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# PHASE PORTRAITS OF A FAMILY OF HAMILTONIAN CUBIC SYSTEMS

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Abstract. We deal with the one-dimensional parameter family of Hamiltonian cubic polynomial differential systems

$$\dot{x} = y - y(y^2 + 3x^2\mu), \qquad \dot{y} = x + x(x^2 + 3y^2\mu),$$

where $(x, y) \in \mathbb{R}^2$ are the variables and $\mu$ is a real parameter. We classify in the Poincaré disc the topological phase portraits of this family of systems when the parameter $\mu$ varies, describing the bifurcations which take place.

## 1. Introduction

A *cubic polynomial differential system* is a system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P$ and $Q$ are polynomials in the variables $x$ and $y$, and the maximum of the degrees of $P$ and $Q$ is three.

The phase portraits of the polynomial differential systems (1) of degree 1, i.e. the *linear differential systems*, are well known. There are more than one thousands papers published on the polynomial differential systems (1) of degree 2, i.e. the so called *quadratic systems*, see for instance the books [2, 16, 19] and the hundreds of references quoted in each of these books. With respect to the polynomial differential systems (1) of degree 3, or simply *cubic systems*, few papers have been published if we compare with the papers dedicated to the quadratic systems, but this is changing see for instance some of the papers dedicated to the cubic systems published in 2020 [3, 4, 6, 7, 8, 9, 10, 11, 12, 17, 18, 20, 21] and their references.

Very few papers are dedicated to classify the phase portraits of some families of cubic systems. In this article we classify all topological

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phase portraits in the Poincaré disc of the Hamiltonian family of cubic polynomial differential systems

$$\dot{x} = y - y(y^2 + 3x^2\mu), \qquad \dot{y} = x + x(x^2 + 3y^2\mu), \tag{2}$$

where $(x, y) \in \mathbb{R}^2$ are the variables and $\mu$ is a real parameter. We denote the vector field associated to this differential system by $X_\mu(x, y) = (y - y(y^2 + 3x^2\mu), x + x(x^2 + 3y^2\mu))$. The Hamiltonian of the system (2) is

$$H = H(x, y) = \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{4}\left(x^4 + y^4\right) + \frac{3}{2}\mu x^2 y^2. \tag{3}$$

Roughly speaking the Poincaré disc is the closed disc $\mathbb{D}^2$ centered at the origin of $\mathbb{R}^2$ of radius one, its interior is identified with $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$, because in $\mathbb{R}^2$ we can go to the infinity in many directions as points has the circle $\mathbb{S}^1$. Then polynomial vector fields $X_\mu$ can be extended to analytic vector fields $p(X_\mu)$ defined in $\mathbb{D}^2$. See Chapter 5 of [5] for the details on the Poincaré compactification.

It is known that two phase portraits in the Poincaré disc $\mathbb{D}^2$ are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ which sends orbits of one of the phase portraits into orbits of the other phase portrait, preserving or reversing the orientation of all orbits.

For studying the local phase portraits at the finite and infinite singular points of the compactified cubic polynomial differential systems we use notations and results summarized in Chapters 1, 2, 3 and 5 from [5], and which are due to many different authors. In order to classify all topological phase portraits of the cubic polynomial differential systems (2) in the plane $\mathbb{R}^2$ extended to infinity we shall use the Theorem 1.43 of [5] due to Markus [13], Neumann [14] and Peixoto [15], which guarantees that we only need to classify all the different configurations that arise in the regions delimited by the separatrices of the compactified cubic polynomial differential systems.

Now we are ready to establish our main result.

**Theorem 1.** *The cubic polynomial differential system* (2) *has a phase portrait in the Poincaré disc topologically equivalent to the phase portraits of Figure* 1 *if* $\mu > -1/3$, *of Figure* 2 *if* $\mu < -1/3$ *and of Figure* 3 *if* $\mu = -1/3$. *So there are only three different topological phase portraits in the Poincaré disc for system* (2), *and a unique bifurcation value* $\mu = -1/3$.

