

ON THE NUMBER OF LIMIT CYCLES FOR TWO CLASSES OF POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. This paper deals with two classes of polynomial differential systems which are generalized rigid systems, i.e. differential systems whose orbits rotate with constant angular velocity. We give the bounds for the maximum number of limit cycles of such polynomial differential systems provided by the averaging theory of first order. These bounds are reached.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In the qualitative theory of planar polynomial differential systems, one of the most important problems is the determination and distribution of their limit cycles. This problem is known as the famous Hilbert’s 16th problem [11, 14]. Since this problem is very difficult and it remains open, many mathematicians pay attention to particular classes of planar polynomial differential systems, as the Liénard systems see [4, 7], or the Z_2 -equivariant systems see [5, 13], ...

A *rigid polynomial differential system* can be written as

$$\frac{dx}{dt} = -y + xF(x, y), \quad \frac{dy}{dt} = x + yF(x, y), \quad (1)$$

where $F(x, y)$ is polynomial with $F(0, 0) = 0$. Using the polar coordinates (r, θ) defined by $x = r \cos \theta$, $y = r \sin \theta$, the rigid system (1) becomes

$$\frac{dr}{dt} = rF(r \cos \theta, r \sin \theta), \quad \frac{d\theta}{dt} = 1.$$

It is worth to note that when the origin $(0, 0)$ of this differential system is a center, this class of rigid systems have been studied extensively, see for instance [8, 9, 10, 15, 22].

In this paper we deal with the polynomial differential systems

$$\frac{dx}{dt} = -y(x^2 + y^2)^{k/2} + xU_l(x, y), \quad \frac{dy}{dt} = x(x^2 + y^2)^{k/2} + yU_l(x, y), \quad (2)$$

where k is an even positive integer, and $U_l(x, y) = \sum_{i+j=l} a_{i,j}x^i y^j$ is a homogeneous polynomial of degree l in the variables x and y .

In polar coordinates the differential system (2) writes

$$\frac{dr}{dt} = r^{l+1}U_l(\cos \theta, \sin \theta), \quad \frac{d\theta}{dt} = r^k. \quad (3)$$

Doing the time scaling $d\tau = r^k dt$ the differential system (3) becomes the rigid system

$$\frac{dr}{d\tau} = r^{l+1-k}U_l(\cos \theta, \sin \theta), \quad \frac{d\theta}{d\tau} = 1.$$

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Therefore we shall say that the differential system (3) and consequently the differential system (2) are *generalized rigid systems*.

In order that the origin $(0, 0)$ of the differential systems (3) or (2) be a center, it is sufficient that

$$\int_0^{2\pi} \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta = 0. \quad (4)$$

Note that condition (4) always holds if l is odd. When l is even in general condition (4) is not satisfied, but for instance it holds if the coefficients $a_{i,j} = 0$ when i and j even.

We denote by $H(n+1)$ the maximum number of limit cycles, which can be obtained using the averaging theory of first order, bifurcating from the periodic orbits surrounding the center $O(0, 0)$ of the differential system (2) when this system is perturbed by a homogeneous polynomial of degree $n+1$ as follows

$$\begin{aligned} \frac{dx}{dt} &= -y(x^2 + y^2)^{(l+1)/2} + xU_l(x, y) + \varepsilon xU_n(x, y), \\ \frac{dy}{dt} &= x(x^2 + y^2)^{(l+1)/2} + yU_l(x, y) + \varepsilon yU_n(x, y), \end{aligned} \quad (5)$$

with $U_n(x, y) = \sum_{i+j=n} b_{i,j} x^i y^j$ if l is odd, and as

$$\begin{aligned} \frac{dx}{dt} &= -y(x^2 + y^2)^{l/2} + xU_l(x, y) + \varepsilon x \sum_{p=0}^n U_p(x, y), \\ \frac{dy}{dt} &= x(x^2 + y^2)^{l/2} + yU_l(x, y) + \varepsilon y \sum_{p=0}^n U_p(x, y), \end{aligned} \quad (6)$$

with $U_p(x, y) = \sum_{i+j=p} b_{i,j} x^i y^j$ if l is even.

Our two main results are the following.

Theorem 1. *Consider the polynomial differential system (5) with $n > l$ and l odd. Then $H(n+1) = (n-l-1)/2$ if n is even, and $H(n+1) = (n-l-2)/2$ if n is odd.*

Theorem 2. *Consider the polynomial differential system (6) with $n > l$ and l even. Assume system (6) satisfies (4). Then $H(n+1) = [n/2]$.*

We must mention that the maximum number of limit cycles of the differential systems (5) and (6) can be larger than the ones provided by the averaging theory of first order in Theorems 1 and 2, respectively. But in this paper we study the maximum number of limit cycles provided by the averaging theory of first order.

