CONNECTED MCMULLEN-LIKE JULIA SETS IN A CHEBYSHEV-HALLEY FAMILY

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ABSTRACT. In this paper we study a one parameter family of rational maps obtained by applying the Chebyshev-Halley root-finding algorithms. We show that the dynamics near parameters where the family presents some degeneracy might be understood from the point of view of singular perturbations. More precisely, we relate the dynamics of those maps with the one of the McMullen family $M_\lambda(z) = z^4 + \lambda/z^2$, using quasiconformal surgery.

Keywords: holomorphic dynamics, singular perturbations, degeneracy parameters, Julia sets, root-finding algorithm.

MSC2020: 30D05, 37F10, 37F31, 37F44

INTRODUCTION

The root-finding algorithms are widely known as iterative dynamical systems. On the chaotic part, the iterative method fails, so it is important to understand the chaotic set and how it varies with the method.

The root-finding algorithms applied to polynomials are rational maps and, in this setting, there is a well defined dichotomy between the tame part, the Fatou set denoted $\mathcal{F}(f)$, and the chaotic part, the Julia set denoted $\mathcal{J}(f)$. On the Fatou set there is eventually some limiting behaviour since $\mathcal{F}(f)$ is the set of points z_0 where the family of iterates $(f^n)_{n \in \mathbb{N}}$ is normal if restricted to some neighbourhood of z_0.

Among the rational cases, the quadratic polynomials—which are also the simplest rational maps—have been well studied and proved to be universal. This follows from the work of many authors including Douady, Hubbard [7], Lyubich [8], and McMullen [11]. Roughly speaking, it means that the Julia sets of those polynomials appear in a lot of families : the dynamics can be restricted so as to look like quadratic. It is much more easy to recognize the Julia set when it is connected, so the Mandelbrot set $M = \{c \in \mathbb{C} \mid \mathcal{J}(z^2 + c) \text{ is connected}\}$ plays a fundamental role in parameter spaces. Its boundary ∂M is the bifurcation locus of the quadratic family: the place where the dynamics changes drastically. One of the first appearances of the universality of the quadratic family was observed by the presence of "copies" of the Mandelbrot set in the family of Newton's method applied to a cubic polynomial [7].

However, there are rational maps whose Julia set is not homeomorphic to any quadratic Julia set. The family of rational maps $M_{n,d,\lambda}(z) = z^n + \lambda/z^d$ firstly introduced by C. McMullen [10] is an example of this phenomenon. Indeed, the

Julia set of $M_{n,d,\lambda}$ could be a Cantor set of circles surrounding the origin or a Sierpinski carpet, among other possibilities [6]. These kind of Julia sets is not occurring for polynomials since, in this case, there is no Fatou component whose boundary is the whole Julia set. The family of maps $M_{n,d,\lambda}$ is referred in the literature as *McMullen family*.

In this article we show that one can find copies of the Julia set of maps in the McMullen family as subsets of the Julia set of a family coming from Chebyshev-Halley root-finding algorithms.

The family of *Chebyshev-Halley root-finding algorithms is* given by the recursive sequence $z_{n+1} = CH_\alpha^f(z_n)$, where α is a complex parameter and

$$CH_\alpha^f(z) = z - \left(1 + \frac{1}{2}\,\frac{L_f\,(z)}{1 - \alpha L_f\,(z)}\right)\frac{f\,(z)}{f'\,(z)} \qquad \text{with} \quad L_f\,(z) = \frac{f\,(z)\,f''\,(z)}{\left(f'\,(z)\right)^2}.$$

When this family is applied to the polynomial $f(z) = z^3 - 1$ and using the new parameter $a = 5 - 4\alpha$, one gets the rational map

$$R_a(z) = \frac{2az^6 + (15 - a)z^3 + 3 - a}{3z^2(5 - a + (1 + a)z^3)}.$$

For $a = 0$ the degree drops from 6 to 5. The goal of this work is to study this family around the singular parameter $a = 0$. The main result is the following.

Theorem. *There exists a neighbourhood Λ of 0 such that for $a \in \Lambda \setminus \{0\}$ the map R_a^2 is McMullen-like : the map R_a^2 is conjugated to a map in the McMullen family in some annulus A_a.*

Theorem A in the §2 is a more detailed statement of this result (see also Theorem 5.3). Moreover, in §6 we prove that

Corollary. *For parameters $a \in \Lambda \setminus \{0\}$ the Julia set $\mathcal{J}(R_a)$ contains the image by some homeomorphism of the Julia set of a map in the McMullen family. Moreover, the three different types of escaping Julia sets of the McMullen family appear in Λ.*

The paper goes as follows. In §1 we give a short introduction to the dynamics of the Chebyshev-Halley family. In §2 we state the properties of the McMullen family, present Theorem A in §2.1, and list the properties to guarantee that a rational map of degree 6 is a McMullen map in §2.2. In §3 and §4 we discuss in detail the dynamical properties of the rational maps R_0 and R_a, respectively. Then, §5 is mainly devoted to prove Theorem A, which is the content of Theorem 5.3. The proof is based on a cut and paste quasiconformal surgery procedure (see [2]) relating the dynamics of R_a^2 with the one of M_λ. Finally, in §6 we prove that the three types of Julia sets described in the Escape Trichotomy Theorem of [6] can be found as subset of the Julia set of R_a for different values of a. Furthermore, we remark that the same ideas work for the Chebyshev-Halley method applied to the polynomial $z^n - 1$ with $n > 3$.

1. CHEBYSHEV-HALLEY FAMILY

The family of Chebyshev-Halley root-finding algorithms is given by the recursive sequence $z_{n+1} = CH_\alpha^f(z_n)$,

$$z_{n+1} = z_n - \left(1 + \frac{1}{2} \frac{L_f(z_n)}{1 - \alpha L_f(z_n)}\right) \frac{f(z_n)}{f'(z_n)} \qquad \text{with} \qquad L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2}$$

and $\alpha \in \mathbb{C}$. This family of root-finding algorithms was already studied in [15] (see also [14]); from the point of view of complex dynamics it also appears in several works (see for instance [5, 4, 12]). In contrast to Newton's method, which has quadratic convergence for simple roots, these algorithms have cubic convergence, i.e. every simple root of a polynomial is a super-attracting fixed point of local degree 3 of CH_α^f. This family contains some well known root-finding algorithms. For example, $\alpha = 0$ corresponds to Chebyshev's method, $\alpha = 1/2$ corresponds to Halley's method, and as α tends to infinity the family converges to Newton's method.

In this paper, we focus on the Chebyshev-Halley method applied to the polynomial $f(z) = z^3 - 1$. It has the expression

$$CH_\alpha^f(z) = \frac{2 - 4\alpha + (-10 - 4\alpha)z^3 + (-10 + 8\alpha)z^6}{6z^2(-2\alpha + (2\alpha - 3)z^3)}.$$

It can be simplified by considering the parameter $a = 5 - 4\alpha$. We obtain then the following one parameter family of rational maps defined on the Riemann sphere $\hat{\mathbb{C}}$ that we call R_a:

$$(1) \qquad R_a(z) = \frac{2az^6 + (15 - a)z^3 + 3 - a}{3z^2(5 - a + (1 + a)z^3)}, \quad a \in \mathbb{C}.$$

For a parameter $a \notin \{0, 3\}$, the rational map $R_a(z)$ exhibits 3 free critical points. Let $\zeta = e^{2\pi i/3}$. Given a choice of a punctual determination of a cubic root, the critical points are

$$(2) \qquad c_{a,j} = \zeta^j \sqrt[3]{\frac{15 - 8a + a^2}{a(a + 1)}}, \quad \text{where } j = 0, 1, 2,$$

and the critical values $v_{a,j} := R_a(c_{a,j})$ are

$$(3) \qquad v_{a,j} = \zeta^j \frac{(25 - 6a + a^2)}{(a - 5)^2(a + 1)} \sqrt[3]{a^2 \frac{15 - 8a + a^2}{a + 1}}.$$

Some basic properties of R_a are related to its symmetry. It is straightforward to check that $R_a(\xi z) = \xi R_a(z)$ for any $\xi \in \mathbb{U}$, where $\mathbb{U} = \{\xi \in \mathbb{C} \mid \xi^3 = 1\}$ is the group of third roots of the unity generated by $\zeta = e^{2\pi i/3}$. As a consequence, this symmetry provides a conjugacy in the dynamical plane. Note that the orbits of the 3 free critical points are symmetric with respect to multiplication by a third root of the unity. We can conclude that the $a-$plane is the natural parameter plane of the family R_a.

For the map R_a, the elements of $\mathbb{U}$ are *super-attracting* fixed points with local degree 3 (since the Chebyshev-Halley methods have order of convergence 3). Hence,

to every $\xi \in \mathbb{U}$ is associated its basin of attraction

$$(4) \qquad\qquad A_a(\xi) = \{z \in \mathbb{C} \mid R_a^n(z) \to \xi \ \text{as} \ n \to \infty\}$$

and its *immediate basin of attraction* $A_a^*(\xi)$ defined as the connected component of $A_a(\xi)$ containing ξ.

The rational map R_a has degree 6, except for $a = 0$ and $a = 3$. The parameter $a = 3$ corresponds to Halley's method, which is relatively simple to study since there are no critical points other than the super-attracting fixed points which correspond to the roots of the polynomial (see [4]).

At the parameter $a = 0$ a singular perturbation happens. Indeed, if $a = 0$ the dynamics at ∞ change drastically: the map is

$$(5) \qquad\qquad R_0(z) = \frac{15z^3 + 3}{3z^2(5 + z^3)}$$

and has the points $\{0, \infty\}$ as a period two super-attracting cycle whereas for $|a| \neq 0$ small enough ∞ is a repelling fixed point. More precisely, the point 0 is in both cases critical and is mapped with degree 2 to ∞, but for $a = 0$ the point $z = \infty$ is sent back with degree 2 to 0 (see Figure 1 (left)), while for $a \neq 0$ the point ∞ becomes a fixed point of multiplier $3(1 + a)/2a$. Hence, infinity is a repelling fixed point when a is close enough to 0, $a \neq 0$.

