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Limit cycles of homogeneous polynomial Kukles differential systems

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ABSTRACT

We study the number of limit cycles which can bifurcate from the periodic orbits of the harmonic oscillator when it is perturbed by homogeneous polynomials of degree n, only in the second differential equation, which corresponds to the so-called Kukles systems. Moreover, the degenerate Hopf bifurcation is also studied for such systems.

1. Introduction

Consider differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials. A *limit cycle* of system (1) is a periodic orbit that is isolated in the set of all the periodic orbits of system (1). The notion of a limit cycle was introduced by Poincaré in his seminal works on differential equations; see [1]. The study of the limit cycles that can occur in a polynomial differential system became an important problem included in Hilbert's list of open problems to be solved in the twentieth century (see [2]). In fact, limit cycles appear in different models of physics, chemistry, biology, and economics, as well as in various phenomena such as synchronization; see for instance [3,4]. Hence, the study of the limit cycles of planar differential systems is one of the main problems in the qualitative theory of differential systems.

The two most common methods for producing limit cycles are the bifurcation of limit cycles from a singular point of focus type (i.e., the Hopf bifurcation) or the perturbation of the periodic orbits of a center. The first method produces the so-called *small amplitude limit cycles*, while the second one generates limit cycles that tend to some periodic orbits as the perturbation goes to zero, called in some works *big limit cycles*; see [5]. We recall that a singular point of system (1) is a center if there exists a neighborhood of it filled with periodic orbits, with the exception of the singular point. These techniques for obtaining limit cycles have been studied intensively in recent decades by several authors; see for instance [3,6,7] and the references therein. One open question is whether the number of small amplitude limit cycles and big ones coincide for a given differential system; see [5]. Indeed, in this latter work, it is proved that these two numbers coincide for a linear center perturbed by homogeneous polynomials.

The methods used for studying the limit cycles that can bifurcate from a period annulus of a center are mainly the Abelian integrals [3], the Melnikov functions method [8,9], and averaging theory [6,7,10]. However, in the plane, all these methods are

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Table 1
Lower bounds for the number of small limit cycles M_n.

n	2	3	4	5
M_n	1	3	4	5

equivalent in the sense that they produce the same results, through the computations may be more effective using one method rather than another; see [11,12].

In general, we are interested in the number of limit cycles that polynomial differential systems of the form

$$\dot{x} = -y + \lambda x, \qquad \dot{y} = x + \lambda y + Q_n(x, y), \tag{2}$$

where Q_n is a polynomial of degree n, can have.

For $n = 3$, Kukles [13] was the first to study the center problem of the differential system

$$\begin{aligned} \dot{x} &= \quad y, \\ \dot{y} &= \quad -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3, \end{aligned} \tag{3}$$

where A, B, C, K, L, M, and N are real constants. Indeed, Kukles did not completely solve the center problem of (3). For many years, it was thought that Kukles' center conditions were necessary and sufficient. For instance, in [14], it was proved that the Kukles conditions were also necessary if $B = 0$. However, some new centers were found afterward; see [15,16]. In [17], the center problem for the class of systems (3) with $N = 0$ was solved, and it was proved that at most five limit cycles bifurcate from the origin. The solution to the center problem for the Kukles system was given independently by Lloyd and Pearson [18] (see also [19]) and by Sadovskii [20] using different methods. The solution of the center problem for system (3) is the following.

Theorem 1. *The origin of system (3) is a center if, and only if, one of the following conditions holds:*

(a) $B = L = N = 0$;
(b) $A = C = L = N = 0$;
(c) $K - C(A + C) = L + B(A + C) = M(A + 2C) + C^2(A + C) = N = 0$;
(d) $AB + BC + L + N = 0$;
$2B^3 - ABC + 2BK + CL + BM - 2AN = 0$;
$6B^2L - ACL - ABM - BCK + KL + LM + A^2N - 2KN = 0$;
$6BL^2 - CKL - BKM - ALM + 2AKN = 0$;
$2L^3 - KLM + K^2N = 0$.

We give the result explicitly, correcting a mistake that appears in [21]. To simplify the computations, in [18], $B \neq 0$ was assumed, and in [20], it was set that $B = N$ if $BN \neq 0$. The cases outside these restrictions are simple and were not considered in those works. Nevertheless, in [22], the solution to the center-focus problem of (3) is given without restrictions on the parameters. Moreover, in this last work, a focus with cyclicity exactly seven was found within the class of cubic Kukles systems. Assuming that system (3) has two fine foci, in [23] it is shown that if they are of the same order then the order is at most two, and that if one fine focus is of order one then the other is of order at most five.