The organization of this paper is as follows. In section 2 we recall the basic results on the averaging theory of first order that we need for proving our results. The proof of Theorems 1 and 2 are given in section 3.

2. THE AVERAGING THEORY OF FIRST ORDER

The first steps on the averaging theory were given by Lagrange and Laplace, but was Fatou[6] in 1928 who started to work with rigor on it, and after improved by Bogoliubov and Krylov [12]. In recent years the averaging theory has been extended into continuous system [2], and discontinuous systems [16]. Roughly speaking the averaging theory gives a qualitative relation between the solutions of a non-autonomous periodic differential system and the solutions of its averaged differential system, which is autonomous. The next theorem provides one of the versions of the averaging theory of first order in ε for a periodic differential system. For more details, see [19, 21].

Consider the following periodic differential equation

$$\frac{dr}{d\theta} = F_0(\theta, r) + \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r, \varepsilon), \quad (7)$$

where ε is sufficiently small, $\varepsilon_0 > 0$, Ω is an open subset of $\mathbb{R}$, the functions $F_0, F_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$ are C^2 and 2π -periodic in the variable θ . The unperturbed system of (7) $_{\varepsilon=0}$ is $dr/d\theta = F_0(\theta, r)$.

Let $r(\theta, r_0)$ be a periodic solution of the unperturbed equation satisfying $r(0, r_0) = r_0$ and consider its variational equation

$$\frac{dr}{d\theta} = \frac{\partial F_0(\theta, r(\theta, r_0))}{\partial r} r. \quad (8)$$

Let $M(\theta, r)$ be the solution of (8). Suppose that there exists an open subset V with its closure $\bar{V} \subset \Omega$ such that for every $r \in \bar{V}$, the solution $r(\theta, r)$ of the unperturbed equation is 2π -periodic in the variable θ . The function $F : \bar{V} \rightarrow \mathbb{R}$ defined by

$$F(r) = \int_0^{2\pi} \frac{F_1(\theta, r(\theta, r))}{M(\theta, r)} d\theta. \quad (9)$$

is called the *averaged function of first order*. Then if for some $a \in V$ we have that $F(a) = 0$ and $F'(a) \neq 0$, the differential equation (7) has a periodic solution $r(\theta, \varepsilon)$, such that $r(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$. This result goes back to Malkin [17] and Roseau [18], for a more clear proof see [3].

From this result we know that the simple zeros of the function $F(z)$ given in (9), provide limit cycles bifurcating from the periodic orbits with initial conditions on V . Hence estimating the maximum number of such zeros we will have the maximum number of limit cycles of the differential equation (7) which can be obtained using the averaged function of first order.

The following two results are crucial in order to bound the number of limit cycles.

Theorem 3 ([20]). *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be vectors of a vectorial space E . Then*

$$G(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) := \begin{vmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \dots & \dots & \dots & \dots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{vmatrix} \geq 0,$$

if and only if the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linear dependence, where $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ denote the inner product.

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [1]).

Theorem 4 (Descartes Theorem). *Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$ with $0 \leq i_1 < i_2 < \dots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \dots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m , then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.*

3. PROOF OF THE MAIN RESULTS

According to the averaging theory of first order the number of simple zeros of the averaged function (9) provide the same number of w limit cycles bifurcating from the periodic orbits surrounding the center $(0, 0)$ of system (7). We must recall that the maximum number

of limit cycles of the differential system (7) can be larger than the ones provided by the averaging theory of first order. But in this paper we study the maximum number of limit cycles provided by the averaging theory of first order.

3.1. Proof of Theorem 1. In polar coordinates the differential system (5) writes

$$\frac{dr}{dt} = r^{l+1}U_l(\cos \theta, \sin \theta) + \varepsilon r^{n+1}U_n(\cos \theta, \sin \theta), \quad \frac{d\theta}{dt} = r^{l+1}, \quad (10)$$

and taking as independent variable the θ the differential system (10) is equivalent to the periodic differential equation

$$\frac{dr}{d\theta} = U_l(\cos \theta, \sin \theta) + \varepsilon r^{n-l}U_n(\cos \theta, \sin \theta) \triangleq F_0(\theta, r) + \varepsilon F_1(\theta, r). \quad (11)$$

The unperturbed system of (11) is

$$\frac{dr}{d\theta} = F_0(\theta, r) = \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta. \quad (12)$$

It is obvious that the solution $r(\theta, r_0)$ of the differential equation (12) satisfying $r(0, r_0) = r_0 > 0$ is

$$r(\theta, r_0) = r_0 + \int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta.$$