In summary, the dynamics of the map R_0 is completely understood. The map R_0 has degree 5 and thus has 8 critical points, which are the following.

- The 3 solutions of the equation $z^3 - 1 = 0$. They correspond to 3 superattracting fixed points of R_0 with local degree 3 and thus these 3 solutions are critical points of multiplicity 2 (so 6 critical points counting multiplicity).
- The points $z = 0$ and $z = \infty$ are simple critical points, which form the superattracting 2-cycle $\{0, \infty\}$.

The case $a = 3$ is also well understood: the Fatou set of R_3 consists of the basin of attraction of the third roots of the unity and there are no critical points other than the roots themselves.

On the other hand, the map R_a, $a \notin \{0, 3\}$, has degree 6 and thus has 10 critical points, which are the following.

- As in the case of R_0, the 3 solutions of the equation $z^3 - 1 = 0$ are critical points of multiplicity 2.
- The point $z = 0$ is a simple critical point that is mapped onto $z = \infty$, which is now a fixed point.
- The 3 critical points $c_{a,j}$, $j = 0, 1, 2$ (see (2)). These critical points converge to $z = \infty$ as a tends to 0.

Moreover, in the case of R_a, $a \notin \{0, 3\}$, the dynamics of the 3 new critical points $c_{a,j}$, $j = 0, 1, 2$, is tied by the symmetry of the family and, hence, the parameter space of R_a is a one dimensional space parametrized by $a \in \mathbb{C}$. See Figures 2 and 7.

Furthermore, we can notice similarities between the dynamical planes of R_0 and part of that of R_a, namely on the closure of the whole basins of attraction under R_0 of the roots ζ^k, $k = 0, 1, 2$, denoted by $\overline{A_0(\zeta^k)}$ (see Figure 1). This follows from the fact $\overline{A_0(\zeta^k)}$ is compact ($A_0(\zeta^k)$ are in the complement of the basin at ∞) and

also that the map R_a converges uniformly on compact sets of $\mathbb{C}$ to R_0 as a tends to 0. Indeed, it follows from the expression (1) of R_a that can be rewritten as

$$R_a(z) = \frac{15z^3 + 3 + a(z^3 - 1)(2z^3 + 1)}{3z^2(z^3 + 5) + a \cdot 3z^2(z^3 - 1)}.$$
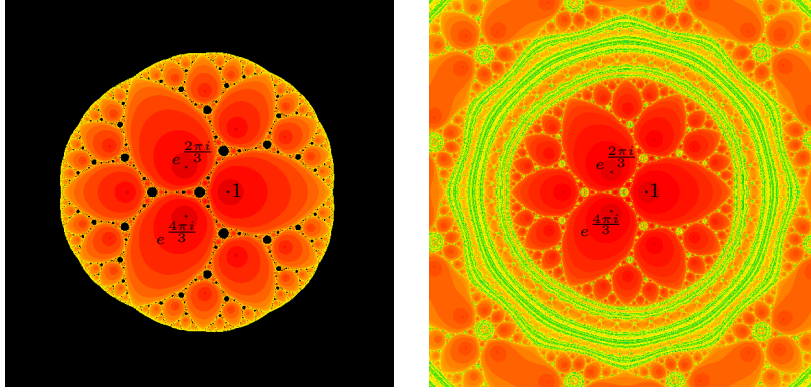

FIGURE 1. Dynamical planes of $R_a(z)$ for $a = 0$ (left) and $a = 0.0001$ (right). In both dynamical planes red points represent points converging towards a third root of unity. For $a = 0$ (left) black points represent points converging to the super-attracting cycle $\{0, \infty\}$. We mark the third roots of unity $1, e^{2\pi i/3}, e^{4\pi i/3}$.

In fact we can construct a holomorphic motion of $\overline{A_0(\zeta^k)}$ (see Lemma 6.2 and Figure 1). Since the concept of holomorphic motion will appear at several places in this paper, we recall it in the next definition.

Definition 1.1. A holomorphic motion of a set $X \subset \widehat{\mathbb{C}}$ parameterized by a domain $\Lambda \subset \mathbb{C}$ and based at $\lambda_0 \in \Lambda$ is a map $H : \Lambda \times X \to \widehat{\mathbb{C}}$ such that

- $H(\lambda_0, z) = z \quad \forall z \in X$
- $z \mapsto H(\lambda, z)$ is injective for each $\lambda \in \Lambda$
- the map $\lambda \mapsto H(\lambda, z)$ is holomorphic for each $z \in X$.

2. MCMULLEN FAMILY AND MAIN RESULT

As mentioned before, the singular perturbations $M_{n,d,\lambda}(z) = z^n + \lambda/z^d$ where introduced by C. McMullen [10] in order to prove the existence of buried Julia components, i.e. connected components of the Julia set which do not intersect the boundary of any Fatou component. More specifically, he provided the first example of rational map whose Julia set is a Cantor set of quasicircles. Afterwards, R. Devaney, D. Look, and D. Uminsky [6] provided the following classification for the Julia set of McMullen maps when the orbit of all critical points tend to ∞ (see Figure 3).

Theorem (Escape Trichotomy, [6])**.** *Assume that all critical points of $M_{n,d,\lambda}$ belong to $A_{M_{n,d,\lambda}}(\infty)$. Then, exactly one of the following occurs.*

- *All critical points of $M_{n,d,\lambda}$ belong to $A^*_{M_{n,d,\lambda}}(\infty)$. Then, the Julia set $M_{n,d,\lambda}$ is a Cantor set of points.*
- *All critical points of $M_{n,d,\lambda}$ are mapped in exactly two iterates into $A^*_{M_{n,d,\lambda}}(\infty)$. Then, the Julia set $\mathcal{J}(M_{n,d,\lambda})$ is a Cantor set of circles.*
- *All critical points of $M_{n,d,\lambda}$ are mapped in exactly $m > 2$ iterates into $A^*_{M_{n,d,\lambda}}(\infty)$. Then, the Julia set $\mathcal{J}(M_{n,d,\lambda})$ is a Sierpinsky carpet.*

2.1. **The main result.** The goal of this paper is to relate the dynamics of R_a with the dynamics of the following McMullen map

$$(6) \qquad\qquad M_\lambda(z) := M_{4,2,\lambda}(z) = z^4 + \frac{\lambda}{z^2}$$

for parameters a close to 0. In Figure 3 we can observe that there appear structures in $\mathcal{J}(R_a)$ similar to the Cantor sets of circles, the Sierpinski carpet and the Cantor set of the family of maps M_λ. In Figure 2 we compare the parameter plane of R_a near the origin to the parameter plane of M_λ.
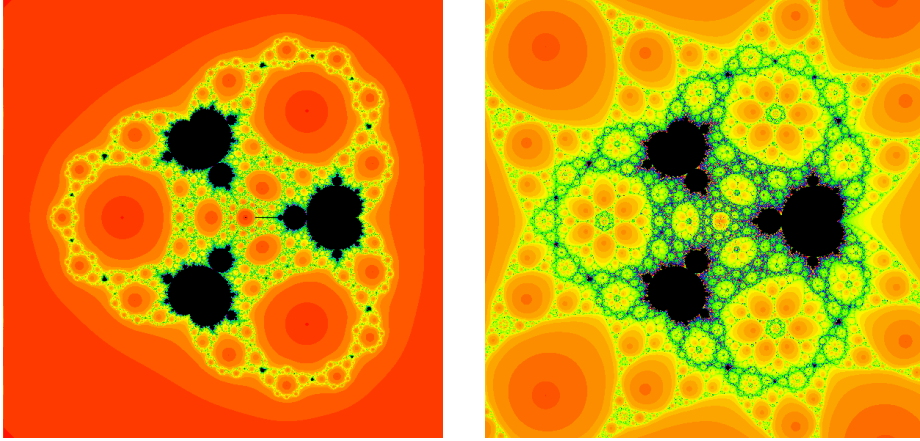

FIGURE 2.   On the left side we plot the parameter plane of M_λ (6) and on the right side the parameter plane of R_a (1) near $a = 0$.

The main result of this work is the following theorem, which basically states that for a in a neighbourhood of the origin (see Section 4 for details) the second iterate of R_a (1) is conjugate to some M_λ (6) in a concrete annulus defined in the dynamical plane. The result implies that a copy of the Julia set of M_λ is contained in the Julia set of R_a.

