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COEXISTENCE OF ANALYTIC AND PIECEWISE ANALYTIC LIMIT CYCLES IN PLANAR PIECEWISE QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract. We study the simultaneous bifurcation of limit cycles in planar piecewise quadratic differential systems separated by a straight line. These limit cycles arise from a degenerate Hopf bifurcation at two equilibrium points in the positive and negative half-planes, as well as from an equilibrium on the separation line. All the limit cycles are of small amplitude. This bifurcation creates a configuration of limit cycles of type $(3, 5, 3)$. Additionally, in each half-plane, the maximum number of small-amplitude hyperbolic limit cycles that a quadratic vector field can have is three.

1. Introduction

The study of periodic orbits in planar polynomial differential systems has been a major focus in recent years, especially the determination of the maximum number of isolated periodic orbits, called limit cycles. This is directly linked to the second part of Hilbert's 16th problem, which asks for the maximum number $H(n)$ of limit cycles when the degree of the polynomial vector fields is n. However, this upper bound problem is still unsolved, and much of the research has focused on finding lower bounds for $H(n)$ or exploring different configurations of their positions. This study can also be applied to piecewise polynomial vector fields in two zones separated by a straight line, called $H_p(n)$. In this case, limit cycles usually cross both zones and are called crossing limit cycles. We also explore the coexistence of analytic and piecewise analytic limit cycles, which we refer to as the *coexistence configuration*.

In recent years, there has been growing interest in piecewise differential systems because they can model many real-world phenomena. These systems are especially important in areas like electrical and mechanical engineering, control theory, and genetic networks, among others (see [1, 13, 15]). Most studies focus on periodic orbits in piecewise systems, particularly crossing-type orbits. Specifically, this interest is centered on systems defined in two regions separated by a straight line. Crossing limit cycles are those that pass through both regions.

It is well known that linear systems do not have limit cycles, so $H(1) = 0$. The question of the upper bound for the number of limit cycles in polynomial vector fields of degree 1 emerged around 30 years ago with the conjecture on the uniqueness of the limit cycle for continuous piecewise linear vector fields, which was proven in [16] after being proposed in [29]. When the continuity condition is removed, the best-known lower bound is $H_p(1) \geq 3$, first observed numerically in [26] and later proven analytically in [28]. These limit cycles are of crossing type and nested. This result can also be proven using higher-order averaging analysis (see [5]). More recently, in a Hopf-type bifurcation near infinity, three limit cycles were found (see [17]). In [25], it is stated that this maximal situation only occurs when the piecewise linear system has both visible and invisible equilibrium points

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of focus type. In [18], it was shown that only two small-amplitude limit cycles appear near an equilibrium point, with the third one being larger. We observe a parallel bifurcation phenomenon between piecewise linear systems and smooth quadratic systems regarding the number of limit cycles. In the smooth case, the maximum number of limit cycles requires two distinct equilibrium points, while in the piecewise linear case, all periodic orbits are nested around a single sliding segment.

As mentioned above, piecewise linear systems only have a single nest of crossing limit cycles. Therefore, the lowest degree scenario for having more nests is the quadratic family. Before presenting our results, we describe the background for the smooth case. The most significant result was presented by Shi in [35], where he showed the existence of a quadratic system with four limit cycles in the configuration $(3,1)$, thus establishing $H(2) \geq 4$. This example has recently been revisited in [19]. Furthermore, studies such as [36, 37] highlight other instances where similar configurations arise in near-integrable systems, each offering insights into the appearance of limit cycles. While various configurations are possible, they typically fall into categories like $(0,i)$ or $(1,i)$ with $i = 1,2,3$, notably excluding $(2,2)$ configurations. For more details on these configurations, refer to [38, 39]. Additionally, the maximum number of limit cycles of small amplitude in the quadratic family, bifurcating from an equilibrium point, is three, as shown in [3]. Consequently, in the maximal configuration, the fourth limit cycle always manifests as one of significant amplitude. To our knowledge, there are few studies on the configurations of limit cycles with two or more nests.

The aim of this work is to analyze the number of limit cycles for a piecewise quadratic vector field in a configuration with three nests: one in each zone and another crossing both zones. Our study addresses the simultaneous existence of crossing-type limit cycles along with the classical ones, i.e., those that are entirely contained in one of the regions where the vector field is differentiable. To our knowledge, this is the first paper to undertake with this analysis.

We have named this new concept *coexistence of analytic and piecewise analytic bifurcation limit cycles*. In this context, all limit cycles will be of small amplitude and bifurcate simultaneously through degenerate Hopf bifurcations, from weak foci located in the half-planes and on the separation line. Clearly, this simultaneous phenomenon cannot occur when the vector fields in each region are linear. Based on the properties described, it is straightforward to provide piecewise quadratic differential systems having (at most) three limit cycles (of small amplitude) entirely contained within each zone. Consequently, the focus is on analyzing how many crossing-type limit cycles (also of small amplitude) can bifurcate from a weak focus located on the separation straight line, if any, while maintaining the maximum number outside of the separation line.

Before stating the results, we will fix notation and provide some preliminary definitions to be more precise. We will consider piecewise quadratic vector fields $Z^{\pm}(x,y)$ defined in two zones separated by a straight line. More precisely, these are associated with the differential systems

$$\begin{cases} \dot{x} = X^{\pm}(x,y), \\ \dot{y} = Y^{\pm}(x,y), \end{cases} \quad \text{if} \ \ (x,y) \in \Sigma^{\pm}, \tag{1}$$

where $X^{\pm}$ and $Y^{\pm}$ are polynomials of degree 2 in $\Sigma^{\pm}$. The separation line is defined by $\Sigma = \{y - \tan(\theta_0)x = 0\}$ for some angle $\theta_0 \in (-\pi/2, \pi/2)$. The two regions are denoted by $\Sigma^{\pm} = \{\pm(y - \tan(\theta_0)x) > 0\}$.

The local trajectories on Σ were described by Filippov in [15] (see Figure 1); see also [22]. The points on Σ where both vector fields simultaneously point outward and inward from Σ define the escaping region (Σ^e) and the sliding region (Σ^s), respectively. The interior of its complement on Σ defines the crossing region (Σ^c), and the boundary of these regions consists of tangential points of $Z^\pm$ with Σ. In this paper, the precise definition of the vector field on Σ^e and Σ^s is not necessary to be stated, although we highlight that the segment Σ^e (resp. Σ^s) is a repelling (resp. attracting) set.

