

ANALYTIC STUDY OF TWO LIMIT CYCLES BIFURCATING FROM A ZERO–HOPF EQUILIBRIUM

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ABSTRACT. In this paper we provide sufficient conditions for the existence of two limit cycles bifurcating from the unique zero–Hopf equilibrium of the differential system

$$\dot{x} = y + a, \quad \dot{y} = -x + z, \quad \dot{z} = -bx^2 + z^2 + c,$$

where a , b and c are real arbitrary parameters. Our study uses the averaging theory. This differential system has been studied previously for some authors because it can exhibit chaotic motion when it has no equilibrium points.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Differential systems appear in many areas, such as, Physics, Biology, Chemistry, Control Theory, Mechanical Oscillators, ... see for instance [1, 2, 12], and their study is important. A limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the differential system. The study of the limit cycles of polynomial differential systems has two main motivations, first they play a relevant role in the dynamics of the differential systems, and also their analysis become motivated by Hilbert’s 16-th problem, see [4, 11].

A way to produce limit cycles is perturbing a differential system having zero–Hopf equilibria with a real small parameter ε . An equilibrium point of a differential system in $\mathbb{R}^3$ such that the eigenvalues of the Jacobian matrix of the differential system at the equilibrium point are 0 , ωi and $-\omega i$, with $\omega > 0$ is called *zero–Hopf equilibrium*. For more details regarding on the zero–Hopf bifurcations see [5, 6, 10].

In the paper [7] the authors investigated the chaotic behavior of the differential system

$$(1) \quad \begin{aligned} \dot{x} &= y + a, \\ \dot{y} &= z - x, \\ \dot{z} &= z^2 - bx^2 + c, \end{aligned}$$

2020 *Mathematics Subject Classification.* 34C07, 34C25, 34C29, 37C27, 37G10, 37G15.
Key words and phrases. limit cycles, averaging theory, zero–Hopf bifurcation.

when it has no equilibria, i.e. when either $(b-1)c < 0$, or $b = 1$. Differential system (1) also is the differential system (21) of the page 64 of the book [8].

The main goal of this paper is to characterize the zero-Hopf equilibria of differential system (1) and to study analytically the limit cycles that can bifurcate from such equilibria, providing an analytic expression of them. For doing this we shall use the averaging theory of first order [3, 9, 13]. The next proposition establishes conditions for the existence of a zero-Hopf equilibrium point in the differential system (1).

Proposition 1. *Differential system (1) has the unique zero-Hopf equilibrium $(0, -a, 0)$ when $c = 0$.*

The following theorem is our main result.

Theorem 1. *Assume that in differential system (1) we have $c = c^*\varepsilon^2$, with $bc^* > 0$ and $c^*(b-1) > 0$. Then, using the averaging theory of first order for $\varepsilon \neq 0$ sufficiently small the differential system (1) has the following two limit cycles bifurcating from its zero-Hopf equilibrium*

$$\begin{aligned} (x_1(t, \varepsilon), y_1(t, \varepsilon), z_1(t, \varepsilon)) &= (-\varepsilon\sqrt{2c^*/b}\cos t, -a + \varepsilon\sqrt{2c^*/b}\sin t, 0) + \mathcal{O}(\varepsilon^2), \\ (x_2(t, \varepsilon), y_2(t, \varepsilon), z_2(t, \varepsilon)) &= \left(\varepsilon\sqrt{c^*/(b-1)}, -a, \varepsilon\sqrt{c^*/(b-1)}\right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Moreover the last limit cycle is unstable.

We prove Proposition 1 and Theorem 1 in Section 3. In what follows we provide an example that illustrates the results obtained in Theorem 1.

Example 1. The differential system

$$\dot{x} = y + 1, \quad \dot{y} = z - z, \quad \dot{z} = z^2 - 2x^2 + \varepsilon^2,$$

is obtained from differential system (1) taking $a = 1$, $b = 2$ and $c = \varepsilon^2$. Theorem 1 ensures the existence of the following two limit cycles

$$\begin{aligned} \gamma_1^*(t, \varepsilon) &= (-\varepsilon\cos t, \varepsilon\sin t - 1, 0) + \mathcal{O}(\varepsilon^2), \\ \gamma_2^*(t, \varepsilon) &= (\varepsilon, -1, \varepsilon) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

bifurcating from the zero-Hopf equilibrium $(0, -1, 0)$ when $\varepsilon = 0$. These two limit cycles are shown in Fig.1 for $\varepsilon = 0.1$.