Theorem A. *There exists a neighbourhood Λ of 0 such that for $a \in \Lambda \setminus \{0\}$ the map R_a^2 is conjugated to a map in the McMullen family M_λ. More precisely, there exists a quasiconformal map $\varphi_a : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi_a\left(R_a^2(z)\right) = M_{\lambda(a)}\left(\varphi_a(z)\right)$ for all z in some annulus A_a with $\varphi_a^{-1}(\mathcal{J}(M_{\lambda(a)})) \subset A_a$, where $a \mapsto \lambda(a)$ is a map defined on Λ.*
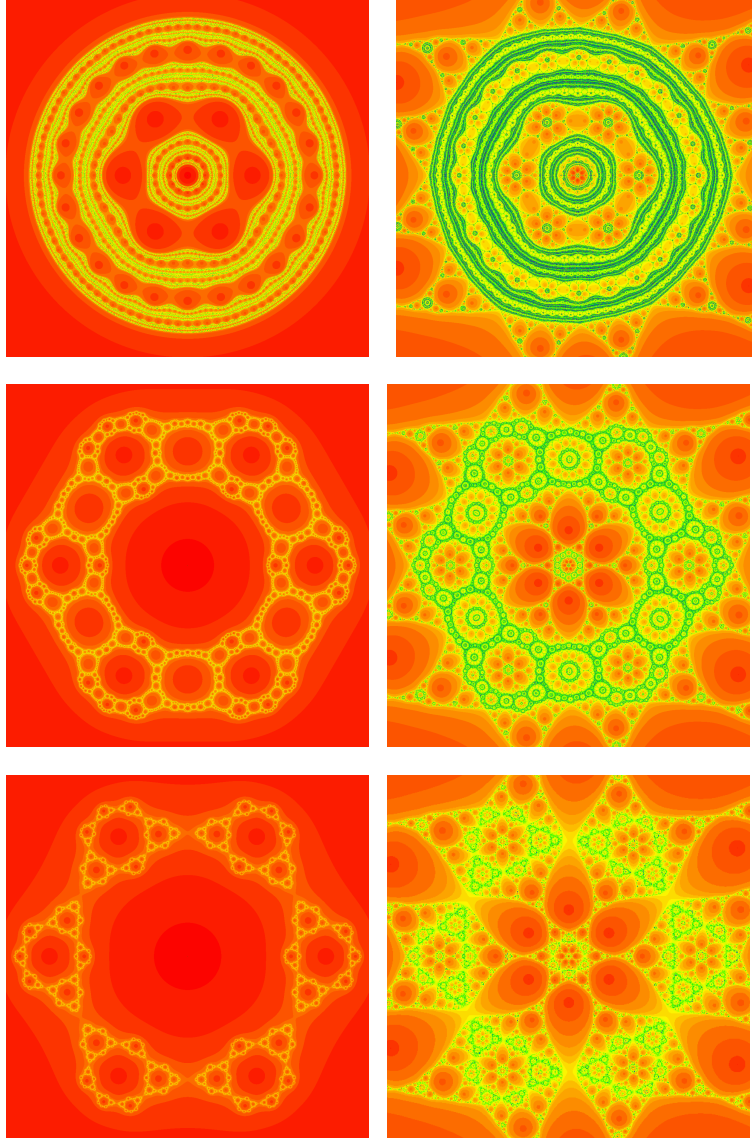
FIGURE 3. In the first column we present the dynamical plane of M_λ (6) for $\lambda = 0.005$ (top) $\lambda = -0.28$ (middle) and $\lambda = -0.455$ (bottom). In the second column the dynamical plane of R_a (see (1)) for $a = -0.0003$ (top) $a = -0.0164$ (middle) and $a = -0.028$ (bottom) with Re(z) $\in$ $(-0.245, 0.245)$ and Im(z) $\in (-0.245, 0.245)$. In the dynamical plane of R_a red points represent points converging to a third root of unity while in the McMullen family M_λ red points represent points converging to infinity.

2.2. **Rigidity of McMullen's family.**

The following proposition is a characterisation for a rational map to be linearly conjugate to McMullen's map $M_\lambda(z) = z^4 + \frac{\lambda}{z^2}$ (6).

Proposition 2.1. *Let $Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree 6 rational map satisfying the following properties:*

a) The point $z = \infty$ is super-attracting of local degree 4;
b) The point $z = 0$ is a double preimage of $z = \infty$.
c) The map Q is symmetric with respect to multiplication by a third root of the unity, i.e. $Q(\xi z) = \xi Q(z)$ where $\xi^3 = 1$.
d) The map Q has exactly 6 different simple critical points (other than $z = 0$ and $z = \infty$) which are mapped under Q onto exactly 3 different critical values.

Then Q is linearly conjugate to

$$M_\lambda(z) = z^4 + \frac{\lambda}{z^2}.$$

Proof. It follows directly from a), b) and c) that the map Q can be written as

$$Q(z) = \frac{az^6 + \tilde{b}z^3 + \tilde{\lambda}}{z^2},$$

where $a, \tilde{b}, \tilde{\lambda} \in \mathbb{C}$ and $a\tilde{\lambda} \neq 0$. This map is linearly conjugate to

$$M_{b,\lambda}(z) = \frac{z^6 + bz^3 + \lambda}{z^2},$$

where $b, \lambda \in \mathbb{C}$, by the map $z \to z/\sqrt[3]{a}$. We now use property d) to prove that $b = 0$. The critical points of $M_{b,\lambda}$ are solutions of

$$4z^6 + bz^3 - 2\lambda = 0.$$

Writing $w = z^3$, we get the equation $4w^2 + bw - 2\lambda = 0$, whose solutions are

$$w_\pm = \frac{-b \pm \sqrt{b^2 + 32\lambda}}{8}.$$

By d) $M_{b,\lambda}$ has 6 different simple critical points (other than $z = 0$ and $z = \infty$), we conclude that $\Delta := \sqrt{b^2 + 32\lambda} \neq 0$. Moreover, we have that $w_+ \cdot w_- = -\lambda/2$.

Let $\zeta = e^{2\pi i/3}$. These 6 critical points of $M_{b,\lambda}$ are labelled:

- $c_{1,+} = \sqrt[3]{w_+}$, $c_{2,+} = \zeta \cdot c_{1,+}$, $c_{3,+} = \zeta^2 \cdot c_{1,+}$;
- $c_{1,-} = \sqrt[3]{w_-}$, $c_{2,-} = \zeta \cdot c_{1,-}$, $c_{3,-} = \zeta^2 \cdot c_{1,-}$.

Notice that the labelling of the critical points depends on arbitrary choices of the square and cubic roots, but this is not important since we find them all. By hypothesis, $M_{b,\lambda}$ maps these 6 critical points to exactly three different critical values. Notice that if $M_{b,\lambda}(c_{j,+}) = M_{b,\lambda}(c_{k,+})$ with $j \neq k$ then, by symmetry, $M_{b,\lambda}(c_{1,+}) = M_{b,\lambda}(c_{2,+}) = M_{b,\lambda}(c_{3,+})$. It would then follow that $M_{b,\lambda}$ maps the six critical points to two critical values, which is not possible by hypothesis. We can conclude that $M_{b,\lambda}(c_{j,+}) = M_{b,\lambda}(c_{k,-})$ for some j and k. For all critical points $c_{j,\pm}$ we have

$$M_{b,\lambda}(c_{j,\pm}) = c_{j,\pm}^4 + b \cdot c_{j,\pm} + \frac{\lambda}{c_{j,\pm}^2} = c_{j,\pm}\left(c_{j,\pm}^3 + b + \frac{\lambda}{c_{j,\pm}^3}\right) = c_{j,\pm}\left(w_\pm + b + \frac{\lambda}{w_\pm}\right) =$$

$$= c_{j,\pm}\left(w_\pm + b - 2w_\mp\right).$$

By taking the third power on both sides of the equality $M_{b,\lambda}(c_{j,+}) = M_{b,\lambda}(c_{k,-})$ we obtain, independently of j and k, the equality:

$$w_+ \left(w_+ + b - 2w_-\right)^3 = w_- \left(w_- + b - 2w_+\right)^3.$$

By using that $w_\pm = (-b \pm \Delta)/8$, where $\Delta := \sqrt{b^2 + 32\lambda} \neq 0$, simple computations yield that the previous equation is equivalent to $432\Delta^3 b = 0$. Since $\Delta \neq 0$, we conclude that $b = 0$, which finishes the proof. □

3. Dynamics of the map R_0

In this section we study the dynamics of the unperturbed map R_0. We start by analyzing the relation of the repelling fixed points of R_0 and the basins of attraction $A_0(\zeta^j)$, where $j = 0, 1, 2$ (see (4)). The maps $R_a(z)$ have 3 fixed points in $\mathbb{C}$ other than the third roots of the unity, given by

$$(7) \qquad x_{a,j} = \zeta^j \sqrt[3]{\frac{a-3}{a+3}}, \quad \text{where } \zeta = e^{2\pi i/3} \text{ and } j = 0, 1, 2.$$

In this case we take the determinacy of the third root such that $\sqrt[3]{-1} = -1$ (which is well defined for $|a|$ small). Notice that for $a = 0$ these three fixed points are given by $x_{0,j} = -\zeta^j$. Recall that the third roots of the unity are ζ^j with $j = 0, 1, 2$.

Lemma 3.1. *Each repelling fixed point $x_{0,j}$ of R_0 belongs to the boundary of the immediate basins of attraction $A_a^*(\zeta^k)$ and $A_a^*(\zeta^{k'})$ where $j, k, k' \in \{0, 1, 2\}$ and $k' \neq j \neq k \neq k'$.*

Proof. If $a = 0$ the critical points are the roots of the unity and the points 0 and ∞, which form a super-attracting 2-cycle. Every root of the unity ζ^j is a super-attracting fixed point of local degree 3. Since its immediate basin of attraction $A_0^*(\zeta^j)$ cannot contain any free critical point, the Böttcher coordinate extends until reaching the boundary of the immediate basin of attraction. It follows that the fixed dynamical rays, of angles 0 and 1/2, land at $\partial A_0^*(\zeta^j)$. They either land at two different fixed points or at a common fixed point.

If they land at different fixed points we are done. Indeed, since there are only 3 fixed points other than the roots of the unity, it follows from the symmetry in the dynamical plane that every $x_{0,j}$ belongs to the boundary of exactly 2 immediate basins of attraction.