Different authors have studied the bifurcations and distributions of limit cycles of the Kukles system (3), see, for instance [24,25]. The global asymptotic stability of the origin of the Kukles systems (3) was studied in [26]. Finally, some global phase portraits for the Kukles systems (3) are determined in a series of works [27–29].

The easy case is when Q_n is a homogeneous polynomial, and it is the case we first study. In [30], the center conditions and the number of small amplitude limit cycles for the cases $n = 4$ and $n = 5$ were studied. For the first family, there are two cases of centers: when $Q_4(-x, y) = -Q_4(x, y)$ and when $Q_4(x, -y) = Q_4(x, y)$. For the second family, the only center case corresponds to $Q_5(-x, y) = -Q_5(x, y)$. Moreover, in the quartic family, it is deduced from the number of independent Poincaré–Lyapunov quantities that four limit cycles can bifurcate from the origin, and in the quintic family, there are five such limit cycles. Here, we will prove these lower bounds again, as previous proofs were incomplete. For the differential system (2) with Q_n homogeneous, the following two open questions were established in [31].

(a) Is it true that the previous system for $n \geq 2$ has a center at the origin if and only if its vector field is symmetric about one of the coordinate axes?
(b) Is it true that the origin is an isochronous center of the previous system, with the exception of the linear center, only if the system has even degree?

Both questions are proved under certain restrictions; see [32,33]. Under the same restrictions, the uniqueness of certain families of centers is also proved in the center variety of differential systems in $\mathbb{R}^3$, see [34]. In the following table, we present the number M_n of small limit cycles that can bifurcate from the origin of system (2) as a function of the degree n when Q_n is a homogeneous polynomial.

The values of M_n are justified by the results of the last section of this work. From the result in Table 1, we wonder if it is true that $M_n = n$ for all $n \geq 3$, but it is very difficult to go further when n increases. Now we are interested in the number of limit cycles

Table 2
Bounds for the number of isolated zeroes of $M_k(h)$.

$n\backslash k$	1	2	3	4	5	6
2	0	1	1	2	2	3
3	1	2	3	4	5	6
4	1	3	4	6	7	9
5	2	4	6	8	10	12

which can bifurcate from the periodic orbits of the center located at the origin of system (2). But first, we consider a more general problem. Consider a fixed system with a non-degenerate center at the origin:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y), \tag{4}$$

where P and Q are real polynomials of degree at most d without constant or linear terms. For system (4), there exists an analytical first integral $H(x, y)$, see [1,3]. The periodic orbits $\gamma_h \subset \{H = h\}$ surrounding the origin of (4) can be parameterized by the values of H. The period annulus $\mathcal{P}$ is defined by

$$\mathcal{P} = \{\gamma_h \,:\, h \in (h_0, h_1)\},$$

where $h_0 \in \mathbb{R}$ corresponds to the inner boundary (i.e. the origin) and $h_1 \in \mathbb{R} \cup \{+\infty\}$ corresponds to the outer boundary. Consider now a family of perturbations of system (4) of the form

$$\begin{aligned} \dot{x} &= -y + P(x, y) + \varepsilon\, p(x, y, \tilde{\lambda}, \varepsilon), \\ \dot{y} &= x + Q(x, y) + \varepsilon\, q(x, y, \tilde{\lambda}, \varepsilon), \end{aligned} \tag{5}$$

where p and q are polynomials in x, y of degree n and analytic functions in the small *bifurcation parameter* ε and in the parameters $\tilde{\lambda} \in \mathbb{R}^m$. We remind that we consider the problem of bifurcation of limit cycles from the period annulus $\mathcal{P}$ of system (4) in the family (5). In order to define what is a Melnikov function, we consider the Poincaré map $\pi(\cdot;\varepsilon) : \Sigma \to \Sigma$ associated with system (5) and the period annulus $\mathcal{P}$, where Σ is a transversal section parameterized by h, passing through the origin and cutting the whole $\mathcal{P}$. We are under the assumption that for $\varepsilon = 0$ system (5) has a center at the origin; thus, we have that $\pi(h;0) = h$ for all $h \in [h_0, h_1)$. By the analyticity of the Poincaré map with respect to parameters, we have the displacement map

$$d(h;\varepsilon) = \pi(h;\varepsilon) - h = M_1(h)\,\varepsilon + M_2(h)\,\varepsilon^2 + \cdots + M_r(h)\,\varepsilon^r + \mathcal{O}(\varepsilon^{r+1}).$$

Depending on the parameters $\tilde{\lambda}$, there exists some $k \geq 1$ such that $M_r(h) \equiv 0$ for any $1 \leq r < k$ and $M_k(h) \not\equiv 0$, i.e.

$$d(h;\varepsilon) = M_k(h)\,\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}).$$

The function $M_k(h)$ is called the Melnikov function of order k. The isolated zeroes of $M_k(h)$ (counted with multiplicity) allow us to study limit cycles of system (5) which bifurcate from the orbits of the period annulus of system (4), see for instance [9].