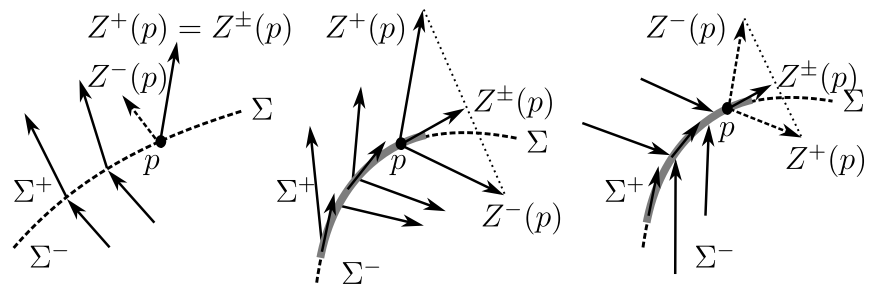

FIGURE 1. Definition of the vector field $Z^\pm$ on Σ following Filippov's convention in the crossing, escaping, and sliding regions, respectively.

As we have previously mentioned, the aim of this paper is showing the coexistence of limit cycles and crossing limit cycles, both of small amplitude type. More specifically, they will appear as small-amplitude bifurcations from three degenerate Hopf type equilibria, one in each region Σ^+, Σ, and Σ^-. We will recall the main techniques in next section. For the classical smooth bifurcation, we refer the reader to [2, 34], and for the piecewise scenario, to [4, 23]. Specifically, we use the cyclicity dependence on parameters for families of piecewise centers. The main tools are the ones developed in [21]. Additionally, the so-called pseudo-Hopf bifurcation also appears. It was initially observed in [15] but this name was first used in [27]. This phenomenon has been recently revisited in [7, 18].

We say that the piecewise differential system (1) has a *3-nested coexistence configuration of limit cycles of type* (m^+, m^0, m^-) if it has $m^\pm$ nested limit cycles fully contained in $\Sigma^\pm$ and m^0 nested crossing limit cycles intersecting Σ. Other coexistence configurations can arise as the system's degree increases, but in the quadratic case, their number is more limited. This article does not aim to classify all possible configurations but rather to identify those that, based on our intuition, yield the highest number of limit cycles when more than two period annuli appear. For this reason, we focus on quadratic families that maximize the number of limit cycles surrounding a single equilibrium point, which is three. Our approach follows [32], but without reversible symmetry, since we work in a piecewise setting where perturbations on both sides generally differ.

Our main result is the following:

Theorem 1.1. *Planar piecewise quadratic differential systems defined in two regions separated by a straight line exhibit a 3-nested coexistence configuration of limit cycles of type* $(3, 5, 3)$.

As mentioned above, all limit cycles are of small amplitude. Thus, we begin in Section 2 by recalling the degenerate Hopf bifurcation and the so-called Lyapunov quantities, obtained from the first non-vanishing coefficients of the return map near a monodromic equilibrium point in both smooth and non-smooth cases. After these preliminaries, we proceed with the proof of our main result in several steps. Until the end of the proof, we restrict all perturbations to prevent the system from having a sliding segment containing the origin. In Section 3, we first establish the coexistence of three centers and prove their alignment. Next, we introduce a canonical class of piecewise quadratic systems, where the separation line is the horizontal x-axis, containing two weak foci of order three on the vertical y-axis and one monodromic equilibrium at the origin. Within this four-parameter family, we show that the origin can be a weak focus of order three and that for certain parameter values, an unfolding with two small-amplitude crossing limit cycles occurs. We then explore how modifying the separation line can increase the weak-focus order at the origin to five by selecting an appropriate point in the parameter space. Additionally, we prove the existence of four small-amplitude crossing limit cycles bifurcating from the origin, ensuring that the condition of having two weak foci in $\Sigma^{\pm}$ remains unbroken. In Section 4, we first confirm that the weak foci outside Σ in our canonical form have order three. Then, we analyze a special perturbation that gives rise to three small-amplitude limit cycles in $\Sigma^{\pm}$. Finally, we show that these bifurcations, initially considered separately, can occur simultaneously. This is discussed in Section 5, where we prove the existence of the three-nested coexistence configurations of types $(3,4,3)$ and $(3,5,3)$, respectively, without and with a sliding segment. This completes the proof of our main result.

2. Preliminaries

In this section, we recall some classical and necessary concepts to prove the results of this paper. First, we present the method for obtaining the first coefficients of the return map for smooth and non-smooth differential systems whose traces of the linear parts are not zero. Second, we introduce the pseudo-Hopf bifurcation, which refers to the birth of a limit cycle of small amplitude when sliding is considered. Finally, we state a result given by Christopher in [9] that allows us to study the degenerate Hopf bifurcation near centers. We also include its natural extension to the piecewise scenario, which was introduced in [23].

We consider a planar analytic differential system written in the canonical form

$$\begin{cases} \dot{x} = ax - by + X(x,y), \\ \dot{y} = bx + ay + Y(x,y), \end{cases} \tag{2}$$

with $b>0$ and X and Y being analytic functions without constant or linear terms. Thus, using the usual polar coordinates, $(x,y) = (r\cos\theta, r\sin\theta)$, its power series can be written as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sum_{j=1}^{\infty} S_j(\theta) r^j, \tag{3}$$

where $S_1 = a/b$ and $S_j(\theta)$ are trigonometric polynomials for $j \geq 2$. For any $0 < r_0 \ll 1$, we denote by $r(\theta, r_0)$ the solution of (3) such that $r = r_0$ when $\theta = 0$, and thus

$$r(\theta, r_0) = r_0 + \sum_{j=1}^{\infty} u_j(\theta) r_0^j.$$

Substituting this solution into (3), we obtain a sequence of recurrence formulas for the coefficients u_j. Due to their size, we only show the first three coefficients:

$$\begin{aligned} u_1(\theta) &= \exp\left(\int_0^{\theta} \frac{a}{b}\, d\theta\right), \\ u_2(\theta) &= \int_0^{\theta} S_2(\theta) u_1^2(\theta)\, d\theta, \\ u_3(\theta) &= \int_0^{\theta} \big(S_3(\theta) u_1^3(\theta) + 2S_2(\theta) u_2(\theta) u_1(\theta)\big)\, d\theta. \end{aligned}$$

From the Poincaré return map $\Pi(r_0) = r(r_0, 2\pi)$, we define the displacement function as

$$d(r_0) = \Pi(r_0) - r_0 = \sum_{j=1}^{\infty} V_j r_0^j, \tag{4}$$

as illustrated in Figure 2.