To finish the proof we have to see that these fixed rays cannot land at a common fixed point. We focus on the basin of attraction of the root $\zeta^0 = 1$. Since R_0 leaves the real line invariant, the map $i(z) = \overline{z}$ conjugates R_0 with itself. We can conclude that if a fixed ray lands at $x_{0,j} \in \partial A_0^*(1)$, then the other fixed ray lands at $\overline{x_{0,j}} \in \partial A_0^*(1)$. Since we are assuming that they land at the same point, we can deduce that they both land at $x_{0,0} = -1$. Using the symmetry with respect to rotation by a third root of the unity, we can conclude that $-1 \in \partial A_0^*(1)$, that $-\zeta \in \partial A_0^*(\zeta)$, and that $-\zeta^2 \in \partial A_0^*(\zeta^2)$. However, this is impossible since then the 3 different basins of attraction would have a non-empty intersection. It also follows from the previous argument that $x_{0,0} \notin \partial A_0^*(\zeta^0)$. By symmetry we conclude that $x_{0,j} \notin \partial A_0^*(\zeta^j)$, $j = 0, 1, 2$. □

We will construct a partition of the dynamical plane of R_0 using dynamical rays. The third roots of the unity are super-attracting fixed points of local degree

3 under the map R_0. Since R_0 has no free critical points, the Böttcher coordinate extends to the whole immediate basin of attraction of each third root of the unity ζ^j, $j = 0, 1, 2$. It follows that each $A_0^*(\zeta^j)$ contains two invariant rays, of angles 0 and 1/2, which land at two different fixed points on $\partial A_0^*(\zeta^j)$ by Lemma 3.1.

Let η denote the curve obtained as the union of these 6 fixed rays (2 fixed rays for each one of the third root of the unity). It follows from Lemma 3.1 that η is a simple closed curve which surrounds $z = 0$ and is invariant under rotation by third roots of the unity (see Figure 4). Moreover, for each the root ζ^j, the curve η separates it from its preimages. Indeed, the root 1 has two different real preimages, one in the interval $(-1, 0)$ and another one in the interval $(-\infty, -1)$. Notice that η contains the repelling fixed point -1.
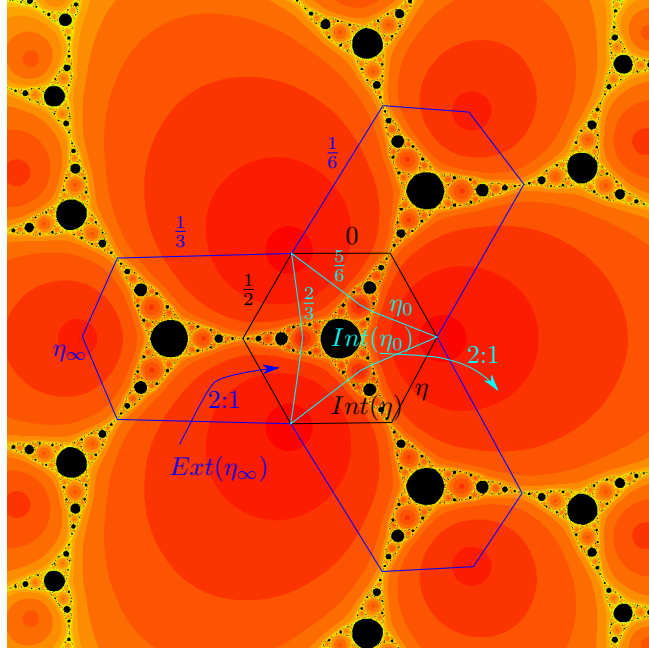

FIGURE 4. Dynamical plane of R_0 and sketch of the dynamical rays involved in the construction

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The preimage of η under R_0 consists of the union of 3 different simple closed curves which intersect at the roots ζ^j (see Figure 4). One of the curves is η, which is mapped with degree 1 onto itself. The other curves are $\eta_\infty \subset \overline{Ext(\eta)}$ and $\eta_0 \subset \overline{Int(\eta)}$. By relabelling the external rays if necessary (for each $A_0^*(\zeta^j)$ we can choose which fixed ray is 0 and which is 1/2), we can assume that η_∞ contains the rays of angles 1/6 and 1/3 and that η_0 contains the rays of angles 2/3 and 5/6. Besides those rays, the curve η_∞ and η_0 contain the preimages of the rays of angles 0 and 1/2 attached at the preimages of the roots of the unity contained in $Ext(\eta)$ and $Int(\eta)$, respectively (see Figure 4). It follows that η_∞ and η_0 are simple closed curves that surround $z = 0$ and are invariant under rotation by a third root of the unity.

Lemma 3.2. $R_0 : Ext(\eta_\infty) \to Int(\eta)$ and $R_0 : Int(\eta_0) \to Ext(\eta)$ are proper maps of degree 2.

Proof. The open set $Ext(\eta_\infty)$ contains no other preimage of $z = 0$ than $z = \infty$. Indeed, $z = 0$ has three preimages under R_0 other than $z = \infty$, which are given by $-\zeta^j \sqrt[3]{1/5}$ for $j = 0, 1, 2$ and are contained in $Int(\eta)$. It follows that $R_0 : Ext(\eta_\infty) \to Int(\eta)$ is a proper map. The degree of this proper map is 2 since $z = \infty$ is mapped 2 to 1 onto $z = 0$ under R_0. Analogously, $R_0 : Int(\eta_0) \to Ext(\eta)$ is a proper map of degree 2. □

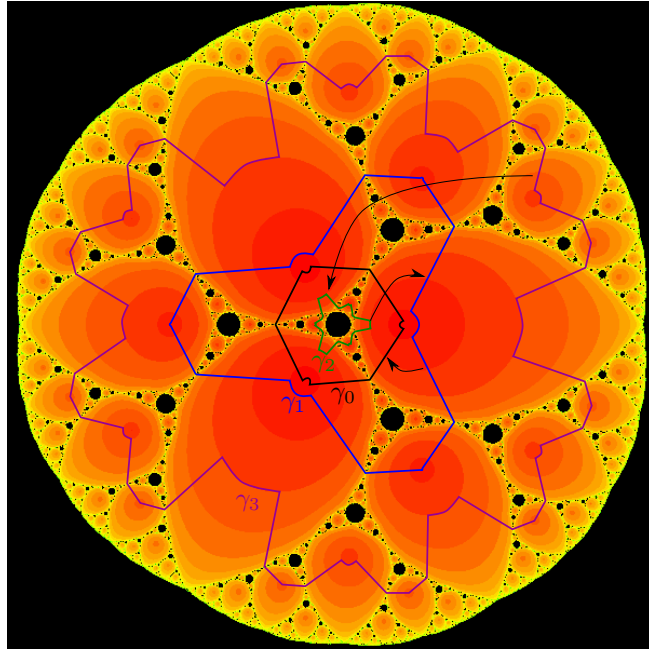FIGURE 5. Configuration of the dynamics of the curve γ_0 and its preimages up to γ_3. the map R_0 maps each curve γ_k onto γ_{k-1} with degree 2 for $k = 1, \ldots, 4$.

The curve η and its preimages are well defined for any parameter a close enough to 0 (see next section). We want to use them to perform a cut and paste surgery (see [2]) to relate the dynamics of R_a with the one of $z^4 + \lambda/z^2$. However, these curves always intersect at the roots of the unity. To avoid this problem we now introduce a modified version of η (see Figure 5). This modified version uses the equipotentials given by the Böttcher coordinates in the immediate basins of attraction of the third roots of the unity.

Definition 3.3. Fix $0 < l_0 < 1$. We define γ_0 to be the simple closed curve obtained by cutting η near the roots of the unity by the equipotentials of level l_0 and joining the cut points through these equipotentials so that $Int(\gamma_0)$ is an open neighbourhood of $z = 0$ not containing the roots of the unity.

By construction γ_0 is invariant with respect to rotation by a third root of the unity. Notice that γ_0 depends on the choice of the level l_0. Notice also that

the rotation with respect to a third root of the unity maps the equipotentials of level l_0 amongst themselves. We can now use the maps $R_0|_{Ext(\eta_\infty)}$ and $R_0|_{Int(\eta_0)}$ (Lemma 3.2) to take preimages of γ_0 (see Figure 5).

Lemma 3.4. *Let γ_1 be the preimage of γ_0 contained in $\overline{Ext(\eta_\infty)}$, γ_2 be the preimage of γ_1 contained in $Int(\eta_0)$, γ_3 be the preimage of γ_2 contained in $Ext(\eta_\infty)$, and γ_4 be the preimage of γ_3 contained in $Int(\eta_0)$. Then the curves γ_k for $k = 1, \ldots, 4$ are simple closed curves which are invariant with respect to rotation by a third root of the unity, surround the origin, and $R_0 : \gamma_k \to \gamma_{k-1}$ has degree 2 for $k = 1, \ldots, 4$. Moreover, we have the inclusions $\gamma_4 \subset Int(\gamma_2)$, $\gamma_2 \subset Int(\gamma_0)$, $\gamma_0 \subset Int(\gamma_1)$, and $\gamma_1 \subset Int(\gamma_3)$.*

Proof. These curves are well defined by the dynamics of $R_0|_{Ext(\eta_\infty)}$ and $R_0|_{Int(\eta_0)}$ (see Lemma 3.2). Their properties also come from the dynamics of $R_0|_{Ext(\eta_\infty)}$ and $R_0|_{Int(\eta_0)}$. Notice that γ_1 coincides with η_∞ except at the preimages of the subintervals replaced by equipotential segments. $\qquad\qquad\square$

The next remark describes properties of the curves γ_1, γ_2 and γ_3 which will be used in the next section. They follow from the dynamics of R_0 (Lemma 3.2).

Remark 3.5. All preimages of γ_2 under R_0, other than γ_3, are compactly contained in $Int(\gamma_3)$. Moreover, the open set $Int(\gamma_1)$ contains all preimages of $z = \infty$ under R_0.

We continue with a lemma showing that the Fatou components of R_0 are quasidisks.

Lemma 3.6. $\partial A_0^*(\infty)$, $\partial A_0^*(0)$ and $\partial A_0^*(\zeta^j)$ for $j = 0, 1, 2$ are quasicircles.

Proof. We firstly consider the 2-cycle $\{0, \infty\}$ of R_0. The triple $(R_0^2; Int(\gamma_2), Int(\gamma_0))$ is a polynomial-like map of degree 4 whose Julia set coincides with the boundary of $A_0^*(0)$. Notice that $z = 0$ is a super-attracting fixed point of local degree 4 under R^2, so the polynomial-like map $(R_0^2, Int(\gamma_2), Int(\gamma_0))$ is hybrid equivalent to a polynomial of the form bz^4, $b \in \mathbb{C}$ and the result follows.

