The Ramanujan Journal (2025) 67:72
https://doi.org/10.1007/s11139-025-01083-w

Infinitely many cubic points on $X_0(N)/\langle w_d\rangle$ with N square-free

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Received: 10 June 2024 / Accepted: 24 March 2025

Abstract

We determine all modular curves $X_0(N)/\langle w_d\rangle$ that admit infinitely many cubic points over the rational field $\mathbb{Q}$, when N is square-free.

Keywords Modular curve · Cubic point · Quadratic forms · Degeneracy maps

Mathematics Subject Classification Primary 11G18 · Secondary 11G30 · 14G05 · 14H10 · 14H25

1 Introduction

A non-singular smooth curve C over a number field K of genus $g_C > 1$ always has a finite set of K-rational points $C(K)$ by a celebrated result of Faltings (here we fix once and for all $\overline{K}$, an algebraic closure of K). We consider the set of all points of degree at most n for C by $\Gamma_n(C, K) = \cup_{[L:K]\leq n} C(L)$ and exact degree n by $\Gamma'_n(C, K) = \cup_{[L:K]=n} C(L)$, where $L \subseteq \overline{K}$ runs over the finite extensions of K. A point $P \in C$ is said to be a point of degree n over K if $[K(P) : K] = n$.

The set $\Gamma_n(C, M)$ is infinite for a certain finite extension M/K if C admits a degree at most n map, defined over M, to a projective line $\mathbb{P}^1$ or an elliptic curve with positive M-rank. The converse is true for $n = 2$ [12], and under certain restrictions it is true for $n = 3$ [1] and $n = 4$ [1, 8] (cf. [19, Theorem 1.2] for a precise statement). If we

Francesc Bars Cortina is supported by PID2020-116542GB-I00.

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Published online: 26 May 2025

fix the number field M in the above results (i.e. an arithmetic statement for $\Gamma_n(C, M)$ with M fixed), we need a precise understanding over M of the set $W_n(C) = \{v \in \mathrm{Pic}^n(C) | h^0(C, \mathcal{L}_v) > 0\}$, where Pic^n is the usual n-Picard group and $\mathcal{L}_v$ the line bundle of degree n on C associated to v. If $W_n(C)$ contains no translates of abelian subvarieties with positive M-rank of $\mathrm{Pic}^n(C)$ then $\Gamma'_n(C, M)$ is finite (under the assumption that C admits no maps of degree at most n to $\mathbb{P}^1$ over M) (cf. [5, Theorem 4.2]).

For $n = 2$ the arithmetic statement for $\Gamma_n(C, K)$ follows from [1] (for a sketch of the proof and the precise statement see [2, Theorem 2.14]).

For $n = 3$, Daeyeol Jeon introduced an arithmetic statement (for modular curves) and its proof in [18] following [1] and [8]. In particular, for a nice curve C/K with a K-rational point, if $g_C \geq 3$ and C has no degree 3 or 2 map to $\mathbb{P}^1$ and no degree 2 map to an elliptic curve over $\overline{K}$, then the set of exact cubic points of C over K, $\Gamma'_3(C, K)$, is an infinite set if and only if C admits a degree 3 map to an elliptic curve over K with positive K-rank (cf. [19, Theorem 1.2] for a more precise statement).

Observe that if $g_C \leq 1$ (with $C(K) \neq \emptyset$ for $g_C = 1$), then C has a degree 3 map over K to $\mathbb{P}^1$, and thus $\Gamma'_3(C, K)$ is always an infinite set. Thus for curves C with $C(K) \neq \emptyset$ we restrict to $g_C \geq 2$ in order to study the finiteness of the set $\Gamma'_3(C, K)$. We now introduce a few more notations.

For any $N \in \mathbb{N}$, let $X_0(N)$ denote the modular curve corresponding to the group $\Gamma_0(N)$. The modular curve $X_0(N)$ is the coarse moduli space over $\mathbb{Q}$ of the isomorphism classes of elliptic curves E with a cyclic subgroup A of order N. For any $d||N$, the d-th (partial) Atkin-Lehner operator $w_d^{(N)}$ (acting on $X_0(N)$) is defined by the action of the matrix $\left(\begin{smallmatrix} dx & y \\ Nu & dv \end{smallmatrix}\right)$ such that $d^2xv - Nuy = d$. Then $w_d^{(N)}$ defines an involution on $X_0(N)$ given by

$$(E, A) \to (E/A_d, (E[d] + A)/A_d),$$

where $A_d := \ker([d] : A \to A)$. Let $X_0^{+d}(N) := X_0(N)/\langle w_d^{(N)} \rangle$ denote the quotient curve. Thus, there is a $\mathbb{Q}$-rational degree 2 quotient mapping $\varpi_d^{(N)} : X_0(N) \to X_0^{+d}(N)$. In [4], the authors computed all the values of N for which the curve $X_0^{+N}(N)$ has infinitely many cubic points over $\mathbb{Q}$. In this article, we determine all pairs (N, d) such that the curve $X_0^{+d}(N)$ has infinitely many cubic points over the rational field $\mathbb{Q}$ when N is a square-free integer and $1 < d < N$. Unless otherwise stated explicitly, throughout the article we always assume that N is a square-free positive integer. Since modular curves $X_0^{+d}(N)$ always have a $\mathbb{Q}$-rational point (cusp), we restrict ourselves to the cases where $g_{X_0^{+d}(N)} := \mathrm{genus}(X_0^{+d}(N)) \geq 2$. For simplicity of notation, we write w_d instead of $w_d^{(N)}$, the level N will be clear from the context. The main result of this article is the following:

Theorem 1.1 *Suppose $g_{X_0^{+d}(N)} \geq 2$. Then $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is infinite if and only if $g_{X_0^{+d}(N)} = 2$ or (N, w_d) is in the following list:*

$g_{X_0^{+d}(N)}$	(N, w_d)
3	$(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7),$ $(82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47}),$
4	$(66, w_{33}), (74, w_{37}), (86, w_{43})$
5	$(106, w_{53}), (158, w_{79})$
6	$(122, w_{61}), (166, w_{83})$
7	$(130, w_{65})$
8	$(178, w_{89})$
9	$(202, w_{101}), (262, w_{131})$

One of the key ideas to decide if the set $\Gamma'_3(X, \mathbb{Q})$ is finite or infinite is to check whether there is a $\mathbb{Q}$-rational degree 3 mapping between X and an elliptic curve E with positive $\mathbb{Q}$-rank. Using the ideas from §2 we can not directly compute the values of N and d such that the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is infinite. For example, we can not determine whether there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+p}(Np) \to E$, where p is a prime, $p \nmid N$ and $\mathrm{Cond}(E) = p$. In this article, based on the ideas of M.Derickx and P.Orlić from [9], in §3 we develop a criterion to check whether there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+p}(Np) \to E$, where p is a prime, $p \nmid N$ and $\mathrm{Cond}(E) = p$.

The MAGMA codes for computing the models of $X_0^{+d}(N)$ can be found at https://github.com/Tarundalalmath/Models-for-X0-N-d and the MAGMA codes for computing the points of $X_0^{+d}(N)$ over finite fields can be found at https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves.

2 General considerations

We recall the notion of gonality of a curve. Given a complete curve C over K, the gonality of C is defined as follows:

$$\mathrm{Gon(C)} := \min\{\deg(\varphi) \mid \varphi : \mathrm{C} \to \mathbb{P}^1 \text{defined over } \overline{\mathrm{K}}\}.$$

The following lemma plays a crucial role in deciding whether the set $\Gamma'_3(C, K)$ is infinite or not (cf. [4, Lemma 2]).

Lemma 2.1 *Suppose C has $\mathrm{Gon}(C) \geq 4$, $P \in C(K)$ and does not have a degree ≤ 2 map to an elliptic curve. If $\Gamma'_3(C, K)$ is an infinite set, then C admits a K-rational map of degree 3 to an elliptic curve with positive K-rank.*

The following results are well known:

Theorem 2.2 *(i).* ([10]) *The values of (N, d) such that $X_0^{+d}(N)$ is hyperelliptic are given in Table 4.*
(ii). ([15]) *The values of (N, d) such that $\mathrm{Gon}(X_0^{+d}(N)) = 3$ are given in Table 5.*
(iii). ([6]) *The values of (N, d) such that $X_0^{+d}(N)$ is bielliptic are given in Table 6.*

We say that a triple (N, d, E), where N is a natural number, $d||N$ and E is an elliptic curve over $\mathbb{Q}$ with positive $\mathbb{Q}$-rank, is admissible if there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+d}(N) \to E$.

Throughout this section we consider the values of N, d such that $\mathrm{Gon}(X_0^{+d}(N)) \geq 4$ and $X_0^{+d}(N)$ has no degree ≤ 2 map to an elliptic curve. By Lemma 2.1, the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is infinite if and only if the triple (N, d, E) is admissible for some elliptic curve E with positive $\mathbb{Q}$-rank.

The following lemma gives a criterion to rule out the triples which are not admissible (cf. [4, Lemma 4]).