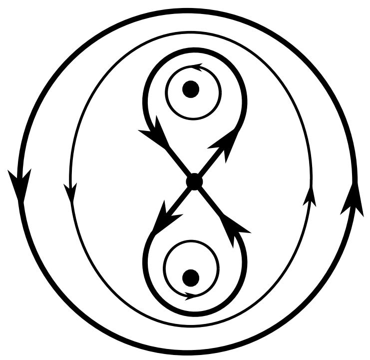


FIGURE 1. Global phase portrait for $\mu > -1/3$.

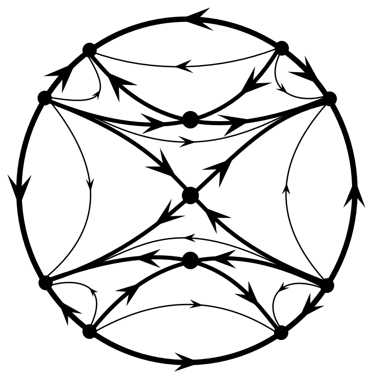


FIGURE 2. Global phase portrait for $\mu < -1/3$.

The rest of this article is dedicated to proof Theorem 1 and for that it is organized as follows. In section 2 we analyze the local phase portraits of the finite singular points. The local phase portraits at the infinite singular points are study in section 3. Finally we prove Theorem 1 in section 4.

## 2. Finite singular points

To study the finite singular points of system (2) we consider the cases $|\mu| < 1/3$, $\mu = 1/3$, $\mu = -1/3$ and $|\mu| > 1/3$.

For $\mu = -1/3$ the two components of the differential system has the common factor $1 + x^2 - y^2$, and the original differential system (2) becomes the differential $\dot{x} = y$, $\dot{y} = x$ doing a rescaling of independent variable by the factor $1 + x^2 - y^2$. Furthermore, in the region $\mathbb{R}^2 \setminus C$ with $C = \{(x, y) \in \mathbb{R}^2 : 1+x^2-y^2 = 0\}$, system (2) has the same orbits

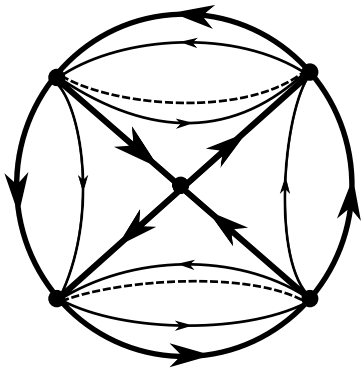


FIGURE 3. Global phase portrait for $\mu = -1/3$. The dashed curve is the hyperbola $y^2 - x^2 = 1$, its points are singularities.

as $\dot{x} = y$, $\dot{y} = x$, but noting that the orbits have reversing orientation where $1 + x^2 - y^2$ is negative. Of course, all the points of the curve $C$ are singular points. The linear part of the differential system at the origin has eigenvalues $\pm 1$, so by Theorem 2.15 of [5] the origin is a hyperbolic saddle.

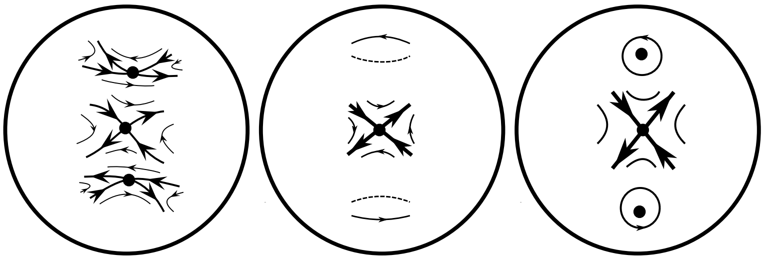
In the rest of this section $\mu \neq -1/3$. Then the finite singular points of the differential system are $(0, -1), (0, 0)$ and $(0, 1)$. Now we shall determine the local phase portraits at these singular points. For this we shall use notations and results stated in section 2.6 from [5]. We denote the linear part associated to system (2) by $DX_\mu(x, y)$, its determinant by $\det(DX_\mu(x, y)) = \det(x, y)$, and its trace by $\operatorname{tr}(DX_\mu(x, y)) = \operatorname{tr}(x, y)$.