We secondly consider the super-attracting fixed points located at ζ^j for $j = 0, 1, 2$. By symmetry, it is enough to prove the result for $\partial A_0^*(1)$. We construct a curve γ which passes through the external rays of angles $1/4$ and $3/4$ in $\partial A_0^*(\zeta^1)$ and $\partial A_0^*(\zeta^2)$. We continue these curves in the immediate basins of attraction of 0 and ∞ by following appropriate external rays and cutting by equipotentials so that it surrounds $z = 1$ (see Figure 6). Moreover, γ is modified to follow equipotentials near ζ^1 and ζ^2 so that it does not contain any of the fixed points (see Figure 6). Let V be the domain bounded by γ (which contains $z = 1$). It is not difficult to show that, if the equipotentials in the basins of $z = 0$ and $z = \infty$ are chosen appropriately, the connected component U of $R_0^{-1}(V)$ which contains $\zeta^0 = 1$ is simply connected, is compactly contained in V and is mapped with degree 3 onto V under R_0. It follows that the triple $(R_0; U, V)$ is a degree 3 polynomial-like mapping. Since $\zeta^0 = 1$ is a super-attracting fixed point of local degree 3, it follows that $R_0|_U$ is quasiconformally conjugate to z^3. This quasiconformal conjugacy maps the immediate basin of attraction of $z = 1$ to the unit disk. Hence, we can conclude that $\partial A_0^*(1)$ is a quasicircle (see Figure 6). $\qquad\qquad\square$
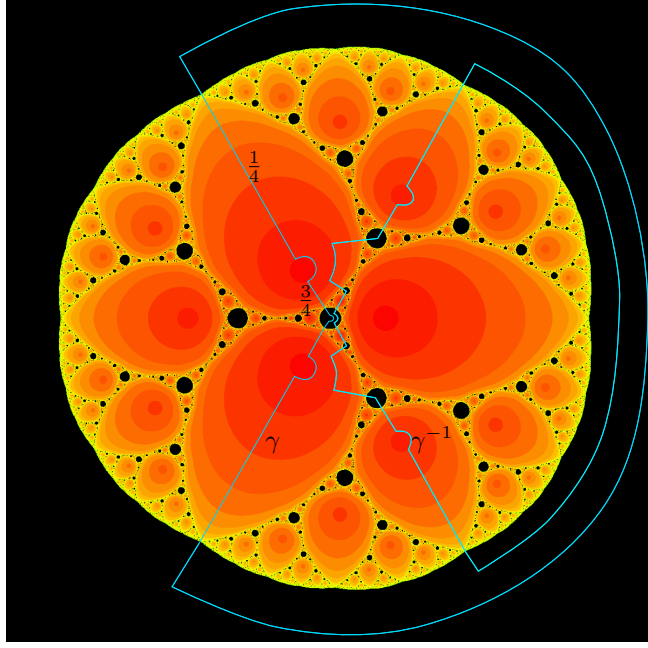
FIGURE 6. Curves that delimit the polynomial-like mapping around $z = 1$.

We finish this section by showing that the curve γ_0 is a quasicircle. Since R_0 has no *free* critical points (all critical points of R_0 are fixed points), it follows that all preimages of γ_0 under R_0 are also quasicircles.

Corollary 3.7. *The curve γ_0 is a quasicircle.*

Proof. The curve γ_0 is built by a finite union of analytic curves (dynamical rays of angles 0 and 1/2 and equipotentials in the basins of attraction of the third roots of the unity). In order to proof that γ_0 is a quasicircle it is enough to show that the analytic curves are joined forming positive angles. This is trivially true at the points where dynamical rays connect with equipotentials. The only problems could happen at the crossing of the dynamical rays at the repelling fixed points $x_{0,j}$. Since $\partial A_0^*(\zeta^j)$, $j = 0, 1, 2$, are quasicircles (Lemma 3.6), it follows that the dynamical rays land at the fixed points $x_{0,j}$, where $j = 0, 1, 2$, with a positive angle. $\qquad\square$

4. DYNAMICS OF THE MAP R_a

In the next section we present a surgery construction relating the rational map R_a^2 (1) and the McMullen map M_λ (6). This construction is based on the fact that the curves γ_0, γ_1, γ_2, γ_3, and γ_4 (see Lemma 3.4) can be defined continuously for $|a|$ small enough keeping the same dynamics. We shall denote these continued curves by $\gamma_0(a)$, $\gamma_1(a)$, $\gamma_2(a)$, $\gamma_3(a)$, and $\gamma_4(a)$. The goal of the next lemmas is to introduce formally these curves. First, we introduce a lemma that allows us to locate the critical values which are the images of the critical points that appear for $a \neq 0$.

Lemma 4.1. *There exists a constant $\mathcal{C}_1 > 0$ that does not depend on a such that if $|a|$ is small enough, $a \neq 0$, then $|R_a(z)| < \mathcal{C}_1|a|^{2/3}$ for all $z \in \mathbb{A}\left(\frac{1}{|a|^{1/3}}, \frac{3}{|a|^{1/3}}\right)$.*

Proof. Every point in $\mathbb{A}\left(\frac{1}{|a|^{1/3}}, \frac{3}{|a|^{1/3}}\right)$ can be written as $\frac{b}{a^{1/3}}$, where $1 < |b| < 3$. We have

$$R_a\left(\frac{b}{a^{1/3}}\right) = \frac{\frac{2b^6}{a} + \frac{15b^3}{a} - b^3 + 3 - a}{\frac{3b^2}{a^{2/3}}\left(5 - a + \frac{b^3}{a} + b^3\right)} = \frac{(15b^3 + 2b^6)a^{2/3} + (3 - b^3)a^{5/3} - a^{8/3}}{3b^5 + 3b^2(5 + b^3 - a)a}.$$

Therefore, we have

$$\left|R_a\left(\frac{b}{a^{1/3}}\right)\right| = \left|\frac{15 + 2b^3}{3b^2}\right||a|^{2/3} + o(|a|^{2/3}).$$

To finish the proof it is enough to take

$$\mathcal{C}_1 = 1 + \max_{1 \leq |b| \leq 3}\left|\frac{15 + 2b^3}{3b^2}\right|.$$

$\square$

The curve γ_0 is defined using the dynamical rays of angles 0 and 1/2 of ζ^j cut through the equipotentials of level l_0 so that $\zeta^j \notin Int(\gamma_0)$. The following lemma establishes the necessary conditions so that this construction can be repeated for $a \neq 0$ and serves as definition of $\gamma_0(a)$.

Lemma 4.2. *There exists $\Lambda_0 \subset \mathbb{C}$ an open and simply connected set of parameters containing $a = 0$ such that the following hold:*

  i) *The critical points $c_{a,j}$ (see (2)) do not lie on the external rays of angles 0 or 1/2 or the equipotentials of level l_0 of the immediate basins of attraction of ζ^j, $j = 0, 1, 2$.*

  ii) *The fixed points $x_{a,j}$ (see (7)) are repelling.*

Moreover, for $a \in \Lambda_0$, the curve γ_0 admits a holomorphic motion whose image is a Jordan curve $\gamma_0(a)$ formed by the dynamical rays of angles 0 and 1/2 and the equipotentials of level l_0. This curve $\gamma_0(a)$ is symmetric with respect to action of $\mathbb{U}$ and is a quasicircle as γ_0.

Proof. The fact that $\gamma_0(a)$ is a Jordan curve that is well defined by dynamical continuation of γ_0 follows directly from i) and ii). Since the curve $\gamma_0(a)$ is defined piecewise by dynamical objects that move holomorphically, it is a holomorphic motion of γ_0. Since γ_0 is a quasicircle (see Corollary 3.7) it follows from the λ-Lemma (see [9]) that $\gamma_0(a)$ is also a quasicircle.

Notice that the $x_{a,j}$ collide at $a = \pm 3$ (see (7)). However, those parameters do not belong to Λ_0. On the one hand, for $a = -3$ the fixed points $x_{-3,j}$ are parabolic. On the other hand, $a = 3$ is a singular parameter for which the degree of R_a decreases to 4, the fixed points $x_{3,j}$ and the critical points $c_{3,j}$ collapse at $z = 0$, and one of the fixed dynamical rays at the basin of attraction of each root of the unity lands at $z = \infty$ (the other one lands at $z = 0$, which is a repelling fixed point for this singular parameter). This last claim is also satisfied for $|a - 3|$ small enough, so $a = 3$ does not belong to $\partial\Lambda_0$. $\square$

Remark 4.3. The fixed points $x_{a,j}$ are repelling for parameters in the complement of the closed disk of centre $a = -5$ and radius 2 (compare [4, Proposition 2.4]).

Once the curve $\gamma_0(a)$ is defined as a holomorphic motion of γ_0 over a set of parameters Λ_0 (Lemma 4.2), we can define recursively $\gamma_i(a)$ as a holomorphic motion of γ_i, $i = 1, 2, 3, 4$. In the next lemma we introduce these curves and describe their basic properties.

Lemma 4.4. *Define $\gamma_i(a)$, $i = 1, 2, 3, 4$, recursively as follows. Let $\Lambda_i \subset \Lambda_{i-1}$ be an open simply connected set of parameters such that $\gamma_{i-1}(a)$ does not contain any critical value. Let $\gamma_i(a)$ to be the connected component of $R_a^{-1}(\gamma_{i-1}(a))$ which is a holomorphic motion of $\gamma_i(a)$. Then, the curves $\gamma_i(a)$ satisfy the following properties:*

- *i) They are quasicircles and are symmetric with respect to rotation by a third root of the unity.*
- *ii) The curve $\gamma_i(a)$, $i = 1, 2, 3, 4$ is mapped 2 to 1 onto $\gamma_{i-1}(a)$ under R_a.*

Moreover, we have the inclusions $\gamma_4(a) \subset Int(\gamma_2(a))$, $\gamma_2(a) \subset Int(\gamma_0(a))$, $\gamma_0(a) \subset Int(\gamma_1(a))$, and $\gamma_1(a) \subset Int(\gamma_3(a))$.