Lemma 2.3 *If (N, d, E) is admissible, then:*

1. *E has conductor M with $M|N$ and for any prime $p \nmid N$ we have $|\overline{X}_0^{+d}(N)(\mathbb{F}_{p^n})| \leq 3|\overline{E}(\mathbb{F}_{p^n})|$ and $|\overline{X}_0(N)(\mathbb{F}_{p^n})| \leq 6|\overline{E}(\mathbb{F}_{p^n})|$, $\forall n \in \mathbb{N}$. $\forall n \in \mathbb{N}$, where $\overline{C}$ denotes the reduction of the curve C modulo the prime p.*
2. *if the conductor of E is N, then the degree of the strong Weil parametrization of E divides 6.*
3. *for any prime $p \nmid N$ we have $\frac{p-1}{12}\psi(N) + 2^{\omega(N)} \leq 6(p+1)^2$, where $\omega(N)$ is the number of prime divisors of N and $\psi(N) = N \prod_{\substack{q|N \\ q \text{ prime}}} (1 + 1/q)$ is the ψ-Dedekind function,*
4. *for any Atkin-Lehner involution w_r of $X_0(N)$ with $r \neq d$, we have $g_{X_0^{+d}(N)} \leq 3 + 2 \cdot g_{X_0^{+d}(N)/\langle w_r \rangle} + 2$.*

Corollary 2.4 *For $N > 623$, the triple (N, d, E) is not admissible.*

Proof The proof is similar to that of [4, Corollary 5]. ◻

We recall the following result (cf. [4, Lemma 6])

Lemma 2.5 *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N and $\varphi : X_0(N) \to E$ be the strong Weil parametrization of degree k defined over $\mathbb{Q}$. Suppose that w_N acts as +1 on the modular form f_E associated to E. Then φ factors through $X_0^{+N}(N)$ (and k is even).*

We now make a minor improvement to [4, Lemma 7].

Lemma 2.6 *Consider a degree k map $\varphi : X \to E$ defined over $\mathbb{Q}$, where X is a quotient modular curve $X_0(N)/W_N$ with W_N a proper subgroup of $B(N)$ ($B(N)$ is the subgroup of $Aut(X_0(N))$ generated by all Atkin-Lehner involutions). Assume that $cond(E) = M$ ($M|N$). Let $d \in \mathbb{N}$ with $d||M$, $(d, N/d) = 1$ and $w_d \notin W_N$, such that w_d acts as $+1$ on the modular form f_E associated to E. Then,*

1. *if E has no non-trivial 2-torsion over $\mathbb{Q}$, then φ factors through $X/\langle w_d \rangle$ and k is even.*
2. *if w_d has a fixed point on X, then φ factors through $X/\langle w_d \rangle$ and k is even.*
3. *if E has non-trivial 2-torsion over $\mathbb{Q}$ and k is odd, then we obtain a degree k map $\varphi' : X/\langle w_d \rangle \to E'$, by taking w_d-invariant to φ, where E' is an elliptic curve isogenous to E.*

Proof The proofs of (1) and (3) can be found in [4, Lemma 7]. Moreover, from the proof of [4, Lemma 7], we get $\varphi \circ w_d = \varphi + P$, where P is a 2-torsion point of $E(\mathbb{C})$. Thus it is easy to see that if w_d has a fixed point on X, then P is the trivial zero of E and φ factors through $X/\langle w_d \rangle$. □

As an immediate consequence of Lemma 2.6(2), we obtain the following result:

Corollary 2.7 *The following triples are not admissible*

$$(130, 2, 65a1), (130, 5, 65a1), (130, 10, 65a1), (130, 13, 65a1), (130, 26, 65a1), (185, 5, 185c1)$$
$$(185, 37, 185c1), (195, 3, 65a1), (195, 5, 65a1), (195, 13, 65a1), (195, 39, 65a1).$$

Proof Consider the triple $(130, 5, 65a1)$. Suppose $f : X_0^{+5}(130) \to 65a1$ is a $\mathbb{Q}$-rational degree 3 mapping. By Riemann-Hurwitz theorem, we see that w_{13} has fixed points on $X_0^{+5}(130)$. By Lemma 2.6 (2), f must factors through $X_0(130)/\langle w_5, w_{13} \rangle$. Which is not possible since $\deg(f)$ is odd. Hence the triple $(130, 5, 65a1)$ is not admissible. A similar argument will work for the other cases (for the triples of the form $(\_, \_, 65a1)$ and $(\_, \_, 185c1)$ apply Lemma 2.6(2) with the involutions w_{65} and w_{185}, respectively). □

After applying Lemma 2.3, Corollary 2.4 and Lemma 2.6, we can conclude that if $\mathrm{Gon}(X_0^{+d}(N)) \geq 4$ and $X_0^{+d}(N)$ has no degree ≤ 2 map to an elliptic curve, then the triple (N, d, E) is admissible possibly only for the following values of N, d and E (cf. Table 3 for a detailed discussion):

N	d	E	N	d	E	N	d	E
106	53	53a1	182	91	91b1	249	83	83a1
122	61	61a1	183	61	61a1	262	131	131a1
129	43	43a1	195	65	65a1	267	89	89a1
130	65	65a1	202	101	101a1	273	91	91b1
158	79	79a1	215	43	43a1	303	101	101a1
166	83	83a1	222	37	37a1	305	61	61a1
178	89	89a1	237	79	79a1	395	79	79a1

To handle the remaining triples (N, d, E) mentioned in the last table, we study the degeneracy maps between quotient modular curves.

3 Degeneracy maps

Let M be a positive integer such that $M|N$. For every positive divisor d of $\frac{N}{M}$, we have $\iota_d \Gamma_0(N) \iota_d^{-1} \subset \Gamma_0(M)$, where $\iota_d := \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Hence ι_d induces a degeneracy map

$$\iota_{d,N,M} : X_0(N) \to X_0(M),$$

where $\iota_{d,N,M}$ acts on $\tau \in \mathbb{H}^*$ in the extended upper half plane as $\iota_{d,N,M}(\tau) = \iota_d \cdot \tau = d\tau$. For any positive integer r with $r||M$ and $r||N$, let $w_r^{(N)}$ denote the r-th (partial) Atkin-Lehner operator acting on $X_0(N)$ and $w_r^{(M)}$ denote the r-th (partial) Atkin-Lehner operator acting on $X_0(M)$. For any positive divisor d of $\frac{N}{M}$ with $(d, r) = 1$, a regular calculation gives the equality of matrices $\iota_d w_r^{(N)} = \gamma w_r^{(M)} \iota_d$, for some $\gamma \in \Gamma_0(M)$. Thus $\iota_{d,N,M}$ induces a degeneracy map

$$\iota_{d,N,M,r} : X_0^{+r}(N) \to X_0^{+r}(M).$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} X_0(N) & \xrightarrow{\iota_{d,N,M}} & X_0(M) \\ \varpi_r^{(N)}\downarrow & & \downarrow\varpi_r^{(M)} \\ X_0^{+r}(N) & \xrightarrow{\iota_{d,N,M,r}} & X_0^{+r}(M). \end{array} \tag{3.1}$$

As an immediate consequence we obtain:

Corollary 3.1 *The modular curves* $X_0^{+53}(106)$, $X_0^{+61}(122)$, $X_0^{+65}(130)$, $X_0^{+79}(158)$, $X_0^{+83}(166)$, $X_0^{+89}(178)$, $X_0^{+101}(202)$, $X_0^{+131}(262)$ *have infinitely many cubic points over* $\mathbb{Q}$.

Proof By the previous discussions, there are natural $\mathbb{Q}$-rational degree 3 mappings

$$\begin{aligned} X_0^{+53}(106) &\to X_0^{+53}(53) \cong 53a1 \\ X_0^{+61}(122) &\to X_0^{+61}(61) \cong 61a1 \\ X_0^{+65}(130) &\to X_0^{+65}(65) \cong 65a1 \\ X_0^{+79}(158) &\to X_0^{+79}(79) \cong 79a1 \\ X_0^{+83}(166) &\to X_0^{+83}(83) \cong 83a1 \\ X_0^{+89}(178) &\to X_0^{+89}(89) \cong 89a1 \\ X_0^{+101}(202) &\to X_0^{+101}(101) \cong 101a1 \\ X_0^{+131}(262) &\to X_0^{+131}(131) \cong 131a1. \end{aligned}$$

Since all the elliptic curves $53a1, 61a1, 65a1, 79a1, 83a1, 89a1, 101a1$ and $131a1$ have positive $\mathbb{Q}$-ranks, we conclude that all the modular curves mentioned in the statement of this corollary have infinitely many cubic points over $\mathbb{Q}$. □

We now introduce some notations. For any curves C, C' over a field k and any morphism $f : C \to C'$, we define the following mappings

$$f_* : J(C) \to J(C') \text{ defined by } f_*([\sum n_i P_i]) = [\sum n_i f(P_i)], \text{ and}$$

$$f^* : J(C) \to J(C') \text{ defined by } f^*([\sum n_i P_i]) = [\sum n_i f^{-1}(P_i)],$$

where $J(C)$ (resp., $J(C')$) denote the Jacobian of C (resp., C'). It is easy to check that $f_* \circ f^* = [\deg(f)]$.

