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On the Evolution Operators of a Class of Time-Delay Systems with Impulsive Parameterizations

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Abstract: This paper formalizes the analytic expressions and some properties of the evolution operator that generates the state-trajectory of dynamical systems combining delay-free dynamics with a set of discrete, or point, constant (and not necessarily commensurate) delays, where the parameterizations of both the delay-free and the delayed parts can undergo impulsive changes. Also, particular evolution operators are defined explicitly for the non-impulsive and impulsive time-varying delay-free case, and also for the case of impulsive delayed time-varying systems. In the impulsive cases, in general, the evolution operators are non-unique. The delays are assumed to be a finite number of constant delays that are not necessarily commensurate, that is, all of them being integer multiples of a minimum delay. On the other hand, the impulsive actions through time are assumed to be state-dependent and to take place at certain isolated time instants on the matrix functions that define the delay-free and the delayed dynamics. Some variants are also proposed for the cases when the impulsive actions are state-independent or state- and dynamics-independent. The intervals in-between consecutive impulses can be, in general, time-varying while subject to a minimum threshold. The boundedness of the state-trajectory solutions, which imply the system's global stability, is investigated in the most general case for any given piecewise-continuous bounded function of initial conditions defined on the initial maximum delay interval. Such a solution boundedness property can be achieved, even if the delay-free dynamics is unstable, by an appropriate distribution of the impulsive actions. An illustrative first-order example is developed in detail to illustrate the impulsive stabilization results.

Keywords: delay differential systems; point delays; evolution operator; impulsive actions; global stability

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1. Introduction

Systems with delayed effects appear in many real-life problems, such as war and peace models, some biological problems—for instance, the sunflower equation, logistic equations, epidemic models, etc.—electrical transmission lines, economic models, heat exchangers, urban traffic, digital control, remote control, tele-operation processes, integro-differential Volterra-type equations, neutral equations, etc. [1–14]. See also some of the references therein. Delays often influence the solution trajectory.

Different types of delays are common and these types sometimes appear in a combined manner. In this context, delays can be internal, that is, in the state dynamics, or external, in the sense that they operate either on the forcing inputs, or controls, or in the measurable outputs, or in both of them. Typically, the delays can be either constant or time-dependent point delays or they can also have a distributed nature. On the other hand, point delays can be commensurate, in the sense that all of them are integer multiples of a base delay, or incommensurate, when they do not have the above mutual proportionally nature—see, for instance, [1]. Distributed delays are typical in certain either deterministic or stochastic classes of Volterra-type integro-differential equations. See, for instance, [11–14] and some of the references therein. In particular, in [12], the stability of a class of integro-differential Volterra-type equations, including the dynamics associated with a finite number of point delays and impulsive effects, is analyzed by means of a Karovskii–Lyapunov analysis.

From the point of view of differential systems, time-delay systems are infinite-dimensional because of their time-interval-dependent memory, and so they possess infinitely many characteristic zeros. The initial conditions of time-delay differential systems are defined by any piecewise-continuous bounded function on the time interval $[-h, 0]$, where h is the maximum size of all the delays present in the system, which coincides with the delay of the maximum size.

It has been seen that, in the event that there is no time-varying dynamics in the delay-free part, and furthermore, there are no delays in the dynamics, then the solution of the differential system is generated by an evolution operator; this is a C_0-semigroup of a time-invariant infinitesimal generator [15–20], which is a one-parameter semigroup that is both strongly continuous and uniformly continuous. The one-parameter semigroup property of the linear time-invariant undelayed typically becomes lost if the dynamics is time-varying and/or it involves delays [21].

On the other hand, impulsive continuous-time- and discrete-time-controlled systems and their stability, asymptotic stability, and stabilization properties under appropriate controls have been studied in the background literature in both delay-free and delayed cases. See, for instance, Refs. [22–28] and some of the references therein. In particular, the global exponential stability of a type of system with control impulses and time-varying delays is studied in [22] by means of an impulsive delay differential inequality. Also, an "ad hoc" impulsive stabilizing controller was synthesized. In [23], an inequality of Razumikhin-type with delays in the impulses is addressed, while the information on the time delay in the impulses is used for stabilization of the delayed systems. In [27], a sliding-mode controller is designed for impulsive stabilization of a class of time-delay systems in the presence of input disturbances. The designed control allows for counteracting the unsuitable perturbation effects. In [28], sufficiency-type conditions for exponential stability are developed for delayed systems where the delay size is not limited by the distribution of the impulsive intervals. Related studies have also been extended to some nonlinear problems involving delays such as Cohen–Grossberg neural networks or actuator-saturated dynamics and to some stochastic control problems involving delays [29–31]. The distribution of the impulses through time is not necessarily periodic since the aperiodicity, and eventual sampling adaptation to the signal profiles, allows for the improvement of the data acquisition performance [32]; improvement of the system dynamics in discrete-time systems [33]; to better fit certain optimization processes [34]; the improvement of the signal filtering and attenuation of the noise effects [35]; or the improvement of behavior prediction [36]. Some strategies of non-periodic sampling are described and discussed in [37–40]. In [41], non-uniform sampling is proposed for the improvement of tracking control against stochastic actuator faults. On the other hand, the adaptation transients can be improved by using adaptive sampling that is, by nature, non-uniform and adapts itself to the signal variations. This allows for reducing the signal overshoot peaks, especially at the beginning of the state and the output responses. See, for instance, Refs. [42–47] and some of the references therein. In particular, non-periodic sampled-data controllers have been proposed for distributed networked control against potential cyber-attacks of a stochastic type [46]. In [48,49], impulsive controls are proposed and synthesized for certain epidemic models in such a way that the controls are not uniformly distributed through time. Typically, the impulsive effects are characterized through Dirac distributions in the differential system, which generate finite jumps in the solution trajectory at the time instants where such impulsive effects take place. Some classical books of systems theory and physics explain both the intuitive ideas behind and the mathematical description of the impulsive effects. See, for instance, [50–52]. Some recent results in the field of time-delay systems are given in [53–56] as follows: In [53], the solvability of a state-dependent integro-differential inclusion is studied together with the existence and uniqueness of solutions of these types of equations when subject to delay nonlocal conditions. In [54], new sufficient and necessary conditions are derived regarding the oscillatory behavior of second-order differential equations with mixed and multiple delays under a canonical operator. In [55], oscillation conditions are presented for fourth-order neutral differential equations. On the other hand, the purpose of the investigation performed in [56] is the study of the asymptotic properties and oscillation regime of neutral differential equations with delays of even order. The given results are based on the Riccati transformation together with a comparison with first- and second-order delay equations.

This paper relies on the analytic forms and some relevant properties of the evolution operators that generate the state-trajectory solutions of a class of eventually impulsive linear time-delay with linear time-varying mixed delay-free dynamics and delayed dynamics. The analysis of the impulsive effects is a main novelty compared with the results given in [21]. It should be pointed out that impulses translate into finite jumps in the solution trajectory. In this article, the impulsive actions are assumed to take place inside the parameterization instead of influencing the forcing terms as is usual in the related previous background literature. The eventual impulses directly influence the parameterization of the undelayed dynamics and, eventually, that of the delayed dynamics in the differential system at the impulsive time instants. The various combined delayed dynamics correspond to a finite number of known constant point delays. In more detail, the evolution operators are made explicit for both the non-impulsive and the impulsive time-varying delay-free cases and also for the impulsive time-varying systems with eventual delayed dynamics. The delays are assumed to be constant and, in general, multiple and arising in a finite number but they are not necessarily commensurate in the sense that they are not required to be an integer multiple of a minimum or base-minimum delay. The impulsive actions are assumed to be state-dependent and to take place at certain isolated time instants on the matrix functions, which define the delay-free and the delayed dynamics. Some variants of the above model are also briefly discussed for the cases when the impulsive actions are either state-independent or both state- and dynamics-independent. It is assumed that the inter-impulse intervals, that is, the time intervals in-between consecutive impulses, can be, in general, time-varying while being subject to a minimum threshold. The boundedness and the associated global stabilization of the state-trajectory solutions for any given piecewise-continuous bounded function of the initial conditions (defined on the initial maximum delay interval) are investigated in the most general case of the proposed time-varying impulsive system with delays. Such a boundedness property can be achieved, even if the delay-free dynamics is unstable, by an appropriate distribution of the impulsive actions. An illustrative first-order example with impulsive parametrization changes is developed in detail to illustrate the boundedness and stabilization results.

The impulsive system under study could be interpreted as a limit case of a switched delay system with instantaneous switching actions from one parameterization to another, and then immediately returning to the former parameterization in the configuration previous to the impulsive action. In that context, the parameterizations are impulsive and seen as instantaneous switching actions occurring at the impulsive time instants.

The paper is organized as follows. Section 2 presents expressions of the evolution operator that generate the state-trajectory solution of a class of linear time-varying differential delay-free systems. The evolution operators used in this article are based on the analysis of the solution of the differential systems with time delays taking into account the impulses in the parameterization of the dynamics that are, in general, state-dependent and lead to finite jumps in the solution of the differential system at the impulsive time instants. In that context, the impulsive time instants might be either effective or ineffective according to their validity to produce, or not, parameterization changes in the dynamics. Intuitively, an impulsive time instant is considered to be ineffective if it is not able to generate a jump in the state being valid to modify the parameterization of the dynamics. The simplest case (but not the only one) of impulsive ineffectiveness happens when the left limit of the trajectory solution is null at the "a priori" claimed impulsive time instant. In such a case, another closely allocated impulsive time instant candidate should be tried. To establish the main results, the impulsive-free part of the matrix function of dynamics of such systems is supposed to be bounded, piecewise-continuous, and Lebesgue-integrable on $[0,\infty)$. The cases of absence and presence of Dirac-type impulses in the system matrix of dynamics are described. In the impulsive case, the evolution operator is seen to be, in general, non-unique. The efficiency of the impulsive time instants in the generation of relevant impulsive actions depends on the particular evolution of the matrix function that describes the dynamics. In that sense, the potential impulsive time instants might be effective or ineffective. Section 3 extends the above results to the presence of delayed dynamics associated with, in general, multiple constant point delays. The impulsive actions at certain time instants can take place both in the delay-free dynamics and in the various matrices of delayed dynamics. The boundedness of the solution trajectory leading, as a result, to the global stability of the system for any bounded initial conditions is also investigated. An appropriate distribution of the impulsive time instants, subject to a minimum inter-impulsive threshold, is proved to be essential for the potential stabilization of the differential system. An illustrative first-order example is discussed in detail to visualize the theoretical results of Section 3. Finally, some conclusions end the paper.

Nomenclature

$$\overline{p} = \{1, 2, \cdots, p\}$$

Commensurate point delays $h_m = h_1 < h_2 < \cdots < h_p = h$ are those that are the integer multiple of a minimum common multiple, which is the minimum, or base, delay $h_m = h_1$, namely, $h_i = ih_m$ for $i \in \overline{p}$, then $h = h_p = ph_m$.

Incommensurate point delays $h_m = h_1 < h_2 < \cdots < h_p = h$ are those that are not the integer multiple of a minimum common multiple.

$\overline{f}_{ta}$ denotes the strip of the function $f : \boldsymbol{R} \to \boldsymbol{R}^n$ on $[t-a,t]$, defined as a function from $\boldsymbol{R}$ to $\boldsymbol{R}^n$ by $\overline{f}_{ta}(\tau) = f(\tau);\ \tau \in [t-a,t],\ \overline{f}_{ta}(\tau) = 0;\ \tau \in \boldsymbol{R} \backslash [t-a,t];\ t \in \boldsymbol{R}$ in such a way that the support of $\overline{f}_{ta}$ is a proper or improper subset of $[t-a,t]$. Thus, $\overline{x}_{th}$ is the strip of the solution $x(t)$ on $[t-h,t]$ of an n-th-order time delay system on a maximum delay period of size h. To abbreviate the notation, it is referred to simply as $\overline{x}_t$ to avoid confusion.

I_n is the n-th identity matrix;

0_n is the n-th zero matrix.

$$\boldsymbol{R}_{0+}=\boldsymbol{R}_{+}\cup\{0\}=\{r\in\boldsymbol{R}:r\geq 0\};\ \boldsymbol{R}_{+}=\{r\in\boldsymbol{C}:r>0\},$$

$$\boldsymbol{R}_{-0}=\boldsymbol{R}_{-}\cup\{0\}=\{r\in\boldsymbol{R}:r\leq 0\};\ \boldsymbol{R}_{-}=\{r\in\boldsymbol{R}:r<0\},$$

where $\boldsymbol{R}$ is the set of real numbers, and

$$\boldsymbol{C}_{0+}=\boldsymbol{C}_{+}\cup\{0\}=\{z\in\boldsymbol{C}:Rez\geq 0\};\ \boldsymbol{C}_{+}=\{z\in\boldsymbol{C}:Rez>0\},$$

$$\boldsymbol{C}_{-0}=\boldsymbol{C}_{-}\cup\{0\}=\{z\in\boldsymbol{C}:Rez\leq 0\};\ \boldsymbol{C}_{-}=\{z\in\boldsymbol{C}:Rez<0\},$$

where $\boldsymbol{C}$ is the set of complex numbers.

In the same way, we can define "mutatis mutandis" the respective subsets $\boldsymbol{Z}_{0+}$, $\boldsymbol{Z}_{+}$, $\boldsymbol{Z}_{-0}$, and $\boldsymbol{Z}_{-}$ of the set $\boldsymbol{Z}$ of integer numbers.

The closures of the real set and subsets are $cl\boldsymbol{R}=\boldsymbol{R}\cup\{\pm\infty\}$,

$$cl\boldsymbol{R}_{0+}=\boldsymbol{R}_{0+}\cup\{+\infty\};\ cl\boldsymbol{R}_{+}=\boldsymbol{R}_{+}\cup\{0,+\infty\},$$

$$cl\boldsymbol{R}_{-0}=\boldsymbol{R}_{-0}\cup\{-\infty\};\ cl\boldsymbol{R}_{-}=\boldsymbol{R}_{-}\cup\{0,-\infty\},$$

where $BPC([-h,0];\boldsymbol{R}^n)$ denotes the set of bounded piecewise-continuous functions from $[-h,0]$ to $\boldsymbol{R}^n$.

$PC^1(\boldsymbol{R}_{+};\boldsymbol{R}^n)$ is the continuous differentiable class of n-real vector functions on $\boldsymbol{R}_{+}$, whose first-derivative is piecewise-continuous but not necessarily continuous. Note that $C^1(\boldsymbol{R}_{+};\boldsymbol{R}^n)\subset PC^1(\boldsymbol{R}_{+};\boldsymbol{R}^n)$.

$\boldsymbol{R}_{0+h}(t)=\{[t-h,t]:t\in\boldsymbol{R}_{0+}\}\subset\boldsymbol{R}$; $\boldsymbol{R}_{+h}(t)=\{[t-h,t]:t\in\boldsymbol{R}_{+}\}\subset\boldsymbol{R}$ are, respectively, non-negative and positive strips of real intervals of the Lebesgue measure h relative to the time instant "t".

An impulsive real function $f:\boldsymbol{R}_{0+}\to\boldsymbol{R}$ is that which has a non-zero set of Dirac-distribution-type impulses $K\delta(\tau-t_i)$ on a finite set of impulsive points $\tau=t_i$ such that there is an impulsive jump $f(t_i^+)-f(t_i^-)=K(t_i^-)\delta(0)$ at $t=t_i$ of size $K(t_i^-)(\neq 0)\in\boldsymbol{R}$. The same idea applies for a vector function $f:\boldsymbol{R}_{0+}\to\boldsymbol{R}^n$ in the sense that $t=t_i\in\boldsymbol{R}_{0+}$ is impulsive if there is at least a component $j\in\overline{n}$ of $f(t_i)$ such that $f_j(t_i^+)-f_j(t_i^-)=K_j(t_i^-)\delta(0)$ with $K_j(t_i^-)\neq 0$. Thus, t_i is an impulsive point of $f(t)$ if $f(t_i^+)-f(t_i^-)=\left(K_1(t_i^-)\delta(0),K_2(t_i^-)\delta(0),\cdots,K_2(t_i^-)\delta(0)\right)^T$ is non-zero, that is, if there is at least one non-zero $K_j(t_i^-)$ for $j\in\overline{n}$. An abbreviated notation for that is $f(t_i^+)-f(t_i^-)=\left(K_j(t_i^-)\delta(0)\right)$. Again, a generalization for the real matrix functions $F:\boldsymbol{R}_{0+}\to\boldsymbol{R}^{n\times n}$ is direct in the sense that t_k is an impulsive point of $F(t)=\left(F_{ij}(t)\right)$ if $F(t_k)-F(t_k^-)=\left(K_{ij}(t_k^-)\delta(0)\right)$ with at least one entry of $K(t_k^-)=\left(K_{ij}(t_k^-)\right)$ being non-zero. Note that if t is non-impulsive, then the impulsive jump amplitude is null. To keep the notation less involved, the right limit t^+ of t is simply denoted by t so that $F(t)$ stands as the notation for $F(t^+)$.