The linear part of $X_\mu(x, y)$ is

$$DX_\mu(x, y) = \begin{pmatrix} -6\mu xy & -3\mu x^2 - 3y^2 + 1 \\ 3x^2 + 3\mu y^2 + 1 & 6\mu xy \end{pmatrix}.$$

Thus we obtain $\det(0, 0) = -1$, then by Theorem 2.15 of [5] the singular point $(0, 0)$ is a saddle for all the values of the parameter $\mu$. At the other singular points $(0, -1)$ and $(0, 1)$, the determinant is $\det(x, y) = 2+6\mu$. Hence for $\mu < -1/3$ the determinants $\det(0, 1)$ and $\det(0, -1)$ are negative, again by Theorem 2.15 of [5] the singular points $(0, 1)$ and $(0, -1)$ are saddles. Now for $\mu > -1/3$ the determinants $\det(0, 1)$ and $\det(0, -1)$ are positive and the traces $\operatorname{tr}(0, 1)$ and $\operatorname{tr}(0, -1)$ are zero. Then the singular points $(0, -1)$ and $(0, 1)$ are weak focus, and as system (2) is Hamiltonian, the existence of the first integral given by the Hamiltonian (3) implies that these weak focus are centers (see also the Liouville's Theorem [1]).

In summary see the local phase portraits at the finite singular points of the differential system (2) in Figure 4 according the different values of the $\mu$ parameter.



$\mu < -1/3$ $\mu = -1/3$ $\mu > -1/3$

FIGURE 4. Local phase portraits of the finite singular points in the Poincaré disc.

## 3. Infinite singular points

In this section we study the infinite singular points of system (2), and as in the previous section here we shall use notations and results from Chapter 5 of [5].

From equation (5.2) in [5] the expression of the Poincaré compactification $p(X_\mu)$ of the differential system (2) in the local chart $U_1$ is

$$\begin{array}{rcl} \dot{u} &=& 1 + 6\mu u^2 + v^2 + u^4 - u^2v^2, \\ \dot{v} &=& 3\mu uv + u^3v - uv^3. \end{array} \tag{4}$$

And from equation (5.3) in [5] the expression of the Poincaré compactification $\tilde{p}(X_\mu)$ of the differential system (2) in the local chart $U_2$ is

$$\begin{array}{rcl} \dot{u} &=& -1 - 6\mu u^2 + v^2 - u^4 - u^2v^2, \\ \dot{v} &=& -3\mu uv - u^3v - uv^3. \end{array} \tag{5}$$

**Proposition 2.** *Consider system* (2) *and its Poincaré compactification.*

(a) *If* $\mu > -1/3$ *system* (2) *has no infinite singular points in the local charts* $U_1$ *and* $V_1$*. So the infinity in the Poincaré disc is a periodic orbit, see Figure 5.*
(b) *If* $\mu < -1/3$ *system* (2) *has eight infinite singular points, two stable nodes at*

$$q_1 = (-\sqrt{-\sqrt{9\mu^2 - 1} - 3\mu}, 0) \quad \text{and} \quad q_3 = (\sqrt{-\sqrt{9\mu^2 - 1} - 3\mu}, 0),$$

*and two unstable nodes at*

$$q_2 = (-\sqrt{\sqrt{9\mu^2-1}-3\mu}, 0) \quad and \quad q_4 = \sqrt{\sqrt{9\mu^2-1}-3\mu}, 0),$$

*in the local chart* $U_1$*, and of course the diametrically opposite four nodes in the local chart* $V_1$*, see Figure 5.*

(c) *If* $\mu = -1/3$ *system* $\dot{x} = y$*,* $\dot{y} = x$ *has four infinite singular points, one unstable node at* $(-1, 0)$ *and one stable node at* $(1, 0)$ *in the local chart* $U_1$*, and of course the diametrically opposite two nodes in the local chart* $V_1$*. Then the hyperbola* $y^2 - x^2 = 1$ *full up with singularities for system* (2) *divides these four nodes in two parabolic sectors one stable and the other unstable, see Figure 5.*

*Proof.* Consider system (4), i.e. the Poincaré compactification $p(X_\mu)$ in the local chart $U_1$. The points $(u, v)$ with $v = 0$ are the points the points at infinity of the local chart $U_1$. Then the infinite singular points $(u, 0)$ are given by the roots $u$ of the polynomial equation $1 + 6\mu u^2 + u^4 = 0$. So the infinite singular points in $U_1$ are $\left(\pm\sqrt{\pm\sqrt{9\mu^2-1}-3\mu}, 0\right)$. These four points are real if and only if $\mu < -1/3$. If $\mu = -1/3$ then $q_1 = q_2 = (-1, 0)$ and $q_3 = q_4 = (1, 0)$. And $q_i$ for $i = 1, 2, 3, 4$ are non-real if $\mu > -1/3$. Hence statement (a) is proved.