Let M be a positive integer. We use the notation $[E, r]_M$ to denote a pair (E, r), where E is an elliptic curve of conductor M and r is a positive integer with $r||M$ such that w_r acts as $+1$ on E, and at least one of the following is true:

1. $r = M$,
2. $r < M$ and E has no non-trivial 2-torsion points,
3. $r < M$ and w_r has a fixed point on $X_0(M)$.

Consider a pair $[E, r]_M$, if $f : X_0(M) \to E$ is the modular parametrization, then by Lemma 2.5 and Lemma 2.6, f induces a mapping

$$f_r : X_0^{+r}(M) \to E, \tag{3.2}$$

and the following diagram commutes:

$$\begin{array}{ccc} X_0(M) & \xrightarrow{f} & E \\ {\scriptstyle \varpi_r^{(M)}}\downarrow & \nearrow_{f_r} & \\ X_0^{+r}(M). & & \end{array} \tag{3.3}$$

Observe that if E' is any elliptic curve isogenous to E and $f' : X_0^{+r}(M) \to E'$ is a mapping, then f' factors through f_r. Also, for any positive divisor d of $\frac{N}{M}$ with $(d, r) = 1$, we have the following commutative diagram:

$$\begin{array}{ccccc} X_0(N) & \xrightarrow{\iota_{d,N,M}} & X_0(M) & \xrightarrow{f} & E \\ {\scriptstyle \varpi_r^{(N)}}\downarrow & & {\scriptstyle \varpi_r^{(M)}}\downarrow & \nearrow_{f_r} & \\ X_0^{+r}(N) & \xrightarrow{\iota_{d,N,M,r}} & X_0^{+r}(M). & & \end{array} \tag{3.4}$$

Let $J_0(N)^{w_r}$ (resp., $J_0(M)^{w_r}$) denote the Jacobian of $X_0^{+r}(N)$ (resp., $X_0^{+r}(M)$) and n be the number of divisors of $\frac{N}{M}$. Considering the pushforward maps of (3.4), we obtain the commutative diagram:

$$\begin{array}{ccccc} J_0(N) & \xrightarrow{\iota_{d,N,M,*}} & J_0(M) & \xrightarrow{f_*} & E \\ {\scriptstyle \varpi_{r,*}^{(N)}}\downarrow & & {\scriptstyle \varpi_{r,*}^{(M)}}\downarrow & \nearrow_{f_{r,*}} & \\ J_0(N)^{w_r} & \xrightarrow{\iota_{d,N,M,r,*}} & J_0(M)^{w_r}. & & \end{array} \tag{3.5}$$

Note that there is a natural mapping $\delta_{N,M,r} : J_0(N)^{w_r} \to (J_0(M)^{w_r})^n$ defined by

$$\delta_{N,M,r} := (\iota_{1,N,M,r,*}, \ldots, \iota_{\frac{N}{M},N,M,r,*}).$$

We recall the notion of optimal B-isogenous quotient (cf. [9, Definition 3.2]).

Definition 3.2 Let A, B be abelian varieties over a field k with B simple. An abelian variety A' together with a quotient map $\pi : A \to A'$ is an optimal B-isogenous quotient if A' is isogenous to B^n for some integer n and every morphism $A \to B'$ with B' isogenous to B^m for some integer m uniquely factors through π.

Proposition 3.3 *Suppose $N < 408$ is a positive integer. For a positive divisor M of N consider a pair $[E, r]_M$, where E is a strong Weil curve over $\mathbb{Q}$ of positive rank and conductor M. Let $f_r : X_0^{+r}(M) \to E$ be the mapping induced by the modular parametrization $f : X_0(M) \to E$. Consider the mapping $\xi_{E,N,r} := (f_{r,*})^n \circ \delta_{N,M,r} : J_0(N)^{w_r} \to E^n$, where n denotes the number of divisors of $\frac{N}{M}$. Then $\xi_{E,N,r}^\vee : E^n \to J_0(N)^{w_r}$ has a trivial kernel (where $\xi_{E,N,r}^\vee$ denotes the dual mapping of $\xi_{E,N,r}$). Hence $\xi_{E,N,r}^\vee : E^n \to J_0(N)^{w_r}$ is a maximal E-isogenous abelian subvariety and $\xi_{E,N,r} : J_0(N)^{w_r} \to E^n$ is an optimal E-isogenous quotient of $J_0(N)^{w_r}$.*

Proof Consider the mapping $\xi_{E,N} := (f_*)^n \circ (\iota_{1,N,M,*}, \ldots, \iota_{\frac{N}{M},N,M,*}) : J_0(N) \to E^n$ (cf. [9, Definition 3.8]).

From the commutative diagram (3.5) we get

$$\xi_{E,N} = \xi_{E,N,r} \circ \varpi_{r,*}^{(N)}. \tag{3.6}$$

Hence $\xi_{E,N}^\vee = (\varpi_{r,*}^{(N)})^\vee \circ \xi_{E,N,r}^\vee$. Since $\xi_{E,N}^\vee$ has trivial kernel (cf. [9, Proposition 3.9]), we conclude that $\xi_{E,N,r}^\vee$ has trivial kernel. The remaining statements follow from [9, Proposition 3.9] and the commutative diagram (3.5). □

Let E be a strong Weil curve of conductor M ($M|N$) and odd $\mathbb{Q}$-rank, with w_M acting as $+1$ on E, and $f_M : X_0^{+M}(M) \to E$ be the mapping induced by the modular parametrization of E. Since $\xi_{E,r,N} : J_0(N)^{w_M} \to E^n$ is an E-isogenous optimal quotient when $N < 408$, every mapping $J_0(N)^{w_M} \to E$ uniquely factors through E^n. Therefore we get that the maps $f_M \circ d_i$, where d_i runs over the degeneracy maps $X_0^{+M}(N) \to X_0^{+M}(M)$, form a basis for $\mathrm{Hom}_{\mathbb{Q}}(J_0(N)^{w_M}, E) \cong \mathrm{Hom}_{\mathbb{Q}}(E^n, E) \cong \mathbb{Z}^n$.

3.1 Degree pairing

We recall the definition of degree pairing from [9].

Definition 3.4 [9, Definition 2.2] Let C, E be curves over a field k, where E is an elliptic curve. The degree pairing is defined on $\mathrm{Hom}(C, E)$ as

$$\begin{aligned} \langle \_, \_ \rangle : \mathrm{Hom}(C, E) \times \mathrm{Hom}(C, E) &\to \mathrm{End}(J(E)) \\ f, g &\to f_* \circ g^*. \end{aligned}$$

Note that with this notation we have $\langle f, f \rangle = [\deg(f)]$ for $f \in \mathrm{Hom}(C, E)$. If $P \in C(k)$, then we can define the degree pairing on $\mathrm{Hom}(J(C), E)$ as follows

$$\begin{aligned} \langle \_, \_ \rangle : \mathrm{Hom}(J(C), E) \times \mathrm{Hom}(J(C), E) &\to \mathrm{End}(J(E)) \\ f, g &\to (f \circ f_P)_* \circ (g \circ g_P)^*, \end{aligned}$$

where the mapping f_P (similarly for g_P) is defined as

$$\begin{aligned} f_P : C &\to J(C) \\ x &\to [x - P]. \end{aligned}$$

Proposition 3.5 *Assume that N is a square-free positive integer and $M|N$. Let E be an elliptic curve of conductor M with newform $\sum_{n=1}^{\infty} a_n q^n$ such that $w_M^{(M)}$ acts as $+1$ on E. Suppose that $f_M : X_0^{+M}(M) \to E$ be the mapping induced by the modular parametrization $f : X_0(M) \to E$. For any positive divisors d_1, d_2 of $\frac{N}{M}$ we define $a := a_{\frac{d_1 d_2}{\gcd(d_1,d_2)^2}}$, then we have the following equality in* $\mathrm{End}(J(E))$*:*

$$\langle f_M \circ \iota_{d_1,N,M,M}, f_M \circ \iota_{d_2,N,M,M}\rangle = [\deg(f_M)]\Big[a\frac{\psi(N)}{\psi(\frac{M\cdot\mathrm{lcm}(d_1,d_2)}{\gcd(d_1,d_2)})}\Big],$$

where $\psi(s) = s\prod_{p|s}(1+\frac{1}{p})$.