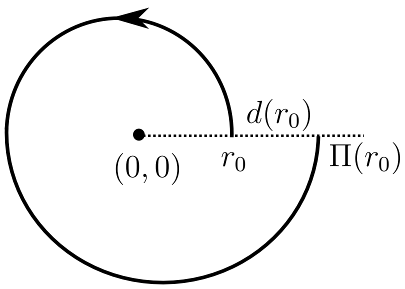

Figure 2. The Poincaré return map $\Pi(r_0)$ and the displacement function $d(r_0)$.

As usual, when $a \neq 0$, the origin is a focus, and its stability is determined by the sign of a. However, when $a = 0$, the stability question becomes more intricate and is related to the center-focus problem. In this context, it is well-known that $V_2 = 0$, and the first non-vanishing coefficient of (4) has an odd subscript. The coefficient V_{2k+1} is called the kth-order Lyapunov quantity of (2) with $a = 0$. We then say that the origin is a weak focus of order k.

When we consider these quantities as functions of the coefficients of X and Y in (2), we can prove that they are polynomials when $a = 0$. See, for example, [10]. An interesting property, described in [34] and proved in [33, 11], of these coefficients is that for each k, we have

$$\langle V_4, \ldots, V_{2k} \rangle \subset \langle V_3, V_5, \ldots, V_{2k-1} \rangle. \tag{5}$$

When $a = 0$ and $V_3 \neq 0$, the stability of the equilibrium point is determined by the sign of V_3. Specifically, the origin is stable when $V_3 < 0$ and unstable when $V_3 > 0$. Consequently, a stable limit cycle of small amplitude bifurcates from the origin when a is a small positive real number, and an unstable limit cycle bifurcates when a is a small negative real number. This bifurcation is known as the classical Hopf bifurcation. The degenerate Hopf bifurcation is the natural generalization, where limit cycles of small amplitude appear similarly from a weak focus of order k. Moreover, the number of such limit cycles is at most k.

We now consider an analytic vector field $Z^{\pm}$ whose associated differential system is

$$\begin{cases} \dot{x} = a^{\pm}x - b^{\pm}y + X^{\pm}(x,y), \\ \dot{y} = b^{\pm}x + a^{\pm}y + Y^{\pm}(x,y), \end{cases} \quad \text{if } (x,y) \in \Sigma^{\pm}, \tag{6}$$

where $\Sigma^{\pm} = \{(x,y) : \pm y > 0\}$, $b^{\pm} > 0$, and $X^{\pm}$, $Y^{\pm}$ are analytic functions without constant or linear terms. Similar to before, we can transform (6) into

$$\frac{\mathrm{d}r^{\pm}}{\mathrm{d}\theta} = \sum_{j=1}^{\infty} S_j^{\pm}(\theta) r^j,$$

where $S_1^{\pm} = a^{\pm}/b^{\pm}$ and $S_j^{\pm}(\theta)$ are trigonometric polynomials for $j \geq 2$. Additionally, for $0 < r_0 \ll 1$, the power series of the piecewise solution, which satisfies $r^{+}(r_0, 0) =$

$r^-(r_0,\pi)=r_0$, is written as

$$r^{\pm}(\theta,r_0)=\begin{cases} r_0+\sum_{j=1}^{\infty}U_j^+(\theta)r_0^j, & \text{if } \theta\in[0,\pi],\\ r_0+\sum_{j=1}^{\infty}U_j^-(\theta)r_0^j, & \text{if } \theta\in[\pi,2\pi].\end{cases}$$

Therefore, we define the positive and negative Poincaré half-return maps as $\Pi^+(r_0)=r^+(r_0,\pi)$ and $\Pi^-(r_0)=r^-(r_0,2\pi)$, respectively. Finally, we define the piecewise Poincaré return map by the composition of the two half-return maps, $\Pi^-(\Pi^+(r_0))$. As usual in the study of limit cycles of small amplitude and its stability, to simplify computations, instead of considering the displacement function introduced above, we will use the equivalent difference map

$$\Delta(r_0)=\left(\Pi^-\right)^{-1}(r_0)-\Pi^+(r_0)=-\sum_{j=1}^{\infty}W_jr_0^j, \tag{7}$$

as illustrated in Figure 3. We have indicated by $\left(\Pi^-\right)^{-1}$ the inverse of the half-return map in the Z^- zone, which is the solution starting at $(\rho_0,0)$ evaluated at $-\pi$ instead of the solution starting at $(-\rho_0,0)$ evaluated at π.

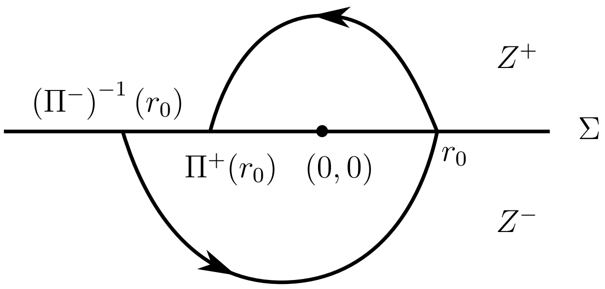

FIGURE 3. The positive and negative half-return maps Π^+ and $\left(\Pi^-\right)^{-1}$, respectively.