The easiest center in system (4) is the linear center $\dot{x} = -y$, $\dot{y} = x$. By perturbing this center within the class of all polynomial differential systems of degree n, and using the Melnikov function of order k, it was shown in [8] that at most $[\frac{k(n-1)}{2}]$ limit cycles we can obtained, where $[\cdot]$ denotes the integer part function. The upper bound for different values of k and n is computed in the following table, which was also presented in [8]. For quadratic perturbations, i.e., for $n = 2$, the results summarized in Table 2 are correct and well known from the results of Bautin and Iliev [35,36] (see also [37]), where it is proved that all functions $M_k(h)$, for $k > 6$, have at most 3 zeros. For $n > 3$, the situation becomes too complicated. It is not clear if the values given in Table 2 are correct because it is not possible, in general, to compute the functions $M_k(h)$ for any perturbation given by an arbitrary polynomial of degree n. In fact, in [38], it was conjectured that the number of functionally independent focal values of the system $\dot{x} = -y + \lambda x + P(x, y)$, $\dot{y} = x + \lambda y + Q(x, y)$ at the origin is $M(n) = n^2 + 3n - 7$. This value has been updated to $M(n) = n^2 + 3n - 6$ in [39]. Hence, taking into account that the number of small limit cycles and the big ones are the same for certain families (see [5]), we can think that the values given in Table 2 are correct. These values are not in contradiction with the above conjecture.

A challenging open problem is determining the value of $k = k(n)$ at which the numbers in Table 2 stabilize. Specifically, this involves identifying the order of the Melnikov function $M_k(h)$ at which subsequent Melnikov functions will have the same maximum number of isolated zeros. To solve this problem, we first need to address the center–focus problem of the considered family, as discussed in [37].

In this work, we focus on the Kukles systems defined by (2). More specifically, we are interested in the *Kukles homogeneous systems*. For this family, the number of small limit cycles coincides with the number of large ones, as it is a particular case of the family studied in Theorem B of [5], which considers the family

$$\dot{x} = -y + P_n(x, y), \qquad \dot{x} = x + Q_n(x, y), \tag{6}$$

where Q_n and P_n are homogeneous polynomials of degree n. The conjecture in [38] states that the number of functionally independent focal values of system (6) at the origin is $M(n) = 2n - 1$. Furthermore, it is known that for $n = 3$, system (6) can have at most 5 limit cycles, and that all functions $M_k(h)$ for $k > 6$ have at most 5 zeros, as shown in [37]. Therefore, the results presented in Table 2 are not optimal (i.e., not realizable) for either system (6) or system (2). In fact, the exact upper bound and feasible limit for system (6) when $n = 3$ and $k = 6$ is given by $M_6(h) = 5$. This result comes from the fact that the maximum number of small limit cycles for $n = 3$ is 5 [40] and from Theorem B of [5] that proves that this number coincides with the number of large ones.

Table 3
Number of isolated zeroes of the averaged functions up to order $k = 6$.

$n\backslash k$	1	2	3	4	5	6
2	0	1	1	1	1	1
3	1	2	3	3	3	3
4	0	1	1	2	2	4
5	1	2	3	4	5	5

2. Limit cycles with averaging

Our first objective in this work is to provide the optimal upper bounds for the number of limit cycles which can be obtained by perturbing the center $\dot{x} = -y$, $\dot{y} = x$, when linear terms and homogeneous nonlinearities of degree n are introduced only in the second equation. This is achieved through the application of averaging theory at first order. In other words, we seek to determine the maximum number of limit cycles of systems of the form

$$\dot{x} = -y + x\sum_{s=1}^{k}\varepsilon^s\lambda_s, \qquad \dot{x} = x + y\sum_{s=1}^{k}\varepsilon^s\lambda_s + \sum_{s=1}^{k}\varepsilon^s\sum_{i=0}^{n}a_{i,s}x^i y^{n-i}, \tag{7}$$

for $\varepsilon \neq 0$ sufficiently small, which bifurcate from the periodic orbits of the center. The number of these limit cycles is determined by using averaging theory up to order k. The answers to this question are given in Theorem 2 for $k = 1, \ldots, 6$.

Theorem 2. *For $\varepsilon \neq 0$ sufficient small, the maximum number of limit cycles of system (7) which bifurcate from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x$ using averaging theory of kth order, is given in Table 3.*

The results of Table 3 are in agreement with Theorem B of [5], which states that the number of small limit cycles equals the number of big limit cycles i.e., limit cycles that bifurcate from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x$, when only the second equation is perturbed by homogeneous polynomials of degree n.