Proof By the definition of the pairing, we have

$$\begin{aligned}
&\langle f_M \circ \iota_{d_1,N,M,M} \circ \varpi_M^{(N)}, f_M \circ \iota_{d_2,N,M,M} \circ \varpi_M^{(N)}\rangle \\
&= (f_M \circ \iota_{d_1,N,M,M} \circ \varpi_M^{(N)})_* \circ (f_M \circ \iota_{d_2,N,M,M} \circ \varpi_M^{(N)})^* \\
&= (f_M)_* \circ (\iota_{d_1,N,M,M})_* \circ (\varpi_M^{(N)})_* \circ (\varpi_M^{(N)})^* \circ (\iota_{d_2,N,M,M})^* \circ (f_M)^* \\
&= (f_M)_* \circ (\iota_{d_1,N,M,M})_* \circ [\deg(\varpi_M^{(N)})] \circ (\iota_{d_2,N,M,M})^* \circ (f_M)^* \\
&= (f_M)_* \circ (\iota_{d_1,N,M,M})_* \circ [2] \circ (\iota_{d_2,N,M,M})^* \circ (f_M)^* \\
&= [2] \circ (f_M)_* \circ (\iota_{d_1,N,M,M})_* \circ (\iota_{d_2,N,M,M})^* \circ (f_M)^* \\
&= [2] \circ \langle f_M \circ \iota_{d_1,N,M,M}, f_M \circ \iota_{d_2,N,M,M}\rangle
\end{aligned}$$

On the other hand, from the diagram (3.4) (note that we have $[E, M]_M$) we have the following equalities inside $\mathrm{End}(E)$:

$$\begin{aligned}
&\langle f_M \circ \iota_{d_1,N,M,M} \circ \varpi_M^{(N)}, f_M \circ \iota_{d_2,N,M,M} \circ \varpi_M^{(N)}\rangle \\
&= \langle f_M \circ \varpi_M^{(M)} \circ \iota_{d_1,N,M}, f_M \circ \varpi_M^{(M)} \circ \iota_{d_2,N,M}\rangle \\
&= \langle f \circ \iota_{d_1,N,M}, f \circ \iota_{d_2,N,M}\rangle.
\end{aligned}$$

By [9, Theorem 2.13], we have

$$\langle f \circ \iota_{d_1,N,M}, f \circ \iota_{d_2,N,M}\rangle = \Big[a\frac{\psi(N)}{\psi(\frac{M\cdot\mathrm{lcm}(d_1,d_2)}{\gcd(d_1,d_2)})}\deg(f)\Big] = \Big[a\frac{\psi(N)}{\psi(\frac{M\cdot\mathrm{lcm}(d_1,d_2)}{\gcd(d_1,d_2)})}\deg(f_M)\Big][2]$$

Consequently,

$$[2]\circ\langle f_M\circ\iota_{d_1,N,M,M},\, f_M\circ\iota_{d_2,N,M,M}\rangle = \langle f\circ\iota_{d_1,N,M},\, f\circ\iota_{d_2,N,M}\rangle$$
$$= [2][\deg(f_M)]\Big[a\frac{\psi(N)}{\psi(\frac{M\cdot \mathrm{lcm}(d_1,d_2)}{\gcd(d_1,d_2)})}\Big].$$

Therefore we conclude that

$$\langle f_M\circ\iota_{d_1,N,M,M},\, f_M\circ\iota_{d_2,N,M,M}\rangle = [\deg(f_M)]\Big[a\frac{\psi(N)}{\psi(\frac{M\cdot \mathrm{lcm}(d_1,d_2)}{\gcd(d_1,d_2)})}\Big].$$

This completes the proof. □

We now explain via some examples how to use the above recipe to study the triples (N, d, E).

4 The remaining cases from §2

In this section we give complete explanations for the cases $(222, 37, 37a1)$ and $(195, 65, 65a1)$. The arguments for the other cases are similar to these.

4.1 The curve $X_0^{+37}(222)$:

We want to check whether there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+37}(222)\to 37a1$. The modular parametrization $f : X_0(37)\to 37a1$ has degree 2 and w_{37} acts as $+1$ on $37a1$. By Lemma 2.6, f factors through $X_0^{+37}(37)$ and the induced map $f_{37} : X_0^{+37}(37)\to 37a1$ has degree 1 (note that $X_0^{+37}(37)\cong 37a1$ and we have $[37a1, 37]_{37}$). The degeneracy maps are $\iota_{1,222,37,37}$, $\iota_{2,222,37,37}$, $\iota_{3,222,37,37}$ and $\iota_{6,222,37,37}$ (note that all these maps have degree 12). Moreover, by Proposition 3.5, we have

$$\langle f_{37}\circ\iota_{1,222,37,37},\, f_{37}\circ\iota_{2,222,37,37}\rangle = [a_2\times\psi(3)\times\deg(f_{37})] = [(-2)\times 4\times 1] = [-8],$$
$$\langle f_{37}\circ\iota_{1,222,37,37},\, f_{37}\circ\iota_{3,222,37,37}\rangle = [a_3\times\psi(2)\times\deg(f_{37})] = [(-3)\times 3\times 1] = [-9],$$
$$\langle f_{37}\circ\iota_{1,222,37,37},\, f_{37}\circ\iota_{6,222,37,37}\rangle = [a_6\times\psi(1)\times\deg(f_{37})] = [6\times 1\times 1] = [6],$$
$$\langle f_{37}\circ\iota_{2,222,37,37},\, f_{37}\circ\iota_{3,222,37,37}\rangle = [a_6\times\psi(1)\times\deg(f_{37})] = [6\times 1\times 1] = [6],$$
$$\langle f_{37}\circ\iota_{2,222,37,37},\, f_{37}\circ\iota_{6,222,37,37}\rangle = [a_3\times\psi(2)\times\deg(f_{37})] = [(-3)\times 3\times 1] = [-9],$$
$$\langle f_{37}\circ\iota_{3,222,37,37},\, f_{37}\circ\iota_{6,222,37,37}\rangle = [a_2\times\psi(3)\times\deg(f_{37})] = [(-2)\times 4\times 1] = [-8].$$

The maps $f_{37}\circ\iota_{1,222,37,37}$, $f_{37}\circ\iota_{2,222,37,37}$, $f_{37}\circ\iota_{3,222,37,37}$ and $f_{37}\circ\iota_{6,222,37,37}$ form a basis for $\mathrm{Hom}_{\mathbb{Q}}(J_0(222)^{w_{37}}, 37a1)$. If $\varphi : X_0^{+37}(222)\to 37a1$ is a $\mathbb{Q}$-rational mapping, then we can write

$$\varphi = x_1(f_{37}\circ\iota_{1,222,37,37}) + x_2(f_{37}\circ\iota_{2,222,37,37}) + x_3(f_{37}\circ\iota_{3,222,37,37}) + x_4(f_{37}\circ\iota_{6,222,37,37}), \quad (4.1)$$

for some $x_1, x_2, x_3, x_4 \in \mathbb{Z}$. For simplicity of notations we write $f_{37,i} := f_{37} \circ \iota_{i,222,37,37}$ for $i \in \{1, 2, 3, 6\}$. Now from (4.1) we have

$$\begin{aligned}\langle \varphi, \varphi\rangle =&[x_1^2]\langle f_{37,1}, f_{37,1}\rangle + [x_2^2]\langle f_{37,2}, f_{37,2}\rangle + [x_3^2]\langle f_{37,3}, f_{37,3}\rangle + [x_4^2]\langle f_{37,6}, f_{37,6}\rangle\\&+ [x_1x_2]\langle f_{37,1}, f_{37,2}\rangle + [x_1x_3]\langle f_{37,1}, f_{37,3}\rangle + [x_1x_4]\langle f_{37,1}, f_{37,6}\rangle\\&+ [x_2x_1]\langle f_{37,2}, f_{37,1}\rangle + [x_2x_3]\langle f_{37,2}, f_{37,3}\rangle + [x_2x_4]\langle f_{37,2}, f_{37,6}\rangle\\&+ [x_3x_1]\langle f_{37,3}, f_{37,1}\rangle + [x_3x_3]\langle f_{37,3}, f_{37,2}\rangle + [x_3x_4]\langle f_{37,3}, f_{37,6}\rangle\\&+ [x_4x_1]\langle f_{37,6}, f_{37,1}\rangle + [x_4x_2]\langle f_{37,6}, f_{37,2}\rangle + [x_4x_3]\langle f_{37,6}, f_{37,3}\rangle.\end{aligned}$$

Since $\langle f_{37,i}, f_{37,i}\rangle = [\deg f_{37,i}]$ and $\langle f_{37,i}, f_{37,j}\rangle = \langle f_{37,j}, f_{37,i}\rangle$, we get

$$\begin{aligned}[\deg\varphi] = [12x_1^2] + [12x_2^2] + [12x_3^2]+[12x_4^2] + [-16x_1x_2] + [-18x_1x_3] + [12x_1x_4] + [12x_2x_3]\\ + [-18x_2x_4] + [-16x_3x_4],\end{aligned}$$

where "$[a]''$ denotes the multiplication by a-mapping on the elliptic curve $E = 37a1$. Hence we must have

$$\deg\varphi = 12x_1^2+12x_2^2+12x_3^2+12x_4^2-16x_1x_2-18x_1x_3+12x_1x_4+12x_2x_3-18x_2x_4-16x_3x_4. \tag{4.2}$$

This shows that $\deg\varphi$ is of the form $2g(x_1, x_2, x_3, x_4)$, so it can not take the value 3. Consequently, the triple $(222, 37, 37a1)$ is not admissible.