The spectral radius of a matrix function of time $A:\boldsymbol{R}_{0+}\to\boldsymbol{R}^{n\times n}$ is denoted by $\rho(A(t))$;

$e_i\in\boldsymbol{R}^n$ is the unit's i-th Euclidean vector whose unique non-zero component is the i-th one that equalizes unity;

"iff" is the usual equivalent abbreviation of the claim "if and only if";

χ_0 denotes the infinity cardinal of a numerable set;

The entry-to-entry definition of a matrix $K\in\boldsymbol{R}^{n\times m}$ is denoted as $K=\left(K_{ij}\right)$; $i\in\overline{n}$, $j\in\overline{m}$.

$RH_{\infty}^{n\times n}$ is the Hardy space of real–rational complex-valued proper transfer matrices $G(s)$ from C to $C^{n\times m}$, analytic in C_{0+}, and of finite norm $\|G(s)\|_{\infty} = \max_{\omega\in R_{0+}} \overline{\sigma}(G(i\omega))$, where $\overline{\sigma}(G(i\omega))$ is the maximum singular value of $G(s)$ for $s = i\omega$; and $i = \sqrt{-1}$ is the complex imaginary unit.

2. Time-Varying Linear Delay-Free Differential Systems and Their Evolution Operators for the Non-Impulsive and Impulsive Cases

This section studies the analytic expressions of the evolution operator that generate the state-trajectory solution of a class of linear time-varying differential delay-free systems. The impulsive-free part of the system matrix function is assumed to be bounded, piecewise-continuous, and Lebesgue-integrable for all time. The cases of absence and presence of Dirac-distribution-type impulses in the entries of the system matrix of dynamics are described and the corresponding expressions of the evolution operators [15,21], which generate the solution trajectories for given initial conditions, are given explicitly. In the impulsive case, the evolution operator is seen to be, in general, non-unique. The efficiency of the impulsive time instants to generate relevant impulsive actions depends on the particular evolution of the matrix function that describes the dynamics. In this sense, the impulsive time instants might be either effective or ineffective. A discussion is also provided for the case when the impulses do not affect the parameterization of the matrix of dynamics while they produce finite state-independent jumps in the solution trajectory at the impulsive time instants.

Consider the subsequent linear time-varying differential system of order n:

$$\dot{x}(t) = A(t)x(t);\ x(0) = x_0 \tag{1}$$

where $A : R_{0+} \to R^{n\times n}$ is bounded and piecewise-continuous Lebesgue-integrable on R_{0+}; and $A(t)$ and $\int_0^t A(\tau)d\tau$ are assumed to commute for all $t \geq 0$.

Theorem 1. *The following properties hold:*

(i) *The unique solution of (1) is given below:*

$$x(t) = e^{\int_0^t A(\tau)d\tau}x_0; t \in R_{0+} \tag{2}$$

so that the unique evolution operator of (1) from R_{0+} *to* $R^{n\times n}$ *is* $\Psi(t) = e^{\int_0^t A(\tau)d\tau}$ *for any* $t \in R_{0+}$.

(ii) *The evolution operator* $\Psi(t) = e^{\int_0^t A(\tau)d\tau}$ *for any* $t \in R_{0+}$ *satisfies the integral equation*

$$\Psi(t) = I_n + \int_0^t A(\tau)\Psi(\tau)d\tau = I_n + \int_0^t A(\tau)e^{\int_0^\tau A(\sigma)d\sigma}d\tau; t \in R_{0+}. \tag{3}$$

Proof. It turns out that if (2) holds then (1) holds. Assume that there exists another solution for the same initial conditions of the following form:

$$x(t) = \Phi(t)x_0;\ t \in R_{0+} \tag{4}$$

so that

$$\dot{x}(t) = \dot{\Phi}(t)x_0 = A(t)\Phi(t)x_0 = A(t)x(t) = A(t)e^{\int_0^t A(\tau)d\tau}x_0;\ t \in R_{0+} \tag{5}$$

for $t \in \boldsymbol{R}_{0+}$, and

$$\dot{\Phi}(t) = A(t)e^{\int_0^t A(\tau)d\tau} = \frac{d}{dt}\left(e^{\int_0^t A(\tau)d\tau}\right);\ t \in \boldsymbol{R}_{0+} \tag{6}$$

Note from (2) and (4) that $e^{\int_0^0 A(\tau)d\tau} = \Phi(0) = I_n$ since $x(0) = x_0$ for any given arbitrary finite x_0. Since $\Psi(0) = e^{\int_0^0 A(\tau)d\tau} = I_n = \Phi(0)$, and

$$\dot{\Phi}(t) = \frac{d}{dt}\left(e^{\int_0^t A(\tau)d\tau}\right) \tag{7}$$

for all $t \in \boldsymbol{R}_{0+}$ from (6), then $\Phi(t) = e^{\int_0^t A(\tau)d\tau}$ for all $t \in \boldsymbol{R}_{0+}$. Thus, the evolution operator $\Psi(t) \equiv \Phi(t) = e^{\int_0^t A(\tau)d\tau}$ of (1) is unique for all $t \in \boldsymbol{R}_{0+}$. Property (i) has been proved. Since $\Psi(t) = e^{\int_0^t A(\tau)d\tau}$ is the unique evolution operator of (1) for all $t \in \boldsymbol{R}_{0+}$, then

$$\Psi(t) = \Psi(0) + \int_0^t \dot{\Psi}(\tau)d\tau = I_n + \int_0^t \frac{d}{d\tau}\left(e^{\int_0^\tau A(\sigma)d\sigma}\right)d\tau = I_n + \int_0^t A(\tau)\Psi(\tau)d\tau;\ t \in \boldsymbol{R}_{0+} \tag{8}$$

and Property (ii) has been proved. □

Define $< A >_t = (1/t)\int_0^t A(\tau)d\tau$ as the average of $A : [0,t) \to \boldsymbol{R}^{n\times n}$, which exists for all $t \in \boldsymbol{R}_{0+}$ since $\left\|\int_0^t A(\tau)d\tau\right\| < +\infty$ for all $t \in \boldsymbol{R}_{0+}$. Note that

$$\Psi(t) = e^{\int_0^t A(\tau)d\tau} = e^{((1/t)\int_0^t A(\tau)d\tau)t} = e^{\overline{A}_t t};\ t \in \boldsymbol{R}_{0+} \tag{9}$$

Thus, the explicit computation of $\Psi(t) = e^{\int_0^t A(\tau)d\tau}$ can be performed by direct or numerical integration from (8) and also as $\Psi(t) = L^{-1}\left\{(s, I_n, -, <, A, >_t)^{-1}\right\}$ via the Laplace inverse transform of $(sI_n - < A >_t)^{-1}$ for each fixed $t \in \boldsymbol{R}_{0+}$ and its associated (constant for such fixed $t \in \boldsymbol{R}_{0+}$) averaged n-matrix $< A >_t = (1/t)\int_0^t A(\tau)d\tau$.

The above considerations may be extended for more general Lebesgue integrals, which eventually include Dirac-distribution-type jumps in the integrand, that is, infinite point jumps at isolated sampling instants in the dynamics of the differential system, as is now discussed.

Note from Theorem 1 that, under the simplifying notation convention $\Psi(t,0) = \Psi(t)$, we can write the following:

$$x(t) = \Psi(t,0)x_0 = \Psi(t,t_0)x(t_0) = \Psi(t,t_0)\Psi(t_0,0)x_0 \text{for any } t \geq t_0 \geq 0 \tag{10}$$

Note that the evolution operator of Theorem 1 is not a C_0-semigroup—which is essentially one-parametric—since, if $t \neq t_0(\neq 0)$ in (10), then the state-solution cannot be expressed, in general, with one parameter $(t - t_0)$ for any t_0 and t because $A(t)$ is time-variant. Consider now the modification of (1) that follows to include impulsive effects in the time derivative of the solution trajectory, which produces proportional instantaneous finite jumps at the impulsive time instants in such a solution trajectory:

$$\dot{x}(t) = (A(t) + \Delta(t))x(t);\ x(0^-) = x_0 \tag{11}$$

$$\Delta(t) = \sum\nolimits_{t_i \in IMP(t)} K(t^-)\hat{A}(t^-)\delta(t - t_i) \tag{12}$$

where $A : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is bounded and piecewise-continuous Lebesgue-integrable on $[0,t]$ for any $t \in \boldsymbol{R}_{0+}$; $\hat{A}_{ij}(t^-) = A_{ij}(t^-)$ if the (i,j)-entry of $A(t^-)$ is impulsive, and $\hat{A}_{ij}(t^-) = 0$, otherwise, for $i,j \in \overline{n}$; $\Delta(t)$ is a Lebesgue-integrable impulsive real square n-matrix on $[0,t]$ for any $t \in \boldsymbol{R}_{0+}$ such that at least one entry of the real non-continuous bounded

n-matrix function $K(t^-)$ is non-zero if, and only if, the time instant t is impulsive—that is, $t \in IMP\left(= \cup_{\tau \in \boldsymbol{R}_{0+}} IMP(\tau)\right)$, where IMP is the total set of impulsive time instants and $IMP(t) = \left\{\tau(\le t) \in \boldsymbol{R}_{0+} : K(\tau^-)\hat{A}(\tau^-) \ne 0\right\}$ is the set of impulsive sampling instants up to the time instant t, which has a cardinal $N(t)$, i.e., the set of impulses on $[0, t]$. Note that the support of the impulsive matrix function is, by definition, a set of isolated time instants in $\boldsymbol{R}_{0+}$. We also define by convenience the impulsive set of time instants $IMP(t^-) = \{\tau(< t) \in \boldsymbol{R}_{0+} : K(\tau) \ne 0\}$, of cardinal $N(t^-)$ (i.e., the set of impulses on $[0, t)$) so that $IMP(t) = IMP(t^-)$ if and only if $t \notin IMP$. The total finite or (denumerable) infinite number of impulsive time instants is $card(IMP) = N_{imp} \le \chi_0$, where χ_0 denotes the infinity numerable cardinality. Note from (11)–(12) that $x(t) - x(t^-) = K(t^-)\hat{A}(t^-)x(t^-)$ if $t \in IMP$ and $K(t) = 0$ so that $x(t) = x(t^-)$ if $t \notin IMP$ and also its right limit is zero if $t \in IMP$. Note that $\hat{A}(t) = A(t)$ and $x(t) - x(t^-) = K(t^-)\hat{A}(t^-)x(t^-)$ if, and only if, all the entries of $A(t)$ are impulsive. It can be noticed that if $x(t^-) \in Ker\left(K(t^-)\hat{A}(t^-)\right)$ then $x(t) = x(t^-)$ and such a time instant t does not have a real, effective impulsive effect in the solution for such an impulsive gain $K(t)$. If, in addition, $x(t^-) \in Ker\left(\hat{A}(t^-)\right)A(t^-) = 0$, and in particular, if $x(t^-) \in Ker\left(\hat{A}(t^-)\right)$, then t does not have a real, effective impulsive effect in the solution irrespective of the impulsive gain $K(t)$. Then, such a time instant "t" is non-effective or ineffective versus the case when it is effective. Related to these ideas, the following definitions are pertinent:

Definition 1. *If $x(t^-) \in Ker\left(K(t^-)\hat{A}(t^-)\right)$ for $t \in IMP$, then t is an ineffective impulsive time instant for the impulsive gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$.*

Definition 2. *If $x(t^-) \notin Ker\left(K(t^-)\hat{A}(t^-)\right)$ for $t \in IMP$, then t is an effective impulsive time instant for the impulsive gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$.*

Definition 3. *The impulsive set of time instants IMP is effective for the impulsive matrix function gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ if $K(t^-)\hat{A}(t^-)x(t^-) \ne 0$ for any $t \in IMP$.*

Note also that if $\hat{A}(t^-)x(t^-) \ne 0$, then there always exists some impulsive matrix function gain of $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ for which $t \in IMP$ is effective. It suffices for this concern that $\hat{A}(t^-)x(t^-) \notin KerK(t^-)$. The following simple result holds:

Proposition 1. ***(i)*** *A time instant $t \in IMP$ is effective for some impulsive matrix function gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ iff $e_i^T\hat{A}(t^-)x(t^-) \ne 0$ for some $i = i(t) \in \overline{n}$.*

(ii) *$e_i^T\hat{A}(t^-)x(t^-) \ne 0$ for some (unique or non-unique) $i = i(t) \in \overline{n}$ implies and it is implied by $\hat{A}(t^-)x(t^-) \ne 0$ (the converse implication does not imply the uniqueness of $i = i(t) \in \overline{n}$ such that $e_i^T\hat{A}(t^-)x(t^-) \ne 0$. Then, Property (i) is fully equivalent to the following assertion: a time instant $t \in IMP$ is effective for some impulsive matrix function gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ iff $e_i^T\hat{A}(t^-)x(t^-) \ne 0$ for some $i = i(t) \in \overline{n}$.*

(iii) *A time instant $t \in IMP$ is ineffective for any impulsive matrix function gain $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ iff $e_i^T\hat{A}(t^-)x(t^-) = 0$ for all $i \in \overline{n}$, equivalently, iff $\hat{A}(t^-)x(t^-) = 0$.*

Proof: If $e_i^T\hat{A}(t^-)x(t^-) \ne 0$ for $t \in IMP$ and some $i = i(t) \in \overline{n}$, then the i-th component of $\hat{A}(t^-)x(t^-)$ is non-zero. Then, it suffices that the i-th column of $K(t)$ has all its components non-zero in order that $x_j(t) \ne x_j(t^-)$ for all $j \in \overline{n}$. This ensures the existence of a (non-unique) $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ such that $t \in IMP$ is effective. The sufficiency part of Property (i) has been proved. The necessity part is followed by contradiction arguments. Assume that $e_i^T\hat{A}(t^-)x(t^-) = 0$ for $t \in IMP$ and all $i = i(t) \in \overline{n}$. Thus, $v(t) = \hat{A}(t^-)x(t^-) = 0$ so that $x(t) = x(t^-)$ irrespective of $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$. Property (i) has been fully proved. The implication parts " $\Rightarrow$ " and " $\Leftarrow$ " of Property (ii) follow directly from Property (i) so that

Property (ii) is proved as equivalent to Property (i). Property (iii) is the dual of Property (i) so its proof is direct under close "ad hoc" arguments. □

The relevance of extending Theorem 1 of the impulsive case relies on keeping the Lebesgue-integrability of $(A(t) + \Delta(t))$ through time when the impulsive contribution $\Delta(t)$ is non-null. In particular, the following cases are of interest:

(a) $card(IMP) < \infty$, that is, there is a finite number of impulsive time instants;
(b) there are infinitely many impulsive time instants but their influences are mutually compensated so that their total contribution to the solution of (11) is bounded for all time;
(c) there are infinitely many impulsive time instants but their respective jumps collapse asymptotically so that $(A(t) + \Delta(t))$ is Lebesgue-integrable on $[0, t]$ for any $t \in \boldsymbol{R}_{0+}$. Because of its modus operandi, its way of operating, at the impulsive time instants, the system (11)–(12) is said to be a proportional instantaneous finite-jumps system.

