standard argument done for instance in [12, 20, 24], but we provide it by completeness. For every $a \in H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$, the compatibility condition (16) and exactness yield $\Psi(i^*(a), i^*(a)) = B(a \cup \delta(i^*(a)))) = 0$. Thus the image of i^* is isotropic. To get maximality, if $b \in H^1(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$ satisfies $\Psi(i^*(a), b)$ for every $a \in H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$, then by (16) and non-degeneracy of the pairing, $\delta(b) = 0$. Therefore b belongs to the image of i^*. The pairing Ψ being skew-symmetric and non-degenerate, its maximal isotropic subspaces are half-dimensional. This proves the Assertion (i) of the lemma.

For the second assertion, the maps of the long exact sequence

$$H^1(\mathcal{O}^3, \partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \tag{17}$$

and

$$H^2(\mathcal{O}^3, \partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \tag{18}$$

are also compatible with the pairing $B(\cdot \cup \cdot)$, similarly to (16). The map (17) is trivial, because the next map in the long exact sequence of the pair, i^*, is injective. Then, using compatibility with the pairing and non-degeneracy, one can prove that the map (18) is also trivial. Next we consider the continuation of the long exact sequence of the pair:

$$0 \to H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^2(\partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \to H^3(\mathcal{O}^3, \partial\mathcal{O}^3\mathfrak{sl}_4(\mathbb{R})).$$

The conclusion then comes from the vanishing of $H^3(\mathcal{O}^3, \partial\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$: it is dual to $H^0(\mathcal{O}^3; \mathfrak{sl}_4(\mathbb{R}))$ that vanishes by Corollary 5.8. □

5.2 Smoothness of varieties of representations

In this subsection we prove one of the implications of Theorem 5.4, relying on the smoothness of varieties of representations on the components of $\partial\mathcal{O}^3$. As in the previous subsection, let

$$\partial\mathcal{O}^3 = \partial_1\mathcal{O}^3_1 \cup \cdots \cup \partial_k\mathcal{O}^3$$

denote the decomposition into connected components of the boundary.

Assume that $\mathcal{O}^3$ is hyperbolic and let $\rho_i : \pi_1(\partial_i\mathcal{O}^3) \to \mathrm{SL}_4(\mathbb{R})$ denote the restriction of its holonomy, composed with the inclusion $\mathrm{SO}(3, 1) \subset \mathrm{SL}_4(\mathbb{R})$.

Lemma 5.11 *If either ρ_i is non-Fuchsian or $\partial_i\mathcal{O}^3$ is a turnover, then* $\hom(\pi_1(\partial_i\mathcal{O}^3), SL_4(\mathbb{R}))$ *is smooth at ρ_i. Equivalently, Goldman's obstructions to integrability of Lemma 2.16 vanish.*

Proof We distinguish three cases, according to the topology of the 2-orbifold $\partial_i\mathcal{O}^3$.

First assume that $\partial_i\mathcal{O}^3$ is Euclidean (ie the corresponding end has finite volume). If $\partial_i\mathcal{O}^3$ is a manifold, then $\partial_i\mathcal{O}^3$ is a 2-torus and in this case [2, Lemma 3.4] Ballas, Danciger, and Lee prove that for a peripheral torus the space of representations in $\mathrm{SL}_4(\mathbb{R})$ is smooth, and all obstructions to integrability in Lemma 2.16 vanish, see also [2, Proof of Theorem 3.2]. If the Euclidean orbifold $\partial_i\mathcal{O}^3$ is not a manifold, then by Bieberbach theorem $\partial_i\mathcal{O}^3$ has a finite regular covering that is a 2-torus T^2, and the cohomology of $\partial_i\mathcal{O}^3$ is equivalent to the equivariant cohomology of T^2. So the claim follows from naturality of the obstruction by using equivariance.

Next assume that $\partial_i\mathcal{O}^3$ is hyperbolic and ρ_i is not Fuchsian. In this case Proposition 2.15 applies to ρ_i, because by Lemma 5.7 ρ_i is $\mathbb{C}$-irreducible. As we are interested in the obstruction to integrability, it is worth recalling the proof of Proposition 2.15 in [10]: by Corollary 5.8 $H^0(\pi_1(\partial_i\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{R})) = 0$ and therefore, by Poincaré duality, $H^2(\pi_1(\partial_i\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{R})) = 0$. Thus there are no obstructions to integrability at all.