We prove the case $n = 3$ for all k, as the proof for the other cases is similar except for the case $n = 5$ and $k = 6$, which yields only 5 isolated zeroes. For this particular case, a detailed discussion is required to understand why the number of zeros does not increase compared to the lower order analysis.

In order to apply the averaging theory described in the previous section, we must first transform the system into the form (10). Let Q_n be a homogeneous polynomial of degree n. The differential system

$$\dot{x} = \lambda x - y, \qquad \dot{y} = x + \lambda y + Q_n(x, y),$$

can be rewritten using polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ in the form

$$\dot{r} = \lambda r + a(\theta)r^n, \qquad \dot{\theta} = 1 + b(\theta)r^n,$$

where

$$a(\theta) = \sin\theta\, Q_n(\sin\theta, \cos\theta),$$
$$b(\theta) = \cos\theta\, Q_n(\cos\theta, \sin\theta).$$

Next, dividing $\dot{r}$ by $\dot{\theta}$, we obtain the differential equation

$$\frac{dr}{d\theta} = \frac{\lambda r + a(\theta)r^n}{1 + b(\theta)r^{n-1}}. \tag{8}$$

Now, setting $\lambda = \sum_{s=1}^{\infty}\varepsilon^s\lambda_s$ and

$$Q_n(x, y) = \sum_{i=0}^{n}a_i x^i y^{n-i},$$

with $a_i = \sum_{s=1}^{\infty}\varepsilon^s a_{i,s}$, in Eq. (8), and expanding it in a power series of r, we obtain a differential equation that has the normal form (10), allowing us to apply the averaging theory up to the sixth order in ε. In (10), we now have $x = r$, $t = \theta$ and $F_k(\theta, r)$ is the coefficient of ε^k in $dr/d\theta$ for $k = 1, \ldots, 6$. While we omit the large expressions for $F_k(\theta, r)$ here, the reader can find them using an algebraic manipulator.

Proof of Theorem 2. We provide detailed proofs for the cases $n = 3$ and $n = 5$ with $k = 6$. The proofs for the other values of n and k can be carried out in a similar way, but we omit them here due to the large expressions for the averaged functions F_{k0}.

Starting with the case $n = 3$, we compute the function $F_{10}(r)$, as defined in Appendix, and we obtain:

$$F_{10}(r) = 2\pi\lambda_1 r + \frac{1}{4}(3a_{01} + a_{21})\pi r^3.$$

The polynomial $F_{10}(r)$ can have at most one positive real root. Since the coefficients of $F_{10}(r)$ are independent, there exist polynomial differential systems (7) for which this root of $F_{10}(r)$ exists. Therefore, the proof follows for $n = 3$ and $k = 1$.

Next, to apply the second-order averaging theory, we need to vanish $F_{10}(r)$ identically. This is achieved by choosing $\lambda_1 = 0$, $a_{21} = -3a_{01}$. We then compute the function $F_{20}(r)$ as defined in Appendix, and finding

$$F_{20}(r) = 2\pi\lambda_2 r + \frac{1}{4}(3a_{02} + a_{22})\pi r^3 + \frac{1}{8}a_{01}(a_{11} + 3a_{31})\pi r^5,$$

Since the coefficients of the polynomial $F_{20}(r)$ are independent, we have that this polynomial can have at most two positive real roots, and there are polynomial differential systems (7) for which they have them. The coefficients of the polynomial $F_{20}(r)$ are independent, because in every coefficient of this polynomial appears some parameter of the initial polynomial differential system (7) which not appear in the other coefficients. Hence the theorem is proved for $n = 3$ and $k = 2$.

We need to vanish $F_{20}(r)$ identically, in order to apply the averaging theory of third order. Thus, we select

$$\lambda_2 = 0, \qquad a_{22} = -3a_{20}, \qquad a_{11} = -3a_{31}.$$

It is worth noting that if, instead of choosing $a_{11} = -3a_{31}$, we set $a_{01} = 0$, this alternative condition for vanishing F_{20} would not give more isolated zeroes for the averaged functions F_{k0} for $k = 3, 4, 5, 6$, as confirmed by the corresponding computations. Next, computing the function $F_{30}(r)$ defined in Appendix, we obtain

$$F_{30}(r) = 2\pi r\lambda_3 + \frac{1}{4}(3a_{03} + a_{23})\pi r^3 + \frac{1}{8}a_{01}(a_{12} + 3a_{32})\pi r^5 - \frac{3}{32}a_{01}(a_{01}^2 + a_{31}^2)\pi r^7.$$

From the expression of the polynomial $F_{30}(r)$, where all its coefficients are independent, it follows that $F_{30}(r)$ can have at most three positive real roots, and that there are polynomial differential systems (7) for which the corresponding F_{30} has 3 positive real roots. Hence the theorem is proved for $n = 3$ and $k = 3$.