Proof. Since the sets Λ_i are chosen so that the curves $\gamma_{i-1}(a)$ do not contain critical values and $\gamma_0(a)$ is a quasicircle (Lemma 4.2) it follows that all connected components of $R_a^{-1}(\gamma_{i-1}(a))$ are quasicircles. The curves $\gamma_i(a)$ are symmetric with respect to rotation by a third root of the unity since $\gamma_0(a)$ also is and this property is preserved by backward iterations of R_a (as long as the set surrounds $z = 0$).

The fact that the curves $\gamma_i(a)$, $i = 1, 2, 3, 4$ are mapped 2 to 1 onto $\gamma_{i-1}(a)$ follows from the fact that they are holomorphic motions of $\gamma_i(a)$ and the curves $\gamma_i(a)$ satisfy the same property (see Lemma 3.4). The final inclusions also come from the corresponding inclusions of the curves $\gamma_i(a)$. □

Notice that in the previous lemma we have defined recursively the sets Λ_i. We would like to point out that it was not strictly necessary to define all those sets due to the inclusions of the curves. Indeed, by Lemma 4.1, since $\gamma_0(a) \subset Int(\gamma_1(a))$ it follows that we can take $\Lambda_1 = \Lambda_2$. Also, since $\gamma_2(a) \subset Int(\gamma_3(a))$ we can take $\Lambda_3 = \Lambda_4$. Using the previous lemma we can now fix the set of parameters on which we will perform the surgery construction, which actually corresponds to Λ_4.

Definition 4.5. We define $\Lambda := \Lambda_4$, i.e. $\Lambda \subset \mathbb{C}$ as an open simply connected set of parameters containing $a = 0$ such that the holomorphic motion of $\gamma_2(a)$ is well defined and $\gamma_2(a)$ contains no critical value.

Even though it is not the goal of this paper, it follows from standard results in holomorphic dynamics that Λ can be taken as indicated in Figure 7. The set $\partial\Lambda$ is chosen so that the critical values $v_{a,j}$ (3) lie on $\gamma_2(a)$. Notice that the chosen Λ is contained in the complement of the disk where the fixed points $x_{a,j}$ (7) are non-repelling (see Remark 4.3).

We have proven that for $a \in \Lambda$ the curves $\gamma_j(a)$ are mapped 2 to 1 onto γ_{j-1} for $j = 1, \ldots, 4$. In this sense, the dynamics of R_0 is preserved after perturbation. The dynamics of R_a restricted to the regions bounded by these curves is also maintained. However, after perturbation, the dynamics of R_a on the unbounded regions delimited by these curves changes. This is described in the next proposition (see Figure 8).

Proposition 4.6. *Let a in $\Lambda \setminus \{0\}$. Then, there exists a preimage $\gamma_2'(a)$ of $\gamma_2(a)$ under R_a which is a simple closed curve that is mapped 1 to 1 onto $\gamma_2(a)$, is invariant with respect to rotation by a third root of the unity, and satisfies $\gamma_3(a) \subset Int(\gamma_2'(a))$. Moreover, the following holds.*
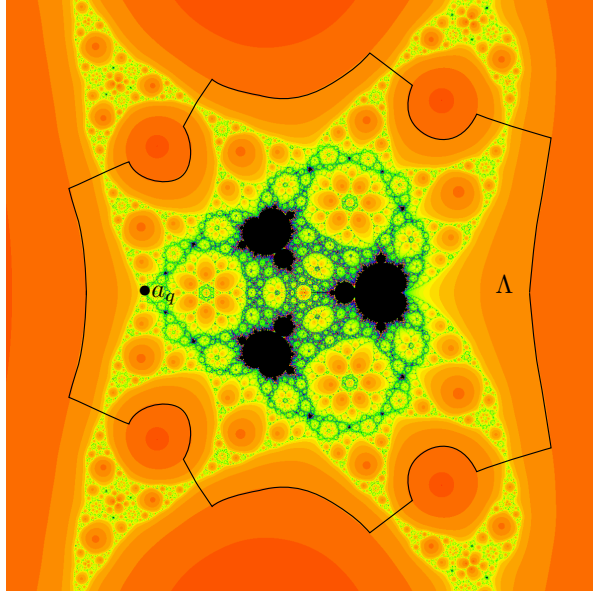
FIGURE 7. Sketch of the domain Λ. The parameters a are such that Re(a) $\in (-0.05, 0.05)$ and Im(a) $\in (-0.05, 0.05)$.

i) The map $R_a : Int(\gamma_2(a)) \to Ext(\gamma_1(a))$ is proper of degree 2.

ii) The map $R_a : A(\gamma_1(a), \gamma_3(a)) \to A(\gamma_2(a), \gamma_0(a))$ is proper of degree 2.

iii) The map $R_a : Ext(\gamma'_2(a)) \to Ext(\gamma_2(a))$ is proper of degree 1.

iv) The map $R_a : A(\gamma'_2(a), \gamma_3(a)) \to Int(\gamma_2(a))$ is proper of degree 3.

In particular, the annulus $A(\gamma'_2(a), \gamma_3(a))$ contains the 3 critical points and the 3 zeros that appear near $z = \infty$ after perturbation.

Proof. Statement i) follows from the fact that $a \in \Lambda$ since, therefore, the dynamics in the region bounded by $\gamma_2(a)$ remain unchanged. In particular, the only pole of R_a in $Int(\gamma_2(a))$ is $z = 0$, which is a pole of order 2. Similarly, it is easy to see that $A(\gamma_1(a), \gamma_3(a))$ is a connected component of $R_a^{-1}(A(\gamma_2(a), \gamma_0(a))$ (notice that, by construction, $A(\gamma_1(a), \gamma_3(a))$ cannot contain neither zeros nor poles). Therefore, $R_a : A(\gamma_1(a), \gamma_3(a)) \to A(\gamma_2(a), \gamma_0(a))$ is proper. By definition of Λ the annulus $A(\gamma_2(a), \gamma_0(a))$ cannot contain critical values. We conclude that the degree of the proper map is achieved on the boundaries of the annulus, and so this degree is 2. This proves ii).

For $a = 0$, the curve γ_3 is mapped with degree 2 onto γ_2. Moreover, all the other preimages of γ_2 lie in the region bounded by γ_3 (see Remark 3.5). Recall that Λ consists of the open connected set of parameters containing $a = 0$ such that no critical values has reached $\gamma_2(a)$. Equivalently, Λ consists of the maximum set of parameters for which all preimages of γ_2 under R_0 can be continued as preimages of $\gamma_2(a)$ under R_a. In particular, all preimages of $\gamma_2(a)$ in $\overline{Int(\gamma_3(a))}$ correspond to holomorphic motions of the preimages of γ_2 under R_0. Since R_0 has degree 5 and R_a has degree 6, it follows that there is a simple closed curve $\gamma'_2(a) \subset Ext(\gamma_3(a)))$ which is mapped 1 to 1 onto $\gamma_2(a)$ under R_a. Moreover, $\gamma'_2(a)$ is invariant under
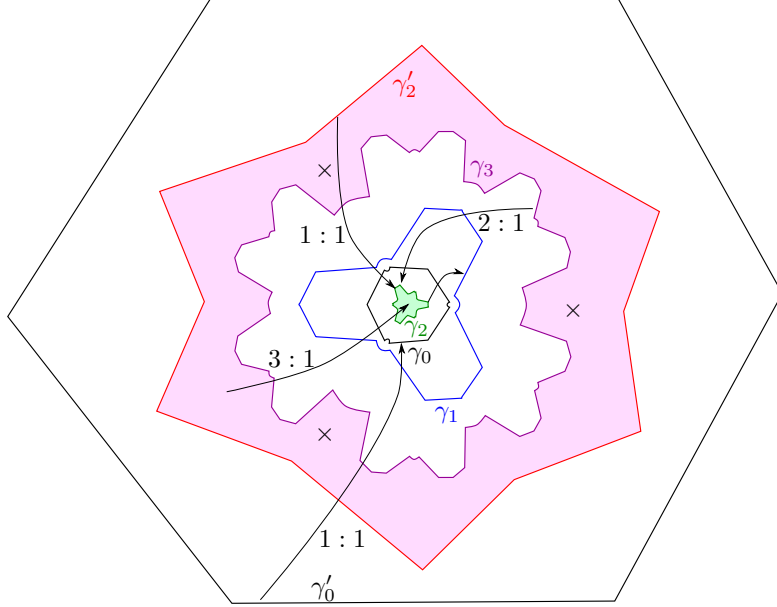
FIGURE 8. Sketch of the dynamics of R_a described in Proposition 4.6. The pink annulus $A(\gamma_2(a), \gamma_0(a))$ contains three critical points (marked with crosses) and is mapped with degree 3 onto the green disc bounded by $\gamma_2(a)$. In order not to overload the figure, the dependence of the curves on a is omitted.

rotation by a third root of the unity since $\gamma_2(a)$ is also invariant. We obtain that $\gamma_3(a) \subset Int(\gamma_2'(a))$.

It follows from Remark 3.5 that $Int(\gamma_1(a))$ contains all the preimages of $z = \infty$ other than itself. We can conclude that $R_a : Ext(\gamma_2'(a)) \to Ext(\gamma_2(a))$ is proper of degree 1. This proves statement iii).