As introduced in the analytical case, the first nonvanishing coefficient in (7), denoted by $W_k\neq 0$, is called the (generalized) kth-order Lyapunov quantity of (6). Note that system (6) has no sliding segment at the origin; indeed, it follows from (7) that $\Delta(0)=0$. In this setting, the origin of (6), which lies on Σ, is a (crossing) weak focus of order k if $W_j=0$, for $1\leq j\leq k-1$, and $W_k\neq 0$. This definition ensures that, in a complete unfolding, k limit cycles of small amplitude bifurcate from the origin. The information provided by this definition of order is analogous to that of the analytic degenerate Hopf bifurcation. The main difference, due to the class of systems considered, is that in this case all coefficients contribute to the complete unfolding, whereas in the smooth case, only half do, due to a symmetry. For a proof of the relation between the coefficients of (7) and those of the Taylor series of the piecewise Poincaré return map, we refer the reader to [20]. Here, following similar arguments as in the analytical case, we obtain at most $k-1$ limit cycles of small amplitude bifurcating from the origin. However, in discontinuous piecewise differential systems, an additional limit cycle may arise due to the presence of a parameter that introduces a stable or unstable region with a sliding segment. For completeness, we include a recent result guaranteeing the existence of a crossing limit cycle in a pseudo-Hopf type bifurcation.

Lemma 2.1 (See [12]). *Let $Z^{\pm}=(P^{\pm}(x,y),Q^{\pm}(x,y))$ be a $\mathcal{C}^1$ piecewise differential system in two zones separated by the straight line $y=0$. Additionally, the origin is a stable monodromic equilibrium point and $\eta=(\partial Q^+/\partial x)|_{(0,0)}>0$. Given a real number μ, we consider the perturbed system $Z_\mu^{\pm}=(P_\mu^{\pm}(x,y),Q_\mu^{\pm}(x,y))$ defined by $P_\mu^{\pm}(x,y)=P^{\pm}(x,y)$, $Q_\mu^-(x,y)=Q^-(x,y)$, and $Q_\mu^+=Q^++\mu$. Then, for μ small enough, system $Z_\mu^{\pm}$ exhibits a pseudo-Hopf bifurcation at $\mu=0$ when $\eta\mu>0$ (see Figure 4).*

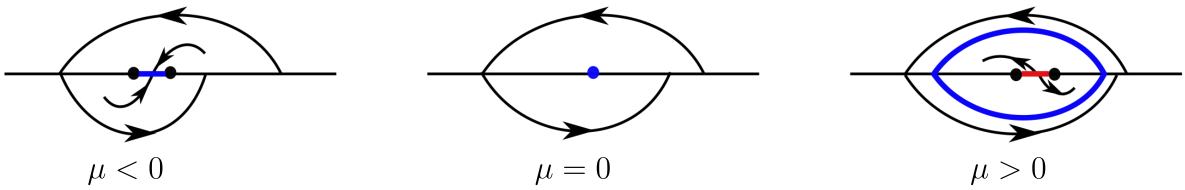

FIGURE 4. Pseudo-Hopf type bifurcation

As we are interested in a simultaneous bifurcation using both types of degenerate Hopf bifurcations, for convenience and to avoid confusion, we will use W_k when dealing with the monodromic points on Σ and V_k when they are outside the separation straight line, i.e., when the monodromic points are entirely contained in $\Sigma^{\pm}$.

For completeness, we recall two results that provide conditions for a complete unfolding of the bifurcation of limit cycles from the Lyapunov quantities computed near centers. The first is due to Christopher ([9]), and the second, which is its generalization to piecewise perturbations of centers, is due to Gouveia and Torregrosa ([23]). Both follow directly from the application of the Implicit Function Theorem. These ideas also appear in [24] for studying the local cyclicity in the Liénard family.

Theorem 2.2. *Suppose c is a point on the center variety in the parameter space and that the first coefficients $V_1, V_3, \ldots, V_{2k+1}$, defined in* (4)*, have independent linear parts (with respect to their expansion with respect to all the perturbation parameters). Then, there exist bifurcations which produce k limit cycles of small amplitude bifurcating from the center corresponding to the parameter value c.*

Theorem 2.3. *Suppose c is a point on the center variety in the parameter space and that the first coefficients $W_1, \ldots, W_k$, defined in* (7)*, have independent linear parts (with respect to their expansion with respect to all the perturbation parameters). Then, considering also the constant terms (which introduce a sliding or escaping segment), there exist bifurcations which produce k limit cycles of small amplitude bifurcating from the center corresponding to the parameter value c.*

Finally, we briefly discuss the different types of degenerate Hopf bifurcations in piecewise vector fields. The generalization of the classical degenerate Hopf bifurcation in smooth vector fields to non-smooth ones is described, for example, in [14, 31], where the pseudo-equilibrium point is of fold-fold type. The parallelism is clear in both scenarios, and the number of limit cycles is related to the non-vanishing coefficients of the return map series, which are half of them, as explained in (5). The degenerate Hopf bifurcation described previously is in the focus-focus case, where, in some sense, the *symmetry* disappears, and all coefficients of the return map series play a role. What is important in our scheme is that we first restrict the analysis to the case where there are no sliding segments, and we keep the foci on the separation line. Afterward, we add an extra limit cycle of crossing type using Lemma 2.1 while maintaining the equilibria on Σ.

3. Centers and limit cycles of small amplitude of crossing type

In this section, we use the theory introduced in the previous section to examine the number and conditions required to obtain small-amplitude limit cycles for a monodromic equilibrium point located on Σ. Specifically, we characterize the centers and analyze the order of the weak foci bifurcating from them, as well as their parameter dependence. For now, we restrict our study to scenarios where sliding segments do not exist.

In a sense, our results can be viewed as an adaptation of the findings in [35] regarding the simultaneous bifurcation analysis. We consider a piecewise quadratic differential system with two weak foci of the highest order in the upper and lower half-planes, along with a monodromic equilibrium point located at the origin, which lies on the separation straight line. Specifically, when $b^{\pm} \neq 0$, we examine the canonical differential equation:

$$\begin{cases} \dot{x} = -\,b^{\pm}y \pm a^{\pm}x \pm (b^{\pm})^2y^2 \mp (4(a^{\pm})^2 + (b^{\pm})^2 + 6)x^2/3, \\ \dot{y} = b^{\pm}x \pm a^{\pm}y - a^{\pm}b^{\pm}y^2 \mp ((a^{\pm})^2 + (b^{\pm})^2 + 1)xy \\ \qquad + a^{\pm}(5(a^{\pm})^2 - 10(b^{\pm})^2 + 9)x^2/(15b^{\pm}), \end{cases} \tag{8}$$

where $(x, y) \in \Sigma^{\pm} = \{(x, y) : \pm(y - x \tan\theta_0) > 0\}$.

