

GLOBAL DYNAMICS ANALYSIS OF A CLASS OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS WITH AN INVARIANT PLANE IN $\mathbb{R}^3$

ALI BAKHSHALIZADEH¹, JAUME LLIBRE², ALEX C. REZENDE³

ABSTRACT. In this paper we study a class of quadratic differential systems in $\mathbb{R}^3$. More precisely, for three distinct families of parameter sets, such differential systems exhibit the property of having an invariant plane. The first family exhibits a first integral in the form of $H(x, y) = ax + by$ ensures the invariance of every plane $ax + by = \text{constant}$. Consequently, we describe the phase portraits of the system on each of such planes in the Poincaré disc. The second family has a Darboux invariant of the form $I(x, y, t) = (d_0 + d_1x + d_2y)e^{-k_0t}$ which will allow us to describe the phase portraits on the Poincaré ball. Unlike the first two families, the third family lacks both a first integral and a Darboux invariant. Nevertheless, we present a detailed analysis of the phase portraits of these systems on the invariant plane using the Poincaré disc.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

This paper aims to investigate the global dynamics of polynomial differential systems in $\mathbb{R}^3$ defined by

$$\dot{x} = X(x, y, z), \quad \dot{y} = Y(x, y, z), \quad \dot{z} = Z(x, y, z), \quad (1)$$

where

$$X(x, y, z) = a_0 + a_1x + a_2y + a_3z,$$

$$Y(x, y, z) = b_0 + b_1x + b_2y + b_3z,$$

$$Z(x, y, z) = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2,$$

with x , y , and z representing real variables, and a_i , b_i , and c_j (for $i = 0, 1, 2, 3$ and $j = 0, 1, \dots, 9$) denoting real parameters.

Let $\mathbb{R}[x, y, z]$ be the ring of the real polynomials in the variables x , y , and z . A polynomial $f = f(x, y, z) \in \mathbb{R}[x, y, z]$ is called a *Darboux polynomial* for system (1) if it satisfies the following partial differential equation

$$X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z} = kf, \quad (2)$$

where $k = k(x, y, z)$ is a real polynomial of degree at most 1, called the *cofactor* of $f(x, y, z)$. If the cofactor is zero, then $f(x, y, z)$ is a *polynomial first integral* for system (1). When f is a Darboux polynomial, the algebraic surface defined by $f(x, y, z) = 0$ in $\mathbb{R}^3$ is an *invariant algebraic surface* of the differential system (1). In other words, if an orbit of system (1) intersects this surface at any point, the entire orbit is contained on it.

A polynomial vector field $\mathcal{X}$ in $\mathbb{R}^n$ can be uniquely extended to an analytic vector field on the sphere $\mathbb{S}^n$ using the Poincaré compactification technique. This extension enables the study of the polynomial vector field in the neighborhood of infinity, corresponding to the equator $\mathbb{S}^{n-1}$ of the sphere $\mathbb{S}^n$. In [7], Poincaré introduced the technique known as Poincaré compactification for studying the behavior of polynomial vector fields in $\mathbb{R}^2$. Its extension to $\mathbb{R}^n$, for $n > 2$, is provided in [5]. The Poincaré compactification technique for $\mathbb{R}^3$, as described in [5], is applied in [10]. Now, with this technique, we can determine the behavior of system (1) at infinity, as outlined in the following theorem.

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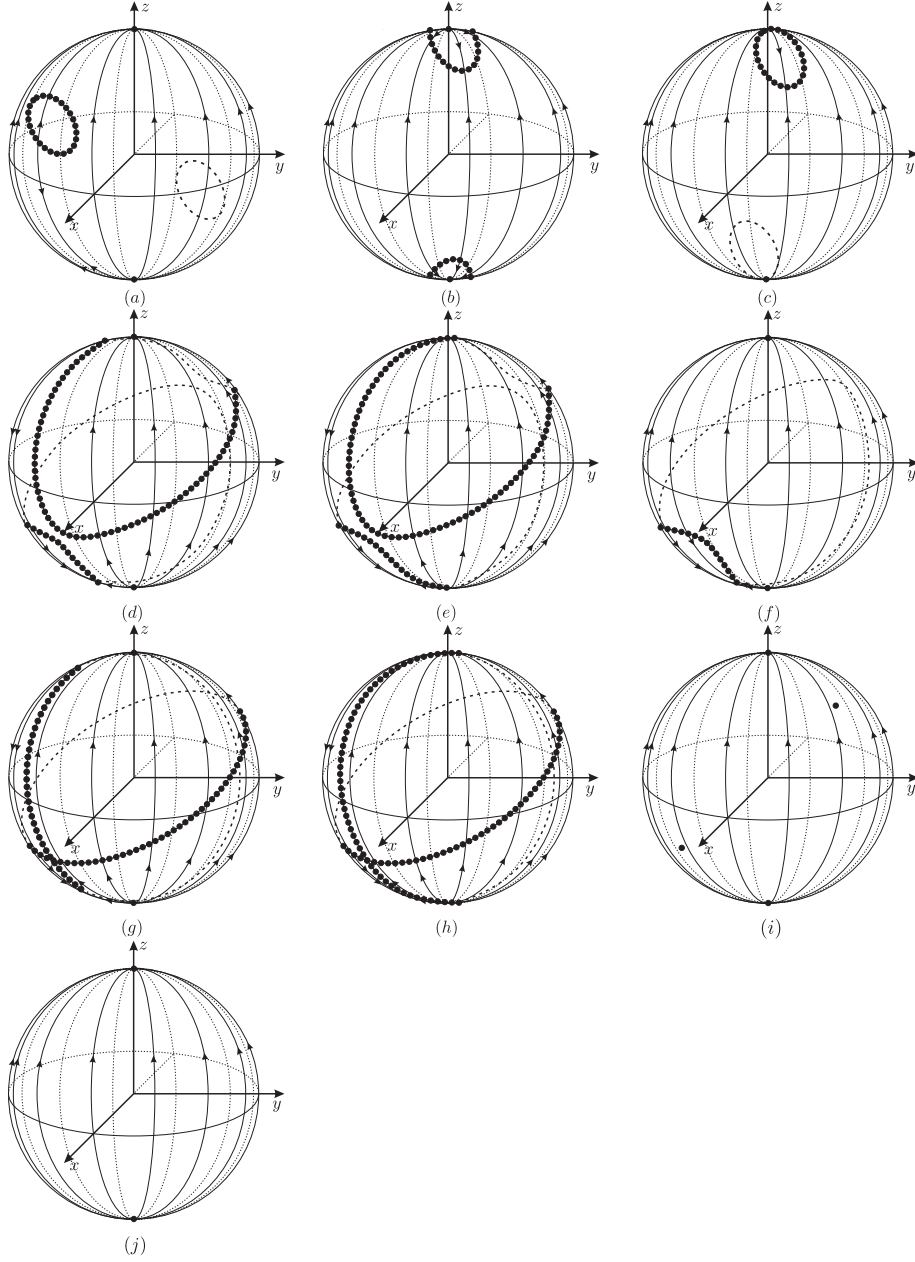


Figure 1. Phase portraits of quadratic system (1) at infinity.

Theorem 1. *Quadratic system (1) exhibits ten topological distinct phase portraits on the sphere $\mathbb{S}^2$ at the infinity, see Figure 1.*

Theorem 1 is proved in Section 2.

Our main focus is on systems that possess an invariant plane. Consequently, in the following theorem, we provide a classification of all differential systems (1) exhibiting an invariant plane.

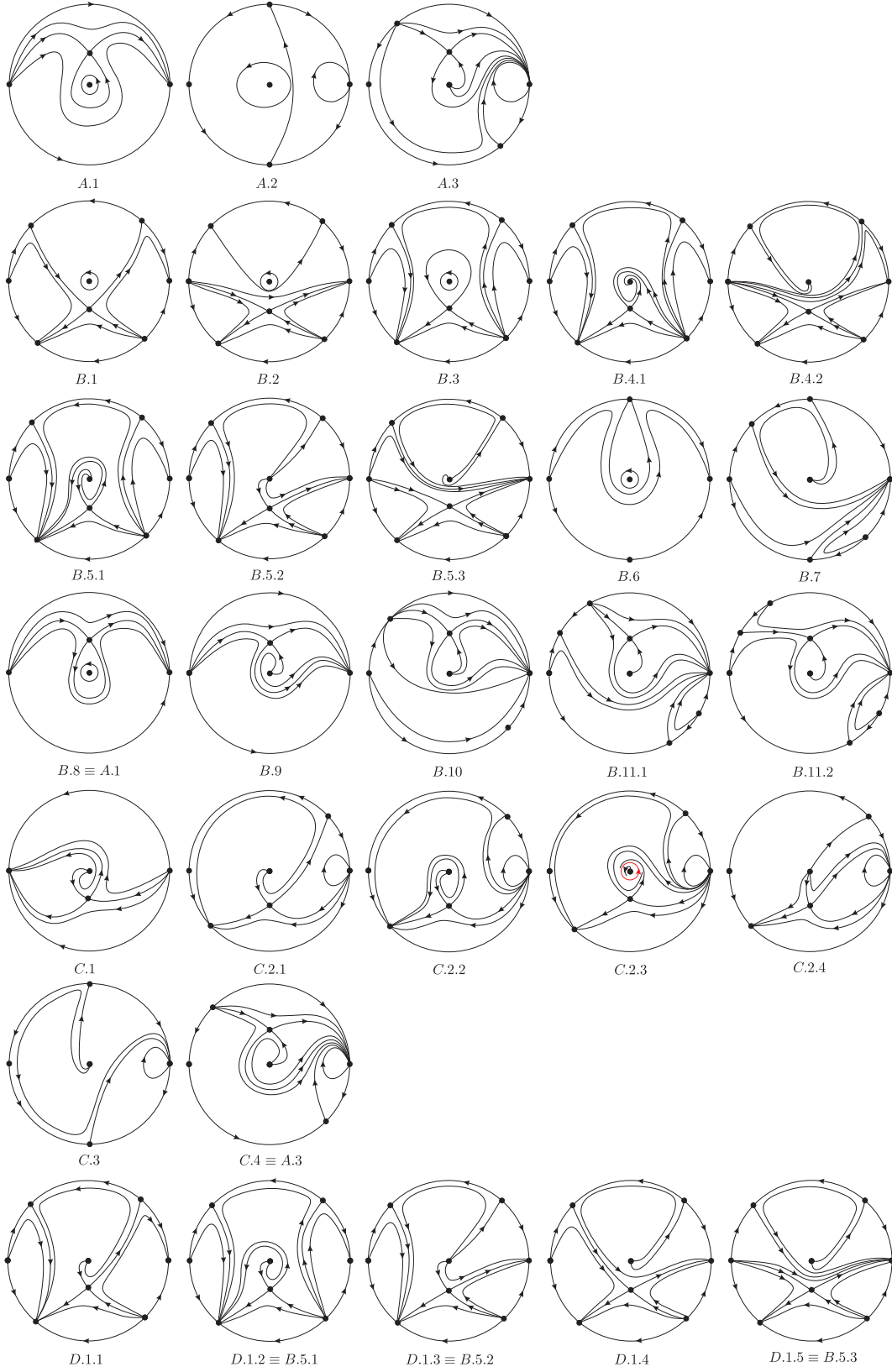


Figure 2. Global phase portraits of quadratic system (I) in Theorem 3 corresponding to the bifurcation diagram in Figure 26.

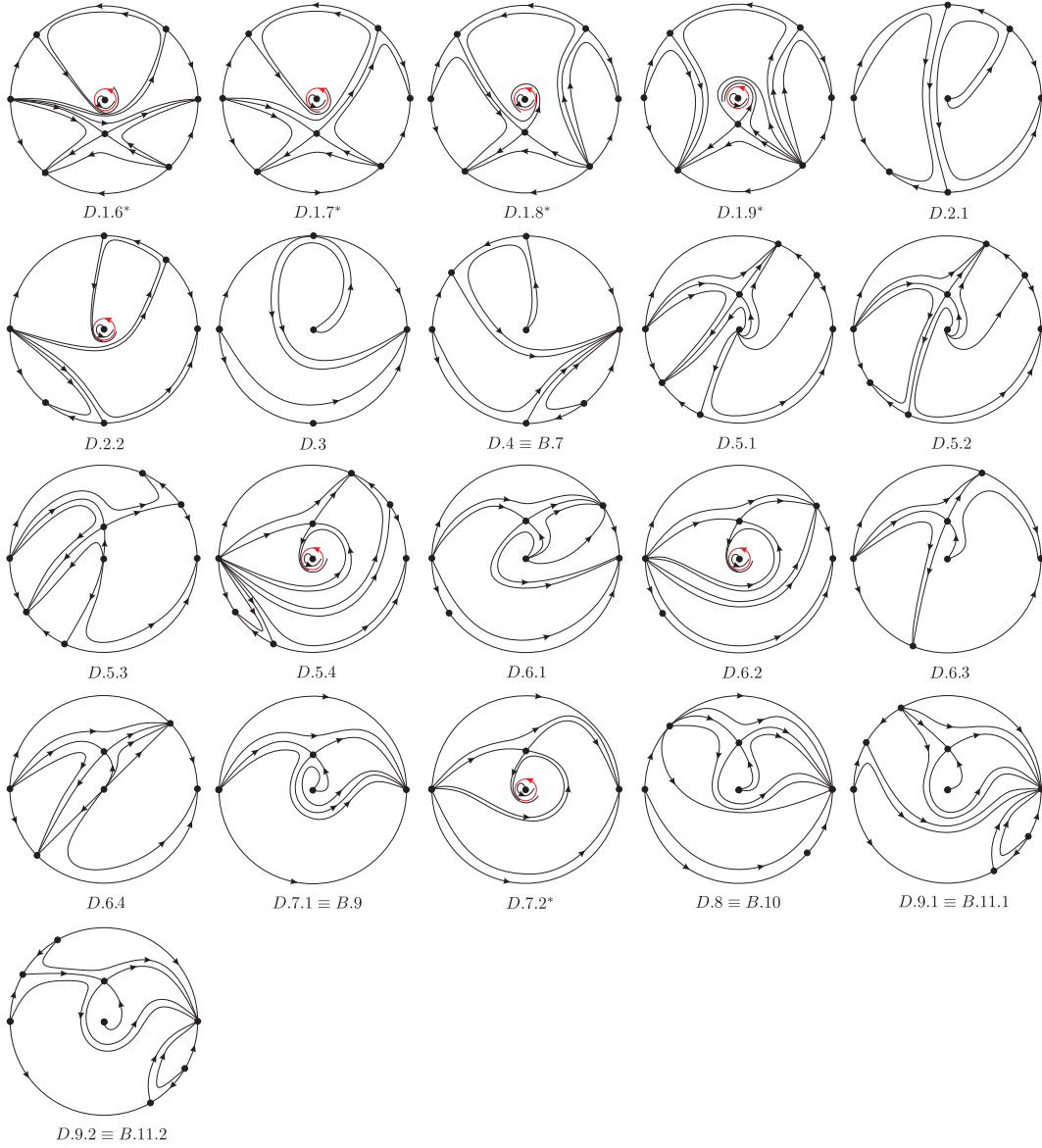


Figure 2. (Continued) Global phase portraits of quadratic system (I) in Theorem 3 corresponding to the bifurcation diagram in Figure 26. Symbols: “ $\equiv$ ” denotes C^0 -equivalence of two global phase portraits, and “*” indicates areas in the bifurcation diagram where a limit cycle can exist.