The uniqueness of the evolution operator is guaranteed along the inter-switching time intervals but such uniqueness is not guaranteed in the general case, at the right limits of the impulsive time instants, since different impulsive gains can generate the same right limit value of the solution trajectory for a given left limit value. This property is formalized in the subsequent results:

Theorem 2. *The following properties hold*:

(i) *The unique solution of the proportional instantaneous finite-jumps system (11)–(12) is given by the expression below:*

$$\begin{aligned} x(t) &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K(t_i^-)\hat{A}(t_i^-)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right)x_0 \\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)x(t^-);\, t \in \boldsymbol{R}_{0+} \end{aligned} \quad (13)$$

where the product of matrices operates from the left to the right in the order of the increasing sequence $\{t_i\}_{i \in \mathbf{Z}_+}$*; and* $IMP(t^-) = \left\{t_1, t_2, \cdots, t_{N(t^-)}\right\}$ *is the impulsive set of time instants on* $[0, t)$ *whose cardinal is* $N(t^-)$*. Thus, the solution of (11)–(12) is given by* $x(t) = \Omega(t)x_0$*;*$t \in \boldsymbol{R}_{0+}$*, where, in general, its non-unique evolution operator is as follows:*

$$\begin{aligned} &\Omega(t) = \left(I_n + K(t^-)\hat{A}(t^-)\right)\Omega(t^-) \\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K(t_i^-)\hat{A}(t_i^-)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right);\, t \in \boldsymbol{R}_{0+} \end{aligned} \quad (14)$$

The evolution operator $\Omega : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ *is unique on* $[0, +\infty)$ *if and only if there is no sequence of impulsive gains of the matrix of dynamics* $\{K'(t_i^-)\}_{t_i \in IMP} \neq \{K(t_i^-)\}_{t_i \in IMP}$ *such that* $x(t_i^-) \in Ker\{(K(t_i^-) - K'(t_i^-))\hat{A}(t_i^-)\}$ *for at least one* $t_i \in IMP$*; equivalently, it is unique if and only if* $x(t_i^-) \notin Ker\{(K(t_i^-) - K'(t_i^-))\hat{A}(t_i^-)\}$ *for all* $t_i \in IMP$*. As a result, if there is at least one* $t_i \in IMP$ *such that* $x(t_i^-) \in Ker\{(K(t_i^-) - K'(t_i^-))\hat{A}(t_i^-)\}$*, with* $K'(t_i^-) \neq K(t_i^-)$*, then* $\Omega : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$ *is non-unique.*

(ii) *If* $t \in IMP$ *then* $t = t_{N(t)}$ *and* $IMP(t) = IMP(t^-) \cup \left\{t_{N(t)}\right\}$ *so that (14) becomes the following:*

$$\begin{aligned} \Omega(t) &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\Omega(t^-) \\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K(t_i^-)\hat{A}(t_i^-)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right) \end{aligned} \quad (15)$$

and

$$\Omega(t^-) = \left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\hat{A}\left(t_i^-\right)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right) \quad (16)$$

If $0 = t_1 \in IMP$*, then*

$$\Omega(t) = \left(I_n + K\left(t^-\right)\hat{A}\left(t^-\right)\right)\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\hat{A}\left(t_i^-\right)\right)\right]\right) \quad (17)$$

If $0 = t_1 \in IMP$ *and* $t \in IMP$*, then* $t = t_{N(t)}$*,* $t_{N(t^-)-1} = t_{N(t)-2}$*, and* $t_{N(t^-)} = t_{N(t)-1}$ *so that*

$$\begin{aligned}\Omega(t) &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\Omega(t^-)\\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\left(\prod_{i=1}^{N(t)-1}\left[e^{\int_{t_i}^{t_{i+1}} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\hat{A}\left(t_i^-\right)\right)\right]\right)\\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)e^{\int_{t_{N(t^-)-1}}^{t} A(\tau)d\tau}\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\hat{A}\left(t_i^-\right)\right)\right]\right)\end{aligned} \quad (18)$$

Proof. It turns out from (11)–(12) that

$$x(t) = e^{\int_{t_{i+1}}^{t} A(\tau)d\tau}x(t_{i+1})\text{if } t \in [t_{i+1}, t_{i+2})\text{and } t_{i+1}, t_{i+2}(> t_{i+1}) \in IMP \quad (19)$$

$$\begin{aligned}x(t_{i+1}) &= x\left(t_{i+1}^-\right) + \int_{t_{i+1}^-}^{t_{i+1}} \dot{x}(\tau)d\tau\\ &= x\left(t_{i+1}^-\right) + \int_{t_{i+1}^-}^{t_{i+1}} K(\tau)\hat{A}\left(t_{i+1}^-\right)x\left(t_{i+1}^-\right)\delta(t_{i+1} - \tau)d\tau\\ &= \left(I_n + K\left(t_{i+1}^-\right)\hat{A}\left(t_{i+1}^-\right)\right)x\left(t_{i+1}^-\right)\\ &= \left(I_n + K\left(t_{i+1}^-\right)\hat{A}\left(t_{i+1}^-\right)\right)e^{\int_{t_i}^{t_{i+1}} A(\tau)d\tau}x(t_i)\end{aligned} \quad (20)$$

The last identity arising since $A(t)$ is not impulsive; thus, (20) into (19) yields if $t \in [t_{i+1}, t_{i+2})$:

$$x(t^-) = e^{\int_{t_{i+1}}^{t} A(\tau)d\tau}x(t_{i+1}) = \left(e^{\int_{t_{i+1}}^{t} A(\tau)d\tau}\left(I_n + K\left(t_{i+1}^-\right)\hat{A}\left(t_{i+1}^-\right)\right)\right)e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}x(t_i) \quad (21)$$

and using recursive calculations for $t \in [t_{i+1}, t_{i+2}]$:

$$\begin{aligned}x(t) &= (I_n + K(t^-)\hat{A}(t^-))e^{\int_{t_{i+1}}^{t^-} A(\tau)d\tau}x(t_{i+1})\\ &= (I_n + K(t^-)\hat{A}(t^-))\left(e^{\int_{t_{i+1}}^{t^-} A(\tau)d\tau}(I_n + K(t_{i+1}^-)\hat{A}(t_{i+1}^-))\right)\left(e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\right)x(t_i)\\ &= (I_n + K(t^-)\hat{A}(t^-))\left[\left(e^{\int_{t_{i+1}}^{t^-} A(\tau)d\tau}(I_n + K(t_{i+1}^-)\hat{A}(t_{i+1}^-))\right)\left(e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}(I_n + K(t_i^-)\hat{A}(t_i^-))\right)\right]\left(e^{\int_{t_{i-1}}^{t_i^-} A(\tau)d\tau}\right)x(t_{i-1})\\ &= (I_n + K(t^-)\hat{A}(t^-))\left[\left(e^{\int_{t_{i+1}}^{t} A(\tau)d\tau}(I_n + K(t_{i+1}^-)\hat{A}(t_{i+1}^-))\right) \times \ldots \times \left(e^{\int_{t_1}^{t_2} A(\tau)d\tau}(I_n + K(t_1^-)\hat{A}(t_1^-))\right)\right]\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right)x_0\end{aligned} \quad (22)$$

which proves (13) for any given finite $x_0 = x(0)$ and any $t \in \boldsymbol{R}_{0+}$. This implies that $x(t) = \Omega(t)x_0$; $t \in \boldsymbol{R}_{0+}$ via the evolution operator (14). The solution is unique for given finite initial conditions because (22) is explicit and unique for finite given initial conditions. It turns out from (13) that the evolution operator that generates this solution for any given finite initial conditions is (14). From Theorem 1, this evolution operator is unique at each time interval $[t_i, t_{i+1})$ for any $t_i, t_{i+1} \in IMP$ and any finite initial condition x_0. It turns out that, as outlined below:

(a) the evolution operator is unique on $\cup_{t_i,t_{i+1}\in IMP}[t_i, t_{i+1})$ if $cardIMP = N_{imp} = \chi_0$, i.e., if there are infinitely many impulsive time instants;

(b) the evolution operator is unique on $\left(\cup_{t_i,t_{i+1}\in IMP}[t_i, t_{i+1})\right) \cup \left[t_{N_{imp}}, +\infty\right)$ if $N_{imp} < \chi_0$, i.e., if there is a finite number of impulsive time instants;

(c) the evolution operator is unique on $[0, +\infty)$ if, and only if, there is no sequence of impulsive gains $\left\{K'\left(t_i^-\right)\right\}_{t_i\in IMP} \neq \left\{K\left(t_i^-\right)\right\}_{t_i\in IMP}$ such that $x\left(t_i^-\right) \in Ker\left\{\left(K\left(t_i^-\right) - K'\left(t_i^-\right)\right)\hat{A}\left(t_i^-\right)\right\}$ for at least a $t_i \in IMP$. Otherwise, assume that there is $t_i \in IMP$ and another impulsive gain matrix $K'\left(t_i^-\right) \neq K\left(t_i^-\right)$ such that, given $x\left(t_i^-\right)$, the right limit of the solution at t_i is $x(t_i) = K\left(t_i^-\right)\hat{A}\left(t_i^-\right)x\left(t_i^-\right) = K'\left(t_i^-\right)\hat{A}\left(t_i^-\right)x\left(t_i^-\right)$. As a result, distinct evolution operators generate an identical solution at t_i and the evolution operator is not unique on $[0, +\infty)$. Then, the evolution operator $\Omega : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is unique on $[0, +\infty)$ if, and only if, for any sequence of impulsive gains $\left\{K'\left(t_i^-\right)\right\}_{t_i\in IMP}$ of the matrix of dynamics, such that $\left\{K'\left(t_i^-\right)\right\}_{t_i\in IMP} \neq \left\{K\left(t_i^-\right)\right\}_{t_i\in IMP}$, one has that $x\left(t_i^-\right) \in Ker\left\{\left(K\left(t_i^-\right) - K'\left(t_i^-\right)\right)\hat{A}\left(t_i^-\right)\right\}$ for all $t_i \in IMP$.

Then, Property (i) has been proved. Property (ii) follows directly for the mentioned particular cases of the evolution operator (14). □

Note that the global non-uniqueness of the evolution operator in Theorem 2 (i) on $[0, +\infty)$ is lost when there is one $t_i \in IMP$ for which there is some impulsive gain $K'\left(t_i^-\right) \neq K\left(t_i^-\right)$ such that $x(t_i) = K\left(t_i^-\right)\hat{A}\left(t_i^-\right)x\left(t_i^-\right) = K'\left(t_i^-\right)\hat{A}\left(t_i^-\right)x\left(t_i^-\right)$. In this case, the same solution is achieved for distinct impulsive gain matrices. Equivalently, the evolution operator is unique if, and only if, $K'\left(t_i^-\right) \neq K\left(t_i^-\right)$, one has that $x\left(t_i^-\right) \notin Ker\left\{\left(K\left(t_i^-\right) - K'\left(t_i^-\right)\right)\hat{A}\left(t_i^-\right)\right\}$ for each $x\left(t_i^-\right)$ for $t_i \in IMP$. We can easily see that, except in some particular cases, the global uniqueness of the evolution operator over time becomes lost.

Note also that the above impulsively parameterized system can be considered as a limit case of a switched system where switches are instantaneous at the switching time instants and the former configuration is re-established immediately after the impulsive actions. In this context, a switched differential system would be of the form $\dot{x}(t) = A_{\sigma(t)}x(t)$. Assume that $t_i \in IMP$, then, for $t \to t_i^-$, one has $\dot{x}\left(t_i^-\right) = A_{\sigma\left(t_i^-\right)}x\left(t_i^-\right)$ and for $t \to t_i^+ (say t_i)$, one has $\dot{x}(t_i) = A_{\sigma(t_i)}x(t_i) = \left[A_{\sigma\left(t_i^-\right)} + \Delta\left(A\left(t_i^-, K\left(t_i^-\right)\right)\right)\right]x(t_i)$.

Remark 1. *Some examples that visualize that the evolution operator of the proportional instantaneous finite-jumps system (11)–(12) is not always unique for all time are as follows*:

(a) *There is some* $t_i \in IMP$ *such that* $x\left(t_i^-\right) = 0$, *then* $x(t_i) = 0$ *for the scheduled impulsive gain* $K(t_i)$ *but also for any other impulsive gain* $K'\left(t_i^-\right) \neq K\left(t_i^-\right)$;

(b) *The order of the system is* $n \geq 2$, *the impulsive gain matrix sequence consists of diagonal matrices and there is one impulsive time instant* t_i *such that the first component is* $x_1\left(t_i^-\right) = 0$. *Then, any other diagonal gain matrix* $K'\left(t_i^-\right)$ *that coincides with the given one in all the diagonal entries except in the first one generates the same* $x(t_i)$ *with* $x_1(t_i) = 0$ *under all such gains, including for* $K\left(t_i^-\right)$.

Now, consider the following alternative impulsive system to (11)–(12), which is a non-proportional instantaneous finite-jumps system depending on the left values of the matrix of dynamics $A(t)$ for $t \in IMP$, while the impulsive actions are, instead, additive to $A(t^-)$ in the differential system:

$$\dot{x}(t) = (A(t) + \Delta(t))x(t);\ x(0) = x_0 \tag{23}$$

$$\Delta(t) = \sum_{t_i \in IMP(t)} K(t^-)\delta(t - t_i) \tag{24}$$

Note that $x(t_i) - x(t_i^-) = K(t_i)x(t_i^-)$ (instead of $K(t_i)\hat{A}(t_i^-)x(t_i^-)$ as it happened in (11)–(12)) if $t_i \in IMP$ and $x(t) = x(t^-)$ if $t \notin IMP$. Note that the increments between the right and the left values of the matrix of dynamics $A(t)$ at impulsive time instants are not proportional to its left value, in contrary to (11)–(12). This implies, in fact, that the feedback solution information from the left limit value of the matrix of the dynamics $A(t)$ value to its right limit value disappears in (23)–(24) with respect to (11)–(12). Thus, the following result is the direct counterpart one for the impulsive system (23)–(24) of Theorem 2:

Theorem 3. *The following properties hold*:

(i) *The unique solution of the instantaneous finite-jumps system (23)–(24) is given by the expression below:*

$$\begin{aligned} x(t) &= (I_n + K(t^-))\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right)x_0 \\ &= (I_n + K(t^-))x(t^-);\, t \in \boldsymbol{R}_{0+} \end{aligned} \tag{25}$$

where $t_0 = 0$; $K(t_0^-) = K(0^-)$; $x_0 = x(0)$; $\{t_i\}_{i\in\mathbf{Z}_+}$; $IMP(t^-) = \left\{t_1, t_2, \cdots, t_{N(t^-)}\right\}$ if $K(0^-) = 0$; and $IMP(t^-) = \left\{t_0 = 0, t_1, \cdots, t_{N(t^-)}\right\}$ if $K(0^-) \neq 0$ is the impulsive set of time instants on $[0, t)$ whose cardinal is $N(t^-)$. Thus, the solution of (23)–(24) is given by $x(t) = \Omega_a(t)x_0$;$t \in \boldsymbol{R}_{0+}$, where, in general, the non-unique evolution operator is now redefined as follows, with respect to its definition $\Omega(t)$ of Theorem 2:

$$\begin{aligned} \Omega_a(t) &= (I_n + K(t^-))\Omega_a(t^-) \\ &= (I_n + K(t^-))\left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right) \end{aligned} \tag{26}$$

The evolution operator $\Omega_a : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is unique on $[0, +\infty)$ if, and only if, there is no sequence of impulsive gains $\left\{K'\left(t_i^-\right)\right\}_{t_i\in IMP} \neq \left\{K\left(t_i^-\right)\right\}_{t_i\in IMP}$ of the matrix of dynamics such that $x\left(t_i^-\right) \in Ker\left\{\left(K\left(t_i^-\right) - K'\left(t_i^-\right)\right)\right\}$ for at least a $t_i \in IMP$. Equivalently, if there is at least a $t_i \in IMP$ such that $x\left(t_i^-\right) \in Ker\left\{\left(K\left(t_i^-\right) - K'\left(t_i^-\right)\right)\right\}$, with $K'\left(t_i^-\right) \neq K\left(t_i^-\right)$, then $\Omega_a : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is non-unique.