Finally assume that $\partial_i\mathcal{O}^3$ is a turnover, we have shown that every representation of $\pi_1(\partial_i\mathcal{O}^3)$ in $\mathrm{SL}_4(\mathbb{R})$ is conjugate to a representation in $\mathrm{SL}_3(\mathbb{R})$ by Corollary 4.4. It follows that the space of representations in $\mathrm{SL}_4(\mathbb{R})$ is smooth, as the space of representations in $\mathrm{SL}_3(\mathbb{R})$ is smooth by [10] (it is the Choi-Goldman space). To describe explicitly Goldman's obstructions, we consider the decomposition of the Lie algebra by $\pi_1(\partial_i\mathcal{O}^3)$-modules of (5):

$$\mathfrak{sl}_4(\mathbb{R}) = \mathfrak{sl}_3(\mathbb{R}) \oplus \mathbb{R}^3_{\rho_i} \oplus \mathbb{R}^3_{\rho_i^3} \oplus \mathbf{d},$$

where $\mathbf{d} \cong \mathbb{R}$ is the subspace of diagonal matrices in $\mathfrak{sl}_4(\mathbb{R})$ that commute with every matrix in $\mathfrak{sl}_3(\mathbb{R})$. The vanishing of the obstructions follows from the fact that $H^1(\partial_i\mathcal{O}^3, \mathbb{R}^3_{\rho_i}) = H^1(\partial_i\mathcal{O}^3, \mathbb{R}^3_{\rho_i^*}) = H^1(\partial_i\mathcal{O}^3, \mathbf{d}) = 0$ (see Corollary 4.4 and the previous paragraph), and $H^2(\partial_i\mathcal{O}^3, \mathfrak{sl}_3(\mathbb{R})) = 0$, by Remark 5.9. □

The following is one of the implications of Theorem 5.4.

Proposition 5.12 *Under the hypothesis of Theorem 5.4, if all ends of int($\mathcal{O}^3$) are either non-Fuchsian or turnovers, then the character of ρ is a smooth point of $X(\mathcal{O}^3, SL_4(\mathbb{R}))$.*

Proof Since Goldman's obstructions to integrability are natural, by Lemmas 5.10 and 5.11 all obstructions to integrability in $H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$ vanish. So we have smoothness by applying a theorem of Artin. More precisely, the vanishing of obstructions yields a formal deformation of ρ in the direction of any cocycle z in the Zariski tangent space, where formal means a power series perhaps not convergent. Then the theorem of Artin asserts that the formal power series can be replaced by a converging series with the same tangent vector [1]. As every vector in the Zariski tangent space is tangent to a path, it is a smooth point. See [11, 14] for more details. □

5.3 Deforming representations in $\mathrm{Isom}^+(\mathbb{H}^3)$.

To continue with the proof of Theorem 5.4, in the next proposition we prove that the holonomy can be perturbed to a representation with similar algebraic properties, such that the image of peripheral subgroups other than turnovers are not conjugate to subgroups of $\mathrm{Isom}^+(\mathbb{H}^2)$ (eg no Fuchsian ends other than turnovers). It is not clear than this perturbed representation is the holonomy of a hyperbolic structure, this would require different techniques and we do not use hyperbolicity. All we need is that the dimension of the Zariski tangent space decreases, so that the initial character is a singular point.

Proposition 5.13 *Let $\mathcal{O}^3$ be a compact orientable 3-orbifold with hyperbolic interior and holonomy $\rho_0 \in \hom(\pi_1(\mathcal{O}^3), Isom^+(\mathbb{H}^3))$. There is a representation $\rho \in \hom(\pi_1(\mathcal{O}^3), Isom^+(\mathbb{H}^3))$ arbitrarily close to ρ_0 such that, for every component $\partial_i\mathcal{O}^3$ of $\partial\mathcal{O}^3$ other than a turnover, $\rho(\pi_1(\partial_i\mathcal{O}^3))$ is not conjugated to a subgroup of $Isom^+(\mathbb{H}^2)$.*

Proof We use the group isomorphisms

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{SO}_0(3,1) \cong \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}(3,\mathbb{C}).$$

The character χ_0 of the holonomy ρ_0 is a smooth point of the variety of characters $X(\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C}))$ with tangent space $H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$, as shown by M. Kapovich in [16, Section 8.8]. Furthermore, Kapovich proves that the restriction

$$\mathrm{res}\colon X(\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C})) \to X(\partial\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C})) = \prod_i X(\partial_i\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C}))$$

is an immersion in a neighborhood of χ_0, with (injective) tangent map

$$\mathrm{res}_*\colon H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \to H^1(\partial\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \cong \bigoplus_i H^1(\partial_i\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})),$$

the natural map induced by the inclusion $\partial\mathcal{O}^3 \subset \mathcal{O}^3$.