Theorem 2. Consider the differential system (1) possessing an invariant plane. Then, the following three cases hold:

(i) When

$$b_0 = -\frac{a_0 d_1}{d_2}, \quad b_1 = -\frac{a_1 d_1}{d_2}, \quad b_2 = -\frac{a_2 d_1}{d_2}, \quad b_3 = -\frac{a_3 d_1}{d_2}, \quad \text{with } d_2 \neq 0,$$

the invariant plane of system (1) is $d_0 + d_1 x + d_2 y = 0$ and its cofactor is zero. Consequently, in this case, system (1) possesses the first integral $H(x, y) = d_1 x + d_2 y$.

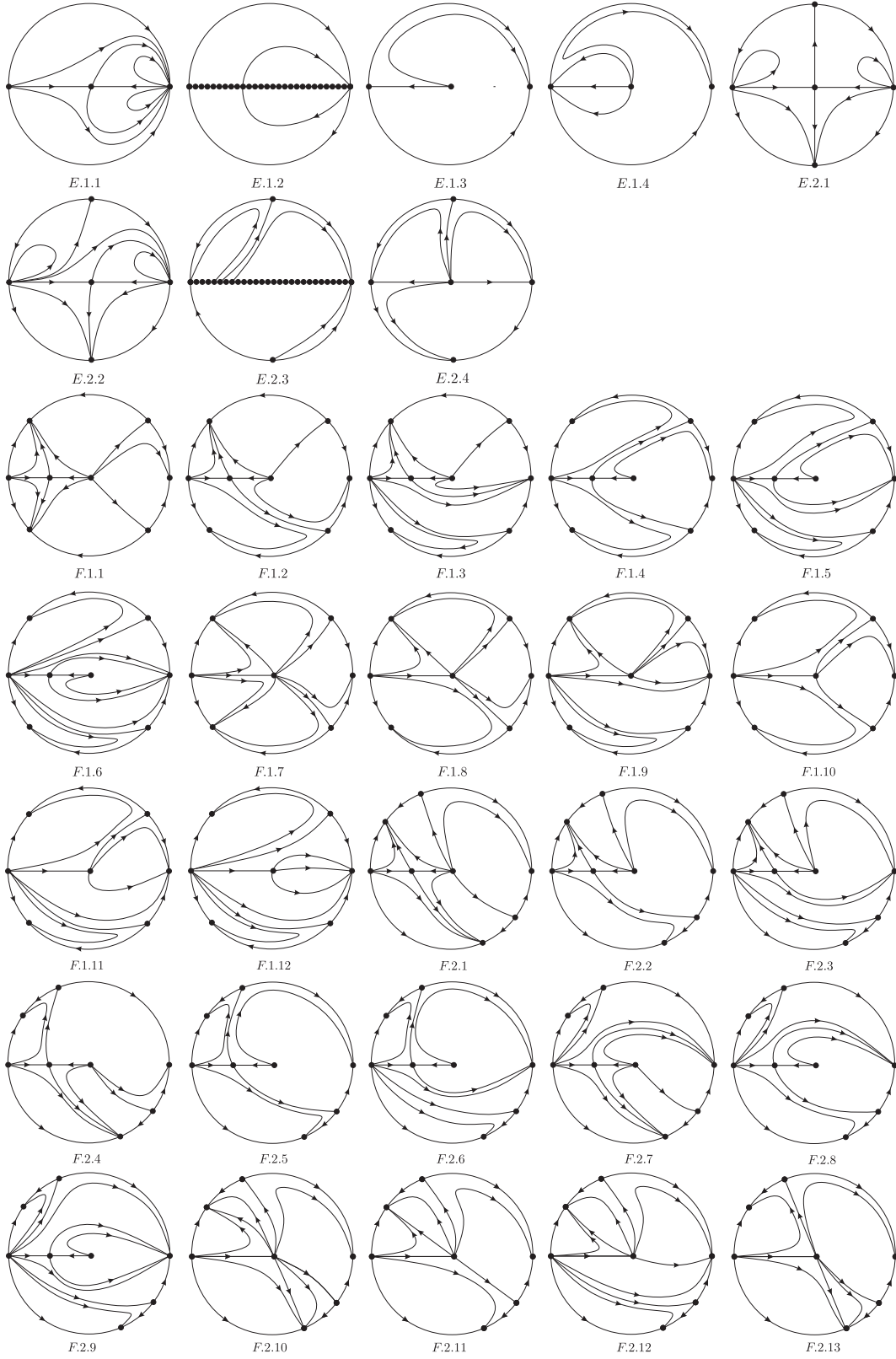


Figure 3. Global phase portraits of quadratic system (II) in Theorem 3.

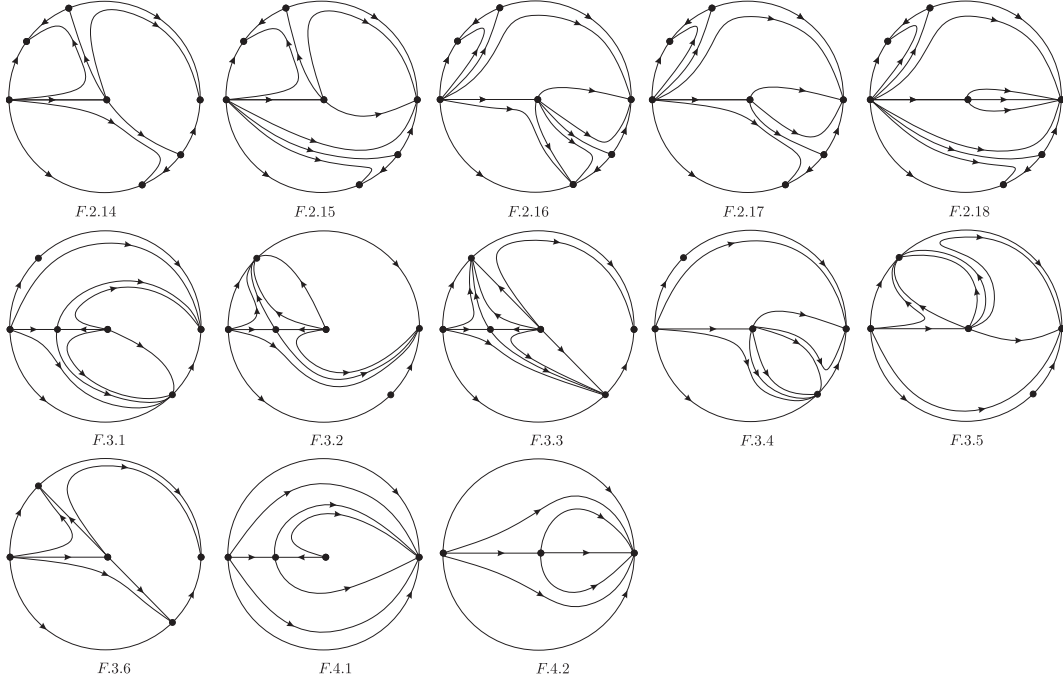


Figure 3. (Continued) Global phase portraits of quadratic system (II) in Theorem 3 corresponding to the regions: $E.1 = \{(b, c, d) \mid b = c = 0, d > 0\}$, $E.2 = \{(b, c, d) \mid b = 0, c > 0, d \geq 0\}$, $F.1 = \{(b, c, d) \mid b > 0, c \geq 0, d < 0\}$, $F.2 = \{(b, c, d) \mid b > 0, c > 0, d \geq 0, c^2 - 4bd > 0\}$, $F.3 = \{(b, c, d) \mid b > 0, c \geq 0, d \geq 0, c^2 - 4bd = 0\}$, $F.4 = \{(b, c, d) \mid b > 0, c \geq 0, d > 0, c^2 - 4bd < 0\}$.

(ii) When

$$b_0 = \frac{-a_0d_1 + d_0k_0}{d_2}, \quad b_1 = \frac{-a_1d_1 + d_1k_0}{d_2}, \quad b_2 = \frac{-a_2d_1 + d_2k_0}{d_2}, \quad b_3 = -\frac{a_3d_1}{d_2}, \quad \text{with } d_2 \neq 0,$$

the invariant plane of system (1) is $d_0 + d_1x + d_2y = 0$, where k_0 is its cofactor. Consequently, in this case, system (1) possesses the Darboux invariant $I(x, y, t) = (d_0 + d_1x + d_2y)e^{-k_0t}$.

(iii) When

$$\begin{aligned} c_0 &= -\frac{a_0d_1 + b_0d_2 - d_0k_0}{d_3}, \quad c_1 = -\frac{a_1d_1 + b_1d_2 - d_1k_0 - d_0k_1}{d_3}, \\ c_2 &= -\frac{a_2d_1 + b_2d_2 - d_2k_0 - d_0k_2}{d_3}, \quad c_3 = \frac{-a_3d_1 - b_3d_2 + d_3k_0 + d_0k_3}{d_3}, \quad c_4 = \frac{d_1k_1}{d_3}, \\ c_5 &= \frac{d_2k_1 + d_1k_2}{d_3}, \quad c_6 = \frac{d_3k_1 + d_1k_3}{d_3}, \quad c_7 = \frac{d_2k_2}{d_3}, \quad c_8 = \frac{d_3k_2 + d_2k_3}{d_3}, \quad c_9 = k_3, \quad \text{with } d_3 \neq 0, \end{aligned}$$

the invariant plane of system (1) is $d_0 + d_1x + d_2y + d_3z = 0$ with the cofactor $k_0 + k_1x + k_2y + k_3z$.

Theorem 2 is proved in Section 2.

Now, we describe the global phase portrait of system (1) with an invariant plane on the Poincaré disc and ball.

Theorem 3. After a change of variables, system (1) under condition (i) of Theorem 2 on the invariant plane $d_0 + d_1x + d_2y = 0$, can be represented in one of the following forms:

$$(I) \quad \dot{x} = -z + ax + bx^2 + cxz + dz^2, \quad \dot{z} = x,$$

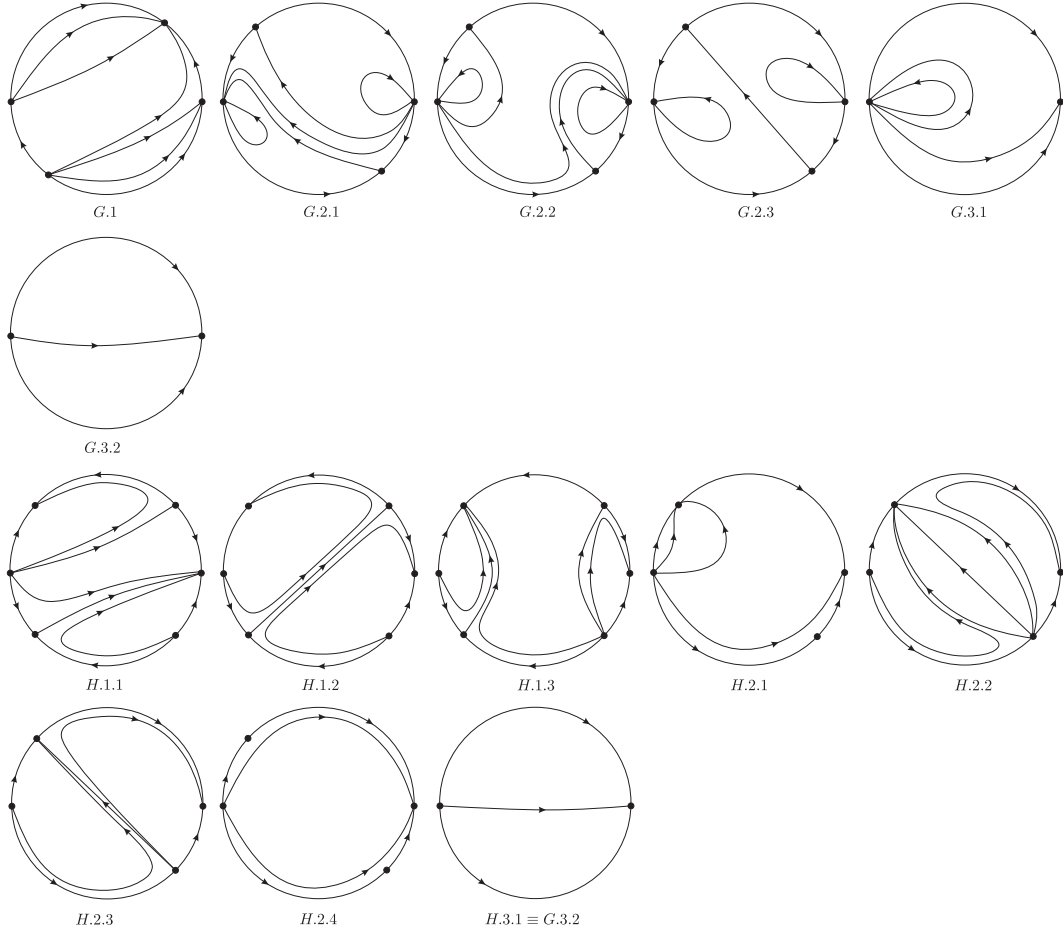


Figure 4. Global phase portraits of quadratic system (III) in Theorem 3 corresponding to the regions: $G.1 = \{(b, c, d) \mid b = 0, c < 0, d \geq 0\}$, $G.2 = \{(b, c, d) \mid b = 0, c > 0, d \geq 0\}$, $G.3 = \{(b, c, d) \mid b = c = 0, d > 0\}$, $H.1 = \{(b, c, d) \mid b > 0, c^2 - 4bd > 0\}$, $H.2 = \{(b, c, d) \mid b > 0, c^2 - 4bd = 0\}$, $H.3 = \{(b, c, d) \mid b > 0, c^2 - 4bd < 0\}$.

- (II) $\dot{x} = qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = z,$
 (III) $\dot{x} = p + qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = 1,$
 (IV) $\dot{x} = p + qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = 0.$

Then, the following statements hold:

- (A) The phase portrait of system (I) with $a = b = 0$ is topologically equivalent to one of the phase portraits shown in Figure 2 (A.1 – A.3).
 (B) The phase portrait of system (I) with $a = 0$ and $b > 0$ is topologically equivalent to one of the phase portraits shown in Figure 2 (B.1 – B.11.2).
 (C) The phase portrait of system (I) with $a > 0$ and $b = 0$ is topologically equivalent to one of the phase portraits shown in Figure 2 (C.1 – A.4).
 (D) The phase portrait of system (I) with $a, b > 0$ is topologically equivalent to one of the phase portraits shown in Figure 2 (D.1.1 – D.9.2).
 (E) The phase portrait of system (II) with $b = 0$ is topologically equivalent to one of the phase portraits shown in Figure 3 (E.1.1 – E.2.4).