(ii) *If $t \in IMP$, then $t = t_{N(t)}$ and $IMP(t) = IMP(t^-) \cup \left\{t_{N(t)}\right\}$ so that (26) becomes the following:*

$$\Omega_a(t) = \left(\prod_{i=1}^{N(t)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right) \tag{27}$$

and

$$\Omega_a\left(t^-\right) = \left(e^{\int_{t_{N(t^-)-1}}^{t^-} A(\tau)d\tau}\right)\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right)\left(e^{\int_0^{t_1^-} A(\tau)d\tau}\right) \tag{28}$$

If $0 = t_1 \in IMP$, *then*

$$\Omega_a(t) = \left(I_n + K\left(t^-\right)\right)e^{\int_{t_{i+1}}^{t^-} A(\tau)d\tau}\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}^-} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right) \quad (29)$$

If $0 = t_1 \in IMP$ *and* $t = t_{N(t)} \in IMP$, *then*

$$\begin{aligned}\Omega_a(t) &= (I_n + K(t^-))\Omega_a(t^-) \\ &= (I_n + K(t^-))\left(\prod_{i=1}^{N(t)-1}\left[e^{\int_{t_i}^{t_{i+1}} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right) \\ &= (I_n + K(t^-))e^{\int_{t_{N(t^-)-1}}^{t} A(\tau)d\tau}\left(\prod_{i=1}^{N(t^-)-1}\left[e^{\int_{t_i}^{t_{i+1}} A(\tau)d\tau}\left(I_n + K\left(t_i^-\right)\right)\right]\right)\end{aligned} \quad (30)$$

3. Impulsive Time-Varying Differential Systems with Constant Point Delays and Their Evolution Operator

The results of Section 2 are reformulated for the case where the dynamics include a finite number of constant point delays, with the associated dynamic matrices being bounded and piecewise-continuous on $\boldsymbol{R}_{0+}$ in the more general setting. The impulsive actions can take place in both the delay-free dynamics and in all or some of the delayed dynamics.

Consider the following impulsive differential system of order n with proportional instantaneous finite jumps and with p, in general, incommensurate constant point delays h_i; $i \in \overline{p}$, subject to $h_i < h_j$ if $j, i(< j) \in \overline{p}$:

$$\dot{x}(t) = (A(t) + \Delta(t))x(t) + \sum_{i=1}^{p}(A_i(t) + \Delta_i(t))x(t - h_i) \quad (31)$$

$$\Delta(t) = \sum_{t_i \in IMPA(t)} K\left(t^-\right)\hat{A}\left(t^-\right)\delta(t - t_i) \quad (32)$$

$$\Delta_i(t) = \sum_{t_i \in IMPA_i(t)} K_i\left(t^-\right)\hat{A}_i\left(t^-\right)\delta(t - t_i) \quad (33)$$

for $i \in \overline{p}$, where the admissible function of the initial conditions $\varphi : \left[-h_p, 0\right] \to \boldsymbol{R}^n$, with $x(0) = \varphi(0) = x_0$, is bounded piecewise-continuous on $\left[-h_p, 0\right]$. The matrix function $A : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ of delay-free dynamics is bounded, piecewise-continuous, and Lebesgue-integrable on $\boldsymbol{R}_{0+}$ and the matrix functions $A_i : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$; $i \in \overline{p}$ associated with the delayed dynamics of the various delays are bounded piecewise-continuous. The matrix function $\hat{A} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is defined in such a way that $\hat{A}_{ij}(t^-) = A_{ij}(t^-)$ if the (i, j)-entry of $A(t^-)$ is impulsive and $\hat{A}_{ij}(t^-) = 0$, otherwise, for $i, j \in \overline{n}$. In the same way, $\hat{A}_{kij}(t^-) = A_{kij}(t^-)$ if the (i, j)-entry of $A_k(t^-)$ is impulsive and $\hat{A}_{kij}(t^-) = 0$, otherwise, for $i, j \in \overline{n}$ and $k \in \overline{p}$.

The above Equations (31)–(33) reflect that impulses also occur for the delayed terms.

It turns out that $K(t^-)\left(\in \boldsymbol{R}^{n\times n}\right) \neq 0$ iff $t \in IMPA$, and $K_i(t^-)\left(\in \boldsymbol{R}^{n\times n}\right) \neq 0$ iff $t \in IMPA_i$, where $IMPA = \left\{t \in \boldsymbol{R}_{0+} : K(t^-)\hat{A}(t^-) \neq 0\right\}$ is the impulsive set of time instants of $A : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$, and $IMPA_i = \left\{t \in \boldsymbol{R}_{0+} : K_i(t^-)\hat{A}_i(t^-) \neq 0\right\}$ is the impulsive set of time instants of $A_i : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ for $i \in \overline{p}$, and $IMPA(t) = \{\tau(\leq t) \in IMPA\}$, $IMPA(t^-) = \{\tau(< t) \in IMPA\}$, $IMPA_i(t) = \{\tau(\leq t) \in IMPA_i\}$, and $IMPA_i(t^-) = \{\tau(< t) \in IMPA_i\}$ for $i \in \overline{p}$.

The impulsive sets on time intervals $[t - \sigma, t]$, $[t - \sigma, t)$, and $(t - \sigma, t)$ of $A(t)$ are denoted, respectively, as $IMPA[t - \sigma, t]$, $IMPA[t - \sigma, t)$, $IMPA(t - \sigma, t]$, and $IMPA(t - \sigma, t)$ such that $IMPA[t - \sigma, t] = IMPA[t - \sigma, t) \cup \{t\}$ if $t \in IMPA$, $IMPA[t - \sigma, t] = IMPA(t - \sigma, t] \cup \{t - \sigma\}$ if $(t - \sigma) \in IMPA$, and $IMPA[t - \sigma, t] = IMPA(t - \sigma, t) \cup \{t - \sigma\} \cup \{t\}$ if $(t - \sigma), t \in IMPA$.

Close definitions apply "mutatis mutandis" for the impulsive set of time instants of $A_i(t)$ for $i \in \overline{p}$. The whole impulsive set for impulses in any matrix of dynamics is $IMP = \{t_k\}_{k\in\overline{cardIMP}} = IMPA \cup \left(\cup_{i\in\overline{p}} IMPA_i\right)$.

3.1. Trajectory Solution of the Differential Impulsive System with Delays

The following result is immediate from (31)–(33):

Proposition 2. *Assume that* $IMP = \{t_k\}_{k\in\overline{cardIMP}} = IMPA \cup \left(\cup_{i\in\overline{p}} IMPA_i\right)$ *with* IMP *can be either finite (i.e., the number of impulses is finite) or infinity numerable (i.e., the number of impulses is infinite but they are located through time) so that* $cardIMP \le \chi_0$. *Then, the unique solution of (31)–(33) for any given admissible function of the initial conditions* $\varphi : \left[-h_p, 0\right] \to \boldsymbol{R}^n$ *with* $x(0) = \varphi(0) = x_0$ *is as follows:*

$$x\left(t^-\right) = e^{\int_{t_k}^{t^-} A(\tau)d\tau} x(t_k) + \sum_{i=1}^{p} \int_{t_k}^{t^-} e^{\int_{\tau}^{t^-} A(\sigma)d\sigma} A_i(\tau) x(\tau - h_i) d\tau \tag{34}$$

$$x(t) = \left(I_n + K\left(t^-\right)\hat{A}\left(t^-\right)\right)x\left(t^-\right) + \sum_{i=1}^{p} K_i\left(t^-\right)\hat{A}_i\left(t^-\right)x\left(t^- - h_i\right) \tag{35}$$

for any $t \in [t_k, t_{k+1}]$, *provided that* $t_k, t_{k+1} \in IMPA \cup IMPA_i$.

In particular, if $t \in IMPA$ *and* $t \notin IMPA_i$ *for some* $i \in \overline{p}$ *then, one has, from (35)*

$$x(t) = \left(I_n + K\left(t^-\right)\hat{A}\left(t^-\right)\right)x\left(t^-\right)$$

If $t \in IMPA \cup (\cup_{i\in\Pi} IMPA_i)$ *for* $\Pi(\subseteq \overline{p})$, *then*

$$x(t) = \left(I_n + K\left(t^-\right)\hat{A}\left(t^-\right)\right)x\left(t^-\right) + \sum_{i\in\Pi} K_i\left(t^-\right)\hat{A}_i\left(t^-\right)x\left(t^- - h_i\right)$$

If $t \in \cup_{i\in\Pi} IMPA_i$ *for* $\Pi(\subseteq \overline{p})$ *and* $t \notin IMPA$, *then from (35),*

$$x(t) = x\left(t^-\right) + \sum_{i\in\Pi} K_i\left(t^-\right)\hat{A}_i\left(t^-\right)x\left(t^- - h_i\right).$$

Note that, since the impulsive matrix of any (delayed or not) particular matrix of dynamics is zero if the impulse does not affect such a matrix $A(.)$ and $A_{(.)}(.)$, then (35) is also valid for all the particular cases of Proposition 2.

The last additive term in (34) reflects the contribution of all the delayed impulsive terms of the form $x(t - h_i)$ on the time interval $[t_k - h_i, t - h_i]$ for all $i \in \overline{p}$. Each of them can be empty, for instance, if there is no impulse associated with $x(\tau)$ in such an interval.

A particular case of the above result is direct under the following assumption of constant time intervals of size in-between consecutive impulsive time instants:

Assumption 1. *Assume that* $IMP = \left\{t_i = iT : x(t_i) \ne x\left(t_i^-\right)\right\}$ *if* $0 \notin IMP$

, and $IMP = \{t_i\}_{i\in\mathbf{Z}_+} = \{(i-1)T\}_{i\in\mathbf{Z}_+}$ *if* $0 \in IMP$, *i.e., there are infinitely many impulses through time with a constant time interval* T *in-between each two consecutive impulses.*

Define $j_i(t) = max(j \in \boldsymbol{Z}_{0+} : jT \le t - h_i)$ for $i \in \overline{p} \cup \{0\}$ with $h_0 = 0$. Then, $t - h_i = j_i(t)T$ iff $(t - h_i) \in IMP$ so that $K(t^- - h_i) = K(j_i(t)T^-)(\ne 0)$ iff $(t - h_i) \in IMP$ and $j_i(t) < j_{i-1}(t)$ for $i \in \overline{p}$ and $t \in \boldsymbol{R}_{0+}$.

Define also for each $t \in \boldsymbol{R}_{0+}$ a proper, improper—which can be empty (if the impulsive set in $\left[t - h_p, t\right]$ is empty)—subset $\overline{p}(t)$ of $\overline{p} \cup \{0\}$ by the following:

$\overline{p}(t) = \{z \in \overline{p} \cup \{0\} : (t - h_z) \in IMP\}$ for $t \in \boldsymbol{R}_{0+}$, which is the indexing set of the impulsive time instants that occurred on $\left[t - h_p, t\right]$. As a result, $K(t^- - h_i) = K(j_i(t)T^-)$ is non-zero iff $i \in \overline{p}(t)$. As a result, $i \in \overline{p}(t)$ $(t - h_i) \in IMP$, implying that $t - h_i = j_i(t)T$.

The following result is an immediate consequence of Proposition 2 and Assumption 1:

Proposition 3. *Assume that* $IMP = \{t_i\}_{i \in \mathbf{Z}_+} = \{iT\}_{i \in \mathbf{Z}_+}$ *(Assumption 1), the unique solution of (31)–(33) for any given admissible function of the initial conditions* $\varphi : [-h_p, 0] \to \mathbf{R}^n$ *with* $x(0) = \varphi(0) = x_0$ *is as follows:*

$$x(t^-) = e^{\int_{kT}^{t^-} A(\tau)d\tau} x(kT) + \sum_{i=1}^{p} \int_{kT}^{t^-} e^{\int_{\tau}^{t^-} A(\sigma)d\sigma} A_i(\tau)x(\tau - h_i)d\tau \tag{36}$$

$$\begin{aligned} x(t) &= \left(I_n + K(t^-)\hat{A}(t^-)\right)x(t^-) + \sum_{i=1}^{p} K_i(t^-)\hat{A}_i(t^-)x(t^- - h_i) \\ &= \left(I_n + K(t^-)\hat{A}(t^-)\right)\left[e^{\int_{kT}^{t^-} A(\tau)d\tau} x(kT) + \sum_{i=1}^{p} \int_{kT}^{t^-} e^{\int_{\tau}^{t^-} A(\sigma)d\sigma} A_i(\tau)x(\tau - h_i)d\tau\right] + \sum_{i=1}^{p} K_i(t^-)\hat{A}_i(t^-)x(t^- - h_i) \end{aligned} \tag{37}$$

forany $t \in [kT, (k+1)T]$

The next assumption relaxes the constraint that the time interval between consecutive impulsive time instants is constant but is still assumed to be an integer multiple of a positive real constant.

Assumption 2. *If* $IMP \ni t_i = j_i T$ *for some positive integer* $j_i = j(i) \le i$ *, then all the impulses on the interval* $[t - h_p, t]$ *take place at time instants* $t_k = j_k T$ *subject to* $t - h_i \le t_k = j_k T \le t$.

Note from Assumption 2 the following:

(1) If $t \in IMP$, then all the impulsive time instants in $[t - h_p, t]$ are in a non-empty set $IMP[t - h_p, t]$ of cardinal $m = m(t) \ge 1$ and the impulsive time instants are of the form

$$\left\{t_i^1 = i_1 T = t_{i_1}, t_i^2 = i_2 T = t_{i_2}, \cdots, t = t_i = i_m T\right\}, \text{ where} i_k, i_j (< i_k) \in \mathbf{Z}_+ \text{for} j < k$$

(2) The impulsive sets in $[t - h_p, t]$ and $[t - h_p, t)$ can be empty and they verify the relations $IMP[t - h_p, t] = \begin{cases} IMP[t - h_p, t) \ if \ t \notin IMP \\ IMP[t - h_p, t) \cup \{t\} \ if \ t \in IMP \end{cases}$.

Note that the above impulsive sets are identical iff $t \notin IMP$ and that they are jointly empty iff there are no impulses in the time interval $[t - h_p, t)$ and $t \notin IMP$.