4.2 The curve $X_0^{+65}(195)$

We want to check whether there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+65}(195) \to 65a1$. Recall that the modular parametrization $f : X_0(65) \to 65a1$ has degree 2, and w_{65} acts as $+1$ on $65a1$. By Lemma 2.6, f factors through $X_0^{+65}(65)$ and the induced map $f_{65} : X_0^{+65}(65) \to 65a1$ has degree 1, since $X_0^{+65}(65) \cong 65a1$ (note that we have $[65a1, 65]_{65}$). Both the degeneracy maps $\iota_{1,195,65,65}$ and $\iota_{3,195,65,65}$ have degree 4. Moreover, we have

$$\langle f_{65}\circ\iota_{1,195,65,65}, f_{65}\circ\iota_{3,195,65,65}\rangle = \left[a_3 \times \frac{\psi(195)}{\psi(65\times 3)} \times \deg(f_{65})\right] = [(-2)\times 1\times 1] = [-2].$$

The maps $f_{65}\circ\iota_{1,195,65,65}$, $f_{65}\circ\iota_{3,195,65,65}$ form a basis for $\mathrm{Hom}_{\mathbb{Q}}(J_0(195)^{w_{65}}, 65a1)$. If $\varphi : X_0^{+65}(195) \to 65a1$ is $\mathbb{Q}$-rational mapping then we can write

$$\varphi = x_1(f_{65}\circ\iota_{1,195,65,65}) + x_2(f_{65}\circ\iota_{3,195,65,65}).$$

Thus we have (the computations are similar as in §4.1)

$$\begin{aligned}[\deg(\varphi)] =&[x_1^2\deg(f_{65}\circ\iota_{1,195,65,65})] + [2x_1x_2]\langle f_{65}\circ\iota_{1,195,65,65}, f_{65}\circ\iota_{3,195,65,65}\rangle\\&+ [x_2^2\deg(f_{65}\circ\iota_{3,195,65,65})],\\=&[4x_1^2] + [x_1x_2][-4] + [4x_2^2],\end{aligned}$$

Table 1 Quadratic forms for remaining cases

N	d	E	$g_{X_0^{+d}(N)}$	Quadratic form
129	43	43a1	7	$4x_1^2 - 4x_1x_2 + 4x_2^2$
182	91	91b1	10	$6x_1^2 + 6x_2^2$
183	61	61a1	10	$4x_1^2 - 4x_1x_2 + 4x_2^2$
195	65	65a1	9	$4x_1^2 - 4x_1x_2 + 4x_2^2$
215	43	43a1	11	$6x_1^2 - 8x_1x_2 + 6x_2^2$
222	37	37a1	18	$2 \cdot g(x_1, x_2, x_3, x_4)$
237	79	79a1	13	$4x_1^2 - 2x_1x_2 + 4x_2^2$
249	83	83a1	8	$4x_1^2 - 2x_1x_2 + 4x_2^2$
267	89	89a1	9	$4x_1^2 - 2x_1x_2 + 4x_2^2$
273	91	91b1	17	$8x_1^2 - 8x_1x_2 + 8x_2^2$
303	101	101a1	10	$4x_1^2 - 4x_1x_2 + 4x_2^2$
305	61	61a1	12	$6x_1^2 - 6x_1x_2 + 6x_2^2$
395	79	79a1	15	$6x_1^2 - 6x_1x_2 + 6x_2^2$

where “$[a]$" denotes the multiplication by a-mapping on the elliptic curve $E = 65a1$.
Hence we must have

$$\deg(\varphi) = 4x_1^2 - 4x_1x_2 + 4x_2^2. \tag{4.3}$$

From (4.3), we see that $\deg(\varphi)$ can not take the value 3. Consequently, $\mathrm{Hom}_{\mathbb{Q}}(J_0(195)^{w_{65}}, 65a1)$ has no element of order 3. Thus there is no $\mathbb{Q}$-rational degree 3-mapping $X_0^{+65}(195) \to 65a1$.

The quadratic forms for the other remaining cases are given in Table 1. It is clear from Table 1 that the triples (N, d, E) appearing in Table 1 are not admissible (note that for each of the triples (N, d, E) appearing in Table 1, w_d acts as $+1$ on E and we have $[E, d]_d$). Consequently, the curves $X_0^{+d}(N)$ have finitely many cubic points over $\mathbb{Q}$ for N, d appearing in Table 1.

5 Hyperelliptic cases

In this section we deal with the pairs (N, w_d) such that $X_0^{+d}(N)$ is hyperelliptic. We first consider the hyperelliptic curves of genus 2.

Theorem 5.1 *The curve $X_0^{+d}(N)$ has infinitely many cubic points over $\mathbb{Q}$ for the following pairs of (N, w_d):*

$(30, w_2), (30, w_3), (30, w_{10}), (33, w_3), (35, w_7), (39, w_{13}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3),$
$(58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (142, w_{71}).$

Proof Using the MAGMA code “`#Points(SimplifiedModel(X0NQuotient(N,[d])):Bound:=10)`", we see that for the above mentioned pairs of (N, w_d), the curve $X_0^{+d}(N)$ has at least three rational points over $\mathbb{Q}$. From [16, Lemma 2.1], we conclude that for each of these cases there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+d}(N) \to \mathbb{P}^1$. Consequently, $X_0^{+d}(N)$ has infinitely many cubic points over $\mathbb{Q}$. □

The remaining genus two cases are $X_0^{+2}(38)$ and $X_0^{+29}(87)$.

Theorem 5.2 *The sets* $\Gamma_3'(X_0^{+2}(38), \mathbb{Q})$ *and* $\Gamma_3'(X_0^{+29}(87), \mathbb{Q})$ *are infinite.*

Proof An affine model of $X_0^{+2}(38)$ is given by

$$X_0^{+2}(38) : y^2 = x^6 - 4x^5 - 6x^4 + 4x^3 - 19x^2 + 4x - 12. \tag{5.1}$$

It is easy to see that $X_0^{+2}(38)$ has two $\mathbb{Q}$-rational points $(1:1:0)$, $(1:-1:0)$ and the hyperelliptic involution permutes these points. From [18, Lemma 2.2], we conclude that there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^{+2}(38) \to \mathbb{P}^1$. Consequently, the set $\Gamma_3'(X_0^{+2}(38), \mathbb{Q})$ is infinite. A similar argument works for the curve $X_0^{+29}(87)$, which has an affine model $y^2 = x^6 - 2x^4 - 6x^3 - 11x^2 - 6x - 3$. □

Now consider the pairs (N, w_d) such that $X_0^{+d}(N)$ is hyperelliptic and $g(X_0^{+d}(N)) \geq 3$. If the set $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ is infinite, then $W_3(X_0^{+d}(N))$ must contain an elliptic curve with positive $\mathbb{Q}$-rank (the arguments are exactly the same as [18, §2.3], since the cusp at infinity is a $\mathbb{Q}$-rational point on $X_0^{+d}(N)$, we can obtain [18, Lemma 2.6, Lemma 2.7] for $X_0^{+d}(N)$ instead of $X_0(N)$).

Thus, by Cremona tables [7] we obtain (because there is no elliptic curve with positive $\mathbb{Q}$-rank for levels dividing N):

Theorem 5.3 *The set* $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ *is finite for the following pairs of* (N, w_d)*:*

$(46, w_2), (51, w_3), (55, w_5), (70, w_{14}), (78, w_{26}), (95, w_{19}), (62, w_2), (66, w_6), (69, w_3), (70, w_{10}),$
$(119, w_{17}), (87, w_3), (95, w_5), (78, w_6), (94, w_2), (119, w_7).$

6 Trigonal cases

It is well known that, if $C/\mathbb{Q}$ is a trigonal curve of genus 3 with a $\mathbb{Q}$-rational point, then the projection from the $\mathbb{Q}$-rational point defines a degree 3 mapping $C \to \mathbb{P}^1$ over $\mathbb{Q}$. Consequently, in these cases the set $\Gamma_3'(C, \mathbb{Q})$ is infinite.