For each hyperbolic component $\partial_i\mathcal{O}^3$ other than a turnover, its Teichmüller space (and hence $X(\partial_i\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C}))$) has positive dimension. We claim the following:

Lemma 5.14 *There exists a tangent vector $v \in H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ such that for each component $\partial_i\mathcal{O}^3$ other than a turnover*

$$(res_i)_*(v) \neq 0,$$

where $(res_i)_*\colon H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \to H^1(\partial_i\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ is the map induced by the restriction to that component.

Proof of Lemma 5.14 We consider the same bilinear form as in Theorem 2.14 and the proof of Lemma 5.10, but with the complex valued Killing form

$$\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}.$$

Namely, for $j = 1, \dots, k$ we denote by

$$(\cdot, \cdot)_j : H^1(\partial_j \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \times H^1(\partial_j \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}$$

the skew-symmetric bilinear form obtained by composing the cup product and the $\mathbb{C}$-Killing form. Its direct sum is denoted by

$$\Psi = \sum (\cdot, \cdot)_j : H^1(\partial \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \times H^1(\partial \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}.$$

The very same argument as in the proof of Lemma 5.10 yields that the image of res_* is a maximal isotropic subspace of Ψ. Since Ψ is a component-wise sum of non-degenerate forms, this implies that the image of each morphism $(\mathrm{res}_i)_* : H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \to H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ is nonzero, provided that $H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) \neq 0$, eg provided that $\partial_i \mathcal{O}^3$ is not a turnover. The last assertion is equivalent to the fact that $\ker(\mathrm{res}_i)_*$ is not the whole space $H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ (whenever $\partial_i \mathcal{O}^3$ is not a turnover). Thus the union of those kernels is not the whole $H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ and the lemma follows. □

We continue the proof of Proposition 5.13. Assume the component $\partial_i \mathcal{O}^3$ is Fuchsian, namely $\rho_0(\pi_1(\partial_i \mathcal{O}^3))$ preserves a totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$. Then after conjugation it is contained in $\mathrm{PSL}(2, \mathbb{R})$, and we have a decomposition

$$H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C})) = H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{R})) \oplus \sqrt{-1}\, H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{R})).$$

After multiplying by a generic complex number, we may assume that, for every Fuchsian component $\partial_i \mathcal{O}^3$ different from a turnover, the vector $v \in H^1(\mathcal{O}^3, \mathfrak{sl}_2(\mathbb{C}))$ of Lemma 5.14 satisfies

$$(\mathrm{res}_i)_*(v) \notin H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_2(\mathbb{R})). \tag{19}$$

We consider a deformation $\{\rho_t\}_{t \in [0, \varepsilon)}$ of the holonomy whose path of characters $\{\chi_{\rho_t}\}_{t \in [0, \varepsilon)}$ in $X(\mathcal{O}^3, \mathrm{PSL}_2(\mathbb{C}))$ is tangent to v.

We claim that for $t > 0$ sufficiently small ρ_t satisfies the conclusion of the proposition. Not being Fuchsian is an open property, so we just need to deal with components $\partial_i \mathcal{O}^3$ different from a turnover that are Fuchsian, namely $\rho_0(\pi_1(\partial_i \mathcal{O}^3))$ is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. If $\partial_i \mathcal{O}^3$ is Fuchsian, but not a turnover, then condition (19) means that the path ρ_t restricted to $\pi_1(\partial_i \mathcal{O}^3)$ is transverse to the orbit by conjugation of $\hom(\pi_1(\partial_i \mathcal{O}^3), \mathrm{PSL}_2(\mathbb{R}))$ inside $\hom(\pi_1(\partial_i \mathcal{O}^3), \mathrm{PSL}_2(\mathbb{C}))$. Thus ρ_t restricted to $\pi_1(\partial_i \mathcal{O}^3)$ does not preserve a totally geodesic hyperbolic plane for $t > 0$. □

5.4 Computing Zariski tangent spaces

In Proposition 5.16 we prove one more piece of Theorem 5.4. The argument is based on a computation of Zariski tangent spaces (Lemma 5.15 below) and Proposition 5.13 in the last subsection.