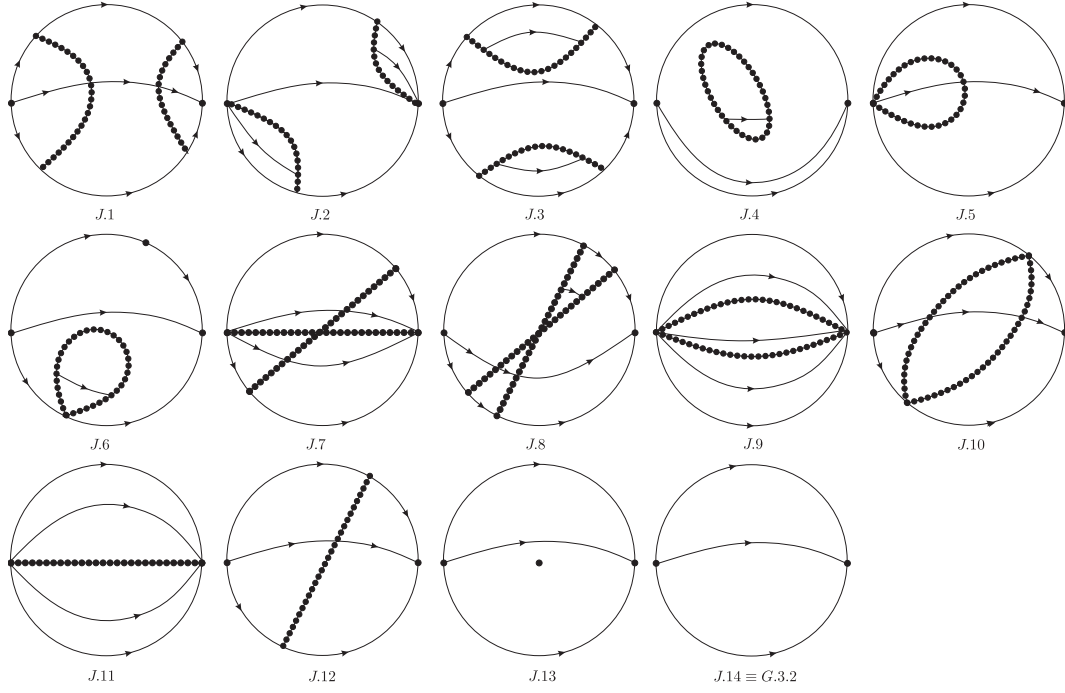


Figure 5. Global phase portraits of quadratic system (IV) in Theorem 3.

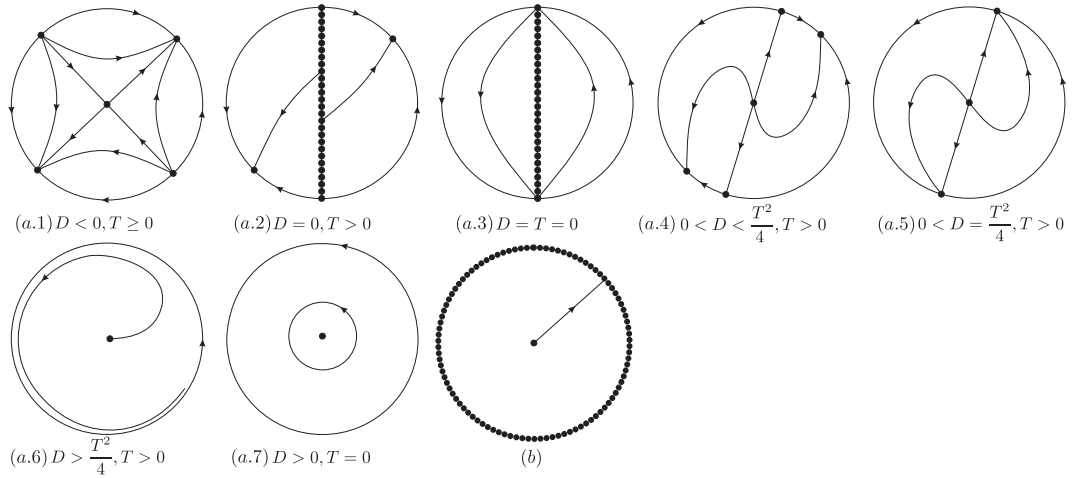


Figure 6. Global phase portraits of the linear systems (a) and (b) in Theorem 6.

- (F) The phase portrait of system (II) with $b > 0$ is topologically equivalent to one of the phase portraits shown in Figure 3 (F.1.1 – F.4.2).
- (G) The phase portrait of system (III) with $b = 0$ is topologically equivalent to one of the phase portraits shown in Figure 4 (G.1 – G.3.2).
- (H) The phase portrait of system (III) with $b > 0$ is topologically equivalent to one of the phase portraits shown in Figure 4 (H.1.1 – H.3.1).
- (J) The phase portrait of system (IV) is topologically equivalent to one of the phase portraits shown in Figure 5 (J.1 – J.14).

Corollary 4. *System (1) under condition (i) of Theorem 2 on the invariant plane $d_0 + d_1x + d_2y = 0$ has 113 topologically different phase portraits.*

Theorem 3 and Corollary 4 are proved in Section 3.

Theorem 5. *After a change of variables, system (1) under condition (ii) of Theorem 2, when restricted to the invariant plane $d_0 + d_1x + d_2y = 0$, can be transformed into one of the systems (I), (II), (III), or (IV) described in Theorem 3. The corresponding phase portraits are shown in Figure 2, leading to 113 distinct global phase portraits in the Poincaré ball. Any trajectory starting on the invariant plane will remain on it indefinitely. For a trajectory γ that does not start on the invariant plane:*

- *If $k_0 > 0$, the trajectory moves towards infinity as $t \rightarrow \infty$ and approaches to the invariant plane as $t \rightarrow -\infty$,*
- *If $k_0 < 0$, the trajectory approaches to the invariant plane as $t \rightarrow \infty$ and moves towards infinity as $t \rightarrow -\infty$,*

where all the possible phase portraits of quadratic system (1) at infinity are given in Figure 1.

Theorem 5 is proved in Section 4.

Theorem 6. *After a change of variables, system (1) under condition (iii) of Theorem 2 on the invariant plane $d_0 + d_1x + d_2y + d_3z = 0$ can be written in one of the following forms*

- (a) $\dot{x} = Tx - Dy, \quad \dot{y} = x,$
- (b) $\dot{x} = x, \quad \dot{y} = y,$
- (c) $\dot{x} = 0, \quad \dot{y} = 0,$

providing 9 topologically different phase portraits. The phase portraits of systems (a) and (b) are shown in Figure 6, while the phase portrait of system (c) is filled with equilibrium points.

Theorem 6 is proved in Section 5.

2. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. In this section we examine the local phase portrait of system (1) in a neighborhood of infinity. We shall use the Poincaré compactification technique discussed in [5, 10]. This involves studying the system in local charts U_i and V_i , where i takes values 1, 2, and 3. This approach allows us to realize the global dynamics of the solutions at infinity

The expression for $p(\mathcal{X})$ of the system in the local chart U_1 is

$$\begin{aligned} \dot{z}_1 &= z_3 (-a_2 z_1^2 - a_3 z_1 z_2 - a_0 z_1 z_3 - a_1 z_1 + b_2 z_1 + b_3 z_2 + b_0 z_3 + b_1), \\ \dot{z}_2 &= -a_2 z_1 z_2 z_3 - a_3 z_2^2 z_3 - a_0 z_2 z_3^2 + c_7 z_1^2 + c_8 z_1 z_2 + c_2 z_1 z_3 + c_9 z_2^2 - a_1 z_2 z_3 + c_3 z_2 z_3 + c_0 z_3^2 \\ &\quad + c_5 z_1 + c_6 z_2 + c_1 z_3 + c_4, \\ \dot{z}_3 &= -z_3^2 (a_2 z_1 + a_3 z_2 + a_0 z_3 + a_1). \end{aligned} \tag{3}$$

Taking $z_3 = 0$, corresponding to the points of the sphere $\mathbb{S}^2$ at infinity, the plane $z_1 z_2$ is invariant under the flow of system (3), therefore the analysis of system (3) reduces to

$$\dot{z}_1 = 0, \quad \dot{z}_2 = f_1(z_1, z_2), \tag{4}$$

where

$$f_1(z_1, z_2) = c_7 z_1^2 + c_8 z_1 z_2 + c_9 z_2^2 + c_5 z_1 + c_6 z_2 + c_4.$$

The phase portrait of this system consists of trajectories on straight lines parallel to z_2 -axis, accompanied by the conic defined by $f_1(z_1, z_2) = 0$, which is filled with equilibrium points. Additionally, the trajectories exhibit increasing behavior when $f_1(z_1, z_2) > 0$ and decreasing behavior when $f_1(z_1, z_2) < 0$.

The expression for $p(\mathcal{X})$ of the system in the local chart U_2 is

$$\begin{aligned} \dot{z}_1 &= z_3 (-b_1 z_1^2 - b_3 z_1 z_2 - b_0 z_1 z_3 + a_1 z_1 - b_2 z_1 + a_3 z_2 + a_0 z_3 + a_2), \\ \dot{z}_2 &= -b_1 z_1 z_2 z_3 - b_3 z_2^2 z_3 - b_0 z_2 z_3^2 + c_4 z_1^2 + c_6 z_1 z_2 + c_1 z_1 z_3 + c_9 z_2^2 - b_2 z_2 z_3 + c_3 z_2 z_3 + c_0 z_3^2 \\ &\quad + c_5 z_1 + c_8 z_2 + c_2 z_3 + c_7, \\ \dot{z}_3 &= -z_3^2 (b_1 z_1 + b_3 z_2 + b_0 z_3 + b_2). \end{aligned} \quad (5)$$

Considering $z_3 = 0$, system (5) is reduced to

$$\dot{z}_1 = 0, \quad \dot{z}_2 = f_2(z_1, z_2), \quad (6)$$

where

$$f_2(z_1, z_2) = c_4 z_1^2 + c_6 z_1 z_2 + c_9 z_2^2 + c_5 z_1 + c_8 z_2 + c_7.$$

Similarly the orbits of this chart are trajectories on straight lines parallel to z_2 -axis, accompanied by the conic defined by $f_2(z_1, z_2) = 0$. This conic is filled with equilibrium points, and the trajectory directions depend on the sign of the conic.

Finally, the expression of $p(\mathcal{X})$ of system in the local chart U_3 is

$$\begin{aligned} \dot{z}_1 &= -c_4 z_1^3 - c_5 z_1^2 z_2 - c_1 z_1^2 z_3 - c_7 z_1 z_2^2 - c_2 z_1 z_2 z_3 - c_0 z_1 z_3^2 - c_6 z_1^2 - c_8 z_1 z_2 + a_1 z_1 z_3 - c_3 z_1 z_3 + a_2 z_2 z_3 \\ &\quad + a_0 z_3^2 - c_9 z_1 + a_3 z_3, \\ \dot{z}_2 &= -c_7 z_2^3 - c_5 z_1 z_2^2 - c_2 z_2^2 z_3 - c_4 z_1^2 z_2 - c_1 z_1 z_2 z_3 - c_0 z_2 z_3^2 - c_8 z_2^2 - c_6 z_1 z_2 + b_2 z_2 z_3 - c_3 z_2 z_3 + b_1 z_1 z_3 \\ &\quad + b_0 z_3^2 - c_9 z_2 + b_3 z_3, \\ \dot{z}_3 &= -z_3 (c_4 z_1^2 + c_5 z_1 z_2 + c_1 z_1 z_3 + c_7 z_2^2 + c_2 z_2 z_3 + c_0 z_3^2 + c_6 z_1 + c_8 z_2 + c_3 z_3 + c_9). \end{aligned} \quad (7)$$

If $z_3 = 0$, then the analysis of system (7) can be simplified to study the dynamics of the following system

$$\dot{z}_1 = -z_1 f_3(z_1, z_2), \quad \dot{z}_2 = -z_2 f_3(z_1, z_2), \quad (8)$$

where

$$f_3(z_1, z_2) = c_4 z_1^2 + c_5 z_1 z_2 + c_7 z_2^2 + c_6 z_1 + c_8 z_2 + c_9.$$

By rescaling the time by $t \mapsto 1/f_3(z_1, z_2)\tau$, this differential system becomes

$$\dot{z}_1 = -z_1, \quad \dot{z}_2 = -z_2.$$

Consequently, the phase portrait of the system in the local chart U_3 resembles that of the star node, with the addition of the conic $f_3(z_1, z_2) = 0$ filled with equilibrium points. This star node is stable if $f_3(z_1, z_2) > 0$ and unstable if $f_3(z_1, z_2) < 0$.

The flow in the local charts V_i for $i = 1, 2, 3$ is the same as the flow in the local charts U_i reversing the time, because the compactified vector field $p(\mathcal{X})$ in V_i coincides with the vector field $p(\mathcal{X})$ in U_i multiplied by -1 . Therefore, system (1) exhibits an unstable star node if $f_3(z_1, z_2) > 0$ and a stable star node if $f_3(z_1, z_2) < 0$ on the infinite sphere at the endpoint of the negative z -semiaxis.

It is well known that the conics $f_i(z_1, z_2) = 0$, for $i = 1, 2, 3$, can be equivalently expressed in matrix form as

$$\mathbf{z} A^{f_i} \mathbf{z}^T = 0,$$

where

$$A^{f_1} = \begin{pmatrix} c_7 & \frac{c_8}{2} & \frac{c_5}{2} \\ \frac{c_8}{2} & c_9 & \frac{c_6}{2} \\ \frac{c_5}{2} & \frac{c_6}{2} & c_4 \end{pmatrix}, \quad A^{f_2} = \begin{pmatrix} c_4 & \frac{c_6}{2} & \frac{c_5}{2} \\ \frac{c_6}{2} & c_9 & \frac{c_8}{2} \\ \frac{c_5}{2} & \frac{c_8}{2} & c_7 \end{pmatrix}, \quad A^{f_3} = \begin{pmatrix} c_4 & \frac{c_5}{2} & \frac{c_6}{2} \\ \frac{c_5}{2} & c_7 & \frac{c_8}{2} \\ \frac{c_6}{2} & \frac{c_8}{2} & c_9 \end{pmatrix},$$

with $\mathbf{z} = (z_1 \ z_2 \ 1)$, see [3]. Given that $\det A^{f_1} = \det A^{f_2} = \det A^{f_3}$, the conics filled with equilibrium points in charts U_i , for $i = 1, 2, 3$, must either be degenerate or non-degenerate. However, due to the symmetry of the Poincaré compactification, the presence of only one conic filled with equilibrium points in each chart, and the compatibility in each chart simultaneously, we have only the following possibilities for having a conic in each chart U_i :

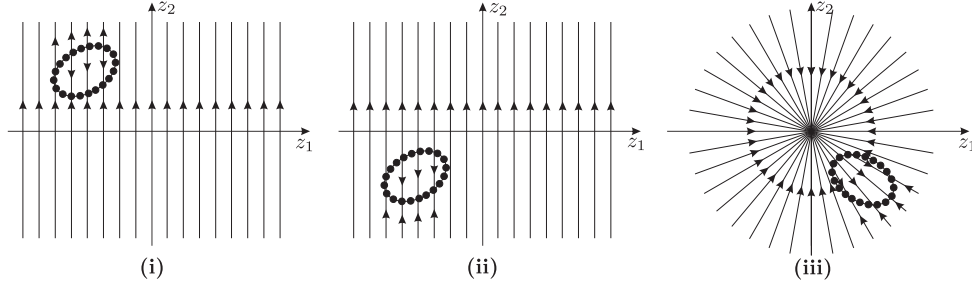


Figure 7. The phase portraits corresponding to systems (4), (6), and (8) under case (a) conditions are depicted in Figures (i), (ii), and (iii), respectively.