Now consider the pairs (N, w_d) such that $X_0^{+d}(N)$ is trigonal and $g(X_0^{+d}(N)) = 4$. A model of $X_0^{+d}(N)$ can be constructed using Petri's theorem. It is well known that a non hyperelliptic curve X (defined over $\mathbb{Q}$) of genus 4 lies either on a ruled surface or on a quadratic cone (defined over either $\mathbb{Q}$, a quadratic field or a biquadratic field) (cf. [14, Page 131]). If the ruled surface or the quadratic cone is defined over $\mathbb{Q}$, then there is a degree 3 mapping $X \to \mathbb{P}^1$ defined over $\mathbb{Q}$. For example, consider the curve

$X_0^{+5}(70)$. Choosing the following basis of weight 2 cusp forms $S_2(\langle\Gamma_0(70), w_5\rangle)$,

$$\begin{aligned} q + q^8 - 2q^9 - q^{10} + q^{11} + O(q^{12}) \\ q^2 + q^6 - 2q^8 - q^{10} + O(q^{12}) \\ q^3 - 3q^5 - 2q^6 - q^7 + 3q^8 - 2q^9 + 3q^{10} + 2q^{11} + O(q^{12}) \\ q^4 - q^5 - q^6 - q^7 + 2q^8 - q^9 + q^{10} + 3q^{11} + O(q^{12}), \end{aligned}$$

and using MAGMA, a model of $X_0^{+5}(70)$ is given by

$$\begin{cases} x^2w + 4xyw - 11xw^2 - y^3 - 3y^2z + 8y^2w - 3yz^2 + 16yzw - 24yw^2 - z^3 + 7z^2w - 9zw^2 + 3w^3, \\ xz + 4xw - y^2 - 4yz + 9yw - 2z^2 + 3zw - w^2. \end{cases}$$

After a suitable coordinate change, the degree 2 homogeneous equation can be written as:

$$-2x^2 + y^2 - 2z^2 + 445w^2 = (y + \sqrt{2}x)(y - \sqrt{2}x) - (\sqrt{2}z + \sqrt{445}w)(\sqrt{2}z - \sqrt{445}w),$$

which is isomorphic to the ruled surface $uv - st$ over $\mathbb{Q}(\sqrt{2}, \sqrt{445})$. Thus $X_0^{+5}(70)$ is not trigonal over $\mathbb{Q}$. The models for the quadratic surfaces for genus 4 curves are given in Table 7. Since the curves $X_0^{+d}(N)$ always has a $\mathbb{Q}$-rational cusp, from the discussions above we conclude that

Theorem 6.1 *Suppose that $g(X_0^{+d}(N)) \geq 3$. Then $X_0^{+d}(N)$ is trigonal over $\mathbb{Q}$ if and only if (N, w_d) is in the following list.*

$g_{X_0^{+d}(N)}$	(N, w_d)
3	$(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7),$ $(82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47})$
4	$(66, w_{33}), (74, w_{37}), (86, w_{43})$

Consequently, for such pairs (N, w_d) the set $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ is infinite.

Now consider the pairs (N, w_d) such that $\text{Gon}(X_0^{+d}(N)) = 3$, but there is no $\mathbb{Q}$-rational degree 3 mapping $X_0^{+d}(N) \to \mathbb{P}^1$. In these cases, if the set $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ is infinite, then $W_3(X_0^{+d}(N))$ contains a translation of an elliptic curve E with positive $\mathbb{Q}$-rank (cf. [18, Theorem 1.1]).

Theorem 6.2 *The set $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ is finite for the following pairs of (N, w_d):*

$(66, w_2), (70, w_5), (74, w_2), (77, w_{11}), (82, w_2), (85, w_5), (85, w_{17}), (91, w_7), (93, w_3), (110, w_{55}),$
$(133, w_{19}), (145, w_{29}), (177, w_{59}).$

Proof For $N = 66, 70, 85, 93, 110, 133, 177$, there is no elliptic curve E of positive $\mathbb{Q}$-rank with $\mathrm{cond}(E)|N$. Hence in these cases, the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is finite. In the remaining cases, the Jacobian decompositions of $X_0^{+d}(N)$ are given by:

$$\begin{aligned} J_0(74)^{\langle w_2\rangle} &\sim^{\mathbb{Q}} 37a1 \times 37b1 \times A_{f,\dim=2} \\ J_0(77)^{\langle w_{11}\rangle} &\sim^{\mathbb{Q}} 77a1 \times 77b1 \times A_{f,\dim=2} \\ J_0(82)^{\langle w_2\rangle} &\sim^{\mathbb{Q}} 82a1 \times A_{f,\dim=3} \\ J_0(91)^{\langle w_7\rangle} &\sim^{\mathbb{Q}} 91a1 \times A_{f,\dim=3} \\ J_0(145)^{\langle w_{29}\rangle} &\sim^{\mathbb{Q}} 145a1 \times A_{f,\dim=3}. \end{aligned}$$

Note that in these cases, $X_0^{+d}(N)$ is bielliptic and there are elliptic curves of positive $\mathbb{Q}$-rank with $\mathrm{cond}(E)|N$. By arguments in [18, Page 353], if there is no $\mathbb{Q}$-rational degree 3 mapping $X_0^{+d}(N) \to E$ where E is an elliptic curve of positive $\mathbb{Q}$-rank and $\mathrm{cond}(E)|N$, then $W_3(X_0^{+d}(N))$ has no translation of an elliptic curve with positive $\mathbb{Q}$-rank. From the Jacobian decomposition, we only need to check whether triples (74,2,37a1), (77,11,77a1), (82,2,82a1), (91,7,91a1), (145,29,145a1) are admissible or not.

Since w_{37} acts as $+1$ on $37a1$ and $37a1$ has no non-trivial 2-torsion over $\mathbb{Q}$, from Lemma 2.6 we conclude that the triple (74,2,37a1) is not admissible. In the remaining cases, if any of the triples (N, d, E) is admissible (consequently, there is a $\mathbb{Q}$-rational degree 6 mapping $X_0(N) \to E$), then the degree of the strong Weil parametrization of E should divide 6. For all the curves $77a1$, $82a1$, $91a1$ and $145a1$ the degree of the strong Weil parametrization is 4. Thus none of the triples is admissible. The result follows. □

7 Bielliptic cases

We are now left to discuss the pairs (N, w_d) such that $X_0^{+d}(N)$ is bielliptic and $\mathrm{Gon}(X_0^{+d}(N)) > 3$. Such pairs (N, w_d) are given in Table 2.

Following [18, Page 353], for any (N, w_d) in Table 2, if the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is infinite, then $W_3(X_0^{+d}(N))$ contains a translation of an elliptic curve E with positive $\mathbb{Q}$-rank, equivalently the triple (N, d, E) is admissible.

Theorem 7.1 *For the pairs (N, w_d) in Table 2, the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is finite.*

Proof For $N = 66, 70, 78, 105, 110$, there is no elliptic curve E of positive $\mathbb{Q}$-rank with $\mathrm{cond}(E) \mid N$. Hence for such values of N and the corresponding values of d as in Table 2, the set $\Gamma'_3(X_0^{+d}(N), \mathbb{Q})$ is finite.

For $N = 118, 123, 141, 142, 143, 145, 155$, the only elliptic curves E with positive $\mathbb{Q}$-rank have conductor equal to N. For such N, E and the corresponding d as in Table 2, if the triple (N, d, E) is admissible, then the degree of the strong Weil

Table 2 Bielliptic remaining cases

$g_{X_0^{+d}(N)}$	(N, w_d)
5	$(66, w_3)$, $(66, w_{22})$, $(70, w_2)$, $(70, w_7)$, $(78, w_3)$, $(86, w_2)$, $(105, w_5)$, $(105, w_{21})$, $(110, w_{11})$, $(111, w_3)$, $(155, w_{31})$í
6	$(78, w_2)$, $(78, w_{13})$, $(111, w_{37})$, $(143, w_{13})$, $(145, w_5)$, $(159, w_{53})$
7	$(105, w_3)$, $(105, w_7)$, $(105, w_{15})$, $(110, w_{10})$, $(118, w_2)$, $(123, w_3)$, $(143, w_{11})$
8	$(110, w_2)$, $(110, w_5)$, $(141, w_3)$, $(155, w_5)$
9	$(142, w_2)$, $(159, w_3)$.

parametrization of E should divide 6. From Cremona's table we see that for elliptic curves E with positive $\mathbb{Q}$-rank of conductors 118, 123, 141, 142, 143, 145 and 155, the degree of the strong Weil parametrization of E does not divide 6. Consequently, for $N = 118, 123, 141, 142, 143, 145, 155$, and the corresponding d as in Table 2, the set $\Gamma_3'(X_0^{+d}(N), \mathbb{Q})$ is finite.

Finally, we are left to check whether the triples $(86, 2, 43a1)$, $(111, 3, 37a1)$, $(111, 37, 37a1)$ and $(159, 3, 53a1)$ are admissible or not (note that for such triples (N, d, E), w_d acts as $+1$ on E). Since the elliptic curves $43a1, 37a1$ and $53a1$ have no non-trivial 2-torsion points, by Lemma 2.6 (1), we conclude that the triples $(86, 2, 43a1)$, $(111, 3, 37a1)$ and $(159, 3, 53a1)$ are not admissible. Consequently, the sets $\Gamma_3'(X_0^{+2}(86), \mathbb{Q})$, $\Gamma_3'(X_0^{+3}(111), \mathbb{Q})$ and $\Gamma_3'(X_0^{+3}(153), \mathbb{Q})$ are finite.