Lemma 5.15 *Let $\mathcal{O}^3$ be a 3-orbifold as in the hypothesis of Theorem 5.4. There exists a neighborhood $U \subset \hom(\pi_1(\mathcal{O}^3), SO(3, 1))$ of the hyperbolic holonomy such that for every*

$\rho \in U$

$$\dim T^{Zar}_{[\rho]} X(\mathcal{O}^3, SL_4(\mathbb{R})) = \frac{1}{2}\sum_{i=1}^{n} \dim X(\partial_i \mathcal{O}^3, SL_4(\mathbb{R})) + f,$$

where f denotes the number of components $\partial_i \mathcal{O}^3$ of $\partial \mathcal{O}^3$ other than turnovers such that the image $\rho(\pi_1(\partial_i \mathcal{O}^3))$ is contained in a conjugate subgroup of $SO(2, 1)$.

Proof Infinitesimal projective rigidity is an open condition in the variety of representations $\hom(\pi_1(\mathcal{O}^3), \mathrm{SO}(3, 1))$. This follows from semi-continuity, see for instance [12, Lemma 3.2]. So we may assume that infinitesimal projective rigidity holds true for any $\rho \in U$ and that, by Lemma 5.10,

$$\dim T^{\mathrm{Zar}}_{[\rho]} X(\mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})) = \dim H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = \frac{1}{2}\sum_{i} \dim H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})).$$

Let ρ_i denote the restriction of ρ to $\pi_1(\partial_i \mathcal{O}^3)$, we aim to relate the dimensions of $H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$ and of $X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R}))$. We distinguish several possibilities, again according to the topology of $\partial_i \mathcal{O}^3$.

When $\partial_i \mathcal{O}^3$ is Euclidean, by [20, Theorem 4.11]

$$\dim H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = \dim X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})).$$

Assume next that $\partial_i \mathcal{O}^3$ is hyperbolic and the image of ρ_i is not conjugate to a subgroup of SO(2, 1). Then by Lemma 5.7 ρ_i is irreducible as a representation in $\mathrm{SL}_4(\mathbb{C})$. In fact Lemma 5.7 is stated for hyperbolic holonomies, but the argument is algebraic. We just need to care about being non-elementary, but this is an open property and we may chose U so that ρ_i is not elementary (if $\partial_i \mathcal{O}^3$ is hyperbolic). Then by Proposition 2.15

$$H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) \cong T^{\mathrm{Zar}}_{[\rho_i]} X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})) = \dim X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})).$$

Finally, assume that $\partial_i \mathcal{O}^3$ is hyperbolic and the image of ρ_i is conjugate to a subgroup of SO(2, 1). By taking U small enough, ρ_i is irreducible in $\mathrm{SL}_3(\mathbb{R})$ and by the results of Sect. 3 we have:

$$\dim H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) = \dim X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})) + 2.$$

More precisely, in Theorem 3.1 we construct an analytic subset

$$\mathcal{S} \subset H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$$

with $T^{Zar}_0 \mathcal{S} = H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R}))$, that is an slice for the action by conjugation in the variety of representations. This set is defined by the vanishing of an obstruction (Theorem 2.17), so $\dim \mathcal{S} = \dim H^1(\partial_i \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{R})) - 1$. Furthermore, the stabilizer of ρ_i has dimension 1, so by Corollary 3.2, $\dim X(\partial_i \mathcal{O}^3, \mathrm{SL}_4(\mathbb{R})) = \dim \mathcal{S} - 1$. □

The following is another of the pieces in the proof of Theorem 5.4.

Proposition 5.16 *Under the hypothesis of Theorem 5.4, if some end of int$(\mathcal{O}^3)$ is Fuchsian and not a turnover, then the character of ρ is a singular point of $X(\mathcal{O}^3, SL_4(\mathbb{R}))$.*

Proof By Proposition 5.13 and Lemma 5.15, ρ can be deformed so that the dimension of the Zariski tangent space strictly decreases. □

5.5 Quadratic singularities

The following concludes the proof of Theorem 5.4.

Proposition 5.17 *The singularity is quadratic.*

Proof To avoid technicalities, we work with the complexification, namely the variety or representations of $\pi_1(\mathcal{O}^3)$ in $\mathrm{SL}_4(\mathbb{C})$ instead of $\mathrm{SL}_4(\mathbb{R})$. This is sufficient to get quadratic singularities, by [11, §3.3].