- (a) If either the conic $f_1(z_1, z_2) = 0$ in the local chart U_1 or the conic $f_2(z_1, z_2) = 0$ in the local chart U_2 , or both forms an ellipse, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(a);
- (b) If the conic $f_3(z_1, z_2) = 0$ with $c_9 < 0$ in the local chart U_3 forms an ellipse, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(b);
- (c) If the conic $f_3(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_3 forms an ellipse, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(c);
- (d) If there is no ellipse in any local chart, and either the conic $f_1(z_1, z_2) = 0$ with $c_9 \neq 0$ in the local chart U_1 or the conic $f_2(z_1, z_2) = 0$ with $c_9 \neq 0$ in the local chart U_2 , or both, form a hyperbola, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(d).
- (e) If there is no ellipse in any local chart, and if the conic $f_1(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_1 , the conic $f_2(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_2 , and/or the conic $f_3(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_3 forms a hyperbola, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(e);
- (f) If there is no ellipse and hyperbola in any local chart, and if either the conic $f_1(z_1, z_2) = 0$ in the local chart U_1 or the conic $f_2(z_1, z_2) = 0$ in the local chart U_2 , or both forms a parabola, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(f);
- (g) If either the conic $f_1(z_1, z_2) = 0$ with $c_9 \neq 0$ in the local chart U_1 or the conic $f_2(z_1, z_2) = 0$ with $c_9 \neq 0$ in the local chart U_2 , or both forms intersecting lines, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(g);
- (h) If the conic $f_1(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_1 , the conic $f_2(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_2 , and/or the conic $f_3(z_1, z_2) = 0$ with $c_9 = 0$ in the local chart U_3 forms a intersecting lines, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(h);
- (i) If there is no intersecting lines, parallel lines and line in any local chart, and if either the conic $f_1(z_1, z_2) = 0$ in the local chart U_1 , or the conic $f_2(z_1, z_2) = 0$ in the local chart U_2 forms a single point, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(i);
- (j) If the conic $f_1(z_1, z_2) = 0$ in the local chart U_1 , and the conic $f_2(z_1, z_2) = 0$ in the local chart U_2 have no real solution but the conic $f_3(z_1, z_2) = 0$ in the local chart U_3 forms a single point, then the phase portrait of system (1) at infinity is topologically equivalent to the phase portrait shown in Figure 1(j).

Here we describe the procedure used to determine this classification. For instance, in case (a), we assume that system (4) in the local chart U_1 exhibits an ellipse filled with equilibrium points located in the second quadrant of the $z_1 z_2$ coordinate plane, then, due to the symmetry of the Poincaré compactification and the

presence of precisely one conic filled with equilibrium points in each chart, it follows that system (6) in the local chart U_2 must possess an ellipse filled with equilibrium points located in the third quadrant of the $z_1 z_2$ coordinate plane. Similarly, the remaining cases can be obtained in the same way, (see Figure 7).

Remark 7. *Note that there exist certain cases where the compatibility in each chart is not possible simultaneously. For instance, if there is a parabola in each local chart U_i , for $i = 1, 2, 3$, it is impossible for the parabola in the local chart U_3 to pass through the north pole. This is because the system of equations $\det A^{f_i} \neq 0$ and $\det A_{33}^{f_i} = 0$ for $i = 1, 2, 3$, with $c_9 = 0$, has no real solution.*

In conclusion, the phase portrait of system (1) is established at infinity, then it follows the proof of Theorem 1. $\square$

Proof of Theorem 2. Assume that $f = d_0 + d_1x + d_2y + d_3z$ is an invariant plane of system (1), and let $k = k_0 + k_1x + k_2y + k_3z$ be its cofactor. By substituting f and k into (2), we obtain

$$\begin{aligned} & a_0d_1 + b_0d_2 + c_0d_3 - d_0k_0 + (a_1d_1 + b_1d_2 + c_1d_3 - d_1k_0 - d_0k_1)x + (c_4d_3 - d_1k_1)x^2 \\ & + (a_2d_1 + b_2d_2 + c_2d_3 - d_2k_0 - d_0k_2)y + (c_5d_3 - d_2k_1 - d_1k_2)xy + (c_7d_3 - d_2k_2)y^2 \\ & + (a_3d_1 + b_3d_2 + c_3d_3 - d_3k_0 - d_0k_3)z + (c_6d_3 - d_3k_1 - d_1k_3)xz \\ & + (c_8d_3 - d_3k_2 - d_2k_3)yz + d_3(c_9 - k_3)z^2 = 0. \end{aligned}$$

To determine the conditions under which this equation holds, we set each coefficient of the monomials involving x , y , and z to zero individually. Considering that the coefficient of the monomial z^2 is $d_3(c_9 - k_3)$, we conclude that either $d_3 = 0$ or $c_9 = k_3$. When $d_3 = 0$, and recognizing that d_1 and d_2 cannot be zero simultaneously to maintain the invariant plane, we deduce that $k_1 = k_2 = k_3 = 0$. Therefore, we only need to solve the following system of equations

$$a_0d_1 + b_0d_2 - d_0k_0 = 0, \quad a_3d_1 + b_3d_2 = 0, \quad a_2d_1 + b_2d_2 - d_2k_0 = 0, \quad a_1d_1 + b_1d_2 - d_1k_0 = 0.$$

When $k_0 = 0$, we have the solutions given in Theorem 2 (i) with cofactor 0, which indicates that system (1) has a first integral. For $k_0 \neq 0$, we have the solutions given in Theorem 2 (ii). Since the cofactor is constant, this system has the Darboux invariant $(d_0 + d_1x + d_2y)e^{-k_0t}$, see Theorem 8.7(vi) of [6].

When $c_9 = k_3$, it is straightforward to find the solution in which all coefficients are equal to zero, so it gives the solutions given in Theorem 2 (iii). This completes the proof of Theorem 2. $\square$

3. PROOF OF THEOREM 3 AND COROLLARY 4

Now the differential system has a first integral of the form $H(x, y) = d_1x + d_2y$. So every plane $d_1x + d_2y = \text{constant}$ is invariant, and we can describe on each of such planes what is the phase portrait of the system in the Poincaré disc. System (1) restricted to the plane $d_1x + d_2y = \text{constant}$, takes the form

$$\begin{aligned} \dot{x} &= d_2((-a_2d_0 + a_0d_2) + (-a_2d_1 + a_1d_2)x + a_3d_2z), \\ \dot{z} &= c_7d_0^2 - c_2d_0d_2 + c_0d_2^2 + (2c_7d_0d_1 - c_5d_0d_2 - c_2d_1d_2 + c_1d_2^2)x + (-c_8d_0d_2 + c_3d_2^2)z \\ &+ (c_7d_1^2 - c_5d_1d_2 + c_4d_2^2)x^2 + (-c_8d_1d_2 + c_6d_2^2)zx + c_9d_2^2z^2. \end{aligned} \quad (9)$$

Considering that the coefficients in the above system are linearly independent, and upon interchanging the roles of x and z in (9), it can be represented as follows

$$\begin{aligned} \dot{x} &= n_0 + n_1x + n_2z + n_3x^2 + n_4xz + n_5z^2, \\ \dot{z} &= m_0 + m_1x + m_2z. \end{aligned} \quad (10)$$

By applying suitable transformations and manipulations, system (10) can be classified into four distinct classes:

- When $m_1 \neq 0$,

$$\dot{x} = -z + ax + bx^2 + cxz + dz^2, \quad \dot{z} = x, \quad (11)$$

- When $m_1 = 0, m_2 \neq 0$,

$$\dot{x} = qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = z, \quad (12)$$

- When $m_1 = m_2 = 0, m_0 \neq 0$,

$$\dot{x} = p + qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = 1, \quad (13)$$

- When $m_1 = m_2 = m_3 = 0$,

$$\dot{x} = p + qz + ax + bx^2 + cxz + dz^2, \quad \dot{z} = 0. \quad (14)$$

These systems correspond to the differential systems labeled as (I), (II), (III), and (IV) in Theorem 3.

We observe the following invariances under specific transformations:

- System (11) is invariant under transformations $(x, z, t, a, b, c, d) \rightarrow (-x, -z, t, a, -b, -c, -d)$, and $(x, z, t, a, b, c, d) \rightarrow (-x, z, -t, -a, b, -c, d)$, which imply that it is sufficient to consider only $a, b \geq 0$.
- System (12) is invariant under transformation $(x, z, t, q, a, b, c, d) \rightarrow (-x, z, t, -q, a, -b, c, -d)$ and $(x, z, t, q, a, b, c, d) \rightarrow (x, -z, t, -q, a, b, -c, d)$, which imply that we only need to consider $b, c \geq 0$.
- System (13) is invariant under transformation $(x, z, t, p, q, a, b, c, d) \rightarrow (-x, z, t, -p, -q, a, -b, c, -d)$ and $(x, z, t, p, q, a, b, c, d) \rightarrow (x, -z, -t, -p, q, -a, -b, c, -d)$, which follows that we only need to consider $a, b \geq 0$.

Remark 8. System (11) also displays additional invariances under certain specific conditions, which we outline as follows:

- When $a = b = 0$, we can see that system (11) is invariant under transformation $(x, z, t, c, d) \rightarrow (-x, z, -t, -c, d)$, and $(x, z, t, c, d) \rightarrow (x, -z, -t, c, -d)$, which imply that it is sufficient to consider only $c, d \geq 0$.
- When $a = 0$ and $b > 0$, system (11) is invariant under transformation $(x, z, t, b, c, d) \rightarrow (-x, z, -t, b, -c, d)$, which follows that we only need to consider $c \geq 0$.
- When $a > 0$ and $b = 0$, system (11) is invariant under transformation $(x, z, t, a, c, d) \rightarrow (-x, -z, t, a, -c, -d)$, which follows that we only need to consider $c \geq 0$. Also, when $a > 0$ and $b = c = 0$, system (11) is invariant under transformation $(x, z, t, a, d) \rightarrow (-x, -z, t, a, -d)$, which follows that we only need to consider $d \geq 0$.

Remark 9. When $b = 0$ and $c \geq 0$, system (12) also exhibits additional invariance under specific transformation $(x, z, t, q, a, c, d) \rightarrow (-x, z, t, -q, a, c, -d)$, which follows that we only need to consider $d \geq 0$.

Remark 10. When $b = 0$ and $a \geq 0$, system (13) also displays additional invariance under specific transformation $(x, z, t, p, q, a, c, d) \rightarrow (-x, z, t, -p, -q, a, c, -d)$, which follows that we only need to consider $d \geq 0$.

3.1. Infinite equilibrium points of systems (11), (12), and (13). Now we aim to investigate the infinite singular points of the first three systems. Thus, we consider the following system

$$\begin{aligned} \dot{x} &= p + qz + ax + bx^2 + cxz + dz^2, \\ \dot{z} &= e + fz + gx, \end{aligned} \quad (15)$$

in the local chart U_1 (see [2, 6]). Note that in system (15), when $p = e = f = 0, q = -1$, and $g = 1$, it corresponds to (11). Similarly, when $p = e = g = 0$ and $f = 1$, it corresponds to (12). Finally, when $f = g = 0$ and $e = 1$, it corresponds to (13).

Let $\mathcal{X}$ be the vector field associated to system (15). Then the expression for $p(\mathcal{X})$ in the local chart U_1 is

$$\begin{aligned} \dot{u} &= v(g - u(a - f + qu)) - u(b + u(c + du)) + v^2(e - pu), \\ \dot{v} &= -v(v(a + pv + qu) + b + cu + du^2). \end{aligned} \quad (16)$$

To compute the equilibrium points of system (16) on $v = 0$, we need to find the solutions of the equation $du^2 + cu + b = 0$. Recall that its discriminant is given by $\Delta = c^2 - 4bd$. With this in mind, we present the following proposition:

Proposition 11. *Consider system (16), where the parameter b is non-negative, under the following three conditions:*

- $p = e = f = 0$, $q = -1$, $g = 1$, and $a \geq 0$,
- $p = e = g = 0$, $f = 1$, and $c \geq 0$,
- $f = g = 0$, $e = 1$, and $a \geq 0$.

Then the following statements regarding the infinite equilibrium points of systems (11), (12), and (13) in the local chart U_1 hold:

- (i) If $\Delta > 0$, $b > 0$, and $d \neq 0$, then system (15) has three infinite equilibrium points: the origin, which is a stable node, and the points

$$p_1 = \left(\frac{-c - \sqrt{\Delta}}{2d}, 0 \right), \quad p_2 = \left(\frac{-c + \sqrt{\Delta}}{2d}, 0 \right),$$

which are saddle-nodes if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$. If $f = g = 0$ and $e = 1$, the point p_1 is a saddle, and the point p_2 is a stable node if $2b + (c(-c + \sqrt{\Delta}))/2d < 0$ and an unstable node if $2b + (c(-c + \sqrt{\Delta}))/2d > 0$.

- (ii) If $\Delta > 0$, $b = 0$, and $d \neq 0$, then system (15) has two infinite equilibrium points: the origin has a local phase portrait consisting of one hyperbolic and one elliptic sector if $g = 1$ (see Figure 8), and a different phase portrait for $g = 0$ as shown in Figures 9 and 10. The point $(-c/d, 0)$ is a saddle-node if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$. If $f = g = 0$, $e = 1$, and $d > 0$, the point $(-c/d, 0)$ is a saddle if $c > 0$, and a stable node if $c < 0$.
- (iii) If $\Delta > 0$, $b > 0$, and $d = 0$, then system (15) has two infinite equilibrium points: the origin which is a stable node, and $(-b/c, 0)$ which is a saddle-node if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$. If $f = g = 0$ and $e = 1$, the point $(-b/c, 0)$ is a saddle if $c < 0$, and an unstable node if $c > 0$.
- (iv) If $\Delta > 0$ and $b = d = 0$, then system (15) has an infinite equilibrium point at the origin whose local phase portrait consists of one hyperbolic and one elliptic sector if $g = 1$, and its local phase portrait for $g = 0$ is given in Figures 9 and 10.
- (v) If $\Delta = 0$ and $c \neq 0$, then system (15) has two equilibria: the origin and another point at $(-(2b)/c, 0)$. The origin is a stable node. For the point $(-(2b)/c, 0)$, we have the following cases:
- When $a \neq f - (cg)/(2b) + (2bq)/c$, it is a saddle-node.
 - When $a = f - (cg)/(2b) + (2bq)/c$:
 - * The local phase portrait is shown in Figure 11 if $p = e = f = 0$, $q = -1$, $g = 1$, and $a > 0$.
 - * The local phase portrait is shown in Figure 12 if $p = e = g = 0$, $f = 1$, and $c > 0$.
 - * The local phase portrait is shown in Figures 13, 14, 16, 17, 18 if $f = g = 0$ and $e = 1$.
- (vi) If $\Delta = 0$ and $c = b = 0$, then system (15) has an infinite equilibrium point at the origin. If $g = 1$, it is a stable node if $d > 0$ and an unstable node if $d < 0$, and its local phase portrait for $g = 0$ is given in Figures 19 and 21.
- (vii) If $\Delta = 0$ and $c = d = 0$, then system (15) has an infinite equilibrium point at the origin which is a stable node.
- (viii) If $\Delta < 0$, then system (15) has an infinite equilibrium point at the origin which is a stable node.

Proof. First we assume that $\Delta > 0$, and then we investigate four distinct cases as follows:

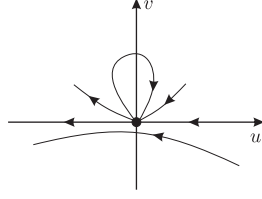


Figure 8. Local phase portrait at the origin of U_1 when $\Delta > 0$, $b = 0$, $d \neq 0$, and $g = 1$.