A similar argument as in §4, shows that if $\varphi : X_0^{+37}(111) \to 37a1$ is a $\mathbb{Q}$-rational mapping then we must have (note that we have $[37a1, 37]_{37}$)

$$\deg(\varphi) = 4x_1^2 - 6x_1x_2 + 4x_2^2, \text{ for some } x_1, x_2 \in \mathbb{Z}. \tag{7.1}$$

Thus $\deg(\varphi)$ can not take the value 3. Consequently, the triple $(111, 37, 37a1)$ is not admissible. This completes the proof. □

A Appendix

Let $N \le 623$. Suppose that $\mathrm{Gon}(X_0^{+d}(N)) \ge 4$ and $X_0^{+d}(N)$ has no degree ≤ 2 map to an elliptic curve. After applying Lemma 2.3 (2 and 3), we are left to check the existence of $\mathbb{Q}$-rational degree 3 mapping $X_0^{+d}(N) \to E$ where E is an elliptic curve with positive $\mathbb{Q}$-rank, only for the following values of N:

106, 114, 122, 129, 130, 154, 158, 159, 166, 174, 178, 182, 183, 185, 195, 202, 215, 222, 231, 237, 246, 249, 258, 259, 262, 265, 267, 273, 282, 285, 286, 301, 303, 305, 326, 371, 393, 395, 407, 415, 427, 445, 473, 481.

Furthermore, the triple (N, d, E) is not admissible for N, d and E appearing in Table 3.

Table 3 Remaining cases after §2

N	d	E	Method	N	d	E	Method
106	$d \neq 53$	53a1	Lemma 2.6(1) with w_{53}	106	all	106b1	Lemma 2.3(2)
111	$d \neq 37$	37a1	Lemma 2.6(1) with w_{37}	114	all	57a1	Lemma 2.6(1) with w_3 or w_{19}
122	$d \neq 61$	61a1	Lemma 2.6(1) with w_{61}	122	all	122a1	Lemma 2.3(2)
129	$d \neq 43$	43a1	Lemma 2.6(1) with w_{43}	129	all	129a1	Lemma 2.3(2)
130	$d \neq 65$	65a1	Corollary 2.7	130	all	130a1	Lemma 2.3(2)
154	all	77a1	Lemma 2.6(1) with w_7 or w_{11}	154	all	154a1	Lemma 2.3(2)
158	$d \neq 79$	79a1	Lemma 2.6(1) with w_{79}	158	all	158a1	Lemma 2.3(2)
158	all	158b1	Lemma 2.3(2)	159	$d \neq 53$	53a1	Lemma 2.6(1) with w_{53}
166	$d \neq 83$	83a1	Lemma 2.6(1) with w_{83}	166	all	166a1	Lemma 2.3(2)
174	all	58a1	Lemma 2.6(1) with w_2 or w_{29}	178	$d \neq 89$	89a1	Lemma 2.6(1) with w_{89}
182	all	91a1	Lemma 2.6(1) with w_7 or w_{13}	182	$d \neq 91$	91b1	Lemma 2.6(1) with w_{91}
183	$d \neq 61$	61a1	Lemma 2.6(1) with w_{61}	185	all	37a1	Lemma 2.3(1) with 3^2
185	all	185a1	Lemma 2.3(2)	185	all	185b1	Lemma 2.3(2)
185	all	185c1	Corollary 2.7	195	$d \neq 65$	65a1	Corollary 2.7
202	$d \neq 101$	101a1	Lemma 2.6(1) with w_{101}	215	$d \neq 43$	43a1	Lemma 2.6(1) with w_{43}
215	all	215a1	Lemma 2.3(2)	222	$d \neq 37$	37a1	Lemma 2.6(1) with w_{37}
231	all	77a1	Lemma 2.6(1) with w_7 or w_{11}	237	$d \neq 79$	79a1	Lemma 2.6(1) with w_{79}
246	all	82a1	Lemma 2.3(1) with 7^2	246	all	123a1	Lemma 2.3(1) with 5^2
246	all	123b1	Lemma 2.3(1) with 7^2	246	all	246d1	Lemma 2.3(2)
249	$d \neq 83$	83a1	Lemma 2.6(1) with w_{83}	249	all	249a1	Lemma 2.3(2)
249	all	249b1	Lemma 2.3(2)	258	all	43a1	Lemma 2.3(1) with 5^2
258	all	129a1	Lemma 2.6(1) with w_3 or w_{43}	258	all	258a1	Lemma 2.3(2)

Table 3 continued

N	d	E	Method	N	d	E	Method
258	all	258c1	Lemma 2.3(2)	259	all	37a1	Lemma 2.3(1) with 2^2
262	$d \neq 131$	131a1	Lemma 2.6(1) with w_{131}	262	all	262a1	Lemma 2.3(2)
262	all	262b1	Lemma 2.3(2)	265	all	53a1	Lemma 2.3(1) with 3^2
265	all	265a1	Lemma 2.3(2)	267	$d \neq 89$	89a1	Lemma 2.6(1) with w_{89}
273	all	91a1	Lemma 2.3(1) with 2^2	273	$d \neq 91$	91b1	Lemma 2.6(1) with w_{91}
273	all	273a1	Lemma 2.3(2)	282	all	141a1	Lemma 2.3(1) with 5^2
282	all	141d1	Lemma 2.6(1) with w_3 or w_{37}	282	all	282b1	Lemma 2.3(2)
285	all	57a1	Lemma 2.3(1) with 2^2	285	all	285a1	Lemma 2.3(2)
285	all	285b1	Lemma 2.3(2)	286	all	143a1	Lemma 2.6(1) with w_{11} or w_{13}
286	all	286b1	Lemma 2.3(2)	286	all	286c1	Lemma 2.3(2)
301	all	43a1	Lemma 2.3(1) with 2^2	303	$d \neq 101$	101a1	Lemma 2.6(1) with w_{101}
303	all	303a1	Lemma 2.3(2)	303	all	303b1	Lemma 2.3(2)
305	$d \neq 61$	61a1	Lemma 2.6(1) with w_{61}	326	all	163a1	Lemma 2.3(1) with 3^2
326	all	326a1	Lemma 2.3(2)	326	all	326b1	Lemma 2.3(2)
371	all	53a1	Lemma 2.3(1) with 3^2	371	all	371a1	Lemma 2.3(2)
393	all	131a1	Lemma 2.3(1) with 5^2	395	$d \neq 79$	79a1	Lemma 2.6(1) with w_{79}
407	all	37a1	Lemma 2.3(1) with 2^2	415	all	83a1	Lemma 2.3(1) with 3^2
427	all	61a1	Lemma 2.3(1) with 3^2	427	all	427b1	Lemma 2.3(2)
427	all	427c1	Lemma 2.3(2)	445	all	89a1	Lemma 2.3(1) with 2^2
473	all	43a1	Lemma 2.3(1) with 2^2	473	all	473a1	Lemma 2.3(2)
481	all	37a1	Lemma 2.3(1) with 2^2	481	all	481a1	Lemma 2.3(2)

Table 4 Hyperelliptic curve $X_0^{+d}(N)$

$g_{X_0^{+d}(N)}$	(N, w_d)
2	(30, w_2), (30, w_3), (30, w_{10}), (33, w_3), (35, w_7), (38, w_2), (39, w_{13}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3), (58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (87, w_{29}), (142, w_{71})
3	(46, w_2), (51, w_3), (55, w_5), (70, w_{14}), (78, w_{26}), (95, w_{19})
4	(62, w_2), (66, w_6), (69, w_3), (70, w_{10}), (119, w_{17})
5	(87, w_3), (95, w_5)
6	(78, w_6), (94, w_2), (119, w_7).

Table 5 $X_0^{+d}(N)$ with $\text{Gon}(X_0^{+d}(N)) = 3$

$g_{X_0^{+d}(N)}$	(N, w_d)
3	(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (77, w_7), (82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47}),
4	(66, w_2), (66, w_{33}), (70, w_5), (74, w_2), (74, w_{37}), (77, w_{11}), (82, w_2), (85, w_5), (85, w_{17}), (86, w_{43}), (91, w_7), (93, w_3), (110, w_{55}), (133, w_{19}), (145, w_{29}), (177, w_{59})

Table 6 Bielliptic curve $X_0^{+d}(N)$

$g_{X_0^{+d}(N)}$	(N, w_d)
2	(30, w_2), (30, w_3), (30, w_{10}), (42, w_3), (42, w_6), (42, w_{21}), (57, w_3), (58, w_{29}), (66, w_{11}), (70, w_{35}), (78, w_{39}), (142, w_{71})
3	(42, w_2), (42, w_7), (57, w_{19}), (58, w_2), (65, w_5), (65, w_{13}), (70, w_{14}), (77, w_7), (78, w_{26}), (82, w_{41}), (91, w_{13}), (105, w_{35}), (118, w_{59}), (123, w_{41}), (141, w_{47})
4	(66, w_2), (66, w_{33}), (70, w_5), (70, w_{10}), (74, w_2), (74, w_{37}), (77, w_{11}), (82, w_2), (86, w_{43}), (91, w_7), (110, w_{55}), (145, w_{29})
5	(66, w_3), (66, w_{22}), (70, w_2), (70, w_7), (78, w_3), (86, w_2), (105, w_5), (105, w_{21}), (110, w_{11}), (111, w_3), (155, w_{31})
6	(78, w_2), (78, w_{13}), (111, w_{37}), (143, w_{13}), (145, w_5), (159, w_{53})
7	(105, w_3), (105, w_7), (105, w_{15}), (110, w_{10}), (118, w_2), (123, w_3), (143, w_{11})
8	(110, w_2), (110, w_5), (141, w_3), (155, w_5)
9	(142, w_2), (159, w_3).