The local analysis near the origin proceeds in three steps. First, in Proposition 3.1, we examine a scenario where (8) exhibits three simultaneous centers on the y-axis, with the x-axis as the separation line ($\theta_0 = 0$). Second, still maintaining $\theta_0 = 0$, in Propositions 3.2 and 3.3, we investigate the center problem and the degenerate Hopf bifurcation of crossing limit cycles in a neighborhood of the origin for our canonical system (8). Finally, in Proposition 3.4, we complete the bifurcation analysis of crossing limit cycles (of small amplitude) by considering a small enough $\theta_0 \neq 0$. Some important properties that we will prove in the next section, but remain open in this one, include the stability, weak-focus order, and degenerate Hopf bifurcation of equilibria outside the separation straight line. Indeed, as we will demonstrate throughout the paper, for almost all values of $a^{\pm}b^{\pm} \neq 0$, system (8) possesses two classical weak foci at $(0, \pm 1/b^{\pm})$ and a crossing focus at the origin. Furthermore, when $\theta_0 = 0$, system (8) is invariant under the change of variables:

$$(x, y, t, a^+, b^+) \to (x, -y, -t, a^-, b^-). \tag{9}$$

We will also explore the role of θ_0 in allowing us to enhance the weakness of the crossing weak focus at the origin for specific parameter values $a^{\pm}, b^{\pm}$.

Proposition 3.1. *For $\theta_0 = a^{\pm} = 0$, and $b^{\pm} > 0$, the piecewise differential system (8) exhibits three centers located at $(0, 0)$ and $(0, \pm 1/b^{\pm})$.*

Proof. When $a^{\pm} = \theta_0 = 0$, it is straightforward to verify that only the three real equilibrium points mentioned in the statement exist, under the necessary condition $b^{\pm} > 0$. Furthermore, each uncoupled system is time-reversible with respect to the y-axis, featuring two centers on the symmetry axis, one at the origin and the other in each zone where it is defined. It can be checked that (8) is coupled in such a way that transforms the origin into a monodromic pseudo-equilibrium. Consequently, owing to the aforementioned symmetry, it also becomes a center. More specifically, this symmetry implies that the half-return maps in the vicinity of the origin adhere to the usual involution, $\Pi(x) = -x$. □

The next result shows that, for $a^{\pm} \neq 0$, the origin can also be a center. However, as indicated by Proposition 4.1, the center property of the equilibria outside Σ is not longer valid.

Proposition 3.2. *For $\theta_0 = 0$, $a^{\pm}b^{\pm} \neq 0$, the piecewise differential system (8) has a center at the origin if and only if $(a^-, b^-) = (a^+, b^+)$ or satisfies the condition:*

$$(a^-, b^-) = (a^+\Psi(a^+, b^+), b^+\Psi(a^+, b^+)) \text{ and } \Phi(a^+, b^+) = 0, \tag{10}$$

where

$$\Psi(a, b) = \frac{3(3a^4 + 29a^2b^2 + 74b^4)}{5(a^6 + 9a^4b^2 + 15a^2b^4 + 7b^6)}$$

and

$$\Phi(a, b) = 5a^6 + 45a^4b^2 + 75a^2b^4 + 35b^6 - 9a^4 - 87a^2b^2 - 222b^4. \tag{11}$$

Proof. To establish the necessity of the conditions stated, we begin by considering that the origin is a monodromic equilibrium point. The displacement function of system (8) is expressed in (7), and we will calculate its coefficients depending on the parameters (a^+, b^+, a^-, b^-). The first coefficient is given by

$$W_1(a^+, b^+, a^-, b^-) = \exp(a^+\pi/b^+) - \exp(a^-\pi/b^-) = E^+ - E^-. \tag{12}$$

Clearly, $W_1 = 0$ when $a^- = a^+b^-/b^+$, for $a^+b^+b^- \neq 0$, yielding

$$W_2(a^+, b^+, b^-) = \frac{a^+E^+(E^+{+}1)(b^+ - b^-)\widetilde{\Psi}(a^+, b^+, b^-)}{15b^-(b^+)^3[(a^+)^2 + 9(b^+)^2][(a^+)^2 + (b^+)^2]}, \tag{13}$$

where

$$\widetilde{\Psi}(a, b, c) = 5c(a^6 + 9a^4b^2 + 15a^2b^4 + 7b^6) - 3b(3a^4 + 29a^2b^2 + 74b^4). \tag{14}$$

When $b^- = b^+$, using (12), we find that $a^- = a^+$, yielding the first family in the statement. The second family arises from the condition $b^- \neq b^+$. In this scenario, we have $\widetilde{\Psi}(a^+, b^+, b^-) = 0$ as a necessary condition to obtain a center. This condition, combined with (12), yields the first equality in (10), and we obtain

$$W_3(a^+, b^+) = -E^+(E^+{+}1)(E^+{-}1)\Phi(a^+, b^+)\Phi_{20}(a^+, b^+)\Phi_{30}(a^+, b^+)/\Phi_{40}(a^+, b^+), \tag{15}$$

where Φ is defined in (11) and

$$\begin{aligned}
\Phi_{20}(a, b) &= 48a^{12} + 852a^{10}b^2 + 7952a^8b^4 + 46424a^6b^6 + 142317a^4b^8 + 172334a^2b^{10} + 7085b^{12},\\
\Phi_{30}(a, b) &= 5a^6 + 45a^4b^2 + 75a^2b^4 + 35b^6 + 9a^4 + 87a^2b^2 + 222b^4,\\
\Phi_{40}(a, b) &= 1800b^2(a^2 + b^2)^3(a^2 + 4b^2)(a^2 + 7b^2)^2(3a^4 + 29a^2b^2 + 74b^4)^2.
\end{aligned}$$

Since the functions Φ_{20}, Φ_{30}, and Φ_{40} only vanish at the origin, the necessary condition $\Phi(a, b) = 0$ provides the second family in the statement.