(i) When $b > 0$ and $d \neq 0$, system (16) has three distinct equilibrium points on $v = 0$, namely

$$p_0 = (0, 0), \quad p_1 = \left(\frac{-c - \sqrt{\Delta}}{2d}, 0 \right), \quad p_2 = \left(\frac{-c + \sqrt{\Delta}}{2d}, 0 \right).$$

The determinant and trace of the linear part of system (16) at the origin are b^2 and $-2b$, respectively. It shows that the origin is a stable node. The determinants of the linear part of system (16) at both points p_1 and p_2 are zero, and the traces at these points are given by

$$\text{tr}(p_1) = 2b + \frac{c(-c - \sqrt{\Delta})}{2d}, \quad \text{tr}(p_2) = 2b + \frac{c(-c + \sqrt{\Delta})}{2d}.$$

Since the trace of these points is non-zero, they are semi-hyperbolic. According to Theorem 2.19 in [6], if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$, these points are saddle-nodes. If $f = g = 0$ and $e = 1$, the point p_1 is a saddle, and the point p_2 is a stable node if $\text{tr}(p_2) < 0$ and an unstable node if $\text{tr}(p_2) > 0$.

(ii) When $b = 0$ and $d \neq 0$, system (16) exhibits two distinct equilibria on $v = 0$: one at the origin and the other at $(-c/d, 0)$. If $g = 1$, both eigenvalues of the Jacobian matrix at the origin are zero but the matrix itself is not linearly zero, indicating that it is a nilpotent point. By applying Theorem 3.5 from [6], we can determine that the local phase portrait of the origin consists of one hyperbolic and one elliptic sector (see Figure 8). If $g = 0$, the Jacobian matrix at the origin is linearly zero, and we need to do blow-ups to understand the local behavior at this point. We perform the u -directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and we obtain

$$\begin{aligned} \dot{u} &= -u^2 (aw + c + du - ew^2 - fw + puw^2 + quw), \\ \dot{w} &= -uw^2(ew + f). \end{aligned}$$

Using a reparametrization of time we eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and we get

$$\begin{aligned} \dot{u} &= -u (aw + c + du - ew^2 - fw + puw^2 + quw), \\ \dot{w} &= -w^2(ew + f). \end{aligned} \tag{17}$$

When either $f = 1$ and $e = 0$ or $f = 0$ and $e = 1$, system (17) has a unique equilibrium point at the origin on $u = 0$. The determinant and trace of the Jacobian matrix at the origin is zero and $-c$, respectively, so it is a semi-hyperbolic point. Using Theorem 2.19 of [6], it is a saddle-node if $f = 1$ and $e = 0$, a stable node if $f = 0$, $e = 1$ and $c > 0$, and a saddle if $f = 0$, $e = 1$ and $c < 0$. Going back through the blow-ups we obtain the local phase portrait at the origin of U_1 shown in Figure 9 and 10. For details on the blow-up technique, see [1, 6]. The determinant and trace of the Jacobian matrix at the point $(-c/d, 0)$ are zero and $-c^2/d$, respectively. Therefore, it is a semi-hyperbolic point. By applying Theorem 2.19 of [6], if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$, it is a saddle-node. When $f = g = 0$, $e = 1$, and $d > 0$, it is a saddle if $c > 0$, and a stable node if $c < 0$.

(iii) When $b > 0$, and $d = 0$, system (16) has two equilibria: one at the origin and the other at $(-b/c, 0)$ when $v = 0$. The origin is a stable node, and the determinant and trace of the Jacobian matrix at the point $(-b/c, 0)$ are zero and b , respectively. This indicates that it is semi-hyperbolic. According to Theorem 2.19

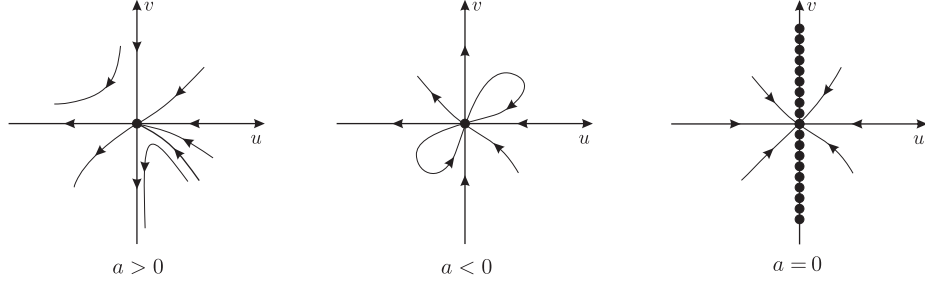


Figure 9. Local phase portrait at the origin of U_1 when $\Delta > 0$, $c > 0$, $b = g = e = p = 0$, and $f = 1$.

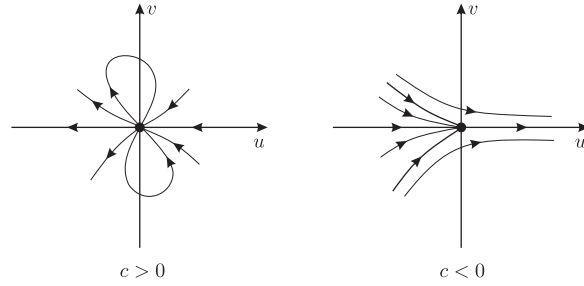


Figure 10. Local phase portrait at the origin of U_1 when $\Delta > 0$, $b = g = f = 0$, and $e = 1$.

of [6], if either $f = 1$ and $g = 0$, or $f = 0$ and $g = 1$, this point is a saddle-node. If $f = g = 0$ and $e = 1$, it is a saddle if $c < 0$, and an unstable node if $c > 0$.

(iv) When $b = d = 0$, the origin is the unique equilibrium point of system (16) on $v = 0$. If $g = 1$, then both the determinant and trace of the Jacobian matrix at this point are zero, but the matrix itself is not identically zero. Furthermore, using Theorem 3.5 from [6], the local phase portrait of the origin is formed by one hyperbolic and one elliptic sector. If $g = 0$, the Jacobian matrix at the origin is identically zero, indicating a need for blow-ups. The blow-up in this case follows the same way as case (ii) where $d = 0$. Therefore, the local phase portrait at the origin of U_1 remains unchanged from case (ii) for $g = 0$, as shown in Figures 9 and 10.

Now we consider the condition $\Delta = 0$, which leads us to investigate three distinct cases as follows:

(v) When $c \neq 0$, system (16) has two equilibria: one at the origin and another at $(-(2b)/c, 0)$ on the line $v = 0$. The origin is a stable node. The Jacobian matrix at $(-(2b)/c, 0)$ is given by

$$J = \begin{bmatrix} 0 & g - \frac{2b(-ac + 2bq + cf)}{c^2} \\ 0 & 0 \end{bmatrix}.$$

This matrix is nilpotent if $a \neq f - (cg)/(2b) + (2bq)/c$. According to Theorem 3.5 from [6], it is a saddle-node point.

If $a = f - (cg)/(2b) + (2bq)/c$, then the Jacobian matrix at the point $(-(2b)/c, 0)$ is linearly zero, so we need to do blow-ups. First, we translate this equilibrium point to the origin. Then, we perform the

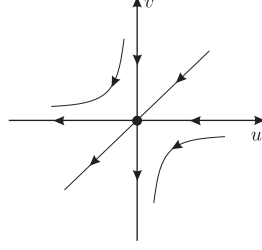


Figure 11. Local phase portrait the point $(-(2b)/c, 0)$ of U_1 when $\Delta = 0$, $c < 0$, $a = -(c^2 + 4b^2)/(2bc)$, $p = e = f = 0$, $q = -1$, and $g = 1$.

u -directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and we get

$$\begin{aligned}\dot{u} &= \frac{1}{4bc}u^2 (8b^2pw^2 + 8b^2qw + 2bc^2 + 4bcew^2 - 4bcpuw^2 - 4bcquw + c^3(-u) + 2c^2gw), \\ \dot{w} &= -\frac{1}{2c}uw (4bpw^2 + 4bqw + c^2 + 2cew^2 + 2cfw).\end{aligned}$$

Using a reparametrization of time we eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and we get

$$\begin{aligned}\dot{u} &= \frac{1}{4bc}u (8b^2pw^2 + 8b^2qw + 2bc^2 + 4bcew^2 - 4bcpuw^2 - 4bcquw + c^3(-u) + 2c^2gw), \\ \dot{w} &= -\frac{1}{2c}w (4bpw^2 + 4bqw + c^2 + 2cew^2 + 2cfw).\end{aligned}\tag{18}$$

Now we consider three different cases as follows:

* If $p = e = f = 0$, $q = -1$, $g = 1$, and $a = -(c^2 + 4b^2)/(2bc) > 0$, system (18) has two equilibrium points on $u = 0$: one at the origin and at $(0, c^2/(4b))$. The determinant of the Jacobian matrix at the origin is $-c^2/4$, indicating it is a saddle point. The determinant and trace of the Jacobian matrix at the point $(0, c^2/(4b))$ are $c^4/(16b^2)$ and $(c/8)(4 + c^2/b^2)$, respectively. So this point is a stable node. Going back through the blow-ups we get the local phase portrait at the point $(-(2b)/c, 0)$ of U_1 shown in Figure 11.

* If $p = e = g = 0$, $f = 1$ and $c > 0$, the number of equilibrium points on $u = 0$ is determined by the solutions of the equation

$$2(2bq + c)w + c^2 = 0.$$

The system (18) has two equilibrium points at the origin and at $p_1 = (0, -c^2/(2(2bq + c)))$ if $2bq + c \neq 0$, and has a unique equilibrium point at the origin if $2bq + c = 0$. The determinant of the Jacobian matrix at the origin is $-c^2/4$, so it is a saddle. The determinant and trace of the Jacobian matrix at the point p_1 are

$$\det(p_1) = \frac{c^3}{4(2bq + c)}, \quad \text{tr}(p_1) = \frac{c(bq + c)}{2bq + c},$$

which shows that the point p_1 is a saddle if $2bq + c < 0$, and an unstable node if $q \geq 0$ or $q < 0$ and $0 < b < -c/(2q)$. Going back through the blow-ups we obtain the local phase portrait at the point $(-(2b)/c, 0)$ of U_1 shown in Figure 12.

* If $f = g = 0$, $e = 1$, and $a \geq 0$, the number of equilibrium points on $u = 0$ depends on the solutions of the equation

$$2(2bp + c)w^2 + 4bqw + c^2 = 0.$$

Introducing the discriminant as 4δ where $\delta = 4b^2q^2 - 2c^2(2bp + c)$, we analyze the following cases:

When $\delta > 0$ and $c \neq -2bp$, system (18) has three equilibrium points on $u = 0$, namely,

$$p_0 = (0, 0), \quad p_1 = \left(0, \frac{-2bq - \sqrt{\delta}}{2(2bp + c)}\right), \quad p_2 = \left(0, \frac{-2bq + \sqrt{\delta}}{2(2bp + c)}\right).$$

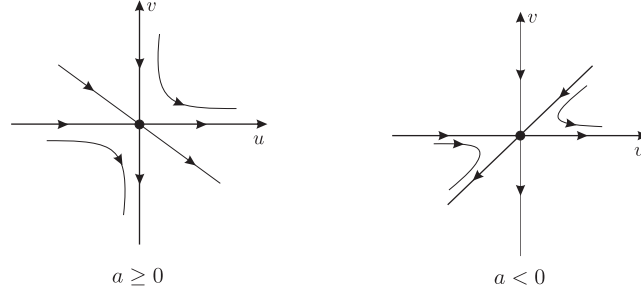


Figure 12. Local phase portrait the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c > 0$, $a = 1 + (2bq)/c$, $p = e = g = 0$, and $f = 1$.

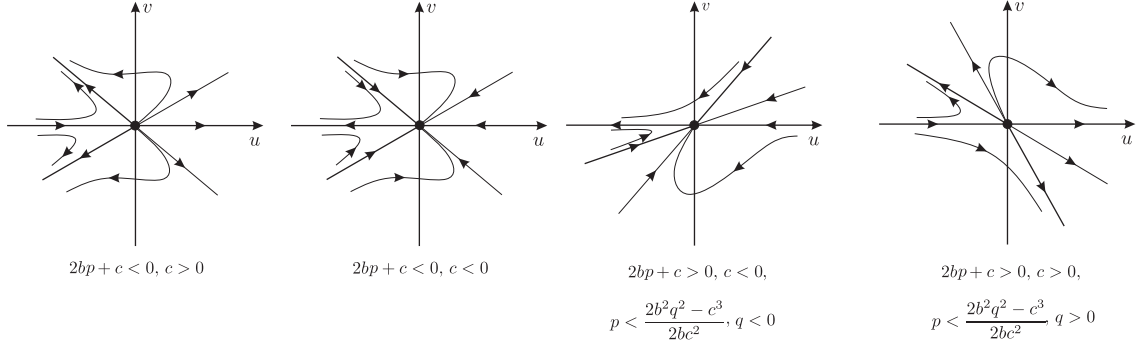


Figure 13. Local phase portrait at the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c \neq 0$, $a = (2bq)/c \geq 0$, $f = g = 0$, $e = 1$, $\delta > 0$, and $c \neq -2bp$.

The determinant of the Jacobian matrix at the origin is $-c^2/4$, so it is a saddle. The determinant at the points p_1 and p_2 is zero, and the trace of these points is given by

$$\text{tr}(p_1) = \frac{c^3 + 2bc^2p - bq(2bq + \sqrt{\delta})}{c(2bp + c)}, \quad \text{tr}(p_2) = \frac{c^3 + 2bc^2p + bq(\sqrt{\delta} - 2bq)}{c(2bp + c)}.$$

These traces are non-zero, indicating that p_1 and p_2 are semi-hyperbolic points. Using Theorem 2.19 of [6], these points are saddle-nodes. Going back through the blow-ups we get the local phase portrait at the point $(-2b/c, 0)$ of U_1 shown in Figure 13.

When $\delta > 0$, $c = -2bp$ and $q \neq 0$, system (18) exhibits two equilibrium points on $u = 0$: one at $p_0 = (0, 0)$ and another at $p_1 = (0, -(bp^2)/q)$. The determinant of the Jacobian matrix at the origin is $-b^2p^2$, indicating it is a saddle point. The determinant and trace of the Jacobian matrix at the point $(0, -(bp^2)/q)$ are 0 and $-bp$, respectively. Thus this point is a semi-hyperbolic point. Again using Theorem 2.19 from [6], it is a saddle-node. Going back through the blow-ups, we obtain the local phase portrait at the point $(-2b/c, 0)$ of U_1 shown in Figure 14.

When $\delta = 0$ and $q \neq 0$, we find that $p = (2b^2q^2 - c^3)/(2bc^2)$. Under these conditions, system (18) exhibits two equilibrium points on $u = 0$: one at the origin and the other one at $(0, -c^2/(2bq))$. The determinant of the Jacobian matrix is $-c^2/4$, which shows it is a saddle. The Jacobian matrix at the point $(0, -c^2/(2bq))$ is identically zero, necessitating the use of blow-ups. We translate this equilibrium point to the origin, and

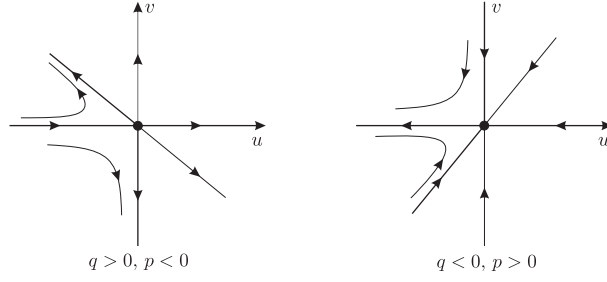


Figure 14. Local phase portrait at the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c = -2bp \neq 0$, $a = -q/p > 0$, $f = g = 0$, $e = 1$, $\delta > 0$, and $q \neq 0$.