Since l is odd, we have $\int_0^{2\pi} \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta = 0$. Therefore $r(\theta, r_0)$ is a 2π -periodic solution. The variational equation of the differential equation (12) on the periodic solution $r(\theta, r_0)$ is $\frac{dr}{d\theta} = 0$. Hence its $M(\theta, r) = 1$. Applying the averaging theory of section 2, the averaged function (9) of the periodic differential equation (11) is

$$\begin{aligned} F(r_0) &= \int_0^{2\pi} F_1(\theta, r(\theta, r_0)) d\theta \\ &= \int_0^{2\pi} r(\theta, r_0)^{n-l} U_n(\cos \theta, \sin \theta) d\theta \\ &= \int_0^{2\pi} \left(r_0 + \int_0^\theta U_l(\cos \theta, \sin \theta) d\theta \right)^{n-l} U_n(\cos \theta, \sin \theta) d\theta \\ &= \int_0^{2\pi} \left(r_0 + \int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right)^{n-l} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta \\ &= \int_0^{2\pi} \sum_{k=0}^{n-l} C_{n-l}^k r_0^{n-l-k} \left(\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right)^k \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta \\ &\triangleq c_0 r_0^{n-l} + c_1 r_0^{n-l-1} + c_2 r_0^{n-l-2} + \dots + c_{n-l-1} r_0 + c_{n-l}, \end{aligned} \quad (13)$$

where

$$\begin{aligned}
C_{n-l}^k &= \binom{n-l}{k}, \\
c_0 &= \int_0^{2\pi} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta; \\
c_1 &= (n-l) \int_0^{2\pi} \left(\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right) \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta; \\
c_2 &= C_{n-l}^2 \int_0^{2\pi} \left(\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right)^2 \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta; \\
&\dots\dots \\
c_{n-l-1} &= (n-l) \int_0^{2\pi} \left(\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right)^{n-l-1} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta; \\
c_{n-l} &= \int_0^{2\pi} \left(\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \right)^{n-l} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta.
\end{aligned}$$

In order to show that the coefficients $c_i, i = 0, 1, \dots, n-l$ are independently, i.e. that they can take arbitrary values, we need the following result.

Proposition 5. Suppose that $M = (m_{i,j})$ is a $\frac{(n-l+1)}{2} \times \frac{(n-l+1)}{2}$ matrix with n even, l odd and

$$m_{i,j} = \int_0^{2\pi} \cos^{2j-2} \theta \sin^{n+2i-2j} \theta d\theta.$$

Then $\det M \neq 0$.

Proof. Let $\mathbf{v}_j = (m_{1,j}, m_{2,j}, \dots, m_{\frac{n-l+1}{2},j})^T$, for $j = 1, 2, \dots, \frac{n-l+1}{2}$, then

$$\begin{aligned}
\mathbf{v}_1 + \mathbf{v}_2 &= \left(m_{1,1} + m_{1,2}, m_{2,1} + m_{2,2}, \dots, m_{\frac{(n-l+1)}{2},1} + m_{\frac{(n-l+1)}{2},2} \right)^T \\
&= \left(\int_0^{2\pi} \sin^{n-2} \theta d\theta, \int_0^{2\pi} \sin^n \theta d\theta, \dots, \int_0^{2\pi} \sin^{2n-l-3} \theta d\theta \right)^T.
\end{aligned}$$

For every $j = 2, \dots, \frac{n-l+1}{2}$, we have

$$\begin{aligned}
&C_{j-1}^0 \mathbf{v}_1 + C_{j-1}^1 \mathbf{v}_2 + \dots + C_{j-1}^{j-1} \mathbf{v}_j \\
&= \left(\int_0^{2\pi} \sin^{n+2-2j} \theta d\theta, \int_0^{2\pi} \sin^{n+4-2j} \theta d\theta, \dots, \int_0^{2\pi} \sin^{2n-l+1-2j} \theta d\theta \right)^T.
\end{aligned}$$

Let $f_i(\theta) = \sin^{\frac{l-3}{2}+2i} \theta$ for $i = 1, 2, \dots, \frac{n-l+1}{2}$, we define the inner product

$$\langle f_i(\theta), f_j(\theta) \rangle = \int_0^{2\pi} f_i(\theta) f_j(\theta) d\theta = \int_0^{2\pi} \sin^{l-3+2i+2j} \theta d\theta = \tilde{m}_{i,j}.$$

Now we consider the matrix of Theorem 3 for the vectors $f_i(\theta) = \sin^{\frac{l-3}{2}+2i} \theta$ varying $i = 1, 2, \dots, \frac{n-l+1}{2}$, i.e.