Finally, we will prove that the above necessary conditions are also sufficient to ensure the existence of a center, thereby completing the proof. The first family clearly corresponds to a symmetric center, using the transformation (9). For the second family, we employ a more intricate symmetry mechanism that transforms the system defined in Σ^+ to Σ^-, along with a change in the direction of the flow. We express the systems in the usual polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and introduce a rational change of coordinates $r = 4R/(4 + A^{\pm}(\theta)R)$, where

$$\begin{aligned}
A^+(\theta) = &\left[\left(5(a^+)^3 + 5a^+(b^+)^2 + 9a^+\right)\cos^3\theta + 5b^+\left((a^+)^2 + (b^+)^2 + 3\right)\sin\theta\cos^2\theta\right.\\
&\left.-15a^+(b^+)^2\cos\theta - 15(b^+)^3\sin\theta\right] / \left[15(b^+)^2\right],\\
A^-(\theta) = &\Big[\left((a^+)^2 + (b^+)^2\right)\left[3\left(5(a^+)^{11} + 85(b^+)^2(a^+)^9 + 470(b^+)^4(a^+)^7 + 950(b^+)^6(a^+)^5\right.\right.\\
&+805(b^+)^8(a^+)^3 + 245(b^+)^{10} + 9(a^+)^9 + 174(b^+)^2(a^+)^7 + 1285(b^+)^4(a^+)^5\\
&\left.+4292(b^+)^6(a^+)^3 + 5476(b^+)^8a^+\right)\cos^3\theta + \left(25b^+(a^+)^{10} + 425(b^+)^3(a^+)^8\right.\\
&+2350(b^+)^5(a^+)^6 + 4750(b^+)^7(a^+)^4 + 4025(b^+)^9(a^+)^2 + 1225(b^+)^{11} + 27(a^+)^8\\
&\left.\left.+522(b^+)^3(a^+)^6 + 3855(b^+)^5(a^+)^4 + 12876(b^+)^7(a^+)^2 + 16428(b^+)^9\right)\sin\theta\cos^2\theta\right]\\
&- 9a^+(b^+)^2\left(3(a^+)^4 + 29(a^+)^2(b^+)^2 + 74(b^+)^4\right)^2\cos\theta - 9(b^+)^3\left(3(a^+)^4\right.\\
&\left.+29(a^+)^2(b^+)^2 + 74(b^+)^4\right)^2\sin\theta\Big] / \Big[15\left((a^+)^2 + (b^+)^2\right)^2\left((a^+)^2 + 7(b^+)^2\right)\left(3(a^+)^4\right.\\
&\left.+29(a^+)^2(b^+)^2 + 74(b^+)^4\right)(b^+)^2\Big].
\end{aligned}$$

In this regard, by transforming the differential systems with homogeneous nonlinearities into equivalent Abel differential equations, we preserve their interaction with Σ. Specifically, we ensure that the condition $A^-(0) - A^+(0) = 0$ holds, as indicated in the second condition of (10). For a comprehensive understanding of how this transformation facilitates the transition from systems with homogeneous nonlinearities to Abel differential equations, we refer the reader to [8]. □

Depending on which center conditions are broken, we obtain different weak-focus orders. The following results provide a complete description of this phenomenon. The second result ensures a higher order than the first. We observe that, although the previous result establishes the existence of centers when $\Phi(a^+, b^+) = 0$ and $\theta_0 = 0$, Proposition 3.4 reveals that W_4 and W_5 can be expressed as a series expansion near $\theta_0 = 0$, thereby increasing the weak-focus order to five.

Proposition 3.3. *Considering system* (8) *with* $\theta_0 = 0$*, the maximum weak-focus order at the origin is three. This maximum order occurs when* $a^+, b^+ \neq 0$, $\Phi(a^+, b^+) \neq 0$*, and* (a^-, b^-) *is defined as in* (10)*. Additionally, there exists an unfolding, considering* (8) *as a family, in which two crossing limit cycles bifurcate from the origin.*

Proof. The nonexistence of weak foci of order higher than or equal to three follows directly from the proof of Proposition 3.2. In addition, it is clear that the family defined in the statement satisfies $W_1 = W_2 = 0$ and $W_3 \neq 0$, as shown by (12), (13), and (15). Therefore, the origin is a weak focus of order exactly three. The unfolding is confirmed through straightforward computations, checking that the Jacobian matrix of (W_1, W_2) with respect to (a^-, b^-) at the point (a^+, b^+) does not vanish. □

Proposition 3.4. *For* $\theta_0 \neq 0$ *small enough, there exist* (a^+, b^+, a^-, b^-) *such that the piecewise system* (8) *exhibits a weak focus of order five at the origin. Furthermore, there exists an unfolding, considering* (8) *as a family, such that four crossing limit cycles bifurcate from the origin.*

Proof. We will assume in the following that $a^\pm b^\pm \theta_0 \neq 0$. For this case, the displacement function (7) is given by

$$-\Delta(r_0) = \Pi^+(r_0) - \left(\Pi^-\right)^{-1}(r_0) = r^+(\theta_0 + \pi, r_0) - r^-(\theta_0 - \pi, r_0).$$

Under the conditions we are dealing with, the coefficients in r_0, which are the Lyapunov quantities, are analytic in the parameters $(a^+, b^+, a^-, b^-, \theta_0)$ because $r^\pm(\theta_0 \pm \pi, r_0)$ are also analytic in θ_0. The approach we will use is based on the Taylor expansions of the Lyapunov quantities around a special value of the center provided by Proposition 3.2, which unfolds a weak-focus point of higher order than the one obtained in Proposition 3.3.

Straightforward computations yield:

$$W_1(a^+, b^+, a^-, b^-) = E^+ - E^-,$$

where $E^\pm$ are defined in (12). We can then isolate a^- by setting the above expression to zero, resulting in:

$$a^- = \frac{a^+ b^-}{b^+}. \tag{16}$$

It is worth noting that, unusually, W_1, coinciding with the expression in (12), does not depend on θ_0. This peculiarity arises from the equilibrium points being of weak-focus type, situated on the separation line, and the linear part of the differential equations being in Jordan normal form.