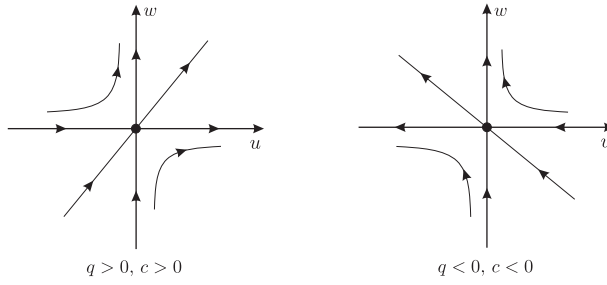


Figure 15. Local phase portrait at the point $(0, -c^2/(2bq))$.

then we perform the u -directional blow-up $(u, w) \mapsto (u, s)$ with $s = w/u$, and we obtain

$$\begin{aligned}\dot{u} &= \frac{1}{8b^3c^3q^2}u^2(16b^5q^4s^2u - 8b^4cq^4s^2u^2 + 4b^2c^4q^2s^2u^2 - 4bc^6qsu + c^8), \\ \dot{s} &= \frac{1}{8b^3c^3q^2}su(-32b^5q^4s^2u + 8b^4c^2q^3s + 8b^4cq^4s^2u^2 - 4b^2c^4q^2s^2u^2 + 4bc^6qsu - c^8).\end{aligned}$$

Using a reparametrization of time we eliminate the common factor u between $\dot{u}$ and $\dot{s}$, and we get

$$\begin{aligned}\dot{u} &= \frac{1}{8b^3c^3q^2}u(16b^5q^4s^2u - 8b^4cq^4s^2u^2 + 4b^2c^4q^2s^2u^2 - 4bc^6qsu + c^8), \\ \dot{s} &= \frac{1}{8b^3c^3q^2}s(-32b^5q^4s^2u + 8b^4c^2q^3s + 8b^4cq^4s^2u^2 - 4b^2c^4q^2s^2u^2 + 4bc^6qsu - c^8).\end{aligned}\tag{19}$$

The system (19) has two equilibrium points on the line $u = 0$: one at the origin and another at $(0, c^6/(8b^4q^3))$. The determinant of the Jacobian matrix at the origin is $-c^{10}/(64b^6q^4)$, indicating that the origin is a saddle point. For the equilibrium point $(0, c^6/(8b^4q^3))$, the determinant and trace of the Jacobian matrix are $c^{10}/(64b^6q^4)$ and $c^5/(4b^3q^2)$, respectively. This implies that the point is an unstable node if $c > 0$ and a stable node if $c < 0$. The local phase portrait of the original system is shown in Figure 15.

Now, by going back through the blow-ups, we get the local phase portrait at the point $(-2b/c, 0)$ of U_1 shown in Figure 16.

When $\delta = 0$, $q = 0$ and $c = -2bp$, system (18) has a unique equilibrium point at the origin on $u = 0$. The determinant of the Jacobian matrix at the origin is $-b^2p^2$, indicating it is a saddle point. Going back through the blow-ups, we obtain the local phase portrait at the point $(-2b/c, 0)$ of U_1 as shown in Figure 17.

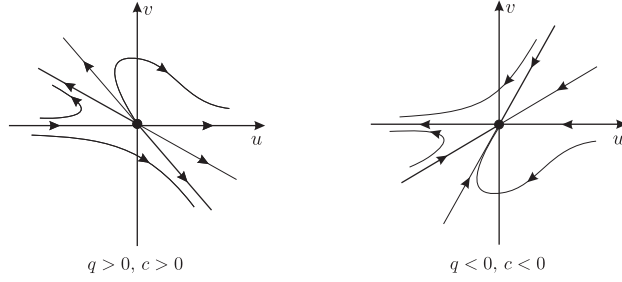


Figure 16. Local phase portrait at the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c \neq 0$, $a = (2bq)/c > 0$, $f = g = 0$, $e = 1$, $\delta = 0$, and $q \neq 0$.

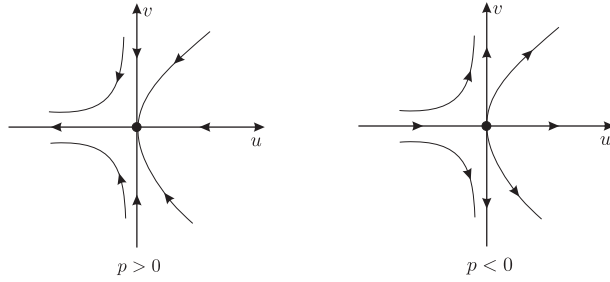


Figure 17. Local phase portrait at the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c = -2bp \neq 0$, $a = 0$, $f = g = 0$, $e = 1$, $\delta = 0$, and $q = 0$.

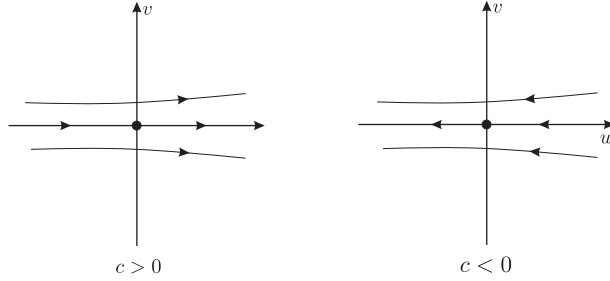


Figure 18. Local phase portrait at the point $(-2b/c, 0)$ of U_1 when $\Delta = 0$, $c \neq 0$, $a = (2bq)/c$, $f = g = 0$, $e = 1$, and $\delta < 0$.

When $\delta < 0$, system (18) has a unique equilibrium point at the origin on $u = 0$. The determinant of the Jacobian matrix at the origin is $-c^2/4$, indicating it is a saddle point. Going back through the blow-ups, we obtain the local phase portrait at the point $(-2b/c, 0)$ of U_1 as shown in Figure 18.

(vi) When $c = b = 0$, system (16) possesses only one equilibrium point at the origin on $v = 0$. This equilibrium point is nilpotent if $g = 1$. According to Theorem 3.5 from [6], it is a stable node if $d > 0$ and an unstable node if $d < 0$. If $g = 0$, the Jacobian matrix is linearly zero and we need to perform blow-ups.

We perform the u -directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and obtain

$$\begin{aligned} \dot{u} &= -u(aw + du - ew^2 - fw + puw^2 + quw), \\ \dot{w} &= -w^2(ew + f), \end{aligned} \tag{20}$$

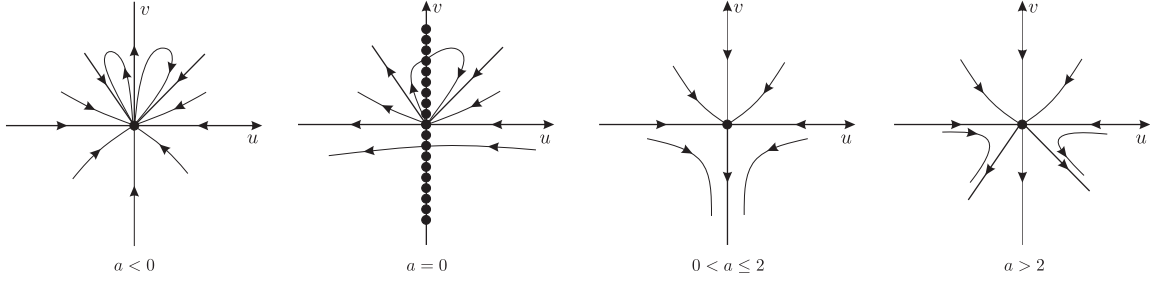


Figure 19. Local phase portrait at the origin of U_1 when $c = b = g = e = p = 0$, $f = 1$, and $d > 0$.

where a common factor u has been canceled. If either $f = 1$ and $e = 0$, or $f = 0$ and $e = 1$, system (20) has a unique equilibrium point at the origin on $u = 0$. The Jacobian matrix at this point is linearly zero, requiring another blow-up. This time, we perform the w -directional blow-up $(u, w) \mapsto (s, w)$ with $s = u/w$ and obtain

$$\begin{aligned}\dot{s} &= -s(a + ds - 2ew - 2f + psw^2 + qsw), \\ \dot{w} &= -w(ew + f),\end{aligned}\tag{21}$$

where a common factor w has been canceled. Now we analyze two different cases:

When $f = 1$, $e = 0$, $p = 0$ and $d > 0$, system (21) has two equilibrium points: one at the origin and another at $((-a + 2)/d, 0)$ for $a \neq 2$. If $a = 2$, there is a unique equilibrium point at the origin. The determinant and trace of the Jacobian matrix at the origin are $-2 + a$ and $1 - a$, respectively. This indicates that the origin is a saddle point if $a < 2$ and a stable node if $a > 2$. For $a = 2$, the origin becomes a semi-hyperbolic point, and according to Theorem 2.19 from [6], it is a saddle-node. The equilibrium point at $((-a + 2)/d, 0)$ has a Jacobian matrix with determinant $2 - a$ and trace $-3 + a$. This point is a stable node if $a < 2$ and a saddle point if $a > 2$. Going back through all the blow-ups, we get the local phase portrait at the origin of U_1 as shown in Figure 19.

When $f = 0$, $e = 1$ and $d > 0$, system (21) has two equilibrium points: one at the origin and another at $(-a/d, 0)$ for $a > 0$. If $a = 0$, there is a single equilibrium point at the origin. The Jacobian matrix at both the origin and $(-a/d, 0)$ has a determinant of zero, with traces $-a$ and a , respectively. For $a > 0$, both the origin and $(-a/d, 0)$ are semi-hyperbolic, and according to Theorem 2.19 from [6], they are saddle-nodes.

If $a = 0$, the Jacobian matrix at the origin is identically zero, and we need to do the blow-ups. We perform the s -directional blow-up $(s, w) \mapsto (s, k)$ with $k = w/s$ and obtain

$$\begin{aligned}\dot{s} &= -s(d + k^2ps^2 + kqs - 2k), \\ \dot{k} &= k(d + k^2ps^2 + kqs - 3k),\end{aligned}\tag{22}$$

where a common factor s has been canceled. System (22) has two equilibrium points: one at the origin and the other at $(0, d/3)$. The determinant of the Jacobian matrix at the origin is $-d^2$, indicating that it is a saddle point. At $(0, d/3)$, the determinant and trace of the Jacobian matrix are $d^2/3$ and $-(4d)/3$, respectively, classifying it as a stable node. Going back through the blow-up, we obtain the phase portrait of system (21) at the origin, see Figure 20

Finally, going back through all the blow-ups we get the local phase portrait at the origin of U_1 as shown in Figure 21.

(vii) When $c = d = 0$, system (16) has only one equilibrium point at the origin on $v = 0$. The determinant and trace of the Jacobian matrix of system (16) at the origin are b^2 and $-2b$, respectively. So it shows that this point is a stable node.

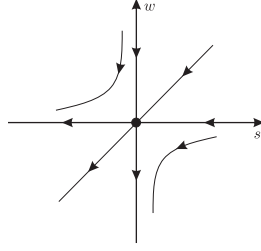


Figure 20. Local phase portrait of system (21) at the origin when $a = 0$.

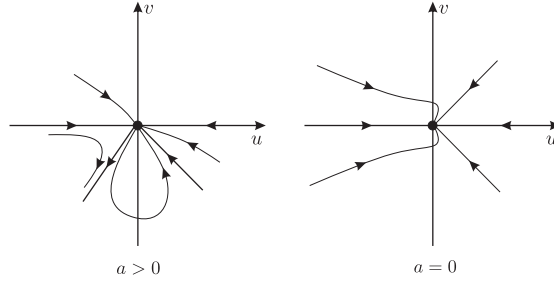


Figure 21. Local phase portrait at the origin of U_1 when $c = b = g = f = 0$, $e = 1$, and $d > 0$.

(viii) When $\Delta < 0$, the origin is the only equilibrium point of system (16) for $v = 0$, and it is a stable node. $\square$

Here, we investigate the infinite singular points of system (15) in the local chart U_2 . The expression for $p(\mathcal{X})$ in the local chart U_2 is

$$\begin{aligned}\dot{u} &= v(au - u(ev + f + gu) + pv + q) + bu^2 + cu + d, \\ \dot{v} &= -v^2(ev + f + gu),\end{aligned}\tag{23}$$

and we have the following proposition:

Proposition 12. *Consider system (23), where the parameter b is non-negative, under the following three conditions:*

- $p = e = f = 0$, $q = -1$, $g = 1$, and $a \geq 0$,
- $p = e = g = 0$, $f = 1$, and $c \geq 0$,
- $f = g = 0$, $e = 1$, and $a \geq 0$.

Then the following statements regarding the origin in the local chart U_2 of systems (11), (12), and (13) hold:

- (i) *When $d = 0$, $f = 1$ and $c \neq 0$, it is a saddle-node.*
- (ii) *When $d = f = 0$, $g = 1$, $q = -1$ and $c \neq 0$, it is a saddle.*
- (iii) *When $d = f = g = 0$, $e = 1$ and $c \neq 0$, it is a saddle if $c > 0$, and a stable node if $c < 0$.*
- (iv) *When $d = c = 0$, $f = 1$, and $q \neq 0$, it is a saddle-node.*
- (v) *When $d = c = f = 0$, $g = 1$, and $q = -1$, it is a saddle.*
- (vi) *When $d = c = f = g = 0$, $e = 1$, and $q \neq 0$, it is a saddle-node.*
- (vii) *When $d = c = q = 0$, its local phase portrait is shown in Figures 22, 24, and 25.*

Proof. The origin is an equilibrium point of system (23) only if $d = 0$. If $c \neq 0$, the origin is a semi-hyperbolic point because the determinant and trace of the Jacobian matrix at the origin are zero and c , respectively.

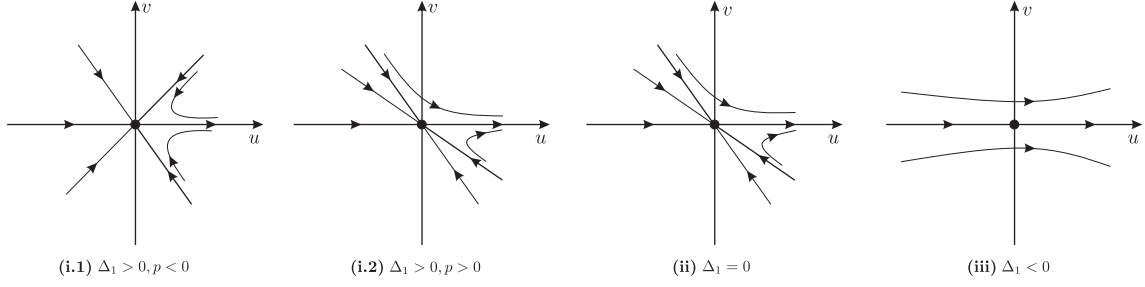


Figure 22. Local phase portrait at the origin of U_2 when $d = c = q = g = f = 0$, $e = 1$, $p \neq 0$.