To continue we vanish the coefficients of F_{30} taking

$$\lambda_3 = 0, \qquad a_{23} = -3a_{03}, \qquad a_{01} = 0.$$

We note again that, taking the other combinations of vanishing F_{30}, we cannot obtain more positive isolated zeros of the averaged functions F_{k0} for $k = 4, 5, 6$. Computing the function $F_{40}(r)$ defined in [6], we obtain

$$F_{40}(r) = 2\pi r\lambda_4 + \frac{1}{4}(3a_{04} + a_{24})\pi r^3 + \frac{1}{8}a_{02}(a_{12} + 3a_{32})\pi r^5 - \frac{3}{32}a_{02}a_{31}^2\pi r^7.$$

In view of expression of F_{40}, we see that no more than 3 isolated zeroes can be obtained at this step. Hence the theorem is proved for $n = 3$ and $k = 4$.

We vanish F_{40} taking

$$\lambda_4 = 0, \qquad a_{24} = -3a_{04}, \qquad a_{02} = 0.$$

Again, by taking the other combinations of vanishing F_{40}, we cannot obtain more positive isolated zeros of the averaged functions F_{k0} for $k = 5, 6$. Next we compute the function $F_{50}(r)$ defined in [6] and we obtain

$$F_{50}(r) = 2\pi r\lambda_5 + \frac{1}{4}(3a_{05} + a_{25})\pi r^3 + \frac{1}{8}a_{03}(a_{12} + 3a_{32})\pi r^5 - \frac{3}{32}a_{03}a_{31}^2\pi r^7.$$

From the expression of F_{50} we see that no more than 3 isolated zeroes can be obtained at this step again. Hence the theorem is proved for $n = 3$ and $k = 5$.

We vanish F_{50} choosing

$$\lambda_5 = 0, \qquad a_{25} = -3a_{05}, \qquad a_{03} = 0.$$

As in the previous cases, the other combinations of vanishing F_{50} do not yield more positive isolated zeros of the averaged functions F_{60}. Finally, we compute the function $F_{60}(r)$ defined in [6] and we obtain

$$F_{60}(r) = 2\pi r\lambda_6 + \frac{1}{4}(3a_{06} + a_{26})\pi r^3 + \frac{1}{8}a_{04}(a_{12} + 3a_{32})\pi r^5 - \frac{3}{32}a_{04}a_{31}^2\pi r^7.$$

Hence, the expression of F_{60} shows that no more than 3 isolated zeroes can be obtained. Hence the theorem is proved for $n = 3$ and $k = 6$.

For instance, for the system

$$\dot{x} = -y - \frac{3\varepsilon^3}{\pi}x,$$
$$\dot{y} = x - \frac{3\varepsilon^3}{\pi}y + \left(\frac{2\cdot 6^{2/3}\varepsilon}{\pi^{1/3}} + \frac{44\varepsilon^3}{\pi}\right)x^2y + \frac{12\cdot 6^{1/3}\varepsilon^2}{\pi^{2/3}}xy^2 - \frac{2\cdot 2^{2/3}\varepsilon}{(3\pi)^{1/3}}y^3,$$

the polynomials $F_{10}(\rho) \equiv F_{20}(\rho) \equiv 0$ and

$$F_{30}(r) = r(r^2 - 1)(r^2 - 2)(r^2 - 3).$$

So this system has three positive real roots. This completes the proof for $n = 3$.

Now we take $n = 5$ and we compute the function $F_{10}(r)$ defined in Appendix, and we get

$$F_{10}(r) = 2\pi\lambda_1 r + \frac{1}{8}(a_{11} + a_{31} + 5a_{51})\pi r^5.$$

and in general

$$F_{k0}(r) = r\sum_{j=0}^{k} P_{j,k}(a_{i,s})r^{4j},$$

where $P_{j,k}$ are polynomials with $P_{0,k} = 2\pi\lambda_k$ and $P_{k,k}$ are homogeneous of degree k depending only on $a_{i,1}$. Indeed, $P_{k,k}$ correspond to the Poincaré–Lyapunov quantities of system (7) at first order in ε. The proof finishes seeing that $P_{6,6}$ vanishes identically when $P_{i,i} \equiv 0$ for $i = 1,\dots,5$. However, the polynomial $P_{6,6}$ does not belong to the ideal $\mathcal{I} = \langle P_{1,1},\dots,P_{5,5}\rangle$ we need to prove that $P_{6,6}|_{\mathcal{V}} \equiv 0$ where $\mathcal{V} = \{P_{1,1} = 0,\dots,P_{5,5} = 0\}$. Although the analysis of the nonlinear system of equations defined by $\mathcal{V}$ is very complicated, we can simplify it by isolating a_{51} from $P_{11} = 0$. Then, we apply the resultant method. First, we compute the following resultants, eliminating the multiple factors:

$$V_{2,k} = \mathrm{Res}(P_{2,2}, P_{k,k}, a_{01}), \text{ for } k = 3,\dots,6.$$

and then

$$V_{3,k} = \mathrm{Res}(V_{2,3}, V_{2,k}, a_{11}), \text{ for } k = 4,\dots,6.$$

where $V_{3,k}$ are polynomials in variables a_{21}, a_{31}, a_{41} with lowest common divisor given by $a_{31}(4a_{41}+3a_{21})(4a_{41}^2 - 4a_{41}a_{21} + 4a_{31}^2 + a_{21}^2)$. It is easy to see that each factor corresponds to the reversible center defined by $\mathcal{V}_C = \{a_{51} = a_{31} = a_{11} = 0\}$, which consequently leads to $P_{6,6} \equiv 0$. The polynomials $V_{3,k}$ for $k = 4,\dots,6$ without common factors are

$$\tilde{V}_{3,4} = (2a_{41} + a_{21})(2a_{41} - 3a_{21})R_{3,4}(a_{21}, a_{31}, a_{41}), \quad \tilde{V}_{3,5} = R_{3,5}(a_{21}, a_{31}, a_{41}),$$

where $R_{3,k}$ are homogeneous polynomials of degrees 11, 18, and 23 respectively. The first two factors of $\tilde{V}_{3,4}$ also correspond to the reversible center, thus leading again $P_{6,6} \equiv 0$. Since we already considered $a_{31} = 0$, we take $a_{31} = 1$ by homogeneity. This reduces the problem to finding a solution for $\{R_{3,4} = 0, R_{3,5} = 0\}$. We then compute the last resultants

$$T_1(a_{21}) = \mathrm{Res}(R_{3,4}, R_{3,5}, a_{41}), \qquad T_2(a_{41}) = \mathrm{Res}(R_{3,4}, R_{3,5}, a_{21})$$

which are polynomials of degree 198, each having two real simple solutions of the form $\pm z_1$ and $\pm z_2$, respectively. This property can be easily checked by Sturm's method, which shows that $z_1 \in (2/5, 1/2)$ and $z_2 \in (5/4, 3/2)$ but $R_{3,4}(z_1, 1, z_2) \neq 0$ as the minimum of $R_{3,4}(z, 1, w)$ is positive in $[2/5, 1/2]\times[5/4, 3/2]$. Thus, there is no real solution in $\mathcal{V}$ except the reversible center variety $\mathcal{V}_C$ described above, and the proof follows. □

Remark 3. *We emphasize that the property whereby the homogeneous polynomials $P_{k,k}$, which depend only on the coefficients $a_{i,1}$, represent the Poincaré–Lyapunov quantities of system (7) at first order in ε, holds for any homogeneous system (see [5]).*

3. Limit cycles by a degenerate Hopf bifurcation

This section is devoted to studying the number of limit cycles bifurcating from the origin for the Kukles homogeneous system of degree $n = 2, 3, 4, 5$

$$\dot{x} = -y + \lambda x, \qquad \dot{y} = x + \lambda y + \sum_{i=0}^{n} a_i x^i y^{n-i}, \tag{9}$$

which is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$ for n even and $(x, y, t) \to (x, -y, -t)$ for n odd. We denote this variety as $\mathcal{V}_R$ for each n. We will see that the number of limit cycles coincides with the results obtained in the previous section, specifically for special points of the center variety; however, the local cyclicity is generically lower.

In general, the cyclicity corresponds to the number of parameters associated to the non-reversible monomials of (9). Using the parameters associated with the reversible monomials, we can increase this number of limit cycles in the non-generic cases, following the arguments in [39].

The number of small amplitude limit cycles is analyzed through the classical degenerate Hopf bifurcation by examining the Taylor series of the return map near the origin. The non-vanishing coefficients of this series are referred to as the Poincaré–Lyapunov quantities, denoted by L_i. As usual, these quantities are defined modulo previous ones; for further details, see [9].