$$\begin{aligned} \det M &= \det \begin{pmatrix} \int_0^{2\pi} \sin^n \theta d\theta & \int_0^{2\pi} \sin^{n-2} \theta d\theta & \cdots & \int_0^{2\pi} \sin^{l+1} \theta d\theta \\ \int_0^{2\pi} \sin^{n+2} \theta d\theta & \int_0^{2\pi} \sin^n \theta d\theta & \cdots & \int_0^{2\pi} \sin^{l+3} \theta d\theta \\ \cdots & \cdots & \cdots & \cdots \\ \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta & \int_0^{2\pi} \sin^{2n-l-3} \theta d\theta & \cdots & \int_0^{2\pi} \sin^n \theta d\theta \end{pmatrix} \\ &= K \det \begin{pmatrix} \int_0^{2\pi} \sin^{l+1} \theta d\theta & \int_0^{2\pi} \sin^{l+3} \theta d\theta & \cdots & \int_0^{2\pi} \sin^n \theta d\theta \\ \int_0^{2\pi} \sin^{l+3} \theta d\theta & \int_0^{2\pi} \sin^{l+5} \theta d\theta & \cdots & \int_0^{2\pi} \sin^{n+2} \theta d\theta \\ \cdots & \cdots & \cdots & \cdots \\ \int_0^{2\pi} \sin^n \theta d\theta & \int_0^{2\pi} \sin^{n+2} \theta d\theta & \cdots & \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta \end{pmatrix} \\ &= KG(f_1, f_2, \dots, f_{(n-l+1)/2}), \end{aligned}$$

where $K = (-1)^{[(n-l)^2-1]/8}$. Since $f_1, f_2, \dots, f_{(n-l+1)/2}$ are linearly independent, from Theorem 3 it follows that $\det M \neq 0$. $\square$

In order to obtain a lower bound for the number of zeros of the averaged function $F(r_0)$, we choose its coefficients $a_{i,j}$ as follows:

$$a_{2k+1, l-1-2k} = C_{\frac{l-1}{2}}^k \text{ for } k = 0, 1, \dots, \frac{l-1}{2}, \text{ and } a_{i,j} = 0, \text{ otherwise.} \quad (14)$$

Then

$$\begin{aligned} \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta &= \sum_{k=0}^{\frac{l-1}{2}} C_{\frac{l-1}{2}}^k \cos^{2k+1} \theta \sin^{l-1-2k} \theta = \cos \theta \sum_{k=0}^{\frac{l-1}{2}} C_{\frac{l-1}{2}}^k \cos^{2k} \theta \sin^{l-1-2k} \theta \\ &= \cos \theta (\cos^2 \theta + \sin^2 \theta)^{\frac{l-1}{2}} = \cos \theta, \end{aligned}$$

and hence

$$\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta = \sin \theta. \quad (15)$$

In the following result we determine the monomials which appear in the polynomial averaged function $F(r_0)$.

Proposition 6. *Assume that l is odd and conditions (14) hold. Then the averaged function is*

$$F(r_0) = \sum_{k=0}^{n-l} C_{n-l}^k r_0^{n-l-k} \int_0^{2\pi} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^{q+k} \theta d\theta.$$

- (i) *If n is even, then the monomials of the polynomial averaged function $F(r_0)$ are the following $(n-l+1)/2$ monomials $\{r_0, r_0^3, \dots, r_0^{n-l-2}, r_0^{n-l}\}$.*
- (ii) *If n is odd, then the monomials of the polynomial averaged function $F(r_0)$ are the following $(n-l)/2$ monomials $\{r_0, r_0^3, \dots, r_0^{n-l-3}, r_0^{n-l-1}\}$.*

Proof. Substituting (14) and (15) into (13) we obtain

$$\begin{aligned}
 F(r_0) &= \int_0^{2\pi} \sum_{k=0}^{n-l} C_{n-l}^k r_0^{n-l-k} (\sin \theta)^k \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^q \theta d\theta \\
 &= \sum_{k=0}^{n-l} C_{n-l}^k r_0^{n-l-k} \int_0^{2\pi} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^{q+k} \theta d\theta \\
 &\triangleq c_0 r_0^{n-l} + c_1 r_0^{n-l-1} + c_2 r_0^{n-l-2} + \dots + c_{n-l-1} r_0 + c_{n-l},
 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
 c_0 &= \sum_{p+q=n} b_{p,q} \int_0^{2\pi} \cos^p \theta \sin^q \theta d\theta; \\
 c_1 &= C_{n-l}^1 \sum_{p+q=n} b_{p,q} \int_0^{2\pi} \cos^p \theta \sin^{q+1} \theta d\theta; \\
 &\dots \\
 c_{n-l-1} &= C_{n-l}^{n-l-1} \sum_{p+q=n} b_{p,q} \int_0^{2\pi} \cos^p \theta \sin^{q+(n-l-1)} \theta d\theta; \\
 c_{n-l} &= \sum_{p+q=n} b_{p,q} \int_0^{2\pi} \cos^p \theta \sin^{q+(n-l)} \theta d\theta.
 \end{aligned}$$