2. PRELIMINARIES

In this section we recall a basic result of the averaging theory of first order that we shall use for proving our results. The following theorem provides periodic solutions of a periodic differential system. See Theorems 11.5 and 11.6 of [13] for a proof.

Consider the differential equation

$$(2) \quad \dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \quad x(0) = x_0,$$

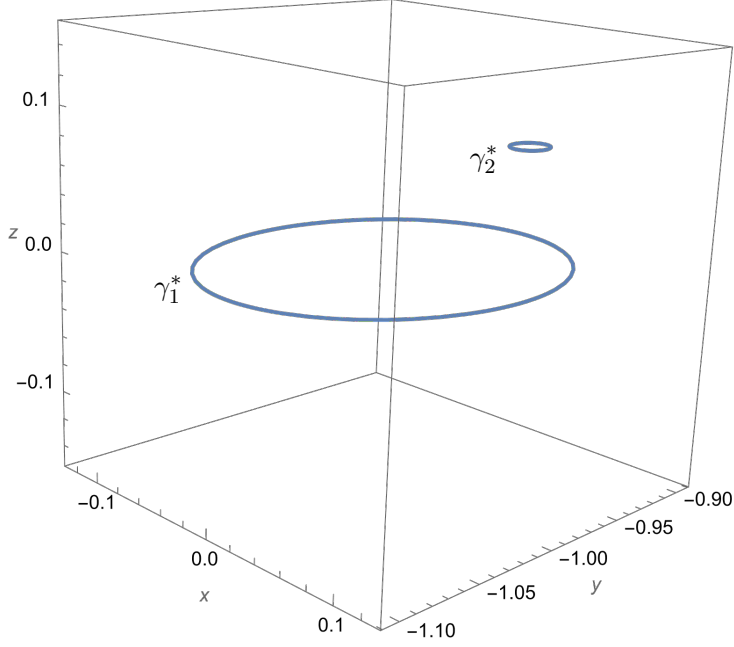


FIGURE 1. The two limit cycles of Example 1.

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, and $t \geq 0$. Moreover we assume that $F(t, x)$ and $R(t, x, \varepsilon)$ are T -periodic in t . Separately, we consider in D the averaged differential equation

$$(3) \quad \dot{y} = \varepsilon f(y), \quad y(0) = x_0,$$

where

$$f(y) = \frac{1}{T} \int_0^T F(t, y) dt.$$

Theorem 2. *Consider the two initial value problems (2) and (3). Suppose:*

- (i) *F , its Jacobian $\partial F/\partial x$, its Hessian $\partial^2 F/\partial x^2$, R and its Jacobian $\partial R/\partial x$ are defined, continuous and bounded by an independent constant of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.*
- (ii) *F and R are T -periodic in t (T independent of ε).*

Then the following statements hold.

- (a) *If p is an equilibrium point of the averaged equation (3) and*

$$\det \left(\frac{\partial f}{\partial y} \right) \Big|_{y=p} \neq 0,$$

then there exists a T -periodic solution $x(t, \varepsilon)$ of equation (2) such that $x(0, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

- (c) If the eigenvalues of the equilibrium point $y = p$ of the averaged equation (3) all have negative real parts, the corresponding periodic solution $x(t, \varepsilon)$ of equation (2) is asymptotically stable for ε sufficiently small. If one of the eigenvalues has positive real part, the periodic solution $x(t, \varepsilon)$ is unstable.

3. PROOF OF PROPOSITION 1 AND THEOREM 1

Proof of Proposition 1. If $c(b-1) \geq 0$ and $b \neq 1$ then differential system (1) has the two equilibrium points

$$(x, y, z) = \left(\pm \sqrt{c/(b-1)}, -a, \pm \sqrt{c/(b-1)} \right).$$

The Jacobian matrix of differential system (1) evaluated at these equilibria are

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ \mp 2b\sqrt{c/(b-1)} & 0 & \pm 2\sqrt{c/(b-1)} \end{pmatrix},$$

and its respective characteristic polynomials are

$$p_{\pm}(\lambda) = \mp 2\sqrt{c(b-1)} - \lambda \pm 2\sqrt{c/(b-1)}\lambda^2 - \lambda^3.$$

In order that these equilibria could be zero-Hopf equilibria it is necessary that $c = 0$. Then the Jacobian matrix of differential system (1) at its unique equilibrium point $(0, -a, 0)$ has eigenvalues 0 and $\pm i$. Hence this equilibrium is a zero-Hopf equilibrium. $\square$

Proof of Theorem 1. We shall study the possible limit cycles bifurcating from the zero-Hopf equilibrium $(0, -a, 0)$ of system (1). So take $c = c^*\varepsilon^2$ in differential system (1) being ε a sufficient small real parameter. Note that when $\varepsilon = 0$ system (1) has the zero-Hopf equilibrium $(0, -a, 0)$.