Theorem 4. *The local cyclicity of $\dot{x} = -y$, $\dot{y} = x + axy$, inside the Kukles homogeneous systems (9) with $n = 2$ is exactly 1 when $a \neq 0$.*

Proof. It is straightforward to observe that the first Poincaré–Lyapunov quantity, when $\lambda = 0$, is given by $L_1 = -a(a_0 + a_2)/2$. The subsequent quantities are zero when $a_0 + a_2 = 0$. Furthermore, when $a_2 = -a_0$, we obtain the system $\dot{x} = -y$, $\dot{y} = x + a_0x^2 + axy - a_0y^2$, which is a center, as it possesses an inverse integrating factor of the form $1 - 2bx - ay + b^2x^2 + abxy - b^2y^2$. Since there is only one element in the Bautin ideal, it is radical, indicating that the local cyclicity is bounded by 1. The existence of a limit cycle is guaranteed by the classical Hopf bifurcation when $\lambda \neq 0$ is sufficiently small and $a_0 + a_2 \neq 0$. □

Theorem 5. *The local cyclicity of $\dot{x} = -y$, $\dot{y} = x + ax^3 + bxy^2$, inside the Kukles homogeneous systems (9) with $n = 3$, is at least 2 when $3a + b \neq 0$ and exactly 3 when $3a + b = 0$ with $a \neq 0$.*

Proof. The linear parts respect a_1 and a_3 of the first Poincaré–Lyapunov quantities, when $\lambda = 0$, are given by $L_1^{[1]} = (a_1 + 3a_3)/3$, $L_2^{[1]} = -(3a + b)a_1/15$ and $L_3^{[1]} = (3a^2 + 2ab + b^2)a_1/105$, modulo the previous ones. The existence of a weak foci curve of order 2, along with the complete unfolding by adding the trace parameter, is clear through the Implicit Function Theorem when $3a + b \neq 0$. Conversely, when $3a + b = 0$ and $a \neq 0$, a more degenerate Hopf bifurcation of order 3 occurs; further details on this technique can be found in [39].

The proof of the upper bound is straightforward: it involves verifying that the Bautin ideal generated by all Poincaré–Lyapunov quantities is radical, with the first three elements serving as generators. The center variety is given by $\mathcal{V}_C = \{a_1 = a_3 = 0\}$. □

Theorem 6. *The local cyclicity of $\dot{x} = -y$, $\dot{y} = x + ax^3y + bxy^3$, inside the Kukles homogeneous systems of degree 4, is at least 3 if $(7a + 3b)(3a + 5b)AB \neq 0$ and at least 4 if $(7a + 3b)(3a + 5b)AC \neq 0$ and $B = 0$, where $A = 3a^3 + 5a^2b - 3ab^2 - 9b^3$, $B = (a+b)(15a^3+40a^2b+11ab^2-26b^3)$, and $C = 291123a^6+1455615a^5b+2579184a^4b^2+1823702a^3b^3+178635a^2b^4-518637ab^5-308854b^6$.*

Proof. The linear parts with respect to a_0, a_2, and a_4 of the first Poincaré–Lyapunov quantities when $\lambda = 0$, modulo the previous ones, are given by

$$
\begin{aligned}
L_1^{[1]} &= -\frac{1}{35}((3a + 7b)a_0 + 3(a + b)a_2 + (7a + 3b)a_4),\\
L_2^{[1]} &= -\frac{2}{715}\frac{(3a + 5b)(a - b)(a^2 + 4ab + 7b^2)}{7a + 3b}a_0 - \frac{2}{2145}\frac{(3a + 5b)A}{7a + 3b}a_2,\\
L_3^{[1]} &= \frac{16}{230945}\frac{a^2b^2B}{A}a_0,\\
L_4^{[1]} &= \frac{16}{58561878375}\frac{a^2b^2C}{A}a_0.
\end{aligned}
$$

When $(7a + 3b)(3a + 5b)A \neq 0$, the linear parts of L_1, L_2, L_3 are linear independent. Consequently, by applying the Implicit Function Theorem, we can generically obtain at least three limit cycles from the reversible center.

In the non-generic case where $(7a + 3b)(3a + 5b)AC \neq 0$ and $B = 0$, the proof follows using the arguments in [39], similar to the previous proof. Indeed, when $B = 0$, we have not vanishing L_3; we only have $L_3^{[1]} = 0$.

It is clear that when $a_0 = 0$, we have a center. Then, $L_3 = 0$ when a_0. Consequently, $L_3 = a_0\tilde{L}_3$ and $\tilde{L}_3 = c_1B + \mathcal{O}(a_0) = \tilde{c}_1(a + b) + \mathcal{O}(a_0)$ with $\tilde{c}_1 \neq 0$ when $a + b = 0$. Again, using the Implicit Function Theorem, we can set L_3 to zero while ensuring $L_4 \neq 0$. □