Assume that n is even. If k is odd, then $p+q+k = n+k$ is odd, and consequently either p or $q+k$ is odd, thus $\int_0^{2\pi} \cos^p \theta \sin^{q+k} \theta d\theta = 0$. Therefore $c_1 = c_3 = \dots = c_{n-l} = 0$, and we have

$$\begin{aligned}
 F(r_0) &= \sum_{\substack{k=0 \\ \text{keven}}}^{n-l} C_{n-l}^k r_0^{n-l-k} \int_0^{2\pi} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^{q+k} \theta d\theta \\
 &\triangleq c_0 r_0^{n-l} + c_2 r_0^{n-l-2} + \dots + c_{n-l-1} r_0.
 \end{aligned}$$

Hence the polynomial $F(r_0)$ has $(n-l+1)/2$ only the monomials $r_0, r_0^3, \dots, r_0^{n-l-2}, r_0^{n-l}$. In the following we prove that the coefficients $c_0, c_2, \dots, c_{n-l-1}$ can be choose arbitrarily. Indeed, we consider the Jacobian matrix

$$\begin{aligned}
 &\frac{\partial(c_0, c_2, \dots, c_{n-l-1})}{\partial(b_{0,n}, b_{2,n-2}, \dots, b_{n-l-1, l+1})} \\
 &= \begin{pmatrix} \int_0^{2\pi} \sin^n \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{n-2} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-1} \theta \sin^{l+1} \theta d\theta \\ C_{n-l}^2 \int_0^{2\pi} \sin^{n+2} \theta d\theta & C_{n-l}^2 \int_0^{2\pi} \cos^2 \theta \sin^n \theta d\theta & \dots & C_{n-l}^2 \int_0^{2\pi} \cos^{n-l-1} \theta \sin^{l+3} \theta d\theta \\ \dots & \dots & \dots & \dots \\ C_{n-l}^1 \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta & C_{n-l}^1 \int_0^{2\pi} \cos^2 \theta \sin^{2n-l-3} \theta d\theta & \dots & C_{n-l}^1 \int_0^{2\pi} \cos^{n-l-1} \theta \sin^n \theta d\theta \end{pmatrix} \\
 &\triangleq A,
 \end{aligned}$$

then

$\det A$

$$= C \det \begin{pmatrix} \int_0^{2\pi} \sin^n \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{n-2} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-1} \theta \sin^{l+1} \theta d\theta \\ \int_0^{2\pi} \sin^{n+2} \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^n \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-1} \theta \sin^{l+3} \theta d\theta \\ \dots & \dots & \dots & \dots \\ \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{2n-l-3} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-1} \theta \sin^n \theta d\theta \end{pmatrix}$$

$$= C \det M,$$

where $C = C_{n-l}^2 C_{n-l}^4 \dots C_{n-l}^{n-l-1}$, recall that $C_{n-l}^{n-l-1} = C_{n-l}^1$. Now from Proposition 5 it follows that $\det A \neq 0$. Thus the coefficients $c_0, c_2, \dots, c_{n-l-1}$ can be chosen arbitrarily.

Assume that n is odd. If k is even, then $p+q+k = n+k$ is odd, and consequently either p or $q+k$ is odd, thus $\int_0^{2\pi} \cos^p \theta \sin^{q+k} \theta d\theta = 0$, and $c_0 = c_2 = \dots = c_{n-l} = 0$. In this case the averaged function is

$$F(r_0) = \int_0^{2\pi} \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-l} C_{n-l}^k r_0^{n-l-k} \sum_{p+q=n} b_{p,q} \cos^p \theta \sin^{q+k} \theta d\theta$$

$$\triangleq c_1 r_0^{n-l-1} + c_3 r_0^{n-l-3} + \dots + c_{n-l-1} r_0.$$

Therefore the polynomial $F(r_0)$ has $(n-l)/2$ the monomials $r_0, r_0^3, \dots, r_0^{n-l-3}, r_0^{n-l-1}$. Now we need to prove that the coefficients $c_1, c_3, \dots, c_{n-l-1}$ can be chosen arbitrarily.