We first translate this equilibrium at the origin of coordinates when $\varepsilon = 0$ by taking the change of coordinates $x = \bar{x}$, $y = \bar{y} - a$ and $z = \bar{z}$. For simplicity we continue denoting the new coordinates $(\bar{x}, \bar{y}, \bar{z})$ by (x, y, z) . Differential system (1) in the new variables becomes

$$(4) \quad \dot{x} = y, \quad \dot{y} = -x + z, \quad \dot{z} = -bx^2 + z^2 + c^*\varepsilon^2.$$

Since we want to study the limit cycles bifurcating from the zero-Hopf equilibrium by using the averaging theory we do the rescaling $x = \varepsilon X$, $y = \varepsilon Y$ and $z = \varepsilon Z$. Thus in the new variables (X, Y, Z) the differential system (4) writes

$$(5) \quad X' = Y, \quad Y' = -X + Z, \quad Z' = \varepsilon(c^* - bX^2 + Z^2).$$

In order to write the differential system (5) in the normal form (2) for applying the averaging theory to it, first we write the linear part of system (5) with $\varepsilon = 0$ in its real Jordan normal form doing the change of coordinates $X = -u + w$, $Y = v$ and $Z = w$. So differential system (5) having its linear part in its real Jordan normal form when $\varepsilon = 0$ is

$$(6) \quad \begin{aligned} \dot{u} &= -v + \varepsilon(c^* - b(u - w)^2 + w^2), \\ \dot{v} &= u, \\ \dot{w} &= \varepsilon(c^* - b(u - w)^2 + w^2). \end{aligned}$$

Now we shall write the differential system (6) in the cylindrical coordinates (r, θ, w) . So taking $u = r \cos \theta$, $v = r \sin \theta$ and $w = w$, we obtain

$$(7) \quad \begin{aligned} \dot{r} &= \varepsilon \cos \theta (c^* + w^2 - b(w - r \cos \theta)^2), \\ \dot{\theta} &= 1 - \varepsilon \sin \theta (c^* + w^2 - b(w - r \cos \theta)^2) / r, \\ \dot{w} &= \varepsilon (c^* + w^2 - b(w - r \cos \theta)^2). \end{aligned}$$

Note that the differential system (7) is invariant under the symmetry

$$(8) \quad (r, \theta, w, t) \rightarrow (-r, -\theta, -w, -t).$$

Considering θ as the new independent variable or time of differential system (7), we get the following differential system

$$(9) \quad \begin{aligned} dr/d\theta &= \varepsilon \cos \theta (c^* + w^2 - b(w - r \cos \theta)^2) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon F_1(\theta, r, w) + \mathcal{O}(\varepsilon^2), \\ dw/d\theta &= \varepsilon (c^* + w^2 - b(w - r \cos \theta)^2) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon F_2(\theta, r, w) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Since the differential system (7) is invariant under the symmetry (8), the differential system (9) is invariant under the symmetry

$$(10) \quad (r, \theta, w) \rightarrow (-r, -\theta, -w).$$

The periodic differential system (9) satisfies the assumptions of Theorem 2, so we apply to it the averaging theory of first order. Computing the averaged function $f(r, w)$ we obtain

$$\begin{aligned} f(r, w) &= \frac{1}{2\pi} \int_0^{2\pi} (F_1(\theta, r, w), F_2(\theta, r, w)) d\theta \\ &= \left(brw, \frac{1}{2} (2c^* - br^2 - 2(b-1)w^2) \right). \end{aligned}$$

The zeros of $f(r, w)$ satisfying $r \geq 0$ are

$$(r_1, w_1) = \left(\sqrt{\frac{2c^*}{b}}, 0 \right), (r_2, w_2) = \left(0, \sqrt{\frac{c^*}{b-1}} \right), (r_3, w_3) = \left(0, -\sqrt{\frac{c^*}{b-1}} \right).$$