Theorem 7. *The local cyclicity of system $\dot{x} = -y$, $\dot{x} = x + ax^5 + bx^3y^2 + cxy^4$, inside the Kukles homogeneous systems of degree 5, is at least 3 if $AB \neq 0$, at least 4 if $AC \neq 0$ and $B = 0$, and at least 5 if $AD \neq 0$ and $BC = 0$ when $B = 0$ and $C = 0$ intersect transversally, where $A = 7c + 9b + 15a$, $B = 1485a^3 + 1593a^2b + 1439a^2c + 351ab^2 + 546abc + 183ac^2 + 27b^3 + 55b^2c + 9bc^2 - 27c^3$, $C = 155925a^4 + 115830a^3b + 89100a^3c + 13365a^2b^2 + 25650a^2bc + 10330a^2c^2 + 162ab^2c + 282abc^2 + 2148ac^3 + 353b^2c^2 + 990bc^3 + 513c^4$, and $D = 1578763851345a^5 + 1001175927759a^4b + 446139679785a^4c + 134078106384a^3b^2 + 169876606848a^3bc + 35853416858a^3c^2 - 5031721546a^2bc^2 + 34431109122a^2c^3 + 7351615416abc^3 + 2974075509ac^4 - 164377341bc^4 + 194007477c^5$.*

Proof. The linear parts with respect to a_0, a_2, and a_4 of the first Poincaré–Lyapunov quantities when $\lambda = 0$, modulo the previous ones, are given by

$$
\begin{aligned}
L_1^{[1]} &= -\frac{1}{5}(a_1 + a_3 + 5a_5),\\
L_2^{[1]} &= -\frac{1}{315}(2(c + 7b + 25a)a_1 + Aa_3),\\
L_3^{[1]} &= \frac{1}{1287}\frac{B}{A}a_1,\\
L_4^{[1]} &= -\frac{4}{984555}\frac{C}{A}a_1,\\
L_5^{[1]} &= \frac{4}{4540293766827}\frac{D}{A}a_1.
\end{aligned}
$$

When $AB \neq 0$, L_1, L_2, and L_3 have independent linear parts and by the Implicit Function Theorem, we obtain at least three limit cycles bifurcating from origin. The proof of the non-generic case $AC \neq 0$ and $B = 0$ follows as in the previous proof to obtain at least four limit cycles of small amplitude. The existence of a weak foci curve of order five and its unfolding for the last non-generic case ($AD \neq 0$ with $BC = 0$) follows using Theorem 3.1 in [39]. Indeed, the transversality is necessary in order to guarantee that we can apply the Implicit Function Theorem adequately to the vanishing of L_3 and L_4. □

CRediT authorship contribution statement

Jaume Giné: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Joan Torregrosa:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

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Appendix. Averaging theory up to order 6

The averaging theory up to any order for studying specifically periodic orbits was developed in [6] for differential equations in dimension one and in [10] for differential equations in arbitrary dimensions. Here, we summarize here the averaging theory up to third order, and we address the reader to [6] for the higher order results.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{10}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D being an open subset of $\mathbb{R}$. Assume that the following hypotheses hold.

(a) $F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1$, $D_x F_2$ are locally Lipschitz with respect to x and R is twice differentiable with respect to ε.
We define $F_{k0} : D \to \mathbb{R}$ for $k = 1,2,3$ as

$$\begin{aligned}
F_{10}(x) &= \frac{1}{T}\int_0^T F_1(s,x)ds,\\
F_{20}(x) &= \frac{1}{T}\int_0^T \left[\frac{\partial F_1}{\partial x}(s,x)\cdot y_1(s,x) + F_2(s,x)\right] ds,\\
F_{30}(x) &= \frac{1}{T}\int_0^T \Big[\ \frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(s,x) y_1(s,x)^2 + \frac{1}{2}\frac{\partial F_1}{\partial x}(s,x) y_2(s,x)\\
&\quad + \frac{\partial F_2}{\partial x}(s,x) y_1(s,x) + F_3(s,x)\ \Big] ds,
\end{aligned}$$

where

$$\begin{aligned}
y_1(s,x) &= \int_0^s F_1(t,x)dt,\\
y_2(s,x) &= 2\int_0^s \left[\frac{\partial F_1}{\partial x}(t,x) y_1(t,x) + F_2(t,x)\right] dt.
\end{aligned}$$

(b) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a \in V$ such that $(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30})(a) = 0$ and

$$\frac{d(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30})}{dx}(a) \neq 0.$$

Then, for sufficiently small $|\varepsilon| > 0$, there exists a T–periodic solution $x(t,\varepsilon)$ of the system such that $x(0,\varepsilon) \to a$ as $\varepsilon \to 0$.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case, the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not, the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case, the previous result provides the *averaging theory of second order*.

If both F_{10} and F_{20} are identically zero, and F_{30} is not, the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ coincide with the zeros of F_{30} for ε sufficiently small. In this case, the previous result provides the *averaging theory of third order*.

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