Consider the Jacobian matrix

$$\frac{\partial(c_1, c_3, \dots, c_{n-l-1})}{\partial(b_{0,n}, b_{2,n-2}, \dots, b_{n-l-2, l+2})}$$

$$= \begin{pmatrix} (n-l) \int_0^{2\pi} \sin^{n+1} \theta d\theta & (n-l) \int_0^{2\pi} \cos^2 \theta \sin^{n-1} \theta d\theta & \dots & (n-l) \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{l+3} \theta d\theta \\ C_{n-l}^3 \int_0^{2\pi} \sin^{n+3} \theta d\theta & C_{n-l}^3 \int_0^{2\pi} \cos^2 \theta \sin^{n+1} \theta d\theta & \dots & C_{n-l}^3 \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{l+5} \theta d\theta \\ \dots & \dots & \dots & \dots \\ C_{n-l}^1 \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta & C_{n-l}^1 \int_0^{2\pi} \cos^2 \theta \sin^{2n-l-3} \theta d\theta & \dots & C_{n-l}^1 \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{n+1} \theta d\theta \end{pmatrix}$$

$$\triangleq \tilde{A},$$

then

$\det \tilde{A}$

$$= \tilde{C} \det \begin{pmatrix} \int_0^{2\pi} \sin^{n+1} \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{n-1} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{l+3} \theta d\theta \\ \int_0^{2\pi} \sin^{n+3} \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{n+1} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{l+5} \theta d\theta \\ \dots & \dots & \dots & \dots \\ \int_0^{2\pi} \sin^{2n-l-1} \theta d\theta & \int_0^{2\pi} \cos^2 \theta \sin^{2n-l-3} \theta d\theta & \dots & \int_0^{2\pi} \cos^{n-l-2} \theta \sin^{n+1} \theta d\theta \end{pmatrix}.$$

where $\tilde{C} = C_{n-l}^1 C_{n-l}^3 \dots C_{n-l}^{n-l-1}$. Similarly we deduce that $\det \tilde{A} \neq 0$, and then $c_1, c_3, \dots, c_{n-l-1}$ are independent. $\square$

Proof of Theorem 1. Assume that n is even. Then, from statement (i) of Proposition 6, the polynomial $F(r_0)$ has $(n-l+1)/2$ distinct monomials whose coefficients are independent. Hence, by Theorem 4 the polynomial $F(r_0)$ can have at most $(n-l-1)/2$ simple zeros, and that this number can be reached due to the independence of the coefficients of $F(r_0)$. So, by the averaging theory of section 2 the differential system (5) has at least $(n-l-1)/2$ limit cycles bifurcating from the periodic orbits surrounding the center of the differential system (2).

Assume that n is odd. Then, from statement (ii) of Proposition 6, the polynomial $F(r_0)$ has $(n-l)/2$ distinct monomials whose coefficients are independent. Hence, by Theorem 4 the polynomial $F(r_0)$ can have at most $(n-l-2)/2$ simple zeros, and that this number can be reached due to the independence of the coefficients of $F(r_0)$. So, by the averaging theory of section 2 the differential system (5) has at least $(n-l-2)/2$ limit cycles bifurcating from the periodic orbits surrounding the center of the differential system (2). $\square$

3.2. Proof of Theorem 2. By assumptions l is even and the origin of the differential system (2) is a center. In polar coordinates the differential system (6) writes

$$\frac{dr}{dt} = r^{l+1}U_l(\cos \theta, \sin \theta) + \varepsilon \sum_{p=0}^n r^{p+1}U_p(\cos \theta, \sin \theta), \quad \frac{d\theta}{dt} = r^l,$$

and taking as the new independent variable the variable θ this system becomes the differential equation

$$\frac{dr}{d\theta} = rU_l(\cos \theta, \sin \theta) + \varepsilon \sum_{p=0}^n r^{p+1-l}U_p(\cos \theta, \sin \theta). \quad (17)$$

The unperturbed system of (17) is:

$$\frac{dr}{d\theta} = r \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta. \quad (18)$$

The solution $r(\theta, r_0)$ such that $r(0, r_0) = r_0$ of the unperturbed system (18) is

$$r(\theta, r_0) = r_0 e^{\int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta} \triangleq r_0 e^{A(\theta)}.$$

Since $A(2\pi) = 0$ this solution is 2π periodic in the variable θ . Then the solution of the variational equation defined by the periodic solution $r(\theta, r)$ is $M(\theta, r) = e^{A(\theta)}$. Then the averaged function (9) in this case is

$$\begin{aligned} F(r_0) &= \int_0^{2\pi} e^{-A(\theta)} \sum_{p=0}^n (r_0 e^{A(\theta)})^{p+1-l} U_p(\cos \theta, \sin \theta) d\theta \\ &= \int_0^{2\pi} \sum_{p=0}^n r_0^{p+1-l} e^{(p-l)A(\theta)} \left(\sum_{m+q=p} b_{m,q} \cos^m \theta \sin^q \theta \right) d\theta \\ &= \sum_{p=0}^n r_0^{p+1-l} \int_0^{2\pi} e^{(p-l)A(\theta)} \left(\sum_{m+q=p} b_{m,q} \cos^m \theta \sin^q \theta \right) d\theta. \end{aligned}$$