Using (16), we can straightforwardly derive the expression for the second Lyapunov quantity $W_2(a^+, b^+, b^-, \theta_0)$, which depends on θ_0. Setting it to zero allows us to isolate

b^- for sufficiently small θ_0, resulting in two solutions. One solution converges to b^+, while the other converges to the expression given in (10) as θ_0 approaches zero. We are particularly interested in the latter solution. Both solutions, derived using the Implicit Function Theorem, originate from the Taylor series expansion in θ_0 of the numerator of the second Lyapunov quantity, given by:

$$\widetilde{W}_2(a^+,b^+,b^-,\theta_0) = \tau\left[\Phi_{21}(a^+,b^+,b^-) + \Phi_{22}(a^+,b^+,b^-)\theta_0\right] + O(\theta_0^2),$$

where

$$\begin{aligned}\Phi_{22}(a^+,b^+,b^-) = 3b^+(b^-+b^+)\big[&5(a^+)^6 + 25(a^+)^4(b^+)^2 + 35(a^+)^2(b^+)^4\\ &+15(b^+)^6\big]\, b^- + 11(a^+)^4 b^+ + 53(a^+)^2(b^+)^3 + 90(b^+)^5,\end{aligned}$$

$\Phi_{21} = a^+(b^+-b^-)\widetilde{\Psi}$, with $\widetilde{\Psi}$ defined as in (14), and $\tau = E^+(E^++1)$.

From the derived expressions for a^- and b^-, the subsequent Lyapunov quantities, which depend on a^+, b^+, and θ_0, can be expanded in Taylor series with respect to θ_0. These expansions yield:

$$\begin{aligned}\widetilde{W}_3(a^+,b^+,\theta_0) =&\tau\left[2\left(E^+-1\right)\Phi(a^+,b^+)\Phi_{31}(a^+,b^+) + \Phi_{32}(a^+,b^+)\theta_0\right] + O\left(\theta_0^2\right),\\ \widetilde{W}_4(a^+,b^+,\theta_0) =&\tau\left[2\left(E^+-1\right)\Phi(a^+,b^+)\Phi_{41}(a^+,b^+) + \Phi_{42}(a^+,b^+)\theta_0\right] + O\left(\theta_0^2\right),\\ \widetilde{W}_5(a^+,b^+,\theta_0) =&\tau\left[2\left(E^+-1\right)\Phi(a^+,b^+)\Phi_{51}(a^+,b^+) + \Phi_{52}(a^+,b^+)\theta_0\right] + O\left(\theta_0^2\right),\end{aligned}$$

where Φ_{i1} and Φ_{i2}, for $i = 3, 4, 5$, are polynomials in a^+ and b^+ of degrees 23, 29, 42, 48, 63, and 69, respectively, and Φ is given by (11). The explicit forms of Φ_{i1} and Φ_{i2} are not presented here due to their size.

Equating the expression for $\widetilde{W}_3$ to zero, we obtain:

$$\Phi(a^+,b^+) = -\frac{\Phi_{32}(a^+,b^+)}{2(E^+-1)\Phi_{31}(a^+,b^+)}\theta_0 + O(\theta_0^2).$$

Substituting this result into the expressions for $\widetilde{W}_4$ and $\widetilde{W}_5$, we find:

$$\begin{aligned}\widetilde{W}_4(a^+,b^+,\theta_0) =&\tau\frac{\Phi_{42}(a^+,b^+)\Phi_{31}(a^+,b^+) - \Phi_{41}(a^+,b^+)\Phi_{32}(a^+,b^+)}{\Phi_{31}(a^+,b^+)}\theta_0 + O\left(\theta_0^2\right),\\ \widetilde{W}_5(a^+,b^+,\theta_0) =&\tau\frac{\Phi_{52}(a^+,b^+)\Phi_{3,1}(a^+,b^+) - \Phi_{51}(a^+,b^+)\Phi_{32}(a^+,b^+)}{\Phi_{31}(a^+,b^+)}\theta_0 + O\left(\theta_0^2\right),\end{aligned}$$

with $\Phi_{31}(a^+,b^+) \neq 0$, for $a^+b^+ \neq 0$, as verified in the proof of Proposition 3.2.

We use precise interval analysis techniques, similar to those in [6, 12], along with the Poincaré–Miranda Theorem ([30]), to confirm the existence of a transversal intersection point (a^*, b^*). This point satisfies the system given by $\Phi = 0$ and

$$\Phi_{42}(a^+,b^+)\Phi_{31}(a^+,b^+) - \Phi_{41}(a^+,b^+)\Phi_{32}(a^+,b^+) = 0,$$

while also ensuring that

$$\Phi_{52}(a^+,b^+)\Phi_{31}(a^+,b^+) - \Phi_{51}(a^+,b^+)\Phi_{32}(a^+,b^+) \neq 0.$$

This approach is preferred over the usual resultant-crossing method due to the complexity of the polynomials involved. The point, approximately located at

$$(a^*, b^*) \approx (-0.555865856, 2.406812582),$$

(see Figure 5), plays a key role in our analysis. By continuity and the Implicit Function Theorem, we establish the existence of a curve $(a^*(\theta_0), b^*(\theta_0))$ near (a^*, b^*) for sufficiently small θ_0. This curve serves as a transversal solution of $\{\widetilde{W}_3 = \widetilde{W}_4 = 0\}$, while ensuring $\widetilde{W}_5 \neq 0$. Consequently, this transversality provides the unfolding of four small-amplitude limit cycles for the weak focus of order five, given the absence of a sliding segment. □

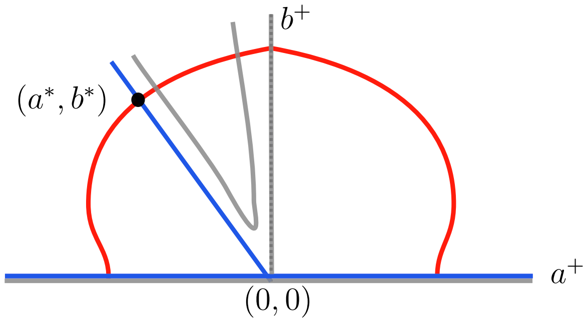

FIGURE 5. The zero level curves of the functions $\widetilde{W}_3$, $\widetilde{W}_4$ and $\widetilde{W}_5$ are drawn in red, blue, and gray, respectively.