As before, let

$$\partial\mathcal{O}^3 = \partial_1\mathcal{O}^3_1 \cup \cdots \cup \partial_k\mathcal{O}^3$$

denote the decomposition into connected components of the boundary, and let $\rho\colon \pi_1(\mathcal{O}^3) \to \mathrm{SO}(3,1) \subset \mathrm{SL}_4(\mathbb{R})$ denote the holonomy of the hyperbolic structure of $\mathcal{O}^3$ and $\rho_i : \pi_1(\partial_i\mathcal{O}^3) \to \mathrm{SL}_4(\mathbb{R})$, its restriction to $\partial_i\mathcal{O}^3$.

By Lemma 5.7 ρ is $\mathbb{C}$-irreducible, and by Theorem 2.6 there exists an analytic slice $\mathcal{S}$ at ρ that satisfies:

$$T^{\mathrm{Zar}}_\rho\mathcal{S} \oplus B^1(\pi_1(\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{C})) = Z^1(\pi_1(\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{C})),$$

hence $T^{\mathrm{Zar}}_\rho\mathcal{S} \cong H^1(\pi_1(\mathcal{O}^3), \mathfrak{sl}_4(\mathbb{C}))$. By [23, Theorem V.A.14] any analytic subvariety of $\mathbb{C}^N$ is equivalent to a subvariety of a (non-singular) $\mathbb{C}$-submanifold that has the same dimension as the Zariski tangent space of the analytic germ. So there exist germs of *non-singular* $\mathbb{C}$-manifolds $M, M_1, \ldots, M_k$, such that, after taking a small enough slice $\rho \in \mathcal{S}$ and choosing neighborhoods $U_i \subset \hom(\pi_1(\partial_i\mathcal{O}^3), \mathrm{SL}_4(\mathbb{C}))$ of ρ_i we have:

$$\begin{aligned} \mathcal{S} &\subset M, & T^{\mathrm{Zar}}_\rho\mathcal{S} &= T_\rho M,\\ U_i &\subset M_i, & T_{\rho|_{\pi_1\partial_i\mathcal{O}^3}}\hom(\pi_1(\partial_i\mathcal{O}^3), \mathrm{SL}_4(\mathbb{C})) &= T_{\rho|_{\pi_1\partial_i\mathcal{O}^3}} M_i. \end{aligned}$$

The restriction map

$$\mathcal{S} \to \prod_i U_i \subset \prod_i \hom(\pi_1(\partial_i\mathcal{O}^3), \mathrm{SL}_4(\mathbb{C}))$$

is analytic, hence it extends to

$$\phi\colon M \to \prod_i M_i \tag{20}$$

and since the restriction induces an injection

$$0 \to H^1(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{C})) \to \bigoplus_i H^1(\partial_i\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{C}))$$

the map $\phi\colon M \to \prod_i M_i$ in (20) is an analytic immersion. The following lemma implies that the singularity of $\mathcal{S}$ at ρ is quadratic, as each $\hom(\pi_1(\partial_i\mathcal{O}^3), \mathrm{SL}_4(\mathbb{C}))$ (or U_i) is quadratic in M_i:

Lemma 5.18 $\phi(\mathcal{S}) = \phi(M) \cap \prod_i \hom(\pi_1(\partial_i\mathcal{O}^3), SL_4(\mathbb{C}))$.

Proof of the lemma By construction we have the inclusion

$$\phi(\mathcal{S}) \subseteq \phi(M) \cap \prod_i \hom(\pi_1(\partial_i\mathcal{O}^3), \mathrm{SL}_4(\mathbb{C}))$$

and we prove equality by contradiction. Assume

$$\phi(\mathcal{S}) \subsetneq \phi(M) \cap \prod_i \hom(\pi_1(\partial_i \mathcal{O}^3), \mathrm{SL}_4(\mathbb{C})).$$

Then for some $k \in \mathbb{N}$ there exists a $(k+1)$-jet in

$$c = c_0 + c_1 t + \cdots + c_{k+1} t^{k+1} \in J^{k+1}_{0,\phi(\rho)}\big(\mathbb{R}, \phi(M) \cap \prod_i \hom(\pi_1(\partial_i \mathcal{O}^3), \mathrm{SL}_4(\mathbb{C}))\big)$$

such that c is not a $(k+1)$-jet of $\phi(\mathcal{S})$, $c \notin J^{k+1}_{0,\phi(\rho)}(\mathbb{R}, \phi(\mathcal{S}))$, but its k-th truncation is:

$$[c]_k = c_0 + c_1 t + \cdots + c_k t^k \in J^k_{0,\phi(\rho)}(\mathbb{R}, \phi(\mathcal{S})).$$

Here we use local coordinates of the analytic variety $\prod_i M_i$ in an open subset of $\mathbb{C}^N$, so that $c_0, \ldots, c_{k+1} \in \mathbb{C}^N$.