In order to study the zeros of $F(r_0)$ we consider the function

$$\begin{aligned} f(r_0) &= r_0^{l-1} F(r_0) = \sum_{p=0}^n r_0^p \int_0^{2\pi} e^{(p-l)A(\theta)} \left(\sum_{m+q=p} b_{m,q} \cos^m \theta \sin^q \theta \right) d\theta \\ &\triangleq c_0 + c_1 r_0 + c_2 r_0^2 + \dots + c_n r_0^n, \end{aligned} \quad (19)$$

Proposition 7. Assume that l is even and k is odd. Then

$$\int_0^{2\pi} e^{(k-l)A(\theta)} \cos^k \theta d\theta = 0, \quad \int_0^{2\pi} e^{(k-l)A(\theta)} \left(\sum_{m+q=k} b_{m,q} \cos^m \theta \sin^q \theta \right) d\theta = 0.$$

Proof. Recall that $A(\theta) = \int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta = 0$.

Assume that i and j are odd, then

$$\begin{aligned}
\int_0^\theta \cos^i \theta \sin^j \theta d\theta &= \int_0^\theta \cos^{i-1} \theta \sin^j \theta d \sin \theta = \int_0^\theta (1 - \sin^2 \theta)^{\frac{i-1}{2}} \sin^j \theta d \sin \theta \\
&= \int_0^\theta \sum_{h=0}^{(i-1)/2} C_{(i-1)/2}^h (-1)^h \sin^{2h+j} \theta d \sin \theta \\
&= \sum_{h=0}^{(i-1)/2} C_{(i-1)/2}^h (-1)^h \frac{\sin^{2h+j+1} \theta}{2h+j+1} \\
&= \sum_{h=0}^{(i-1)/2} C_{(i-1)/2}^h (-1)^h \frac{\sin^{2h+j+1} \theta}{2h+j+1} (\cos^2 \theta + \sin^2 \theta)^{\frac{l-(2h+j+1)}{2}} \\
&= \sum_{i+j=l} d_{i,j} \cos^i \theta \sin^j \theta.
\end{aligned}$$

Assume that i and j are even, then

$$\begin{aligned}
\cos^i \theta \sin^j \theta &= \cos^{2\tilde{i}} \theta \sin^{2\tilde{j}} \theta = (\cos^2 \theta)^{\tilde{i}} (\sin^2 \theta)^{\tilde{j}} \\
&= \left(\frac{1 + \cos 2\theta}{2}\right)^{\tilde{i}} \left(\frac{1 - \cos 2\theta}{2}\right)^{\tilde{j}} = c_1 P(\cos 2\theta),
\end{aligned} \tag{20}$$

where c_1 is a constant, and $P(\cos 2\theta)$ is a polynomial of $\cos 2\theta$ of degree $l/2$.

We claim that $\cos^k 2\theta = \sum_{\substack{p=0 \\ p \text{ even}}}^{2k} c_p \cos p\theta$, where $c_p, p = 0, 2, \dots, 2k$ are constant. We prove

this claim by induction. It is obvious that the claim holds for $k = 1$. Assume that the claim holds for $k = k_1$, and we shall prove it for $k = k_1 + 1$. Indeed

$$\begin{aligned}
\cos^{k_1+1} 2\theta &= \cos^{k_1} 2\theta \cos 2\theta = \sum_{\substack{p=0 \\ p \text{ even}}}^{2k_1} c_p \cos p\theta \cos 2\theta = \sum_{\substack{p=0 \\ p \text{ even}}}^{2k_1} c_p \frac{\cos(p\theta + 2\theta) + \cos(p\theta - 2\theta)}{2} \\
&= \sum_{\substack{p=0 \\ p \text{ even}}}^{2k_1} c_p \frac{\cos((p+2)\theta) + \cos((p-2)\theta)}{2} = \sum_{\substack{p=0 \\ p \text{ even}}}^{2k_1+2} \tilde{c}_p \cos p\theta.
\end{aligned}$$

Hence the claim is proved.

From (20) and the claim, we have $\cos^{\frac{l}{2}} 2\theta = \sum_{\substack{p=0 \\ p \text{ even}}}^l \tilde{c}_p \cos p\theta$, and hence

$$\sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta = a_0 + a_2 \cos 2\theta + a_4 \cos 4\theta + \dots + a_l \cos l\theta.$$

Since $\int_0^{2\pi} \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta = 0$, we have $a_0 = 0$.