They are simple solutions in the sense that the Jacobian of $f(r, w)$ does not vanish at them. The Jacobian of f at (r_1, w_1) is $2bc^* > 0$ and it is equal

to $-2bc^*$ at (r_2, w_2) and (r_3, w_3) . So, from Theorem 2 each one of these three zeros must provide a limit cycle $(r_k(\theta, \varepsilon), w_k(\theta, \varepsilon))$ for $k = 1, 2, 3$ of differential system (9) such that

$$\begin{aligned}(r_1(0, \varepsilon), w_1(0, \varepsilon)) &= \left(\sqrt{\frac{2c^*}{b}}, 0 \right) + \mathcal{O}(\varepsilon), \\(r_2(0, \varepsilon), w_2(0, \varepsilon)) &= \left(0, \sqrt{\frac{c^*}{b-1}} \right) + \mathcal{O}(\varepsilon), \\(r_3(0, \varepsilon), w_3(0, \varepsilon)) &= \left(0, -\sqrt{\frac{c^*}{b-1}} \right) + \mathcal{O}(\varepsilon).\end{aligned}$$

But due to the symmetry (10) the limit cycles $(r_2(\theta, \varepsilon), w_2(\theta, \varepsilon))$ and $(r_3(\theta, \varepsilon), w_3(\theta, \varepsilon))$ are the same, because the symmetry (10) interchange the initial conditions $(r_2(0, \varepsilon), w_2(0, \varepsilon))$ and $(r_3(0, \varepsilon), w_3(0, \varepsilon))$.

Since the Jacobian of $f(r, w)$ at (r_2, w_2) is negative, an eigenvalue of the Jacobian matrix of $f(r, w)$ at (r_2, w_2) has the real part positive and therefore it follows from statement (c) of Theorem 2 that the limit cycle $(r_2(\theta, \varepsilon), w_2(\theta, \varepsilon))$ is unstable.

The eigenvalues of the Jacobian matrix of $f(r, w)$ at (r_1, w_1) are $\pm i\sqrt{2bc^*}$ and in this case the averaging theory of first order does not provide any information about the stability or instability of the limit cycle $(r_1(\theta, \varepsilon), w_1(\theta, \varepsilon))$.

In system (7) the limit cycle $(r_1(\theta, \varepsilon), w_1(\theta, \varepsilon))$ becomes

$$(r_1(t, \varepsilon), \theta_1(t, \varepsilon), w_1(t, \varepsilon)) = \left(\sqrt{2c^*/b}, t, 0 \right) + \mathcal{O}(\varepsilon).$$

This limit cycle in system (6) writes

$$(u_1(t, \varepsilon), v_1(t, \varepsilon), w_1(t, \varepsilon)) = \left(\sqrt{2c^*/b} \cos t, \sqrt{2c^*/b} \sin t, 0 \right) + \mathcal{O}(\varepsilon).$$

In system (5) this limit cycle becomes

$$(X_1(t, \varepsilon), Y_1(t, \varepsilon), Z_1(t, \varepsilon)) = \left(-\sqrt{2c^*/b} \cos t, \sqrt{2c^*/b} \sin t, 0 \right) + \mathcal{O}(\varepsilon).$$

In system (4) with the zero-Hopf equilibrium at the origin of coordinates this limit cycle writes

$$(x_1(t, \varepsilon), y_1(t, \varepsilon), z_1(t, \varepsilon)) = \left(-\varepsilon \sqrt{2c^*/b} \cos t, \varepsilon \sqrt{2c^*/b} \sin t, 0 \right) + \mathcal{O}(\varepsilon^2).$$

Finally, in system (1) with the zero-Hopf equilibrium at $(0, -a, 0)$ this limit cycle is

$$(x_1(t, \varepsilon), y_1(t, \varepsilon), z_1(t, \varepsilon)) = \left(-\varepsilon \sqrt{2c^*/b} \cos t, -a + \varepsilon \sqrt{2c^*/b} \sin t, 0 \right) + \mathcal{O}(\varepsilon^2).$$

This is the first limit cycle of the statement of Theorem 1. In a similar way, we can obtain the other limit cycle of Theorem 1. This completes the proof of Theorem 1. $\square$

ACKNOWLEDGMENTS

The first author is partially supported by the Agencia Estatal de Investigación of Spain grant PID2022-136613NB-I00, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Rieal Acadèmia de Ciències i Arts de Barcelona.

The second author is partially supported by PROFMAT/CAPES grant 848749/2017.

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