Assumption 2 implies that the impulsive time instants are (in general, non-consecutive) integer multiples of a constant time interval T rather than aperiodic [42–44]. The delays in that case are still, in general, incommensurate since they are not necessarily integer multiples of a minimum base delay h. However, the case of commensurate delays can be dealt with simply as a particular case with $h_i = ih$ for $i \in \overline{p}$. The following result follows directly from Proposition 2 and Assumption 2:

Proposition 4. *If Assumption 2 holds, then one has for any two consecutive* $t_k (= j_k T)$ *,* $t_{k+1} (= j_{k+1} T) \in IMP$ *that*

$$\begin{aligned} x(j_{k+1}T) &= (I_n + K(j_{k+1}T^-)\hat{A}(j_{k+1}{}^-))x(j_{k+1}T^-) + \sum_{i=1}^{p} K_i(j_{k+1}T^-)\hat{A}_i(j_{k+1}T^-)x(j_{k+1}T^- - h_i) \\ &= (I_n + K(j_{k+1}T^-)\hat{A}(j_{k+1}{}^-)) \\ &\times \left[e^{\int_{j_k T}^{j_{k+1}T^-} A(\tau)d\tau} x(j_k T) + \sum_{i=1}^{p} \int_{j_k T}^{j_{k+1}T^-} e^{\int_{\tau}^{j_{k+1}T^-} A(\sigma)d\sigma} A_i(\tau)x(\tau - h_i)d\tau\right] + \sum_{i=1}^{p} K_i(j_{k+1}T^-)\hat{A}_i(j_{k+1}T^-)x(j_{k+1}T^- - h_i) \end{aligned} \tag{38}$$

And, if $t \in [t_k, t_{k+1})$*, then*

$$x(t) = e^{\int_{j_k T}^{t} A(\tau)d\tau} x(j_k T) + \sum_{i=1}^{p} \int_{j_k T}^{t} e^{\int_{\tau}^{t} A(\sigma)d\sigma} A_i(\tau)x(\tau - h_i)d\tau \tag{39}$$

The evolution operator, which gives the solution for any given function of the initial conditions, follows directly from Proposition 2:

Proposition 5. *Assume that $IMP = \{t_k\}_{k\in \mathbf{Z}_+}$. Then, in general, the non-unique evolution operator that generates the solution (34)–(35) of the differential system (31)–(32) on $[t_k, t]$, with $t_k, t_{k+1} \in IMP$ being consecutive impulsive time instants and $t \in [t_k, t_{k+1}] \in \mathbf{R}_{0+}$, is defined recursively as follows:*

$$Z\left(t^-, t_k\right) = e^{\int_{t_k}^{t^-} A(\tau)d\tau} Z(t_k, 0) + \sum_{i=1}^{p} \int_{t_k}^{t^-} e^{\int_{\tau}^{t^-} A(\sigma)d\sigma} A_i(\tau) Z(\tau - h_i, 0)d\tau \tag{40}$$

$$Z\left(t, t^-\right) = \left(I_n + K\left(t^-\right)\hat{A}\left(t^-\right)\right) Z\left(t^-, t_k\right) + \sum_{i=1}^{p} K_i\left(t^-\right)\hat{A}_i\left(t^-\right) Z\left(t^- - h_i, 0\right) \tag{41}$$

and $Z(t,t) = I_n$, $Z(t,\tau) = 0_n$ for $t \in \mathbf{R}_{0+}$, and $Z(t, t^-) = I_n$ if $t \notin IMP$ for any $t, (\tau > t) \in \mathbf{R}_{0+}$. The evolution operator is, in general, clearly non-unique and time-differentiable in the open real interval $\cup_{t_i \in IMP}(t_i, t_{i+1})$. If the impulsive set of time instants is finite such that $t_M = \{max t : t \in IMP\} < +\infty$, then the evolution operator is time-differentiable in the open real interval $\left(\cup_{t_i\in IMP}(t_i, t_{i+1})\right) \cup (t_M, +\infty)$.

The non-uniqueness of the evolution operator can be addressed under similar considerations as those used in the proof of Theorem 2 based on the fact that, except for particular cases of the choices of the impulsive gains, the right limits of the solution at impulsive time instants are achievable for more than one impulsive gain from the given reached values of the solution of their left limits. Particular versions of the evolution operator of Proposition 2 follow directly from Propositions 3 and 4 under Assumptions 1 and 2. The particular evolution operators arising for the particular cases discussed in Proposition 2 follow directly from such a result and they are then not displayed explicitly in Proposition 5.

3.2. Some Results on the Solution Boundedness and the Global Stability

The subsequent result relies on the global boundedness of the solution of the impulsive time-delay system in the event that the matrix $A(t)$ is a stability matrix for all time, provided that a minimum threshold in-between consecutive impulsive time instants is respected under sufficient smallness in the norm terms of the matrices of dynamics and that of the impulsive matrix gain. A second part of the result relies on the global stabilization of the differential system by the appropriate choice of the impulsive matrix gain and the set of impulsive time instants, even if the matrix of delay-free dynamics is not stable for any time. The mechanism to achieve stabilization is that the impulses occur at a sufficiently fast rate with the impulsive gain entries being of appropriate signs and of sufficiently large amplitudes. For simplicity of the subsequent exposition, it is assumed that $IMPA_i = \varnothing$ for $i \in \overline{p}$ so that $IMP = IMPA$. This simplification does not imply an essential loss in generality since the resulting most general result could be easily reformulated.

Theorem 4. *The following properties hold*:

(i) *Assume the following:*

(1) $IMP = IMPA$;

(2) *for any $t \in \mathbf{R}_{0+}$, $T_{impt} = \sup_{t_k, t_{k+1}(\le t)\in IMP} (t_{k+1} - t_k) \le +\infty$ is the largest interval between consecutive impulsive time instants, $t_{k+1} \ge t_k + T_{km}$ up until the time instant "t" for some minimum k-inter-impulsive time interval threshold $T_{km}(\ge T_m) \in \mathbf{R}^+$ for any consecutive $t_k, t_{k+1}(> t_k) \in IMP$ with k-inter-impulse time interval $T_k = t_{k+1} - t_k$;*

(3) $a_t = \sup_{0 \le \tau < +\infty} \|A(\tau)\| < +\infty$, *with the supremum spectral radius of* $A(t)$ *satisfying* $\sup_{0 \le \tau < +\infty} \rho(A(\tau)) < 1$, *that is, all the eigenvalues of* $A(t)$ *have a negative real part for any* $t \in \mathbf{R}_{0+}$, $a_{dt} = \max_{i \in \overline{p}} \sup_{0 \le \tau \le t} \|A_i(\tau)\| < +\infty$, $k_t = \sup_{0 \le \tau \le t} \|K(\tau)\| < +\infty$, *and*

$$\underline{\theta}_k(t) = \left\| e^{\int_{t_k}^{t_{k+T_{km}}} A(\tau)d\tau} \right\| \le \theta_t = \sup_{t_k, (t_k < t_{k+1} \le t) \in IMP} \left(\left\| e^{\int_{t_k}^{t_{k+1}} A(\tau)d\tau} \right\| \right) < +\infty \tag{42}$$

Then, a sufficient condition for $\sup_{t \in \mathbf{R}_{0+}} \|x(t)\| < +\infty$ *for any given finite admissible function of the initial conditions is outlined below:*

$$\limsup_{t \to \infty} \left[\theta_t (1 + k_t a_t) \left(1 + p a_{dt} T_{impt} (1 + k_t) \right) \right] \le 1, \tag{43}$$

In addition, (43) holds under the stronger sufficiency-type condition

$$\theta_t \le \frac{1}{(1 + k_t a_t) \left(1 + p a_{dt} T_{impt} (1 + k_t) \right)} \tag{44}$$

for all $t \ge t_0$ *and some finite* $t_0 \in \mathbf{R}_+$.

In order for (44) to hold, a necessary condition is that $t_{k+1} \ge t_k + T_{km}$ *with the inter-impulsive minimum time-interval threshold* T_{km} *being sufficiently small, satisfying:*

$$\left\| e^{\int_{t_k}^{t_{k+T_{km}}} A(\tau)d\tau} \right\| (1 + k_t a_t)(1 + p a_{dt} T_{km} (1 + k_t)) < 1 \tag{45}$$

$$t_k, t_{k+1} (> t_k) \in IMP \text{ with } t_{k+1} = \max(\tau (< t) \in IMP)$$

(ii) *Assume that the conditions of Property (i) hold, except the one invoking that the constraint* $\sup_{0 \le \tau < +\infty} \rho(A(\tau)) < 1$ *holds; furthermore, assume, that for each* $t \in \mathbf{R}_{0+}$, *there exists at least a time instant* $\xi_{t_k} \in \left[t_k + T_{km}, t_k + T_{km} + h_p \right]$ *such that* $\hat{A}\left(\xi_{t_k}^-\right)$ *is non-singular and the maximum inter-impulse time interval* $\left(T_{km} + h_p\right)$ *is sufficiently small. The set of impulsive time instants is effective for an admissible impulsive matrix function gain, that is, if* $t_k \in IMP$, *then* $\hat{A}\left(\xi_{t_k}^-\right) x\left(\xi_{t_k}^-\right) \ne 0$. *Then, the solution can be made globally bounded for all time for any given admissible function of the initial condition—so that the differential system is globally stable—by means of appropriate choices of a sufficiently short choice of the impulsive set of time instants and the sign and amplitudes of the entries of the impulsive matrix function gain* $K : \mathbf{R}_{0+} \to \mathbf{R}^{n \times n}$. *It can also be achieved under more stringent conditions that* $\lim_{t_k (\in IMP)} x(t_k) = \lim_{t_k (\in IMP)} x\left(t_k^-\right) = 0$.

Proof. For any finite time instant $t \in \mathbf{R}_+$, define $x_t^- = \sup_{0 \le \tau \le t^-} \|x(\tau)\|$ and $x_t = \sup_{0 \le \tau \le t} \|x(\tau)\|$, and also define $\boldsymbol{x}^- = \sup_{0 \le t \le +\infty} \left| x_t^- \right|$ and $\boldsymbol{x} = \sup_{0 \le \tau \le +\infty} |x_t|$. Those four amounts depend on the parameterization, impulses, and initial conditions. One has from Proposition 2, and Equations (35) and (34) that

$$x_t \le (1 + k_t a_t) x_t^- \le (1 + k_t a_t) \theta_t \left(1 + p a_{dt} T_{impt} (1 + k_t) \right) x_{\alpha_k(t)} \tag{46}$$

$$x_t^- \le \theta_t \left(1 + p a_{dt} T_{impt} (1 + k_t) \right) x_{\alpha_k(t)} \le \theta_t \left(1 + p a_{dt} T_{impt} (1 + k_t) \right) (1 + k_t a_t) x_{\alpha_k(t)}^- \tag{47}$$

where $\alpha_k(t) = min(t - h_1, t_k(t))$ with $t_k(t) = max(t_k(< t) \in IMP)$. Note that $x_t \geq x_{\alpha_k(t)}$ and that $x_t^- \geq x_{\alpha_k(t)}^-$ by construction of the supremum since $0 < (t - \alpha_k(t)) \leq \gamma$ for all $t \in \boldsymbol{R}_{0+}$. It turns out from (46)–(47) that $\boldsymbol{x}^- < +\infty$ and $\boldsymbol{x} < +\infty$ if (43) holds, guaranteed by (44) for some finite $t_0 \in \boldsymbol{R}_+$, for which a necessary condition is that T_{km} should be sufficiently small according to (45) since $\theta_t \geq \underline{\theta}_k(t)$ and $T_{impt} \geq T_{km}$ since

$$\begin{aligned} \left\| e^{\int_{t_k}^{t_k+T_{km}} A(\tau)d\tau} \right\| &= \underline{\theta}_k(t) \leq \theta_t = \sup_{t_k,(t_{k+1} \leq t) \in IMP} \left(\left\| e^{\int_{t_k}^{t_{k+1}} A(\tau)d\tau} \right\| \right) \\ &< \frac{1}{(1 + k_t a_t)\big(1 + pa_{dt} T_{impt}(1 + k_t)\big)} \leq \frac{1}{(1 + k_t a_t)\big(1 + pa_{dt} T_{km}(1 + k_t)\big)} \end{aligned} \tag{48}$$

that is, only if (45) holds. Property (i) has been proved.

Now, Property (iii) is proved. For the entry-to-entry definition of the matrices $K(t) = \big(K_{ij}(t)\big)$ and $\hat{A}(t^-) = \big(\hat{A}_{ij}(t^-)\big)$, rewrite (35) equivalently, as follows, provided that $\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-)$ is non-zero for some $\updownarrow = \updownarrow(t) \in \overline{n}$ and each $t \in IMP$ so that

$$\begin{aligned} x_i(t) &= x_i(t^-) + \textstyle\sum_{j=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-) \\ &= x_i(t^-) + K_{i\updownarrow}(t^-)\textstyle\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-) + \sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-) \end{aligned} \tag{49}$$

that $x_i(t) = \lambda_i(t)x_i(t^-)$ with

$$\lambda_i(t) = 1 + K_{i\updownarrow}(t^-)\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-) + \sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-) \tag{50}$$

and $0 < |\lambda_i(t)| \leq \varsigma_i(t) \leq 1$ if

$$\begin{aligned} &-\Big(\varsigma_i(t) + 1 + \textstyle\sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-)\Big) \\ &\leq K_{i\updownarrow}(t^-)\textstyle\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-) \\ &\leq \varsigma_i(t) - 1 - \textstyle\sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-) \end{aligned} \tag{51}$$

that is,

$$\begin{aligned} &-\frac{\varsigma_i(t) + 1 + \sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-)}{\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-)} \leq K_{i\updownarrow}(t) \\ &\leq \frac{\varsigma_i(t) - 1 - \sum_{j(\neq \updownarrow)=1}^n \sum_{k=1}^n K_{ij}(t^-)\hat{A}_{jk}(t^-)x_k(t^-)}{\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-)} \end{aligned} \tag{52}$$

Now, note the following:

(1) In the event that $\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-) = 0$ for some $t \in IMP$ and all $\updownarrow = \updownarrow(t) \in \overline{n}$, then the formula (52) has division by zero and one has, equivalently, $\hat{A}(t^-)x(t^-) = 0$ and $x(t) = x(t^-)$ so that t is then an ineffective impulsive time instant (Definition 1, Proposition 1) irrespective of the value of the impulsive matrix gain;

(2) If the impulsive set IMP is effective irrespective of any non-zero matrix function gain $K(t)$, then $\hat{A}(t^-)x(t^-) \neq 0$ for all $t \in IMP$ (Definition 2, Definition 3, and Proposition 1). Thus, by virtue of the given hypothesis, $\hat{A}(t^-)x(t^-) \neq 0$ for all $t \in IMP$, equivalently $\sum_{k=1}^n \hat{A}_{\updownarrow k}(t^-)x_k(t^-) \neq 0$ so that (52) is well-posed without division by zero;

(3) Since $+\infty \geq T_{kM} = t_{k+1} - t_k \geq h_p$ and then

$$h_p \leq T_M = \sup_{t \in \boldsymbol{R}_{0+}} T_{impt} = \sup_{t \in \boldsymbol{R}_{0+}} \sup_{t_k, t_{k+1}(\leq t) \in IMP} (t_{k+1} - t_k) \leq +\infty$$

Thus, it follows as a result that if $x(\tau) = 0$ for $\tau \in [t, t + T_M]$, then $x(t) = 0$ for $t > T_M$, and the differential system has then been stabilized in finite time;

(4) If $x(\tau)$ is not identically zero on $[t_k + T_{km}, t_k + T_{km} + h_p]$ for any $t_k \in IMP$, then there is always (at least) an eligible effective impulsive time instant $\xi_{t_k}(= t_{k+1}) \in [t_k + T_{km}, t + T_{km} + h_p] \cap IMP$ subsequent to "t_k" by hypothesis and provided that $x\left(\xi_{t_k}^-\right) \neq 0$ since for each $t \in \boldsymbol{R}_{0+}$, there exists at least one such a time instant ξ_{t_k}, such that $\hat{A}\left(\xi_{t_k}^-\right)$ is non-singular and then $\hat{A}\left(\xi_{t_k}^-\right)x\left(\xi_{t_k}^-\right) \neq 0$ if $x\left(\xi_t^-\right) \neq 0$. Then, $K(t)$ can be defined so that at ξ_{t_k}, $K\left(\xi_t^-\right)\hat{A}\left(\xi_t^-\right)x\left(\xi_t^-\right) \neq 0$ with $\xi_{t_k} \in IMP$ being an effective impulsive time instant (see Definition 3 and Proposition 1) under the impulsive matrix gain $K(t)$ fulfilling the constraint (52) with the replacement $t \to \xi_{t_k}$. Thus, given an effective $t_k \in IMP$, the next $t_{k+1} \in IMP$ can be fixed as $t_{k+1} = t_k + T_k$ such that $T_k \geq T_{km}$ fulfils the following constraint:

$$\begin{aligned}&\max_{1\le i\le \overline{p}}\left|x_i\left(t_{k+1}^-\right)\right| = \left\|x\left(t_{k+1}^-\right)\right\|_\infty \le \\ &\max_{1\le i\le \overline{p}}\left(\left(\lambda_{ki} - \varsigma_i\left\|e_i^T e^{\int_{t_k}^{t_k^-+T_k} A(\tau)d\tau}\right\|_\infty\right)\left\|x\left(t_k^-\right)\right\|_\infty + \left\|\sum_{i=1}^p e_i^T\int_{t_k}^{t_k^-+T_k} e^{\int_\tau^{t_k^-+T_k} A(\sigma)d\sigma}A_i(\tau)x(\tau-h_i)d\tau\right\|_\infty\right)\end{aligned} \tag{53}$$

provided that $0 < \varsigma_i < min\left(1, \lambda_{ki}/\left\|e_i^T e^{\int_{t_k}^{t_k^-+T_k} A(\tau)d\tau}\right\|_\infty\right)$ for $i \in \overline{p}$, from (34)–(35) with $K_i \equiv 0$, since $IMPA_i = \varnothing$; $i \in \overline{p}$, by hypothesis, and provided that $K\left(t_k^-\right)$ fulfils the constraint (52). It turns out that (53) implies $\left\|x\left(t_{k+1}^-\right)\right\|_\infty \le \left(\max_{1\le i\le \overline{p}}\lambda_{ki}\right)\left\|x\left(t_k^-\right)\right\|_\infty$. Then, one has the following features:

(1) If the real sequence $\left\{\max_{1\le i\le \overline{p}}\lambda_{ki}\right\}_{k=1}^\infty \subset (0, 1]$, then the sequence $\left\{\left\|x\left(t_k^-\right)\right\|_\infty\right\}$ is bounded for any admissible function of the initial conditions. Since $\left\{\left\|K\left(t_k^-\right)\right\|\right\}_{k=1}^\infty$ is bounded, the sequence $\{\|x(t_k)\|_\infty\}$ is also bounded. Since the parameterization of the differential system is bounded on each inter-impulse time interval $[t_k, t_{k+1})$, then $\|x(t)\|_\infty$ is bounded in $\cup_{t_{k,}t_{k+1}\in IMP}[t_k, t_{k+1}]$ and then also in $cl\boldsymbol{R}_{0+}$. This proves Property (ii);

(2) If the real sequence $\left\{\max_{1\le i\le \overline{p}}\lambda_{ki}\right\}_{k=1}^\infty$ satisfies $\limsup_{k\to\infty}\left(\max_{1\le i\le \overline{p}}\lambda_{ki}\right) \le \lambda$ for some $\lambda \in (0, 1)$, then the sequence $\left\{\left\|x\left(t_k^-\right)\right\|_\infty\right\}$ is bounded for any admissible function of the initial conditions and it converges asymptotically to zero. Since $\left\{\left\|K\left(t_k^-\right)\right\|\right\}_{k=1}^\infty$ is bounded, then the sequence $\{\|x(t_k)\|_\infty\}$ is also bounded and it converges asymptotically to zero. Since the parameterization matrices of the differential system are bounded on each inter-impulse interval $[t_k, t_{k+1})$, which is finite unless the impulsive parameterization ends in finite time, then $\|x(t)\|_\infty$ is bounded in $\cup_{t_{k,}t_{k+1}\in IMP}[t_k, t_{k+1}]$ and then also in $cl\boldsymbol{R}_{0+}$. □

Theorem 4 can be directly extended to the general case when $\cup_{i\in\overline{p}}IMPA_i \neq \varnothing$ by an "ad hoc" re-statement of its proof by considering in (35) a non-null impulsive contribution $\sum_{i=1}^p K_i(t^-)\hat{A}_i(t^-)x(t^- - h_i)$ for $t \in IMP$ with $IMP = IMPA \cup \left(\cup_{i\in\overline{p}}IMPA_i\right)$.

Remark 2. *Assume for simplicity of the subsequent explanation that $IMP = IMPA$. It turns out that the impulsive system (31)–(33) provides relative jumps at the right-hand-side of impulsive time instants, related to its value at the left limit of the time instant, according to $x(t) = x(t^-) + K(t^-)\hat{A}(t^-)x(t^-)$ defined by the matrix $\left(I_n + K(t^-)\hat{A}(t^-)\right)$ but $t \in IMP$ is* *ineffective if* $K(t^-)\hat{A}(t^-)x(t^-)$ *and this concern depends on the solution itself. A way to be able to choose effective impulsive time instants in any case is the implementation of absolute jumps at the impulsive time instants independent of the values at the left limits as follows. If* $t \in IMP$*, then* $x(t) = x(t^-) + K(t^-)\hat{A}(t^-)$ *with the impulsive gain being a bounded real vector function* $K : \mathbf{R}_{0+} \to \mathbf{R}^n$ *with a support set consisting of isolated real points. In this case, any component of* $x(t)$ *can differ from its counterpart* $x(t^-)$ *by choosing to be non-zero for all the components of* $K(t^-)$ *provided that* $\hat{A}(t^-)$ *is not identically zero. This also facilitates the stabilization by an impulsive action through time by relaxing some of the conditions of Theorem 4. The finite jumps* $x(t) = x(t^-) + K(t^-)\hat{A}(t^-)$ *in the solution come from the subsequent modification of the impulsive differential system (31)–(33) with* $IMP = IMPA$:

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{p} A_i(t)x(t-h_i) + u(t);\ u(t) = \sum_{t_i \in IMPA(t)} \hat{A}(t^-)K(t^-)\delta(t-t_i)$$

where $u(t)$ *is an impulsive open-loop (i.e., without using feedback) control, contrarily to (31)–(33) [50], where, again, if* $IMP = IMPA$*, it can be rewritten as follows:*

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{p} A_i(t)x(t-h_i) + u(t); u(t) = \sum_{t_i \in IMPA(t)} K(t^-)\hat{A}(t^-)\delta(t-t_i)x(t^-)$$

where now $u(t)$ *is an impulsive state-feedback control.*

Remark 3. *Note that the extension of the impulsive delay-free differential system (23)–(24) to its delayed version might be viewed also as an impulsive open-loop controlled system of the form:*

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{p} A_i(t)x(t-h_i) + u(t); u(t) = \sum_{t_i \in IMPA(t)} K(t^-)\delta(t-t_i)$$

Note that the impulses are not state-dependent and are not related to abrupt modifications in the matrix of delay-free dynamics at the impulsive time instants.

Remark 4. *In the event that the system becomes time-invariant in a finite time* t_f*, then the impulsive actions can be removed in a finite time while keeping the stability as it is well-known. Assume that for* $t \geq t_f$ *the differential system is time-invariant as follows:*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} A_i x(t-h_i) + u(t) = \left(A + \sum_{i=1}^{p} A_i\right)x(t) + \sum_{i=1}^{p} A_i(x(t-h_i) - x(t)) + u(t)$$

It follows that it is globally asymptotically stable for any given finite admissible function of the initial conditions if there are no impulses for $t \geq t_f$, i.e., $u(t) = 0$ *for* $t \geq t_f$*, and, furthermore,*

(a) A *is a stability matrix and* $\sum_{i=1}^{p} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A} \right\|_2 < 1$ *(guaranteed if* $\max_{i \in \overline{p}} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A} \right\|_2 < p^{-1}$*);*

(b) Or $\left(A + \sum_{i=1}^{p} A_i\right)$ *is a stability matrix and* $\sum_{i=1}^{p} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A - \sum_{i=1}^{p} A_i} \right\|_2 < \frac{1}{2}$ *(guaranteed if* $\max_{i \in \overline{p}} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A - \sum_{i=1}^{p} A_i} \right\|_2 < \frac{1}{2p}$*).*

The above conditions guarantee that all the zeros of the characteristic equation $det\left(sI_n - A - \sum_{i=1}^{p} A_i e^{-h_i s}\right) = 0$ *are in* $Res < 0$*—that is, equivalently, that* $\left(sI_n - A - \sum_{i=1}^{p} A_i e^{-h_i s}\right)^{-1}$ *exists for* $Res \geq 0$*. As a result, the transfer matrix* $G(s) = \left(sI_n - A - \sum_{i=1}^{p} A_i e^{-h_i s}\right)^{-1} \in \mathbf{RH}_\infty^{n \times n}$*—that is, it is strictly stable. See, for instance, [1,4].*

The above remark is induced to guess that the global asymptotic stability of the time-varying differential system can be achieved in the absence of impulsive monitored parameterizations, or under a finite number of them, if a nominal time-invariant limit differential system of reference is globally asymptotically stable and the parametrical deviations from it of the current differential system are sufficiently small through time. The subsequent result relies on this feature.

Theorem 5. *Assume that the three hypotheses below hold*:

(1) $\widetilde{A}_i(t) = A_i(t) - A_i$; $i \in \overline{p} \cup \{0\}$ *are bounded piecewise-continuous for* $t \in \mathbf{R}_{0+}$, *where* $A_0 = A$, $\widetilde{A}_0(t) = \widetilde{A}(t) = A(t) - A = A_0(t) - A_0$ *for* $t \in \mathbf{R}_{0+}$ *and some given constant matrices* $A_i \in \mathbf{R}^{n\times n}$; $i \in \overline{p} \cup \{0\}$;

(2) *For some finite* $t_f \in \mathbf{R}_{0+}$ *and all* $t \geq t_f$, $\sup_{t \geq t_f} \left| \widetilde{A}_{i_{jk}}(t) \right| \leq E_{i_{jk}}$; $i \in \overline{p} \cup \{0\}$; $j,k \in \overline{n}$ *and* $E_i = \left(E_{i_{jk}}\right)$ *for* $i \in \overline{p} \cup \{0\}$; $j,k \in \overline{n}$;

(3) *Any of the two conditions below holds:*

(3.1) *A is a stability matrix and* $\sum_{i=0}^{p} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i + E_i}{\mathbf{i}\omega I_n - A} \right\|_2 < 1$, *or if* $\max_{i \in \overline{p}} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i+E_i}{\mathbf{i}\omega I_n - A} \right\|_2 < p^{-1}$;

(3.2) $\left(A + \sum_{i=1}^{p} A_i\right)$ *is a stability matrix and* $\sum_{i=0}^{p} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i + E_i}{\mathbf{i}\omega I_n - \sum_{i=0}^{p}(A_i + E_i)} \right\|_2 < \frac{1}{2}$ *or if* $\max_{i \in \overline{p}} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i + E_i}{\mathbf{i}\omega I_n - A - \sum_{i=1}^{p} A_i} \right\|_2 < \frac{1}{2p}$

where $E_i = \left(E_{i_{jk}}\right)$ *for* $i \in \overline{p} \cup \{0\}$.

Then, the following properties hold:

(i) *The nominal time-invariant differential system is globally asymptotically stable.*

$$\dot{x}_L(t) = Ax_L(t) + \sum_{i=1}^{p} A_i x_L(t - h_i);\ x_L(t) = \varphi(t)$$

with bounded piecewise-continuous initial conditions $x(t) = \varphi(t)$ *for* $t \in [-h_p, 0]$.
Also, the auxiliary differential system is as follows:

$$\dot{\overline{x}}(t) = A\overline{x}(t) + \sum_{i=0}^{p}(A_i + E_i)\overline{x}(t - h_i) = A\overline{x}(t) + \sum_{i=1}^{p} A_i\overline{x}(t - h_i) + \sum_{i=0}^{p} E_i\overline{x}(t - h_i)$$

with bounded piecewise-continuous initial conditions $x(t) = \varphi(t)$ *for* $t \in [-h_p, 0]$, $h_0 = 0$, $A_0 = 0$, $\widetilde{A}_0(t) = \widetilde{A}(t)$, *and* $E_i = \left(E_{i_{jk}}\right)$ *for* $i \in \overline{p} \cup \{0\}$; $j,k \in \overline{n}$ *is globally asymptotically stable;*

(ii) *The current differential system*

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{p}\left(A_i + \widetilde{A}_i(t)\right)x(t - h_i)$$

is globally asymptotically stable;

(iii) *Assume that* $E_i = \lambda_i \|A_i\|_2 I_n$ *for some constants* $\lambda_i \in \mathbf{R}_{0+}$; $i \in \overline{p} \cup \{0\}$ *with* $\lambda = \max_{i \in \overline{p} \cup \{0\}} \lambda_i$.
Then, Properties (i)–(ii) hold if any of the conditions below hold:

(3.1′) *A is a stability matrix and* $\sum_{i=0}^{p} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A} \right\|_2 < \frac{1}{1+\lambda}$ *(which is guaranteed under the sufficient condition* $\max_{i \in \overline{p}} \sup_{\omega \in \mathbf{R}_{0+}} \left\| \frac{A_i}{\mathbf{i}\omega I_n - A} \right\|_2 < \frac{1}{p(1+\lambda)}$*);*

(3.2′) $\left(A+\sum_{i=1}^{p}A_i\right)$ *is a stability matrix and* $\sum_{i=0}^{p}\sup_{\omega\in\mathbf{R}_{0+}}\left\|\frac{A_i}{\mathbf{i}\omega I_n-\sum_{i=0}^{p}(A_i+E_i)}\right\|_2<\frac{1}{2(1+\lambda)}$ *(which is guaranteed if* $\max_{i\in\overline{p}}\sup_{\omega\in\mathbf{R}_{0+}}\left\|\frac{A_i}{\mathbf{i}\omega I_n-A-\sum_{i=1}^{p}A_i}\right\|_2<\frac{1}{2p(1+\lambda)}$*).*

Proof. Property (i) follows since any of the conditions (3.1) or (3.2) guarantee: (a) the global asymptotic stability of the auxiliary system; and (b) the global stability of the time-invariant nominal system since it satisfies the constraints $0=\sup_{t\geq t_f}\left|\widetilde{A}_{i_{jk}}(t)\right|\leq E_{i_{jk}}$; $i\in\overline{p}\cup\{0\}$, $j,k\in\overline{n}$ included in any of the conditions (3.1) or (3.2). Property (i) has been proved.

To prove Property (ii), note that for $t\geq t_f$, one has, for any vector norm and corresponding induced matrix norm,

$$\left\|\dot{\overline{x}}(t)\right\|\leq\|A\|\|\overline{x}(t)\|+\sum_{i=1}^{p}\|A_i\|\|\overline{x}(t-h_i)\|+\sum_{i=0}^{p}\|E_i\|\|\overline{x}(t-h_i)\|$$

$$\begin{aligned}\|\dot{x}(t)\|&\leq\|A\|\|x(t)\|+\sum_{i=1}^{p}\|A_i\|\|x(t-h_i)\|+\sum_{i=0}^{p}\left\|\widetilde{A}_i(t)\right\|\|x(t-h_i)\|\\&\leq\|A\|\|x(t)\|+\sum_{i=1}^{p}\|A_i\|\|x(t-h_i)\|+\sum_{i=0}^{p}\|E_i\|\|x(t-h_i)\|\end{aligned}$$

Since the function of the initial conditions is the same for the current differential system and the auxiliary differential system, i.e., $x(t)=\overline{x}(t)=\varphi(t)$ for $t\in\left[-h_p,0\right]$, then

$$\|\dot{x}(t)\|\leq\left\|\dot{\overline{x}}(t)\right\|=\|A\|\|\overline{x}(t)\|+\sum_{i=1}^{p}\|A_i\|\|\overline{x}(t-h_i)\|+\sum_{i=0}^{p}\|E_i\|\|\overline{x}(t-h_i)\|\text{for}t\in\left[-h_p,0\right]$$

and by complete induction based on the above differential inequality, one has the following:

$$\left[(x(t)=\overline{x}(t))\wedge\left(\|\dot{x}(t)\|\leq\left\|\dot{\overline{x}}(t)\right\|\right);t\in\left[-h_p,0\right]\right]\Rightarrow\left(\|x(t)\|\leq\|\overline{x}(t)\|;t\geq-h_p\right)$$

Since from Property (i) $\overline{x}(t)\to 0$ and $\dot{\overline{x}}(t)\to 0$ as $t\to\infty$, and both the trajectory solution and its time first-derivative are bounded for all time, then the current differential system is also globally asymptotically stable—that is, the solution trajectory and its first time-derivative are bounded for all time and converge asymptotically to zero as time tends to infinity. Property (ii) has been proved. Property (iii) is a direct conclusion as a particular case of Properties [(i)–(ii)] under any of the conditions (3.1′) or (3.2′), which are "ad hoc" modifications of the conditions (3.1)–(3.2). □

A particular interest of the above result is when $A_i(t)\to A_i$ as $t\to\infty$ for $i\in\overline{p}\cup\{0\}$, that is, when the nominal differential system of reference is a limiting system of the current time-varying one.

A simple first-order example with a single point delay is described in detail to illustrate the mechanism of stabilization under impulsive monitored parameterizations. The example is illustrative to see, in an analytic way, the effects of the impulsive parameterization in the solution with the main focus on the stability context and how closed-loop stabilization can be achievable by an appropriate selection of the sign and amplitude of the impulsive gains at suitable chosen effective impulsive time instants. The role in the stabilization of the involvement of a finite or infinite number of impulsive time instants is also seen.