Direct computation shows

$$\begin{aligned}
& \int_0^\theta \sum_{i+j=l} a_{i,j} \cos^i \theta \sin^j \theta d\theta \\
&= \int_0^\theta (a_2 \cos 2\theta + a_4 \cos 4\theta + \dots + a_l \cos l\theta) d\theta \\
&= \frac{a_2}{2} \sin 2\theta + \frac{a_4}{4} \sin 4\theta + \dots + \frac{a_l}{l} \sin l\theta \\
&= \frac{a_2}{2} \sin 2\theta (\cos^2 \theta + \sin^2 \theta)^{(l-2)/2} + \frac{a_4}{4} \sin 4\theta (\cos^2 \theta + \sin^2 \theta)^{(l-4)/2} + \dots + \frac{a_l}{l} \sin l\theta \\
&= \sum_{i+j=l} d_{i,j} \cos^i \theta \sin^j \theta.
\end{aligned}$$

In summary, we have proved $A(\theta) = \sum_{i+j=l} d_{i,j} \cos^i \theta \sin^j \theta$. Applying the Taylor expansion

to $\int_0^{2\pi} e^{(k-l)A(\theta)} \cos^k \theta d\theta$, we have

$$\begin{aligned}
\int_0^{2\pi} e^{(k-l)A(\theta)} \cos^k \theta d\theta &= \sum_{h=0}^{+\infty} \int_0^{2\pi} \frac{(k-l)^h}{h!} \left(\sum_{i+j=l} d_{i,j} \cos^i \theta \sin^j \theta \right)^h \cos^k \theta d\theta \\
&= \sum_{h=0}^{+\infty} \frac{(k-l)^h}{h!} \int_0^{2\pi} \sum_{m_1+m_2=lh} d_{m_1,m_2} \cos^{m_1} \theta \sin^{m_2} \theta \cos^k \theta d\theta \\
&= \sum_{h=0}^{+\infty} \frac{(k-l)^h}{h!} \int_0^{2\pi} \sum_{m_1+m_2=lh} d_{m_1,m_2} \cos^{m_1+k} \theta \sin^{m_2} \theta d\theta.
\end{aligned}$$

Since l is even, lh is even. If k is odd, then $m_1 + m_2 + k = lh + k$ is odd, thus either $m_1 + k$ or m_2 is odd, and $\int_0^{2\pi} \cos^{m_1+k} \theta \sin^{m_2} \theta d\theta = 0$. Therefore $\int_0^{2\pi} e^{(k-l)A(\theta)} \cos^k \theta d\theta = 0$. $\square$

Proposition 8. Assume that l is even and let $f(r_0)$ be the function defined in (19).

- (i) If n is even, then the polynomial $f(r_0)$ has the monomials $\{1, r_0^2, \dots, r_0^{n-2}, r_0^n\}$, and the coefficients of these monomials are independent.
- (ii) If n is odd, then the polynomial $f(r_0)$ has the monomials $\{1, r_0^2, \dots, r_0^{n-1}\}$, and the coefficients of these monomials are independent.

Proof. From Proposition 7, if n is even and k is odd, we have that the coefficient c_k of the function $f(r_0)$ is zero. So $f(r_0) = c_0 + r_0^2 c_2 + r_0^4 c_4 + \dots + r_0^n c_n$. Since the determinant of the Jacobian matrix

$$\begin{aligned}
& \frac{\partial(c_0, c_2, \dots, c_n)}{\partial(b_{0,0}, b_{2,0}, \dots, b_{n,0})} \\
&= \begin{pmatrix} \int_0^{2\pi} e^{-lA(\theta)} d\theta & 0 & \dots & 0 \\ 0 & \int_0^{2\pi} e^{(2-l)A(\theta)} \cos^2 \theta d\theta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \int_0^{2\pi} e^{(n-l)A(\theta)} \cos^n \theta d\theta \end{pmatrix},
\end{aligned}$$

is not zero, the coefficients c_k for k even are independent.

The same proof works for the case n odd. $\square$

Proof of Theorem 2. From the statement (i) of proposition 8 and Theorem 4, we know that the polynomial $f(r_0)$ and consequently the polynomial $F(r_0)$ has at most $n/2$ positive zeros

and that this number of zeros can be reached when l and n are even. Hence, by the averaging theory of section 2, we obtain that $H(n+1) = n/2$.

When l is even and n is odd, similarly we know that the polynomial $f(r_0)$ and consequently the polynomial $F(r_0)$ has at most $(n-1)/2$ positive zeros and that this number of zeros can be reached. Again, by the averaging theory of section 2, we obtain that $H(n+1) = (n-1)/2$.

In summary, we can conclude that $H(n+1) = [n/2]$. So Theorem 2 is proved. $\square$

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