Since the curves $\gamma_2'(a)$ and $\gamma_3(a)$ are mapped onto $\gamma_2(a)$ with degree 1 and 2, respectively, and the annulus $A(\gamma_2'(a), \gamma_3(a))$ contains no preimage of $z = \infty$, it follows that $R_a : A(\gamma_2'(a), \gamma_3(a)) \to Int(\gamma_2(a))$ is proper of degree 3. This proves statement iv). The final claim follows from the Riemann-Hurwitz formula (see for instance [13]) since 3 critical points, counting multiplicity, are required to map a doubly connected domain onto a simply connected domain via a proper map of degree 3. $\qquad\square$

5. SURGERY CONSTRUCTION FROM R_a^2 TO $z^4 + \lambda/z^2$

In this section we relate the dynamics of R_a with the one of $z^4 + \lambda/z^2$. To do so we will perform a cut and paste surgery (see [2]). In this sense, the first step is to build a 'rational-like map' which can be used to define the cut and paste surgery. This rational-like configuration (see Figure 10) is defined for R_a^2 and is based on the dynamics of R_a described in Proposition 4.6 (compare with Figure 8 and Figure 9). However, in order to make the main surgery construction easier to understand, since we are looking at R_a^2, we will introduce a new notation for some of the curves.

Proposition 5.1. *Let a in $\Lambda \backslash \{0\}$. There exist quasicircles $\beta_1^{in}(a)$, $\beta_2^{in}(a)$, $\beta_0^{out}(a)$, $\beta_1^{out}(a)$ and $\beta_2^{out}(a)$ which are analytic except on a finite set of points, surround $z = 0$, are invariant with respect to rotation by a third root of the unity, and such that the following hold:*

 i) *The curves $\beta_1^{out}(a)$ and $\beta_2^{out}(a)$ are mapped with degree 4, under R_a^2, onto $\beta_0^{out}(a)$ and $\beta_1^{out}(a)$, respectively.*

 ii) *The curves $\beta_1^{in}(a)$ and $\beta_2^{in}(a)$ are mapped with degree 2, under R_a^2, onto $\beta_0^{out}(a)$ and $\beta_1^{out}(a)$, respectively.*

 iii) *We have the inclusions*
   - *$\beta_1^{in}(a) \subset Int(\beta_2^{in}(a))$;*
   - *$\beta_2^{in}(a) \subset Int(\beta_2^{out}(a))$;*
   - *$\beta_2^{out}(a) \subset Int(\beta_1^{out}(a))$;*
   - *$\beta_1^{out}(a) \subset Int(\beta_0^{out}(a))$.*

 iv) *The map R_a^2 satisfies:*
   - *$R_a^2 : A(\beta_2^{in}(a), \beta_2^{out}(a)) \rightarrow Int(\beta_1^{out}(a))$ is proper of degree 6.*
   - *$R_a^2 : A(\beta_1^{in}(a), \beta_1^{out}(a)) \rightarrow Int(\beta_0^{out}(a))$ is proper of degree 6.*

Proof. We define $\beta_0^{out}(a) := \gamma_0(a)$, $\beta_1^{out}(a) := \gamma_2(a)$, and $\beta_2^{out}(a) := \gamma_4(a)$. Notice that, by definition, $\beta_0^{out}(a)$ is a quasicircle which is analytic except at a finite set of points. This property is also satisfied by all its iterated preimages (as long as they do not contain a critical point). Statement $i)$ follows directly from Lemma 4.4. By point $iii)$ of Proposition 4.6 , there exists a simple closed curve $\gamma_0'(a) \subset Ext(\gamma_2'(a))$ that is mapped with degree 1 onto $\beta_0^{out}(a) = \gamma_0(a)$, separates $\gamma_2'(a)$ from $z = \infty$ and is symmetric with respect to rotation by a third root of the unity (see Figure 8).
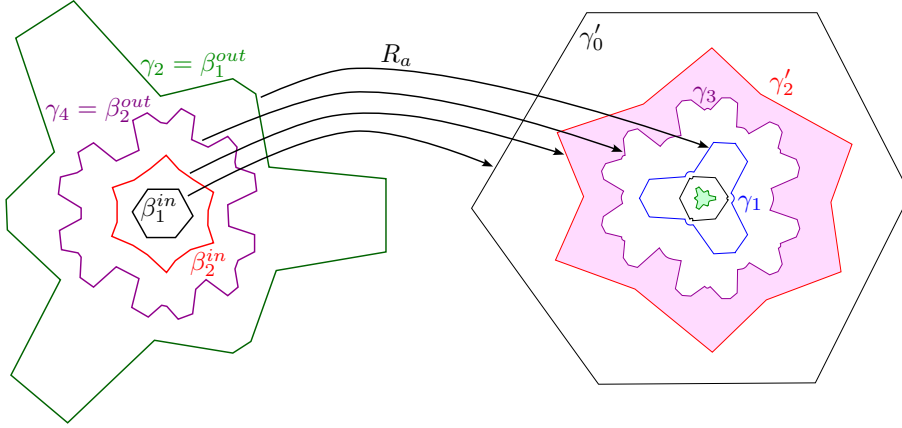

FIGURE 9. Sketch of how the curves $\beta_1^{in}(a)$, $\beta_2^{in}(a)$, $\beta_2^{out}(a)$ and $\beta_1^{out}(a)$ are defined using the dynamics of R_a described in Proposition 4.6. Each of the curves is mapped with degree 2 onto its image. The dependence of the curves on a is omitted.

The curves $\beta_1^{in}(a)$ and $\beta_2^{in}(a)$ are obtained by $i)$ of Proposition 4.6, taking the respective preimage of $\gamma_0'(a)$ and $\gamma_2'(a)$ contained in $Int(\gamma_2(a))$ (see Figure 9). Recall here that, by $i)$ of Proposition 4.6 $R_a : Int(\gamma_2(a)) \rightarrow Ext(\gamma_1(a))$ is proper of degree 2, so $\beta_1^{in}(a)$ and $\beta_2^{in}(a)$ are mapped 2 to one onto $\beta_0^{out}(a)$ and $\beta_1^{out}(a)$,

respectively, under R_a^2. Notice also that since $\gamma_3(a)$ separates $\gamma_2'(a)$ from $\gamma_1(a)$ and $\beta_2^{out}(a) = \gamma_4(a)$ is a preimage of $\gamma_3(a)$, we have that $\beta_1^{in}(a) \subset Int(\beta_2^{in}(a))$ and $\beta_2^{in}(a) \subset Int(\beta_2^{out}(a))$. Together with Lemma 4.4, this finishes the proof of $i)$, $ii)$ and $iii)$.

Finally, we prove $iv)$. By $i)$ and $iii)$ of Proposition 4.6 we know that the maps $R_a : Int(\gamma_2(a)) \to Ext(\gamma_1(a))$ and $R_a : A(\gamma_2'(a), \gamma_3(a)) \to Int(\gamma_2(a))$ are proper of degree 2 and 3, respectively. Recall that the curves $\beta_2^{in}(a)$ and $\beta_2^{out}(a) = \gamma_4(a)$ are the respective preimages of the curves $\gamma_2'(a)$ and $\gamma_3(a)$ in $Int(\gamma_2(a)) = Int(\beta_1^{in}(a))$. So we can conclude that $R_a^2 : A(\beta_2^{in}(a), \beta_2^{out}(a)) \to Int(\beta_1^{out}(a))$ is proper of degree 6. This proper map can be extended to a degree 6 proper map $R_a^2 : A(\beta_1^{in}(a), \beta_1^{out}(a)) \to Int(\beta_0^{out}(a))$. This follows directly from Proposition 4.6. This finishes the proof.   □
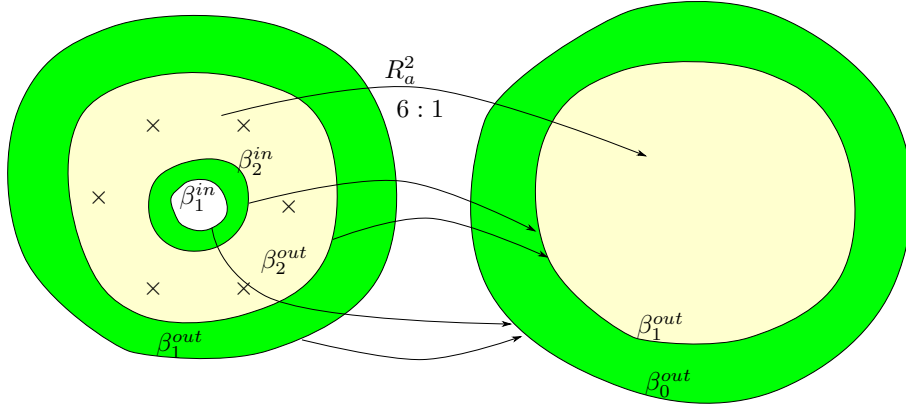

FIGURE 10. Sketch of the dynamics of the curves $\beta_1^{in}(a)$, $\beta_2^{out}(a)$, $\beta_2^{out}(a)$, and $\beta_1^{out}(a)$. The yellow annulus $A(\beta_2^{in}(a), \beta_2^{out}(a))$ is mapped under R_a^2 with degree 6 into the yellow disc $Int(\beta_1^{out}(a))$. We also mark the 6 critical points in the annulus with crosses. The dependence of the curves on a is omitted.

Remark 5.2. It follows from the previous statement that $A(\beta_1^{in}(a), \beta_1^{out}(a))$ contains exactly 6 critical points. Indeed, by Proposition 4.6 $i)$, $A(\beta_2^{in}(a), \beta_2^{out}(a))$ is mapped 2 to 1 onto $A(\gamma_2'(a), \gamma_3(a))$, which contains the 3 critical points $c_{a,j}$, $j = 0, 1, 2$, of R_a. We can conclude that if $a \in \Lambda$ the maps $R_a^2|_{A(\beta_1^{in}(a), \beta_1^{out}(a))}$ have exactly 6 different critical points which are mapped under iteration of R_a^2 onto exactly 3 critical values. Notice that the 6 critical points cannot be mapped onto exactly one critical value since such critical value would have 12 preimages under $R_a^2|_{A(\beta_1^{in}(a), \beta_1^{out}(a))}$, counting multiplicity. This is impossible since $R_a^1|_{A(\beta_1^{in}(a), \beta_1^{out}(a))}$ is a degree 6 proper map.