Example 1. *Consider the subsequent differential equation*:

$$\dot{x}(t)=a(t)x(t)+a_d(t)x(t-h)\tag{54}$$

With a finite delay h and impulsive effects in the function of delay-free dynamics, the impulsive set of time instants is $IMP = \{t_k\}_{k=1}^{\infty}$, $a : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is bounded piecewise-continuous in all the inter-impulsive intervals (t_k, t_{k+1})—with $t_{k+1} = t_k + T_k$—and $a_d : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is bounded piecewise-continuous. The function of the initial conditions $\varphi : [-h, 0] \to \boldsymbol{R}$ is bounded piecewise-continuous with $\varphi(0) = x(0) = x_0$. The solution is the following one for $t \in [t_k, t_{k+1})$ is given by the following:

$$\begin{aligned} x(t^-) &= e^{\int_{t_k}^{t^-} a(\tau)d(\tau)} x(t_k) + \int_{t_k}^{t^-} e^{\int_{\tau}^{t^-} a(\sigma)d\sigma} a_d(\tau) x(\tau - h) d\tau \\ &= \left(1 + k\left(t_k^-\right) a\left(t_k^-\right)\right) e^{\int_{t_k}^{t^-} a(\tau)d(\tau)} x\left(t_k^-\right) + \int_{t_k}^{t^-} e^{\int_{\tau}^{t^-} a(\sigma)d\sigma} a_d(\tau) x(\tau - h) d\tau \end{aligned} \tag{55}$$

Thus,

$$x\left(t_{k+1}^-\right) = \left(1 + k\left(t_k^-\right) a\left(t_k^-\right)\right) e^{\int_{t_k}^{t_{k+1}^-} a(\tau)d(\tau)} x\left(t_k^-\right) + \int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_{k+1}^-} a(\sigma)d\sigma} a_d(\tau) x(\tau - h) d\tau \tag{56}$$

Assume that the impulsive sequence of gains is claimed to guarantee that the matching objective $x\left(t_{k+1}^-\right) = \rho(t_k, t_{k+1}) x\left(t_k^-\right)$ at the left limits of the impulsive time instant holds, where $\{\rho(t_k, t_{k+1})\}_{k=1}^{\infty}$ is bounded and $\underset{t_k \to \infty}{limsup} \rho(t_k, t_{k+1}) \leq \rho$, $\underset{t_k \to \infty}{liminf} \rho(t_k, t_{k+1}) \geq -\rho$ where $\rho(\leq 1) \in \boldsymbol{R}_+$. Note that if $\rho = 1$, then there exists a finite $\lim_{t_k \to \infty} \left|x\left(t_k^-\right)\right|$. Then,

$$\left(\rho(t_k, t_{k+1}) - \left(1 + k\left(t_k^-\right) a\left(t_k^-\right)\right) e^{\int_{t_k}^{t_{k+1}^-} a(\tau)d(\tau)}\right) x\left(t_k^-\right) = \int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_{k+1}^-} a(\sigma)d\sigma} a_d(\tau) x(\tau - h) d\tau \tag{57}$$

equivalently,

$$k\left(t_k^-\right) a\left(t_k^-\right) e^{\int_{t_k}^{t_{k+1}^-} a(\tau)d(\tau)} x\left(t_k^-\right) = \left(\rho(t_k, t_{k+1}) - e^{\int_{t_k}^{t_{k+1}^-} a(\tau)d(\tau)}\right) x\left(t_k^-\right) - \int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_{k+1}^-} a(\sigma)d\sigma} a_d(\tau) x(\tau - h) d\tau \tag{58}$$

and equivalently, if $t_k \in IMP$, is effective, then

$$\begin{aligned} k\left(t_k^-\right) &= k\left(t_k^-, T_k\right) \\ &= a^{-1}\left(t_k^-\right) \left(\left(\rho(t_k, T_k) e^{-\int_{t_k}^{t_k^- + T_k} a(\tau)d(\tau)} - 1 \right) - x^{-1}\left(t_k^-\right) e^{-\int_{t_k}^{t_k^- + T_k} a(\tau)d(\tau)} \int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_k^- + T_k} a(\sigma)d(\sigma)} a_d(\tau) x(\tau - h) d\tau \right) \end{aligned} \tag{59}$$

Claim 1: *$\left\{\left|x\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ is bounded.*

Proof. Assume, on the contrary, that $\left\{\left|x\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ is unbounded. Then, for some large $M \in \boldsymbol{R}_+$, there exists a strictly increasing subsequence $\left\{\left|x\left(t_{k_j}^-\right)\right|\right\}_{j=1}^{\infty}$ of $\left\{\left|x\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ with $\left|x\left(t_{k_j}^-\right)\right| \leq cM$ for $c(\geq 1) \in \boldsymbol{R}_+$ but then $\left|x\left(t_{k_{j+1}}^-\right)\right| \leq \rho^{k_{j+1}-k_1}\left|x\left(t_{k_j}^-\right) \leq cM\right|\rho^{k_{j+1}-k_1}$ and $\underset{j \to \infty}{limsup}\left|x\left(t_{k_{j+1}}^-\right)\right| \leq cM \lim_{j \to +\infty} \rho^{k_{j+1}-k_1} (\in [0, cM])$ since $\rho \leq 1$ is a contradiction. Then, $\left\{\left|x\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ is bounded. □

Claim 2: *If $\rho = 1$, $\left\{\left|x\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ is bounded and it does not converge to zero. Furthermore, $\left\{\left|k\left(t_k^-\right)\right|\right\}_{k=1}^{\infty}$ is bounded, $\{|x(t_k)|\}_{k=1}^{\infty}$ is bounded, and $x : [-h, +\infty) \to \boldsymbol{R}$ is bounded if $cardIMP = \chi_0$ (i.e., if there are infinitely many impulses) and $0 < T_m < T_M = \underset{t_k, t_{k+1} \in IMP}{sup} (t_{k+1} - t_k) < +\infty$. Then, the differential Equation (54) is globally stable for any given admissible function of the initial conditions.*

Proof. The following $\{|x(t_k^-)|\}_{k=1}^{\infty}$ is bounded if $\rho = 1$ since it is bounded for $\rho \le 1$ (Claim 1). Furthermore, since each $t_k \in IMP$ is effective, $\{|x(t_k^-)|\}_{k=1}^{\infty}$ neither converges to zero nor has some null element. On the other hand, $\{|k(t_k^-)|\}_{k=1}^{\infty}$ is bounded from (59), $\{|x(t_k)|\}_{k=1}^{\infty}$ is bounded since $\{|k(t_k^-)|\}_{k=1}^{\infty}$ and $\{|x(t_k^-)|\}_{k=1}^{\infty}$ are bounded. Finally, since there are infinitely many impulses, and the time interval in-between any two consecutive impulses is finite by the hypotheses, then $x : [-h, +\infty) \to \boldsymbol{R}$ is bounded in $\cup_{t_k, t_{k+1} \in IMP}(t_k, t_{k+1})$ since $\{|x(t_k^-)|\}_{k=1}^{\infty}$ and $\{|x(t_k)|\}_{k=1}^{\infty}$ are bounded from the boundedness and piecewise-continuity of $a, a_d : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ in $\cup_{t_k, t_{k+1} \in IMP}(t_k, t_{k+1})$. The global stability of (54) follows as a result. □

Remark 5. *Note that if $cardIMP = \chi_0$ and the objective $x\left(t_{k+1}^-\right) = \rho(t_k, t_{k+1})x(t_k^-)$, where $\{\rho(t_k, t_{k+1})\}_{k=1}^{\infty}$ is bounded with $\underset{t_k \to \infty}{limsup}\rho(t_k, t_{k+1}) \le \rho$, $\underset{t_k \to \infty}{liminf}\rho(t_k, t_{k+1}) \ge -\rho$ is fixed with $\rho < 1$, then $\{|x(t_k^-)|\}_{k=1}^{\infty} \to 0$ and $\{|k(t_k^-)|\}_{k=1}^{\infty} \to +\infty$ from (59) since $\{|x(t_k^-)|\}_{k=1}^{\infty} \to 0$. This also implies that the members of IMP lose effectiveness as they tend to infinity (that is, the impulsive time instants tending to infinity are not effective). One then concludes that, in the case of injecting infinitely many impulses, the matching objective $x\left(t_{k+1}^-\right) = \rho(t_k, t_{k+1})x(t_k^-)$ with $\rho < 1$ is not appropriate in practice since the sequence of impulsive gains diverges.*

Claim 3: *Assume that either*

(1) *$cardIMP = \chi_0$ (there are infinitely many impulses) and either the matching objective $x\left(t_{k+1}^-\right) = \rho(t_k, t_{k+1})x(t_k^-)$, where $\{\rho(t_k, t_{k+1})\}_{k=1}^{\infty}$ is bounded, $\underset{t_k \to \infty}{limsup}\rho(t_k, t_{k+1}) = -\underset{t_k \to \infty}{liminf}\rho(t_k, t_{k+1}) = 1$ and the inter-impulses intervals are subject to finite lower-bounds and upper-bounds;*

(2) *Or the above matching objective is not fixed, $cardIMP < \chi_0$ (there is a finite number of impulses), there is a finite time instant T for which $max(t \in \boldsymbol{R}_{0+} : t \in IMP) \le T$.*

Then, the set IMP can be defined with all the impulsive time instants being effective provided that there is no $t \in \boldsymbol{R}_{0+}$ such that $x(\tau) \equiv 0$ in the time interval $[t, t+h]$.

Proof. Assume that $t_k \in IMP$ and take $\vartheta \ge T_m$ such that

$$x(t_k^- + \vartheta) = e^{\int_0^{\vartheta} e^{a(t_k+\tau)}d\tau}x(t_k) + \int_0^{\vartheta} e^{\int_{\tau}^{\vartheta} e^{a(t_k+\vartheta-\sigma)}d\sigma} a_d(t_k+\tau)x(t_k+\tau-h)d\tau = 0 \quad (60)$$

so that $t_{k+1}^0 = t_k + \vartheta$ is not an effective impulsive time instant. Now, if $T_M = T_m + h$, then the above constraint holds for all $\theta \in [T_m, T_m + h]$ and there is no impulsive action on $[t_k, t_k + T_m + h]$, and then $x(t) \equiv 0$ for $t \ge t_k + T_m$. Thus, if there is no $t \in [t_k + T_m, t_k + T_m + h]$ such that $x(t) \equiv 0$ for $t \in [t_k, t_k + T_m + h]$, then it is guaranteed that there is some $t_{k+1} \in [t_k + T_m, t_k + T_m + h]$ such that $x\left(t_{k+1}^-\right) \ne 0$. Such a t_{k+1} is eligible as the next impulsive time instant to $t_k \in IMP$ since $T_M = T_m + h$ is finite and $t_{k+1} \in [T_m, T_M]$ and it is compatible with the matching objective for $\rho = 1$ (see Claim 2). □

Remark 6. *Note that Claim 3 relies on the fact that, if there are infinitely many impulses, then all of them can be chosen to be effective under the first set of constraints of the claim. In the case of a finite number of impulses, they can be chosen to be effective.*

It has been seen that, if $\rho < 1$, then $|x(t_k^-)| \to 0$ as $k \to +\infty$ asymptotically at an exponential rate. Then, $\{x(t_k^-)\}_{k=1}^{\infty}$ and $\{x(t_k)\}_{k=1}^{\infty}$ are bounded but the boundedness of the sequence of impulsive gains $\{k(t_k^-)\}_{k=1}^{\infty}$ fails. This problem is avoided in the case of absolute jumps at impulsive time instants of the form $x(t_k) = x(t_k^-) + k(t_k^-)a(t_k^-)$ in the solution generated from absolute impulses in the parameterization of $a(t)$ (see Remark 2). In this case, (59) is replaced with

$$k\left(t_k^-\right) = a^{-1}\left(t_k^-\right)\left(\left(\rho(t_k, T_k)e^{-\int_{t_k}^{t_k^-+T_k} a(\tau)d(\tau)} - 1\right) - e^{-\int_{t_k}^{t_k^-+T_k} a(\tau)d(\tau)}\int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_k^-+T_k} a(\sigma)d(\sigma)} a_d(\tau)x(\tau-h)d\tau\right) \tag{61}$$

and the reasoning on the boundedness of the sequences $\left\{x\left(t_k^-\right)\right\}_{k=1}^{\infty}$, $\{x(t_k)\}_{k=1}^{\infty}$, and $\left\{k\left(t_k^-\right)\right\}_{k=1}^{\infty}$ of Claim 2 remain valid even if $\rho < 1$ since the requirement for the impulsive time instants to be effective is no longer needed.

The solution of (54) might be re-stated in an equivalent form to (55) on the right limits of the time instants as follows. If $t_k \in IMP$ and $t \in [t_k, t_{k+1}]$, then $t_{k+1} \in IMP$ is the next consecutive time instant to t_k. Then, one has that

$$\begin{aligned} x(t_{k+1}) &= \left(1 + k\left(t_{k+1}^-\right)a\left(t_{k+1}^-\right)\right)x\left(t_{k+1}^-\right) = \left(1 + \left|k\left(t_{k+1}^-\right)\right|\left|a\left(t_{k+1}^-\right)\right| sgnk\left(t_{k+1}^-\right) sgna\left(t_{k+1}^-\right)\right)x\left(t_{k+1}^-\right) \\ &= \left(1 + k\left(t_{k+1}^-\right)a\left(t_{k+1}^-\right)\right)\left[e^{\int_{t_k}^{t_{k+1}^-} a(\tau)d(\tau)} x(t_k) + \int_{t_k}^{t_{k+1}^-} e^{\int_{\tau}^{t_{k+1}^-} a(\sigma)d\sigma} a_d(\tau)x(\tau-h)d\tau\right] \end{aligned} \tag{62}$$

with sgn(0) defined as zero and $sgn(x) = x/|x|$ if $x \neq 0$. It is direct to see that if $a\left(t_{k+1}^-\right) \neq 0$ and $k\left(t_{k+1}^-\right)$ is chosen with $sgnk\left(t_{k+1}^-\right) = sgna\left(t_{k+1}^-\right)$, then $|x(t_{k+1})| > \left|x\left(t_{k+1}^-\right)\right|$ and if $sgnk\left(t_{k+1}^-\right) = -sgna\left(t_{k+1}^-\right)$, and then $|x(t_{k+1})| < \left|x\left(t_{k+1}^-\right)\right|$. Define $T_k = t_{k+1} - t_k \in [T_m, T_{kM}]$ for $t_k, t_{k+1} \in IMP$, and

$$\alpha(t_k, \delta) = (1 + k(t_k + \delta)a(t_k + \delta))$$

$$\beta(t_k, \delta) = \beta(\overline{x}_{t_k}, \delta) = e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)} x(t_k) + \int_{t_k}^{t_k^-+\delta} e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma} a_d(\tau)x(\tau-h)d\tau$$

for $\delta \in (0, T_k)$. As a result, one has that

$$x(t_k + \delta) = \alpha(t_k, \delta)\beta(t_k, \delta)$$

Note also that

$$\overline{\beta}(t_k, T_k) = \sup_{0<\delta<T_k}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \max_{t_k\le\tau\le t_k+\delta}|a_d(\tau)|\right) \sup_{t_k\le\tau<t_k+\delta}|x(\tau)| \geq \sup_{0<\delta<T_k} \beta(t_k, \delta)$$