According to Theorem 2.19 of [6], if $f = 1$, it is a saddle-node, if $f = e = 0$, $g = 1$, $q = -1$, it is a saddle, and if $f = g = 0$, $e = 1$, it is a saddle if $c > 0$, and a stable node if $c < 0$. If $c = 0$ and $q \neq 0$, the origin is a nilpotent point. Using Theorem 3.5 of [6], if $f = 1$, the origin is a saddle-node, if $f = 0$, $g = 1$, $q = -1$, it is a saddle, and if $f = g = 0$, $e = 1$, it is a saddle-node. If $c = q = 0$, the Jacobian matrix at the origin is linearly zero, so blow-ups are needed to analyze the dynamics. First, we set $g = 0$ since in the first case $q = -1$. Next, we perform the u -directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and obtain

$$\begin{aligned}\dot{u} &= u (aw + b - ew^2 - fw + pw^2), \\ \dot{w} &= -w (pw^2 + aw + b),\end{aligned}\tag{24}$$

where a common factor u has been canceled. The equilibrium points of system (24) for $u = 0$ are determined by the solutions of the equation $pw^2 + aw + b = 0$. The discriminant of this quadratic equation is given by $\Delta_1 = a^2 - 4bp$. It is important to note that $b > 0$ because if $b = 0$, the main system would be linear. If $p \neq 0$, $f = 0$ and $e = 1$, then we have the following cases:

When $\Delta_1 > 0$, system (24) has three equilibrium points on $u = 0$, namely,

$$p_0 = (0, 0), \quad p_1 = \left(0, \frac{-a - \sqrt{a^2 - 4bp}}{2p}\right), \quad p_2 = \left(0, \frac{-a + \sqrt{a^2 - 4bp}}{2p}\right).$$

The determinant of the Jacobian matrix of system (24) at the origin is $-b^2$, indicating it is a saddle. The determinants of the Jacobian matrix at the points p_1 and p_2 are zero, and their traces are given by

$$\text{tr}(p_1) = 2b + \frac{a(-a - \sqrt{a^2 - 4bp})}{2p}, \quad \text{tr}(p_2) = 2b + \frac{a(-a + \sqrt{a^2 - 4bp})}{2p},\tag{25}$$

which are non-zero, so these points are semi-hyperbolic. Using Theorem 2.19 from [6], they are saddle-nodes. Going back through the blow-ups we obtain the local phase portrait at the origin of U_2 as shown in Figure 22 (i.1) and (i.2).

When $\Delta_1 = 0$, we find that $p = a^2/(4b)$. The system (24) then has two equilibrium points along $u = 0$: one at the origin and the other at $(0, -(2b)/a)$. At the origin, the determinant of the Jacobian matrix is $-b^2$, indicating that this point is a saddle. For the point $(0, -(2b)/a)$, the Jacobian matrix is linearly zero, so we need further blow-ups to analyze this equilibrium point. We translate this equilibrium point to the origin, and then perform the u -directional blow-up $(u, w) \mapsto (u, s)$ with $s = w/u$ and we have

$$\begin{aligned}\dot{u} &= \frac{1}{4a^2b} u (a^4 s^2 u - 4a^2 b s^2 u^2 + 16ab^2 s u - 16b^3), \\ \dot{s} &= \frac{1}{2a^2b} s (a^4 (-s^2) u + a^3 b s + 2a^2 b s^2 u^2 - 8ab^2 s u + 8b^3),\end{aligned}\tag{26}$$

where a common factor u has been canceled. System (26) has two equilibrium points at the origin and at $(0, -(8b^2)/a^3)$ on $u = 0$. The determinant of the Jacobian matrix at the origin is $-(16b^4)/a^4$, indicating it is a saddle point. At the point $(0, -(8b^2)/a^3)$, the determinant of the Jacobian matrix is $(16b^4)/a^4$ and

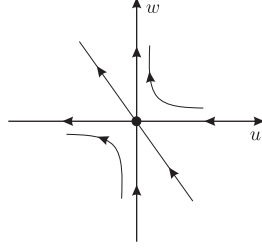


Figure 23. Local phase portrait of system (24) the point $(0, -(2b)/a)$.

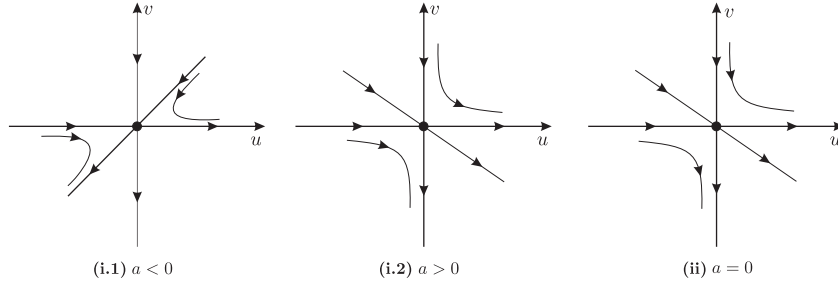


Figure 24. Local phase portrait at the origin of U_2 when $d = c = q = g = p = 0$ and $f = 1$.

the trace is $-(8b^2)/a^2$, so it is a stable node. Going back through the blow-up, we can find the local phase portrait of system (24) at the point $(0, -(2b)/a)$ as shown in Figure 23.

Then, going back through all the blow-ups we obtain the local phase portrait at the origin of U_2 as given in Figure 22 (ii).

When $\Delta_1 < 0$, system (24) has a unique equilibrium point at the origin on $u = 0$. The determinant of the Jacobian matrix at the origin is $-b^2$, so it is a saddle. Going back through the blow-ups we obtain the local phase portrait at the origin of U_2 as given in Figure 22 (iii).

If $p = 0$, then system (24) has two equilibrium points on $u = 0$: one at the origin and another at $(0, -b/a)$. The determinant of the linear part at the origin is $-b^2$, indicating it is a saddle point. When $f = 1$, the determinant and trace of the linear part at the point $(0, -b/a)$ are b^2/a and $(1+1/a)b$, respectively. Therefore, this point is a saddle if $a < 0$ and an unstable node if $a > 0$. When $f = 0$ and $e = 1$, the determinant of the linear part at the point $(0, -b/a)$ is zero, but the trace is b , making this point semi-hyperbolic. According to Theorem 2.19 from [6], it is a saddle-node. Now going back through the blow-ups we have the local phase portrait at the origin of U_2 as shown in Figure 24 (i.1), (i.2) and Figure 25 (i), respectively.

If $p = a = 0$, then system (24) has a unique equilibrium point at the origin on $u = 0$. The determinant of the linear part at this point is $-b^2$, so it is a saddle point. Going back through the blow-ups we have the local phase portrait at the origin of U_2 as shown in Figures 24 (ii) and 25 (ii).

□

Remark 13. We note that due to the even degree of the system, and regarding Proposition 11 and 12, the equilibrium points diametrically opposite in the local chart V_1 and V_2 exhibit opposite stability.

3.2. Finite equilibrium points and potential limit cycle of system (11). We now analyze the finite equilibrium points for the differential system (11). Also, we will investigate the potential limit cycles for the differential system (11).

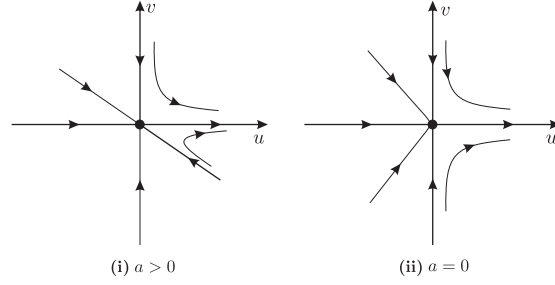


Figure 25. Local phase portrait at the origin of U_2 when $d = c = q = g = p = f = 0$ and $e = 1$.

Proposition 14. Consider system (11), where a and b are non-negative parameters, and c and d are free parameters. When $d \neq 0$, the system has two equilibrium points: the origin $(0,0)$ and the point $(0,1/d)$. When $d = 0$, the system has a single equilibrium point at the origin $(0,0)$. The point $(0,1/d)$ is always a saddle point, and the stability of the origin depends on the parameters as follows:

- (i) It is a center if $a = 0$ and $c(b+d) = 0$;
- (ii) It is a stable focus if $a = 0$ and $c(b+d) < 0$;
- (iii) It is an unstable focus if $c(b+d) > 0$;
- (iv) It is an unstable focus if $0 < a < 2$;
- (v) It is an unstable node if $a \geq 2$.

Proof. System (11) has two distinct equilibrium points at the origin and at $(0,1/d)$, when d is a non-zero real number. However, if $d = 0$, the system has only one equilibrium point, which is at the origin. The determinants of the Jacobian matrix of system (11) at the origin and at $(0,1/d)$ are 1 and -1 , respectively. Also, the trace of the Jacobian matrix at the origin is a . As a result, if $0 < a < 2$ the origin is an unstable focus; if $a \geq 2$ it is an unstable node. However, if $a = 0$, it is either a center or a weak focus, according to Theorem 12.4 of [14], it is a center when $c(b+d) = 0$; otherwise, it is a stable weak focus if $c(b+d) < 0$ and unstable weak focus if $c(b+d) > 0$. The point $(0,1/d)$ is always a hyperbolic saddle. $\square$

We know that in a quadratic system, if there is a limit cycle, it surrounds a unique equilibrium point that it is a focus (see [4]). From Theorem 12.4 of [14], it is shown that when $a = 0$, there is no limit cycle around the origin. According to Theorem 12.5 of [14], if $0 < a < 2$, we can conclude that when $c(b+d) = 0$, there are no limit cycles around the origin. Similarly, when $c(b+d) > 0$, there are also no limit cycles. However, if $c(b+d) < 0$ and a is sufficiently small, system (11) will have a unique limit cycle because the stability of the origin changes when a increases from zero. Hence from Theorem 3.7 of [14] we know that limit cycles will appear near the origin. Also, Theorem 12.6 in [14] states that for any values of c , b , and d , system (11) can have at most one limit cycle around the origin.

When a varies, system (11) forms a complete family of generalized rotated vector fields parameterized by a . If $c(b+d) < 0$, then as a increases from 0, the stability of the origin transitions from stable to unstable. Therefore, according to Theorem 3.7 of [14], it also represents a stable limit cycle.

3.3. Invariant lines passing through the saddle point. Here we emphasize additional crucial details necessary for constructing the global phase portraits of system (11).

- (i) When two infinite equilibria located diametrically opposite from each other are connected by a curve, the connection is due with an invariant straight line [13].

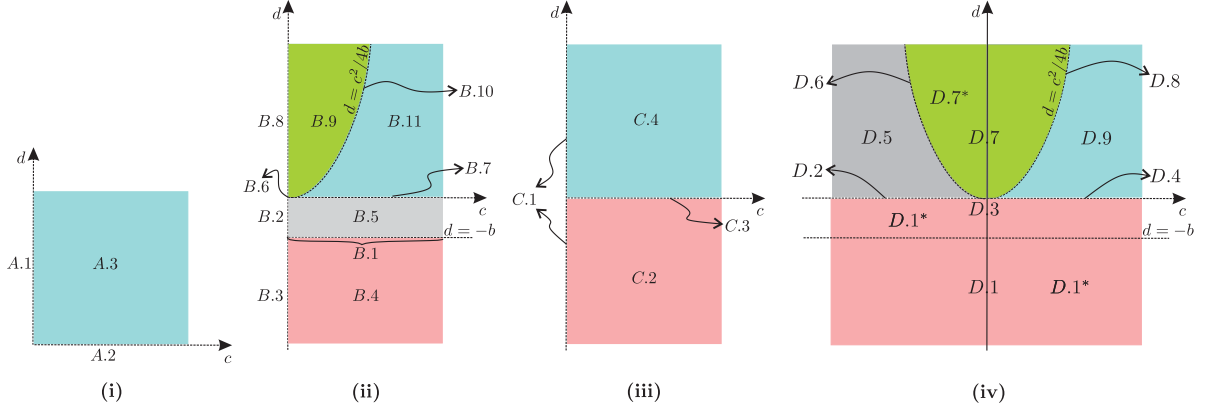


Figure 26. Bifurcation diagrams; (i) system (11) with $a = b = 0$, (ii) system (11) with $a = 0$ and $b > 0$, (iii) system (11) with $a > 0$ and $b = 0$, (iv) system (11) with $a, b > 0$.

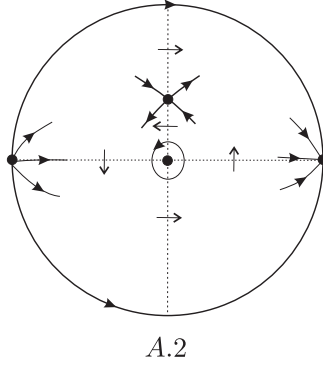


Figure 27. Local phase portraits of system (11) for case A.2.

- (ii) When $a = 0$ and $b > 0$, system (11) possesses an invariant straight line only if $d = -b$. More precisely, when $b = -d$, the separatrices of the saddle point are the two invariant straight lines

$$z = \left(\frac{c \pm \sqrt{c^2 + 4b^2}}{2b} \right) x - \frac{1}{b}.$$

Consequently, it follows that the saddle-node points located diametrically opposite from each other must be linked by these separatrices.

- (iii) When $a > 0$ and $b = 0$, system (11) possesses an invariant straight line only if $a = -b/c$ and $c > 0$. In fact, when $a = -b/c$, one of the separatrices of the saddle point is the invariant line $z = (1/d)(-cx + 1)$.
- (iv) When $a, b > 0$, system (11) possesses an invariant straight line only if

$$bd \neq 0 \quad \text{and} \quad a = \frac{(b+d)(-c \pm \sqrt{\Delta})}{2bd}.$$

In fact, one of the separatrices of the saddle point is the invariant line $z = -(ab)/(ac + b + d)x + 1/d$.

3.4. Global phase portrait of system (11). Now, we describe how we can get the global phase portrait of system (11) on the Poincaré disc. We only describe the global phase portrait of system (11) for case A.2 (please see Figure 2) because the other cases can be found in a similar way.

The local phase portraits of system (11) for case A.2 on the Poincaré disc is shown in Figure 27. Here the unstable separatrix of the saddle in the first quadrant goes to the origin of U_1 , while its stable separatrix in the second quadrant comes from the origin of V_1 . If the other unstable separatrix of the saddle point on the second quadrant converges to the origin of U_1 , then the remaining stable separatrix in the first quadrant lacks a source. Then, it indicates the presence of a homoclinic loop around the center. This completes the description of the global phase portrait of system (11) for case A.2.

3.5. Finite equilibrium points of system (12). Here, we study the finite equilibrium points for the differential system (12). We mention that system (12) cannot exhibit a limit cycle because its equilibrium points lie on an invariant line.

Proposition 15. *Consider system (12), where b and c are non-negative parameters, and q , a , and d are free parameters. When $b > 0$ and $a \neq 0$, the system has two equilibrium points: the origin $(0,0)$ and the point $(-a/b, 0)$. When either $b = 0$ and $a \neq 0$, or $b > 0$ and $a = 0$, the system has a unique equilibrium point at the origin $(0,0)$.*

The stability of the equilibrium points is as follows:

- (i) *The origin $(0,0)$ is a saddle if $a < 0$, an unstable node if $a > 0$, and a saddle-node if $a = 0$.*
- (ii) *The point $(-a/b, 0)$ is an unstable node if $a < 0$, and a saddle if $a > 0$.*

Also, the line $z = 0$ is filled with equilibrium points if $a = b = 0$.