We note that we can add an extra small-amplitude limit cycle by introducing a perturbation that includes the constant terms, such that a sliding segment appears. However, before considering this perturbation, we must first understand the bifurcation of small limit cycles from the separation straight line. Once this is established, we will explore the coexistence of both phenomena, which will be discussed in the following sections.

4. LIMIT CYCLES ON THE UPPER AND LOWER ZONES

In this section, we apply the theory introduced in Section 2 to investigate the bifurcation of small-amplitude limit cycles near equilibria located outside the straight line Σ. Specifically, we rely on Bautin's work [3], where he proved that only three limit cycles can bifurcate. However, it is important to note that the perturbation, as explained in the following section, must meet specific criteria to remain consistent with the bifurcation phenomenon described earlier. Using symmetry, we focus exclusively on the upper system for simplicity. Throughout this section, we use a^+ and b^+ as a and b, respectively, for brevity.

Proposition 4.1. *When $ab \neq 0$, system (8) exhibits a weak focus of order three at $(0, 1/b)$. With an appropriate quadratic perturbation, this equilibrium gives rise to three limit cycles of small amplitude.*

Proof. It can be readily verified that system (8) has an equilibrium point at $(0, 1/b)$ of weak-focus type, where the linear part has a zero trace and the determinant equal to one. By performing the translation $(x, y) \to (u, v + 1/b)$ followed by the Jordan transformation $(u, v) \to (-by, ay + x)$, (8) transforms into the following system:

$$\begin{cases} \dot{x} = -y + \dfrac{2}{5}aby^2 - b(a^2 + b^2 + 1)xy + p_{11}xy, \\ \dot{y} = x + bx^2 + 2abxy - \dfrac{1}{3}b(a^2 + b^2 + 6)y^2 + q_{01}y + q_{11}xy, \end{cases}$$

when $p_{11} = q_{01} = q_{11} = 0$. Using the algorithm outlined in Section 2,we find that the first non-vanishing Lyapunov quantity of the unperturbed system is:

$$\begin{aligned} V_7 = &\frac{\pi}{54000} ab^6(a^2 + 25\, b^2)(25\, a^6 + 75\, a^4\, b^2 + 75\, a^2\, b^4 + 25\, b^6 \\ &+ 90\, a^4 + 240\, a^2 b^2 + 150\, b^4 + 81\, a^2 + 225\, b^2). \end{aligned}$$

By vanishing the trace (i.e $q_{01} = 0$), the complete unfolding follows directly verifying that the linear Taylor developments of V_3 and V_5 with respect to p_{11}, q_{11}, denoted as $V_3^{[1]}$ and $V_5^{[1]}$ respectively, are linearly independent:

$$V_3^{[1]} = \frac{\pi}{60} b\, [6a\, p_{11} + 5(a^2 + b^2 + 3)q_{11}],$$

$$V_5^{[1]} = \frac{\pi}{32400} b^3 [(1950\, a^5 + 3900\, a^3 b^2 + 1950\, ab^4 + 8244\, a^3 + 6300\, ab^2 + 8910\, a) p_{11} + (875\, a^6 + 2625\, a^4 b^2 + 2625\, a^2 b^4 + 875\, b^6 + 9045\, a^4 + 14670\, a^2 b^2 + 5625\, b^4 + 25605\, a^2 + 16425\, b^2 + 22275) q_{11}].$$

□

5. The coexistence configuration

This final section is focused on proving our main result, which shows that limit cycles and crossing limit cycles of small amplitude can coexist. We simultaneously analyze the different bifurcation phenomena discussed in the previous sections.

Proposition 5.1. *For sufficiently small θ_0, there exist parameter values of the piecewise quadratic polynomial system*

$$\begin{cases} \dot{x} = -b^\pm y \pm \left[\left(a^\pm b^\pm \mp q_{01}^\pm b^\pm + q_{11}^\pm\right)/b^\pm\right] x \pm (b^\pm)^2 y^2 - q_{11}^\pm\, xy, \\ \quad \mp \left[\left(4b^\pm (a^\pm)^2 + (b^\pm)^3 + 3a^\pm q_{11}^\pm + 6b^\pm\right)/3b^\pm\right] x^2 \\ \dot{y} = \left[\left(p_{11}^\pm \mp (b^\pm)^3 - a^\pm b^\pm q_{01}^\pm \pm a^\pm q_{11}^\pm\right)/(b^\pm)^2\right] x \pm a^\pm y - a^\pm b^\pm y^2 \\ \quad + \left[\left(p_{11}^\pm \mp b^\pm \pm a^\pm q_{11}^\pm \mp (b^\pm)^3 \mp b^\pm (a^\pm)^2\right)/b^\pm\right] xy \\ \quad \pm a^\pm \left[a^\pm \left(15 p_{11}^\pm \pm 9b^\pm \pm 15 a^\pm q_{11}^\pm \mp 10 (b^\pm)^3 \pm 5b^\pm (a^\pm)^2\right)/(15(b^\pm)^2)\right] x^2, \end{cases} \tag{17}$$

defined in $\Sigma^\pm = \{(x, y) : \pm(y - \tan(\theta_0)x)) > 0\}$, *such that it has the coexistence configuration* $(3, 4, 3)$.

Proof. We note that (17) is symmetric with respect to the x-axis. We start by setting $p_{11}^\pm = q_{11}^\pm = q_{10}^\pm = 0$, making the unperturbed system equivalent to (8). Under these conditions, by Proposition 3.4, there exist parameter values a^+, b^+, a^-, b^-, and a sufficiently small θ_0, such that four crossing hyperbolic limit cycles of small amplitude bifurcate from the origin. Proposition 4.1, combined with the symmetry and taking $p_{11}^\pm, q_{11}^\pm, q_{10}^\pm$ to be small enough, supports this statement. We emphasize that the crossing limit cycles surrounding the origin persist due to their hyperbolicity. □

Proof of Theorem 1.1. The proof follows directly from Proposition 5.1 by adding constant perturbation terms, which lead to a pseudo-Hopf type bifurcation due to the existence of a sliding segment. □

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