The linear part of system (4) at $q_1$ and $q_3$ (resp. $q_2$ and $q_4$) have two negative (resp. positive) eigenvalues, so by Theorem 2.15 of [5] they are stable (resp. unstable) nodes. Of course, we have the diametrically opposite four nodes in the local chart $V_1$. Hence statement (b) is proved.

For $\mu = -1/3$ the singular points in the local chart $U_1$ are $(1, 0)$ and $(-1, 0)$. These singular points are linearly zero. By the equivalence of the original system (2) to $\dot{x} = y, \dot{y} = x$ at $\mathbb{R}^2 \setminus C$ it is possible to obtain the local phase portrait of system (2) at $(-1, 0)$ and $(1, 0)$ at the local chart $U_1$, thus they are an unstable and stable node respectively. We also have the diametrically opposite two nodes in the local chart $V_1$. Hence, statements (c) is proved. □

## 4. Proof of Theorem 1

In this section we prove Theorem 1. We shall use the study of the local phase portraits at the finite singular points in section 2 and at the infinite singular points in section 3 for obtaining all the possible global

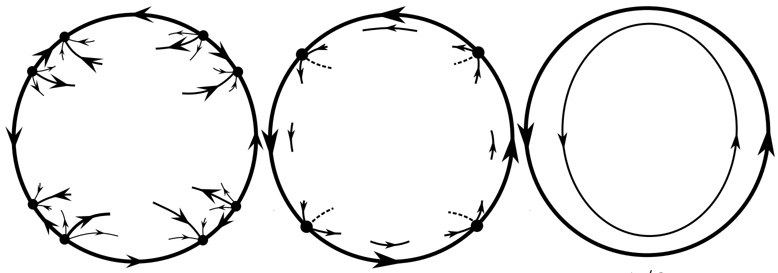



FIGURE 5. Local phase portraits at the infinite singular points in the Poincaré disc.

phase portraits of the differential system (2) in the Poincaré disc when the parameter $\mu$ varies. For doing that we know that is sufficient to consider all the possible $\alpha-$ and $\omega-$ limits of the separatrices of system (2), see Theorem 1.43 of [5].

We recall that the separatrices of a polynomial differential system in the Poincaré disc are the limit cycles, all the orbits at infinity, all the finite singularities, and all the separatrices of the hyperbolic sectors of the finite and infinite singular points.

Our system (2) has no limit cycles because the system is Hamiltonian and the Hamiltonian is polynomial. We know for system (2) the orbits at the infinity of the Poincaré disc, see Proposition 2; and we know its finite singularities. So in order to obtain the possible phase portraits of system (2) we only need to study the $\alpha-$ and $\omega-$ limits of the four separatrices of the finite saddles of the system. For a better understanding we will separate this analysis in the following three cases: $\mu > -1/3$, $\mu = -1/3$ and $\mu < -1/3$.

*Case* $\mu > -1/3$: In this case the finite singular points are the saddle at $(0, 0)$ and two centers, one at $(0, 1)$ and another one at $(0, -1)$, and the infinity is a periodic orbit. Moreover using that the curve $H(x, y) = 0$ which contains the four separatrices of the saddle $(0, 0)$ of system (2) because the Hamiltonian $H(x, y)$ given in (3) is a first integral, the phase portrait for system (2) in this case is topologically equivalent to the one of Figure 1.

*Case* $\mu < -1/3$: In this case the three finite singular points $(0, 0)$, $(0, 1)$ and $(0, -1)$ are saddles. At infinity we have four stable nodes and four unstable nodes see Figure 5. Then using the curves $H(x, y) = 0$ and $H(x, y) = 3/4$ which contain all the separatrices of the three finite saddles, the phase portrait of system (2) in this case is topologically equivalent to the one of Figure 2.