Table 7 Models and Quadratic surface for $X_0^{+d}(N)$ with $g_{X_0^{+d}(N)} = 4$

Curve	Petri's model and Quadratic surface
$X_0^{+2}(66)$	$\begin{cases} 12x^2w - 8xyw - 4xw^2 - 3y^3 + 11y^2z - 2y^2w + 6yz^2 - 2yzw + 3yw^2 - 72z^3 \\ \qquad +18z^2w + 15zw^2 + 2w^3, \\ 12xz - 8xw - 3y^2 + 8yz - 2yw - 12zw + w^2. \end{cases}$ Diagonal form: $-12x^2 - 15y^2 + 540z^2 - 4428w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-41})$
$X_0^{+33}(66)$	$\begin{cases} x^2z - xy^2 + 3y^2z - 2y^2w + 9yz^2 + 3yzw + 4yw^2 + 8z^3 + 9z^2w + 2zw^2 - 2w^3, \\ xw - yz - 2z^2 - 3zw. \end{cases}$ Diagonal form: $-x^2 - y^2 + z^2 + w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^{+5}(70)$	$\begin{cases} x^2w + 4xyw - 11xw^2 - y^3 - 3y^2z + 8y^2w - 3yz^2 + 16yzw - 24yw^2 - z^3 \\ \qquad +7z^2w - 9zw^2 + 3w^3, \\ xz + 4xw - y^2 - 4yz + 9yw - 2z^2 + 3zw - w^2. \end{cases}$ Diagonal form: $-2x^2 + y^2 - 2z^2 + 445w^2$ lies over a ruled surface over $\mathbb{Q}(\sqrt{2}, \sqrt{445})$.
$X_0^{+2}(74)$	$\begin{cases} x^2w + xyw - 4xw^2 - y^3 - 4y^2z + 5y^2w - 10yz^2 + 7yzw - 8yw^2 - 20z^3 \\ \qquad +3z^2w - 2zw^2 + 6w^3, \\ xz + xw - y^2 - yz + 2yw - 4z^2 - 2zw - 2w^2. \end{cases}$ Diagonal form: $15x^2 - 60y^2 - 4z^2 - 7w^2$, lies over a ruled surface over $\mathbb{Q}(\sqrt{-7})$.
$X_0^{+37}(74)$	$\begin{cases} x^2w + xyw + 4xw^2 - y^3 - 3y^2z - 2yz^2 + 3yw^2 - z^2w - 6zw^2 - 2w^3, \\ xz + xw - y^2 - yz + 2yw - z^2 - w^2 \end{cases}$ Diagonal form: $3x^2 - 3y^2 - z^2 + w^2$, lies over a ruled surface over $\mathbb{Q}$.

Table 7 continued

Curve	Petri's model and Quadratic surface
$X_0^{+11}(77)$	$\begin{cases} 2x^2w - 4xyw + 20xw^2 - y^3 + 3y^2z + 4y^2w - 3yz^2 - 12yzw + 8yw^2 + z^3 \\ \qquad +18z^2w + 44zw^2 + 8w^3, \\ xz - 4xw - y^2 + 2yz - 2yw - 2z^2 - 6zw - 2w^2 \end{cases}$
	Diagonal form: $x^2 - 2y^2 - 2z^2 - 49w^2$, lies over a ruled surface over $\mathbb{Q}(i)$.
$X_0^{+2}(82)$	$\begin{cases} x^2w + 4xyw - 12xw^2 - 8y^3 - 24y^2z + 24y^2w - 44yz^2 + 68yzw - 8yw^2 \\ \qquad -40z^3 + 128z^2w - 108zw^2 + 31w^3, \\ xz + xw - 2y^2 - 4yz - 8z^2 + 9zw - 3w^2. \end{cases}$
	Diagonal form $6x^2 - 24y^2 - 8z^2 - 18w^2$, lies over a ruled surface over $\mathbb{Q}(i)$.
$X_0^{+5}(85)$	$\begin{cases} 72x^2w + 4xyw - 28xw^2 - 18y^3 + 25y^2z + 75y^2w - 81yz^2 - 58yzw \\ \qquad -107yw^2 + 81z^3 + 252z^2w + 144zw^2 + 71w^3, \\ 18xz + 2xw - 9y^2 - yz + 15yw - 18z^2 - 27zw - 16w^2 \end{cases}$
	Diagonal form: $1886652x^2 - 11646y^2 - 18z^2 - 14623740w^2$,
	lies on a ruled surface over $\mathbb{Q}(\sqrt{2}, \sqrt{-5015})$.
$X_0^{+17}(85)$	$\begin{cases} 18x^2w - 6xyw + 40xw^2 - 9y^3 + 3y^2z + 12y^2w - 15yz^2 - 32yzw \\ \qquad -118yw^2 + 21z^3 + 80z^2w + 202zw^2 + 212w^3, \\ 3xz - 2xw - 3y^2 + yz - yw - z^2 + 13zw + 2w^2 \end{cases}$
	Diagonal form: $297x^2 - 11y^2 - z^2 + 29835w^2$
	lies on ruled surface over $\mathbb{Q}(\sqrt{3}, \sqrt{1105})$.

Table 7 continued

Curve	Petri's model and Quadratic surface
$X_0^{+43}(86)$	$\begin{cases} x^2z - xy^2 + y^2z - 2y^2w + 5yz^2 + 3yzw + 4yw^2 + 4z^3 + 4z^2w + 2zw^2 - 2w^3, \\ xw - yz - z^2 - zw \end{cases}$ Diagonal form: $-x^2 + 3y^2 - 3z^2 + w^2$, lies over a ruled surface over $\mathbb{Q}$.
$X_0^{+7}(91)$	$\begin{cases} 72x^2w - 60xyw + 52xw^2 - 18y^3 + 57y^2z - 93y^2w + 75yz^2 - 146yzw \\ \quad +25yw^2 + 48z^3 - 109z^2w + 81zw^2 - 8w^3, \\ 6xz - 10xw - 3y^2 + 5yz - 7yw + 4z^2 - 3zw - 4w^2 \end{cases}$ Diagonal form: $-7884x^2 - 292y^2 + 4z^2 + 6588w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-3}, \sqrt{61})$.
$X_0^{+3}(93)$	$\begin{cases} 4500x^2w - 1050xyw + 25xw^2 - 180y^3 + 30y^2z + 270y^2w - 96yz^2 \\ \quad +319yzw - 274yw^2 + 30z^3 + 563z^2w - 412zw^2 + 282w^3, \\ 30xz - 35xw - 6y^2 + 7yz + 2yw - z^2 + 8zw - 6w^2 \end{cases}$ Diagonal form: $-5400x^2 + 25y^2 - z^2 + 1171800w^2$ lies on a ruled surface over $\mathbb{Q}(\sqrt{6}, \sqrt{217})$.
$X_0^{+55}(110)$	$\begin{cases} x^2w - xyw + xw^2 - y^3 + y^2w + 3yzw + yw^2 + z^2w + zw^2 - w^3, \\ xz - xw - y^2 + yz + 2yw + 2zw - w^2 \end{cases}$ Diagonal form: $-x^2 - 3y^2 + 3z^2 + 13w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{13})$
$X_0^{+19}(133)$	$\begin{cases} 54x^2w - 9xyw - 18xw^2 - 6y^3 - 3y^2z + 3y^2w - 4yz^2 + 10yzw - 7yw^2 \\ \quad -4z^3 + 16z^2w - 16zw^2 + 12w^3, \\ 6xz - 3xw - 2y^2 + yz - yw + 2zw - 3w^2 \end{cases}$ Diagonal form: $-6x^2 - 282y^2 + 3384z^2 - 504w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{3}, \sqrt{-7})$.

Table 7 continued

Curve	Petri's model and Quadratic surface
$X_0^{+29}(145)$	$\begin{cases} x^2w - 2xyw - xw^2 - y^3 + y^2z - 2y^2w + 5yz^2 - 4yzw + 2z^3 - 2z^2w, \\ xz - 2xw - y^2 + 2yz - yw + 2z^2 - 3zw + w^2 \end{cases}$
	Diagonal form: $-3x^2 - 6y^2 + 2z^2 + 21w^2$,
	lies on a ruled surface over $\mathbb{Q}(\sqrt{3}, \sqrt{7})$.
$X_0^{+59}(177)$	$\begin{cases} x^2w - xw^2 - y^3 - y^2z - yz^2 + w^3, \\ xz - y^2 - yw - zw - w^2 \end{cases}$
	Diagonal form: $-x^2 - y^2 + z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-3})$.

Acknowledgements The authors are very grateful to the anonymous referee for providing many valuable comments and suggestions, which improved the article considerably. The first author is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), and through MTM project PID2020-116542GB-I00.

Author contributions Both we wrote and contributed in all the parts of the submitted paper.

Funding Open Access Funding provided by Universitat Autonoma de Barcelona.

Data availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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