Since we assume that $[c]_k \in J^k_{0,\phi(\rho)}(\mathbb{R}, \phi(\mathcal{S}))$, we may write $[c]_k = \phi(\rho_k)$ for some k-jet $\rho_k \in J^k_{0,\rho}(\mathbb{R}, \mathcal{S})$, in $\mathcal{S}$, that in its turn we write as

$$\rho_k = \exp(a_1 t + \cdots a_k t^k)\rho \in J^{k+1}_{0,\rho}(\mathbb{R}, \mathcal{S})$$

for some maps (or 1-cochains) $a_i : \pi_1(\mathcal{O}^3) \to \mathfrak{sl}_4(\mathbb{C})$. By Lemma 2.16, the obstruction to extend ρ_k to a $(k+1)$-jet is a cohomology class in $H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{C}))$. By Lemma 5.10 the restriction map induces an isomorphism $H^2(\mathcal{O}^3, \mathfrak{sl}_4(\mathbb{C})) \cong H^2(\partial \mathcal{O}^3, \mathfrak{sl}_4(\mathbb{C}))$. Thus, by naturality of the obstruction and since $[c]_k$ is the truncation of the $(k+1)$-jet c (in the product of varieties of representations of the components $\partial_i \mathcal{O}^3$), this obstruction vanishes in $H^2(\mathcal{O}^3, \mathfrak{sl}(4, \mathbb{C}))$. Therefore ρ_k is the k-truncation of a $(k+1)$-jet in the variety of representations:

$$\rho_{k+1} = \exp(a_1 t + \cdots a_k t^k + a_{k+1} t^{k+1})\rho \in J^{k+1}_{0,\rho}\big(\mathbb{R}, \hom(\pi_1 \mathcal{O}^3, \mathrm{SL}_4(\mathbb{C}))\big).$$

Next we want to conjugate ρ_{k+1} so that the result is not only a jet in $\hom(\pi_1 \mathcal{O}^3, \mathrm{SL}_4(\mathbb{C}))$ but in the slice $\mathcal{S}$. For that purpose we use Theorem 2.6 (ii): a neighborhood of ρ in $\hom(\pi_1 \mathcal{O}^3, \mathrm{SL}_4(\mathbb{C}))$ is obtained by conjugating the slice $\mathcal{S}$ by a neighborhood of the origin in $\mathrm{SL}_4(\mathbb{C})$. Thus we write

$$\rho_{k+1} = \mathrm{Ad}_{\exp(t^{k+1} b)}\, \rho'_{k+1}$$

with $b \in \mathfrak{sl}_4(\mathbb{C})$ and ρ'_{k+1} a jet in the slice $\mathcal{S}$ (whose k truncation is ρ_k):

$$\rho'_{k+1} = \exp(a_1 t + \cdots a_k t^k + a'_{k+1} t^{k+1})\rho \in J^{k+1}_{0,\rho}(\mathcal{S}).$$

Notice that $\phi(\rho'_{k+1})$ and c are two $(k+1)$-jets in $\phi(M)$ whose k-truncations are the same, hence their difference is $t^{k+1} v$ for some $v \in T_{\phi(\rho)}\phi(M)$, namely $v = \phi_*(u)$ for some $u \in T_\rho M$. As $T_\rho M = T_\rho S$, we use u to modify the $(k+1)$-the term of ρ'_{k+1} and define:

$$\rho''_{k+1} = \exp(t^{k+1} u)\rho'_{k+1} = \exp(a_1 t + \cdots a_k t^k + (a'_{k+1} + u)t^{k+1})\rho \in J^{k+1}_{0,\rho}(\mathcal{S}).$$

This $(k+1)$-jet satisfies $\phi(\rho''_{k+1}) = c$, hence a contradiction. □

By the properties of the slice in Theorem 2.6, the singularity at the varieties of characters is quadratic. □

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