*Case* $\mu = -1/3$: System (2) has a commom factor $1 + x^2 - y^2$, and eliminating from the system this common factor doing a rescaling of the independent variable, we get the system $\dot{x} = y, \dot{y} = x$. Therefore at $\mathbb{R}^2 \setminus C$ system (2) has the same orbits as the system $\dot{x} = y$, $\dot{y} = x$, but these orbits reverse their orientation in the region where $1 + x^2 - y^2$ is negative. At infinity of system (2) there are the four singular points described in statement (c) of Proposition 2. Again using the curve $H(x, y) = 0$ we obtain that the phase portrait in this case is topologically equivalent to the one Figure 3. This completes the proof of Theorem 1.

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## References

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd edition, Springer-Verlag, New-York, 1989.

[2] J.C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems: A global classification in the quadratic case*, to be published by Birkhäuser, 2020.

[3] L. Barreira, J. Llibre and C. Valls, *Linear type global centers of cubic Hamiltonian systems symmetric with respect to the $x$-axis*, Electron. J. Differential Equations **2020**, Paper No. 57, 14 pp.

[4] M. Dukaric, W. Fernandes and R. Oliveira, *Symmetric centers on planar cubic differential systems*, Nonlinear Anal. **197** (2020), 111868, 14 pp.

[5] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Springer–Verlag, Berlin, Heidelberg, 2006.

[6] L.F.S. Gouveia and J. Torregrosa, 24 *crossing limit cycles in only one nest for piecewise cubic systems*, Appl. Math. Lett. **103** (2020), 106189, 6 pp.

[7] M. Hu, T. Li and X. Chen, *Bi-center problem and Hopf cyclicity of a cubic Liénard system*, Discrete Contin. Dyn. Syst. Ser. B **25** (2020), 401–414.

[8] B. Huang, *On the limit cycles for a class of discontinuous piecewise cubic polynomial differential system*, Electron. J. Qual. Theory Differ. Equ. **2020**, Paper No. 25, 24 pp.
[9] I.D. Iliev, C. Li and J. Yu, *On the cubic perturbations of the symmetric 8-loop Hamiltonian*, J. Differential Equations **269** (2020), 3387–3413.
[10] F. Li, Y. Jin, Y. Tian and P. Yu, *Integrability and linearizability of cubic $Z_2$ systems with non-resonant singular points*, J. Differential Equations **269** (2020), 9026–9049.
[11] F. Li, Y. Liu, Y. Liu and P. Yu, *Complex isochronous centers and linearization transformations for cubic $Z_2$-equivariant planar systems*, J. Differential Equations **268** (2020), 3819–3847.
[12] J. Llibre and D. Xiao, *On the configurations of centers of planar Hamiltonian Kolmogorov cubic polynomial differential systems*, Pacific J. Math. **306** (2020), 611–644.
[13] L. Markus, *Global structure of ordinary differential equations in the plane:* Trans. Amer. Math Soc. **76** (1954), 127–148.
[14] D. A. Neumann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.
[15] M.M. Peixoto, *Dynamical Systems. Proccedings of a Symposium held at the University of Bahia*, 389–420, Acad. Press, New York, 1973.
[16] J. Reyn, *Phase portraits of planar quadratic systems*, Mathematics and Its Applications **583**, Springer, New York, 2007.
[17] A.P. Sadovskii, *Existence of complex cubic systems with a 14th-order focus*, Differ. Equ. **56** (2020), 140–139.
[18] P. Yang and J. Yu, *The number of limit cycles from a cubic center by the Melnikov function of any order*, J. Differential Equations **268** (2020), 1463–1494
[19] Ye Yanqian, *Theory of limit cycles*, Translations of Math. Monographs, Amer. Math. Soc., Vol 66, 1986.
[20] L. Zhang, C. Wang and Z. Hu, *Limit Cycle Bifurcations from an Order-3 Nilpotent Center of Cubic Hamiltonian Systems Perturbed by Cubic Polynomials*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **30** (2020), no. 9, 2050126, 11 pp.
[21] Z. Zhou, V.G. Romanovski and J. Yu, *Centers and limit cycles of a generalized cubic Riccati system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **30** (2020), no. 2, 2050021, 10 pp.

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