Now, given $t_k \in IMP$ and $\zeta_k \in (0, \zeta]$, if one chooses the next impulsive time instant $t_{k+1}(\in IMP) = t_k + T_k$ with $T_k = inf\left(\delta \geq T_m : |\alpha(t_k, \delta)|\left(\overline{\beta}(t_k, T_k)/\sup_{t_k\le\tau<t_k+\delta}|x(\tau)|\right) \leq \zeta_k\right)$ and then $|x(t_{k+1})| \leq \zeta_k \sup_{0\le\tau<T_k} |x(t_k + \tau)|$ and also, as a result, if $\limsup_{t_k\to+\infty}\zeta_k \leq 1$, then $\sup_{0\le t<+\infty}(|x(t)|) < +\infty$. The choice of $k\left(t_{k+1}^-\right)$ is made as follows in order to satisfy the constraint:

$$\left|1 + k\left(t_{k+1}^-\right)a\left(t_{k+1}^-\right)\right| \leq \zeta_k\left[1/\sup_{0<\delta<T_k}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \max_{t_k\le\tau\le t_k+\delta}|a_d(\tau)|\right)\right]$$

in such a way that

(a) If $\zeta_k \leq \underset{0<\delta<T_k}{sup}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \underset{t_k\leq\tau\leq t_k+\delta}{max}|a_d(\tau)|\right)$, then $k\left(t_{k+1}^-\right)$ is chosen to satisfy the following:

$$sgn\left(k\left(t_{k+1}^-\right)\right) = -sgn\left(a\left(t_{k+1}^-\right)\right) \tag{63}$$

$$\frac{1}{\left|a\left(t_{k+1}^-\right)\right|}\left(1 - \frac{\zeta_k}{\underset{0<\delta<T_k}{sup}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \underset{t_k\leq\tau\leq t_k+\delta}{max}|a_d(\tau)|\right)}\right) \leq \left|k\left(t_{k+1}^-\right)\right| \leq 1 \tag{64}$$

or

(b) If $\zeta_k \geq \underset{0<\delta<T_k}{sup}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \underset{t_k\leq\tau\leq t_k+\delta}{max}|a_d(\tau)|\right)$, then $k\left(t_{k+1}^-\right)$ is chosen to satisfy:

$$sgn\left(k\left(t_{k+1}^-\right)\right) = sgn\left(a\left(t_{k+1}^-\right)\right) \tag{65}$$

$$\left|k\left(t_{k+1}^-\right)\right| \geq \frac{1}{\left|a\left(t_{k+1}^-\right)\right|}\left(\frac{\zeta_k}{\underset{0<\delta<T_k}{sup}\left(\left|e^{\int_{t_k}^{t_k^-+\delta} a(\tau)d(\tau)}\right| + \delta\left|e^{\int_{\tau}^{t_k^-+\delta} a(\sigma)d\sigma}\right| \underset{t_k\leq\tau\leq t_k+\delta}{max}|a_d(\tau)|\right)} - 1\right) \tag{66}$$

Note that the constraint $\underset{t_k\to+\infty}{limsup}\zeta_k \leq 1$ can also be guaranteed with the appropriate choice of the impulsive gain sequence while fixing a sufficiently small inter-impulse minimum threshold interval T_m and the choice of T_k approaching to T_m as k tends to infinity. This threshold can be fixed greater as $a(t)$ becomes negative on relevant intervals of time.

4. Conclusions

The first part of this study has derived explicit expressions for the evolution associated with the state-trajectory solution of a class of linear time-varying differential delay-free systems. The impulsive-free part of its matrix function of dynamics of such systems has been assumed bounded and piecewise-continuous Lebesgue-integrable for all time.

The cases of the absence and the presence of impulsive actions in the system matrix of dynamics are described. In the impulsive case, the evolution operator is seen to be, in general, non-unique. Subsequently, the above obtained results are extended to the presence of delayed dynamics associated with constant point delays, and the evolution operators that generate the trajectory solution are given in an explicit fashion, while they are non-unique, in general, in the impulsive case.

The parameterization impulsive actions at certain time instants can take place in the delay-free dynamics and also in the various matrices of delayed dynamics followed by an immediate return to the previous configuration. The impulsive actions are interpreted as an instantaneous abrupt switching change in the parameterization. Furthermore, the parameterization impulsive actions might be, in general, non-unique in the sense that, depending on the left limits of the solution values at impulsive time instants, the necessary impulsive gains for monitoring the instantaneous switched parameterizations can be non-unique for the achievement of a certain suitable right limit of the solution trajectory.

Next, the boundedness of the solution trajectory of the delayed impulsive differential system is also investigated. It has been seen that an appropriate distribution of the impulsive time instants is relevant for the potential stabilization of the delayed differential system, even in the case when the delay-free dynamics is unstable.

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References

1. Niculescu, S.I. *Delay Effects on Stability. A Robust Control Approach*; Springer: Berlin/Heidelberg, Germany, 2001.
2. Michiels, W.; Van Assche, V.; Niculescu, S.-I. Stabilization of time-delay systems with a Controlled time-varying delay and applications. *IEEE Trans. Autom. Control* **2005**, *50*, 493–504. [CrossRef]
3. De la Sen, M. About Robust Stability of Caputo Linear Fractional Dynamic Systems with Time Delays through Fixed Point Theory. *Fixed Point Theory Appl.* **2011**, *2011*, 867932. [CrossRef]
4. Delasen, M. On some structures of stabilizing control laws for linear and time-invariant systems with bounded point delays and unmeasurable states. *Int. J. Control* **1994**, *59*, 5298–5541.
5. Zhang, X.; Han, Q.; Seuret, A.; Gouaisbaut, F.; He, Y. Overview of recent advances in stability of linear systems with time-varying delays. *IET Control. Theory Appl.* **2019**, *13*, 1–16. [CrossRef]
6. Shen, J.; Lam, J. On ℓ_∞ and L_∞ gains for positive systems with bounded time-varying delays. *Int. J. Syst. Sci.* **2015**, *46*, 1953–1960. [CrossRef]
7. Guajardo-Benavides, E.J.; Arteaga, M.A. On the finite time force estimation for bilateral teleoperation of robot manipulators with time varying delays. *Control Eng. Pract* **2023**, *138*, 105622. [CrossRef]
8. Feng, Z.; Zhang, X.; Lam, J.; Fan, C. Estimation of reachable set for switched singular systems with time-varying delay and state jump. *Appl. Math. Comput.* **2023**, *456*, 128132. [CrossRef]
9. Qian, M.; Sun, C.; Wang, C.; Yan, X.; Bo, C. Control of time-delay interconnected nonlinear systems under input saturations: A broad learning system-based approach. *Int. J. Syst. Sci.* **2023**, *55*, 1037–1055. [CrossRef]
10. Zhang, C.-K.; He, Y.; Jiang, L.; Wang, Q.-G.; Wu, M. Stability Analysis of Discrete-Time Neural Networks With Time-Varying Delay via an Extended Reciprocally Convex Matrix Inequality. *IEEE Trans. Cybern.* **2017**, *47*, 3040–3049. [CrossRef]
11. Burton, T.A. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*; Dover Publications Inc.: Mineola, NY, USA, 1985.
12. De la Sen, M. A study of the stability of integro-differential Volterra-type systems of equations with impulsive effects and point delay dynamics. *Mathematics* **2024**, *12*, 960. [CrossRef]
13. Yeniçerioğlu, A.F. Stability of linear impulsive neutral delay differential equations with constant coefficients. *J. Math. Anal. Appl.* **2019**, *479*, 2196–2213. [CrossRef]
14. Jin, S. Existence and uniqueness of solutions for the stochastic Volterra-Levin equation with variable delays. *Open Math.* **2022**, *20*, 769–782. [CrossRef]
15. Pani, A.K. Semigroup Theory and Evolution Equations, Internet Publication. Available online: https://www.researchgate.net/profile/Amiya-Pani/publication/303569377_Semigroup_Theory_and_Evolution_Equations/links/5748fd8208ae2e0dd30169c3/Semigroup-Theory-and-Evolution-Equations.pdf (accessed on 20 December 2024).

16. Beffa, F. *Weakly Nonlinear Systems with Applications in Communication Systems*; Springer Nature Switzerland AG: Cham, Switzerland, 2024.
17. Engel, K.G.; Nagel, R.; Brendle, S. *One-Parameter Semigroups for Linear Evolution Equations*; Springer: Berlin/Heidelberg, Germany, 2013.
18. Missier, S.P.; Rodrigo, P.A. Strongly α^*-continuous functions in topological spaces. *IOSR J. Math.* **2014**, *10*, 55–60. [CrossRef]
19. Janković, D.S.; Konstadilaki-Savvopoulou, C. On α-continuous functions. *Math. Bohem.* **1992**, *117*, 259–270. [CrossRef]
20. Chang, S.S.; Cho, Y.J.; Zhou, H. Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings. *J. Korean Math. Soc.* **2001**, *38*, 1245–1260.
21. De la Sen, M. On the evolution operators of a class of linear time-delay systems. *Mathematics* **2024**, *12*, 3575. [CrossRef]
22. Yang, Z.; Xu, D. Stability Analysis and Design of Impulsive Control Systems With Time Delay. *IEEE Trans. Autom. Control* **2007**, *52*, 1448–1454. [CrossRef]
23. Li, X.; Li, P. Stability of time-delay systems with impulsive control involving stabilizing delays. *Automatica* **2021**, *124*, 10933. [CrossRef]
24. Li, X.; Song, S. Stabilization of Delay Systems: Delay-Dependent Impulsive Control. *IEEE Trans. Autom. Control.* **2016**, *62*, 406–411. [CrossRef]
25. Zhang, Y.; Sun, J.; Feng, G. Impulsive Control of Discrete Systems With Time Delay. *IEEE Trans. Autom. Control* **2009**, *54*, 830–834. [CrossRef]
26. Sau, N.H.; Thuan, M.V.; Phuong, N.T. Exponential stability for discrete-time impulsive positive singular system with time delays. *Int. J. Syst. Sci.* **2024**, *55*, 1510–1527. [CrossRef]
27. Chen, W.-H.; Xu, W.; Zheng, W.X. Sliding-mode-based impulsive control for a class of time-delay systems with input disturbance. *Automatica* **2024**, *164*, 111633. [CrossRef]
28. Yu, Z.; Ling, S.; Liu, P.X. Exponential stability of time-delay systems with flexible delayed impulse. *Asian J. Control.* **2024**, *26*, 265–279. [CrossRef]
29. Zhang, J.; Nie, X. Multiple exponential stability for short memory fractional impulsive Cohen-Grossberg neural networks with time delays. *Appl. Math. Comput.* **2024**, *486*, 129066. [CrossRef]
30. Zhu, C.; Han, X.; Li, X. Analysis and Design of Time-Delay Impulsive Systems Subject to Actuator Saturation. *IEEE/CAA J. Autom. Sin.* **2024**, *11*, 196–204. [CrossRef]
31. Xu, J.; Liu, Y.; Meng, Q.; Cao, J.; Abdel-Aty, M. Unified stability criteria for impulsive stochastic delayed systems. *Nonlinear Anal. Hybrid Syst.* **2024**, *52*, 101474. [CrossRef]
32. Elmi, M. Advanced Non-Uniform Sampling Techniques for Energy-Efficient Data Acquisition. Ph.D. Thesis, University of Alberta, Edmonton, AB, Canada, 2024.
33. Wen, Y.; Vanden-Eijnden, E.; Peherstorfer, B. Coupling parameter and particle dynamics for adaptive sampling in Neural Galerkin schemes. *Phys. D Nonlinear Phenom.* **2024**, *462*, 134129. [CrossRef]
34. Xie, Y.; Bollapragada, R.; Byrd, R.; Nocedal, J. Constrained and composite optimization via adaptive sampling methods. *IMA J. Numer. Anal.* **2024**, *44*, 680–709. [CrossRef]
35. Cao, J.; Yang, Z.; Chen, X. Delay Coprime Sampling: A Simplified Sub-Nyquist Sampling for Noisy Multi-Sinusoidal Signals. *IEEE Signal Process. Lett.* **2024**, *31*, 1720–1724. [CrossRef]
36. Desmars, N.; Bonnefoy, F.; Grilli, S.; Ducrozet, G.; Perignon, Y.; Guérin, C.-A.; Ferrant, P. Experimental and numerical assessment of deterministic nonlinear ocean waves prediction algorithms using non-uniformly sampled wave gauges. *Ocean Eng.* **2020**, *212*, 107659. [CrossRef]
37. Zhang, X.-M.; Han, Q.-L.; Ge, X.; Ning, B.; Zhang, B.-L. Sampled-data control systems with non-uniform sampling: A survey of methods and trends. *Annu. Rev. Control* **2023**, *55*, 70–91. [CrossRef]
38. Khan, M.S. Strategies for Non-Uniform Rate Sampling in Digital Control Theory. Doctoral's Dissertation, Loughborough University, Loughborough, UK, 2010.
39. Khan, S.; Goodall, R.M.; Dixon, R. Non-uniform sampling strategies for digital control. *Int. J. Syst. Sci.* **2013**, *44*, 2234–2254. [CrossRef]
40. Albertos, P.; Salt, J. Non-uniform sampled-data control of MIMO systems. *Annu. Rev. Control* **2011**, *35*, 65–76. [CrossRef]
41. Hu, S.; Yue, D.; Du, Z.; Liu, J. Reliable $H\infty$ non-uniform sampling tracking control for continuous-time non-linear systems with stochastic actuator faults. *IET Control Theory Appl.* **2012**, *6*, 120–129. [CrossRef]
42. Delasen, M. A method for improving the adaptation transient using adaptive sampling. *Int. J. Control* **1984**, *40*, 639–665.
43. Delasen, M. Adaptive sampling for improving the adaptation transients in hybrid adaptive-control. *Int. J. Control* **1985**, *41*, 1189–1205. [CrossRef]
44. Delasen, M. Application of the non-periodic sampling to the identifiability and model-matching problems in dynamic-systems. *Int. J. Syst. Sci.* **1983**, *14*, 367–383.

45. Berberich, J.; Wildhagen, S.; Hertneck, M.; Allgöwer, F. Data-driven analysis and control of continuous-time systems under aperiodic sampling. *IFAC-PapersOnLine* **2021**, *54*, 210–215. [CrossRef]
46. Bansal, K.; Mukhija, P. Aperiodic sampled-data control of distributed networked control systems under stochastic cyber-attacks. *IEEE/CAA J. Autom. Sin.* **2020**, *7*, 1064–1073. [CrossRef]
47. Yan, Z.; Huang, X.; Cao, J. Variable-sampling-period dependent global stabilization of delayed memristive neural networks based on refined switching event-triggered control. *Sci. China Inf. Sci.* **2020**, *63*, 1–16. [CrossRef]
48. Rattanakul, C.; Chaiya, I. A mathematical model for predicting and controlling COVID-19 transmission with impulsive vaccination. *AIMS Math.* **2024**, *9*, 6281–6304. [CrossRef]
49. Rao, R.; Zhu, Q. Synchronization for Reaction–Diffusion Switched Delayed Feedback Epidemic Systems via Impulsive Control. *Mathematics* **2024**, *12*, 447. [CrossRef]
50. Kailath, T. *Linear Systems*; Prentice-Hall Ltd.: Englewood Cliffs, NJ, USA, 1980.
51. Zadeh, L.A.; Desoer, C.A. *Linear System Theory: The State Space Approach*; McGraw-Hill: New York, NY, USA, 1963.
52. Messiah, A. *Quantum Mechanics*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1961; Volume I.
53. Hassan, T.S.; Ahmed, R.G.; El-Sayed, A.M.A.; El-Nabulsi, R.A.; Moaaz, O.; Mesmouli, M.B. Solvability of a State–Dependence Functional Integro-Differential Inclusion with Delay Nonlocal Condition. *Mathematics* **2022**, *10*, 2420. [CrossRef]
54. Santra, S.S.; El-Nabulsi, R.A.; Khedher, K.M. Oscillation of Second-Order Differential Equations with Multiple and Mixed Delays under a Canonical Operator. *Mathematics* **2021**, *9*, 1323. [CrossRef]
55. Moaaz, O.; El-Nabulsi, R.A.; Bazighifan, O. Oscillatory behavior of fourt-order differential equations with neutral delay. *Symmetry* **2020**, *12*, 371. [CrossRef]
56. Moaaz, O.; El-Nabulsi, R.A.; Bazighifan, O.; Muhib, A. New Comparison Theorems for the Even-Order Neutral Delay Differential Equation. *Symmetry* **2020**, *12*, 764. [CrossRef]