Proof. When $b > 0$ and $a \neq 0$, system (12) has two equilibrium points: one at the origin and another at $(-a/b, 0)$, both on the invariant line $z = 0$. At the origin, the linear part of system (12) has a determinant of a and a trace of $1 + a$. It is a saddle if $a < 0$, or an unstable node if $a > 0$. At $(-a/b, 0)$, the determinant is $-a$ and the trace is $1 - a$. Depending on a , it is an unstable node if $a < 0$, or a saddle if $a > 0$.

When $b > 0$ and $a = 0$, system (12) has a unique equilibrium point at the origin. The determinant of the linear part is 0 and the trace is 1. According to Theorem 2.19 in [6], this equilibrium point is a saddle-node.

When $b = 0$ and $a \neq 0$, system (12) also has a unique equilibrium point at the origin. The determinant and trace of the linear part are a and $1 + a$, respectively. It is a saddle if $a < 0$, or an unstable node if $a > 0$. $\square$

3.6. Global study of system (14). The system (14) has a common factor in the form of a conic, $p + qz + ax + bx^2 + cxz + dz^2$. Thus, we study the reduced system

$$\dot{x} = 1, \quad \dot{z} = 0.$$

On the local chart U_1 , the differential system is

$$\dot{u} = -u, \quad \dot{v} = -v,$$

which has a stable node at the origin. On the local chart U_2 , the differential system is

$$\dot{u} = 1, \quad \dot{v} = 0,$$

which has no equilibrium points. To complete the study of system (14), we need to add the conic of equilibrium points to the phase portraits of the original system (see Figure 5).

The proof of Theorem 3 is now complete. Additionally, according to Theorem 4 in [2], the phase portraits $B.8, C.5, D.1.2, D.1.3, D.1.5, D.4, D.7.1, D.8, D.9.1$, and $D.9.2$ are topologically equivalent to the phase portraits $A.2, A.4, B.5.1, B.5.2, B.5.3, B.7, B.9, B.10, B.11.1$, and $B.11.2$, respectively. Furthermore, the phase portraits $H.3.1$ and $J.14$ are topologically equivalent to the phase portrait $G.3.2$. This observation implies Corollary 4.

4. PROOF OF THEOREM 5

This class, defined by the invariant plane $\mathcal{P} : d_0 + d_1x + d_2y = 0$, possesses the Darboux invariant $I(x, y, t) = (d_0 + d_1x + d_2y)e^{-k_0t}$. This invariant function enables the description of phase portraits on the Poincaré ball.

Consider constants b_0, b_1, b_2 , and b_3 as given in Theorem 2 (ii). System (1), restricted to the invariant plane $d_0 + d_1x + d_2y = 0$, becomes

$$\begin{aligned}\dot{x} &= d_2((-a_2d_0 + a_0d_2) + (-a_2d_1 + a_1d_2)x + a_3d_2z), \\ \dot{z} &= c_7d_0^2 - c_2d_0d_2 + c_0d_2^2 + (2c_7d_0d_1 - c_5d_0d_2 - c_2d_1d_2 + c_1d_2^2)x + (-c_8d_0d_2 + c_3d_2^2)z \\ &\quad + (c_7d_1^2 - c_5d_1d_2 + c_4d_2^2)x^2 + (-c_8d_1d_2 + c_6d_2^2)zx + c_9d_2^2z^2.\end{aligned}$$

Therefore, similar to the proof of Theorem 3, our focus shifts to analyzing the dynamics of the differential system

$$\begin{aligned}\dot{x} &= -z + ax + bx^2 + cxz + dz^2, \\ \dot{z} &= x,\end{aligned}\tag{27}$$

where $a, b \geq 0$, and c and d are free parameters. The phase portraits of the quadratic system (27) are thoroughly studied in Section 3, and their representations on the Poincaré disc are given in Figure 2.

Since the plane $\mathcal{P}$ is invariant, meaning any trajectory that starts on this plane remains on it for all time. For trajectories γ that do not start on $\mathcal{P}$, the behavior is determined by the invariant function

$$I(x, y, t) = (d_0 + d_1x + d_2y)e^{-k_0t}.$$

This invariant function indicates that

$$(d_0 + d_1x + d_2y) = Ce^{k_0t},$$

where C depends on the initial conditions of the trajectory.

If $k_0 > 0$: As $t \rightarrow \infty$, e^{k_0t} grows exponentially to infinity. Consequently, $d_0 + d_1x + d_2y$ must move away from the plane $\mathcal{P}$ and can approach infinity, leading to $\omega(\varphi) \subset \mathbb{S}^2$. Conversely, as $t \rightarrow -\infty$, e^{k_0t} tends to zero, so $d_0 + d_1x + d_2y$ must approach zero, implying $\alpha(\varphi) \subset \mathcal{P}$.

If $k_0 < 0$: As $t \rightarrow \infty$, e^{k_0t} tends to zero, causing $d_0 + d_1x + d_2y$ to decrease to zero, resulting in $\omega(\varphi) \subset \mathcal{P}$. Conversely, as $t \rightarrow -\infty$, e^{k_0t} grows exponentially, causing $d_0 + d_1x + d_2y$ to grow to infinity, leading to $\alpha(\varphi) \subset \mathbb{S}^2$.

This characterization shows how the sign of k_0 influences the trajectory's behavior relative to the invariant plane $\mathcal{P}$ in the phase space. Furthermore, Figure 1 displays all potential phase portraits of system (1) at infinity.

5. PROOF OF THEOREM 6

This class with the invariant plane $d_0 + d_1x + d_2y + d_3z = 0$ has neither a Darboux invariant, and nor a first integral, and we will describe the phase portraits of these systems on that invariant plane using the Poincaré disc.

Consider the conditions under Theorem 2 (iii). By performing the change of variables

$$u = x, \quad v = y, \quad w = d_0 + d_1x + d_2y + d_3z,$$

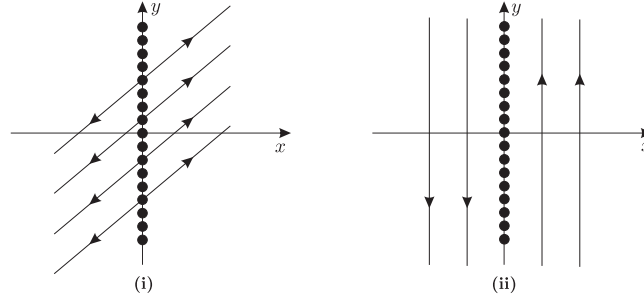


Figure 28. Local phase portraits of system (30) for case (iv) and (v).

system (1), when restricted to the invariant plane $d_0 + d_1x + d_2y + d_3z = 0$, becomes

$$\begin{aligned}\dot{x} &= a_0 - \frac{a_3d_0}{d_3} + \left(a_1 - \frac{a_3d_1}{d_3}\right)x + \left(a_2 - \frac{a_3d_2}{d_3}\right)y, \\ \dot{y} &= b_0 - \frac{b_3d_0}{d_3} + \left(b_1 - \frac{b_3d_1}{d_3}\right)x + \left(b_2 - \frac{b_3d_2}{d_3}\right)y.\end{aligned}\tag{28}$$

After translating the equilibrium point to the origin, then system (28) transforms into the following form

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy,\tag{29}$$

where

$$a = a_1 - \frac{a_3d_1}{d_3}, \quad b = a_2 - \frac{a_3d_2}{d_3}, \quad c = b_1 - \frac{b_3d_1}{d_3}, \quad d = b_2 - \frac{b_3d_2}{d_3}.$$

By applying the suitable transformation, system (10) can be reduced into three distinct classes:

- When $c \neq 0$,

$$\dot{x} = Tx - Dy, \quad \dot{y} = x, \quad \text{with } T = a + d \quad \text{and} \quad D = ad - bc,\tag{30}$$

- When $c = 0$ and $d \neq 0$,

$$\dot{x} = x, \quad \dot{y} = y,\tag{31}$$

- When $c = d = 0$,

$$\dot{x} = 0, \quad \dot{y} = 0.$$

The global phase portrait of the last case $\dot{x} = \dot{y} = 0$ is trivial, characterized by a phase portrait filled with equilibrium points. Therefore, our focus will be on studying the global phase portraits of the differential systems (30) and (31).

It is easy to see that system (30) is invariant under transformations $(x, y, t, T, D) \rightarrow (-x, y, -t, -T, D)$, so it is sufficient to consider only $T \geq 0$. According to [6, 8], we can invoke the following theorem:

Theorem 16. *Consider the linear system given by (30). Then, the subsequent statements hold:*

- (i) *If $D < 0$, then the system has a saddle at the origin;*
- (ii) *If $D > 0$ and $T = 0$, then the system has a center at the origin;*
- (iii) *If $D > T^2/4$ and $T > 0$, then the system has an unstable focus at the origin, and if $D \leq T^2/4$ and $T > 0$, then it has an unstable node at the origin;*
- (iv) *If $D = 0$ and $T > 0$, then the local phase portrait of system at the origin is topologically equivalent to Figure 30 (i);*
- (v) *If $D = 0$ and $T = 0$, but A is not identically zero, then the local phase portrait of system at the origin is topologically equivalent to Figure 28 (ii);*

Now, we investigate the infinite equilibrium points of system (30) in the local chart U_1 . Let $\mathcal{X}$ be the vector field associated to system (30). Hence, the expression for $p(\mathcal{X})$ in the local chart U_1 becomes

$$\begin{aligned}\dot{u} &= Du^2 - Tu + 1, \\ \dot{v} &= v(Du - T),\end{aligned}\tag{32}$$

where the equilibrium points of system (32) on $v = 0$ depend on the real solutions of the equation $Du^2 - Tu + 1 = 0$. Hence, we have the following proposition:

Proposition 17. *Consider system (30), where D is a real parameter and T is a non-negative real parameter. Then, the following cases hold in the local chart U_1 :*

(i) *If $D < T^2/4$ and $D \neq 0$, then system (32) has two infinite equilibrium points*

$$p_0 = \left(\frac{T - \sqrt{T^2 - 4D}}{2D}, 0 \right), \quad p_1 = \left(\frac{T + \sqrt{T^2 - 4D}}{2D}, 0 \right),$$

where the point p_0 is a stable node, and the point p_1 is an unstable node if $D < 0$, and a saddle if $D > 0$;

(ii) *If $D = T^2/4$ and $D \neq 0$, then system (32) has an infinite equilibrium point at $(2/T, 0)$, which is a saddle-node;*

(iii) *If $D > T^2/4$ and $D \neq 0$, then system (32) has no infinite equilibrium point;*

(iv) *If $D = 0$ and $T > 0$, then system (32) has an infinite equilibrium point at $(1/T, 0)$, which is a stable node;*

(v) *If $D = T = 0$, then system (32) has no infinite equilibrium point.*

Proof. (i) When $D < T^2/4$ and $D \neq 0$, system (32) has two equilibrium points on $v = 0$, namely

$$p_0 = \left(\frac{T - \sqrt{T^2 - 4D}}{2D}, 0 \right), \quad p_1 = \left(\frac{T + \sqrt{T^2 - 4D}}{2D}, 0 \right),$$

The determinant and trace of the linear part of system (32) at the points p_0 and p_1 are given by

$$\begin{aligned}\det(p_0) &= -2D + \frac{1}{2}T \left(T + \sqrt{T^2 - 4D} \right), \quad \text{tr}(p_0) = \frac{1}{2} \left(-T - 3\sqrt{T^2 - 4D} \right), \\ \det(p_1) &= -2D + \frac{1}{2}T \left(T - \sqrt{T^2 - 4D} \right), \quad \text{tr}(p_1) = \frac{1}{2} \left(-T + 3\sqrt{T^2 - 4D} \right).\end{aligned}$$

It follows that the point p_0 is a stable node, and the point p_1 is an unstable node if $D < 0$, and a saddle point if $D > 0$.

(ii) When $D = T^2/4$ and $D \neq 0$, system (32) has an equilibrium point at $(2/T, 0)$ on $v = 0$. The determinant and trace of the linear part of system (32) at this point are 0 and $-T/2$, respectively. Therefore, it is a semi-hyperbolic point. Using the theorem in [6], it is a saddle-node.

(iii) When $D > T^2/4$ and $D \neq 0$, it is clear that system (32) has no equilibrium point.

(iv) When $D = 0$ and $T > 0$, system (32) has an equilibrium point at $(1/T, 0)$ on $v = 0$. The determinant and trace of the linear part of system (32) at this point are T^2 and $-2T$, respectively. Therefore, it is a stable node.

(v) When $D = T = 0$, system (32) reduces to $\dot{u} = 1$, $\dot{v} = 0$, which has no equilibrium point on $v = 0$.

□

Next, we examine the infinite equilibrium points of system (30) in chart U_2 . The expression for $p(\mathcal{X})$ in this chart is given by

$$\begin{aligned}\dot{u} &= u(T - u) - D, \\ \dot{v} &= -uv,\end{aligned}\tag{33}$$

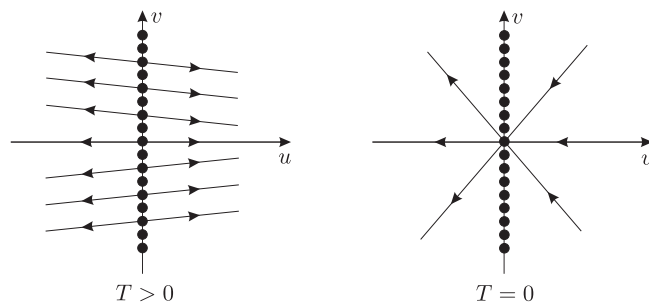


Figure 29. Local phase portraits of system (33) for $D = 0$ at the origin.

and we present the following proposition:

Proposition 18. *Consider of the origin comprises one hyperbolic of the origin comprises one hyperbolic system (30), where D is a real parameter and T is a non-negative real parameter. When $D = 0$, the local phase portrait of the origin in the chart U_2 is shown in Figure 29.*

Proof. From (33), we observe that the origin is an equilibrium point of the system (33) only if $D = 0$. In this case, system (33) has a common factor u . After canceling this factor, the system reduces to the form

$$\dot{u} = T - u, \quad \dot{v} = -v, \quad (34)$$

where the origin is not an equilibrium point for $T > 0$ and is a stable node for $T = 0$. Going back to the main system, we obtain the local phase portrait at the origin of U_2 (see Figure 29). $\square$

Now, we study the differential system (31). It is evident that it possesses a unique equilibrium point at the origin, which is an unstable star node.

In both local charts U_1 and U_2 , the differential system is described by:

$$\dot{u} = 0, \quad \dot{v} = -v, \quad (35)$$

where all points on $v = 0$ are equilibrium points. This observation indicates that the boundary of the Poincaré disc is filled with equilibrium points. This completes the proof of Theorem 6 (see Figure 6).

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^{1,3} DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE SÃO CARLOS, SÃO CARLOS, BRAZIL.

Email address: `alibb@ufscar.br`

Email address: `alexcr@ufscar.br`

² DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

Email address: `jaumellibre@uab.cat`