Once we have the rational-like configuration (Proposition 5.1), we can proceed to prove Theorem A. This is the content of Theorem 5.3. This theorem relates the dynamics of the maps R_a with the one of the McMullen maps $M_\lambda(z) = z^4 + \lambda/z^2$ within the annulus $A(\beta_2^{in}(a), \beta_2^{out}(a))$.

Theorem 5.3. *Let a in $\Lambda \setminus \{0\}$. Then, there exists a quasiconformal map $\varphi_a :$ $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi_a \circ R_a^2(z) = M_{\lambda(a)} \circ \varphi_a(z)$ for all $z \in A(\beta_2^{in}(a), \beta_2^{out}(a))$, where $a \mapsto \lambda(a)$ is a map defined on Λ. Moreover, $\varphi_a^{-1}(\mathcal{J}(M_{\lambda(a)})) \subset A(\beta_2^{in}(a), \beta_2^{out}(a))$.*

Proof. The idea of the proof is to perform a cut and paste surgery (see [2]). More specifically, we will build a model map that coincides with R_a^2 over $A(\beta_2^{in}(a), \beta_2^{out}(a))$, has the dynamics of z^4 and $1/z^2$ in $Ext(\beta_1^{out})$ and $Int(\beta_1^{in})$, respectively, and is globally quasisymmetric. Finally, we will use the Measurable Riemann Mapping Theorem ([2]) and Proposition 2.1 to conclude that the model map is quasiconformally conjugated to a map of the family M_λ.

We first explain how to glue the dynamics of z^4 in $Ext(\beta_1^{out})$ with the one of R_a^2 in $A(\beta_2^{in}(a), \beta_2^{out}(a))$. Pick $\rho > 1$. Let

$$\Phi_{out} : Ext(\beta_1^{out}(a)) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_{\rho^4}$$

be the Riemann map that has positive real derivative at $z = \infty$ and fixes it. Since the curve $\beta_1^{out}(a)$ is invariant under rotation by a third root of the unity, it follows that $\Phi_{out}(\xi \cdot z) = \xi \cdot \Phi_{out}(z)$ for any third root of the unity ξ and $z \in Ext(\beta_1^{out}(a))$. Indeed, since the Riemann map fixing $z = \infty$ is unique up to rotation and $\xi^{-1}\Phi_{out}(\xi \cdot z)$ also has positive real derivative at $z = \infty$, it follows that $\xi^{-1}\Phi_{out}(\xi \cdot z) = \Phi_{out}(z)$. This property is also satisfied by the power map $\Phi_{out}^4(z)$.

Since $\beta_1^{out}(a)$ is a quasicircle, the map Φ_{out} extends to the boundary as a quasisymmetric map (see [2, Thm. 2.9]). Moreover, since $\beta_1^{out}(a)$ is a finite union of analytic curves, this quasisymmetric maps is analytic except at a finite set of points (see [2, Remark 2.12]). Let $\psi_{1,out} : \beta_1^{out}(a) \to \mathbb{S}_{\rho^4}^1$ be the extension map. Since $\psi_{1,out} \circ R_a^2|_{\beta_2^{out}}(a) : \beta_2^{out}(a) \to \mathbb{S}_{\rho^4}^1$ has degree 4, we can choose a quasisymmetric lift $\psi_{2,out} : \beta_2^{out}(a) \to \mathbb{S}_\rho^1$ which is analytic except in a finite set of points so that $\psi_{1,out}(R_a^2(z)) = (\psi_{2,out}(z))^4$. This lift $\psi_{2,out}$ can be chosen so that $\psi_{2,out}(\xi \cdot z) = \xi \cdot \psi_{2,out}(z)$ for any third root of the unity ξ. By [2, Proposition 2.30] there exists a quasiconformal map

$$\psi_{out} : \overline{A}(\beta_2^{out}(a), \beta_1^{out}(a)) \to \overline{\mathbb{A}}(\rho, \rho^4)$$

such that $\psi_{out}|_{\beta_2^{out}(a)} = \psi_{2,out}$ and that $\psi_{out}|_{\beta_1^{out}(a)} = \psi_{1,out}$. Moreover ψ_{out} can be chosen so that $\psi_{out}(\xi \cdot z) = \xi \cdot \psi_{out}(z)$ for any third root of the unity ξ. Indeed, [2, Proposition 2.30] is based on [2, Proposition 2.28], which extends quasisymmetric boundary maps on a straight annulus to a quasiconformal map on the annulus, together with a uniformization map. It is not difficult to see that the quasiconformal map built in [2, Proposition 2.28] is symmetry with respect to rotation by a third root of the unity if the boundary maps are also symmetric. As is the case with the Riemann map, the uniformization map sending a non straight annulus to a straight annulus can also be chosen to be symmetric.

We can now define a quasiregular map in $Ext(\beta_2^{in}(a))$ as:

$$F_a(z) = \begin{cases} \Phi_{out}^{-1}\left(\Phi_{out}^4(z)\right) & \text{for} \quad z \in Ext(\beta_1^{out}(a)) \\ \Phi_{out}^{-1}\left(\psi_{out}^4(z)\right) & \text{for} \quad z \in \overline{A}(\beta_2^{out}(a), \beta_1^{out}(a)) \\ R_a^2(z) & \text{for} \quad z \in A(\beta_2^{in}(a), \beta_2^{out}(a)). \end{cases}$$

To complete the model we have to glue the dynamics of $1/z^2$ in $Int(\beta_2^{in}(a))$. The construction is completely analogous to the previous case, so we skip some details.

Let

$$\Phi_{in} : Int(\beta_1^{in}(a)) \to \mathbb{D}_{1/\rho^4}$$

be the Riemann map that has positive real derivative at $z = 0$ and fixes it. The map Φ_{in} is symmetric with respect to rotation by a third root of the unity. Let $\psi_{1,in} : \beta_1^{in}(a) \to \mathbb{S}_{1/\rho^4}^1$ be the quasisymmetric extension of Φ_{in}. Since $\psi_{1,out} \circ R_a^2|_{\beta_2^{in}(a)} : \beta_2^{in}(a) \to \mathbb{S}_{\rho^4}^1$ has degree 2, there exists a quasisymmetric lift $\psi_{2,in} : \beta_2^{in}(a) \to \mathbb{S}_{1/\rho^2}^1$ such that $\psi_{1,out}(R_a^2(z)) = 1/(\psi_{2,in}(z))^2$. By [2, Proposition 2.30] there exists a quasiconformal map

$$\psi_{in} : \overline{A}(\beta_1^{in}(a), \beta_2^{in}(a)) \to \overline{\mathbb{A}}(1/\rho^4, 1/\rho^2)$$

such that $\psi_{in}|_{\beta_1^{in}(a)} = \psi_{1,in}$ and that $\psi_{in}|_{\beta_2^{in}(a)} = \psi_{2,in}$. As before, ψ_{in} can be taken to be symmetric with respect to rotation by a third root of the unity. Finally, we can define our model map in the whole Riemann Sphere as:

$$F_a(z) = \begin{cases} \Phi_{out}^{-1}\left(\Phi_{out}^4(z)\right) & \text{for} \quad z \in Ext(\beta_1^{out}(a)) \\ \Phi_{out}^{-1}\left(\psi_{out}^4(z)\right) & \text{for} \quad z \in \overline{A}(\beta_2^{out}(a), \beta_1^{out}(a)) \\ R_a^2(z) & \text{for} \quad z \in A(\beta_2^{in}(a), \beta_2^{out}(a)) \\ \Phi_{out}^{-1}\left(\frac{1}{\psi_{in}^2(z)}\right) & \text{for} \quad z \in \overline{A}(\beta_1^{in}(a), \beta_2^{in}(a)) \\ \Phi_{out}^{-1}\left(\frac{1}{\Phi_{in}^2(z)}\right) & \text{for} \quad z \in Int(\beta_1^{in}(a)), \end{cases}$$

The map $F_a(z)$ is quasiregular, is symmetric with respect to rotation by a third root of the unity, and has topological degree 6 by construction. Moreover, F_a is holomorphic in $\widehat{\mathbb{C}} \setminus \{A(\beta_1^{in}(a), \beta_2^{in}(a)) \cup A(\beta_2^{out}(a), \beta_1^{out}(a))\}$. We continue by defining an F_a-invariant complex structure σ. Notice that the orbit of a point z can go at most once through $A(\beta_1^{in}(a), \beta_2^{in}(a)) \cup A(\beta_2^{out}(a), \beta_1^{out}(a))$. Denote $A_n = \{z \mid F_a^n(z) \in A(\beta_1^{in}(a), \beta_2^{in}(a)) \cup A(\beta_2^{out}(a), \beta_1^{out}(a))\}$. Thus, it is enough to define

$$\sigma_a = \begin{cases} \psi_\infty^* \sigma_0 & \text{for} \quad z \in A(\beta_2^{out}(a), \beta_1^{out}(a)) \\ \psi_0^* \sigma_0 & \text{for} \quad z \in A(\beta_1^{in}(a), \beta_2^{in}(a)) \\ (F_a^n)^{\circledast} \sigma_a & \text{for} \quad z \in A_n \\ \sigma_0 & \text{elsewhere,} \end{cases}$$

where σ_0 denotes the standard complex structure and * the pull-back operation. By construction, $F_a^* \sigma_a = \sigma_a$. Since F_a is holomorphic outside $A(\beta_1^{in}(a), \beta_2^{in}(a)) \cup A(\beta_2^{out}(a), \beta_1^{out}(a))$, σ has bounded dilatation. Let ξ denote any third root of the unity and let $O_\xi(z) = \xi \cdot z$. Since F_a satisfies $O_\xi \circ F_a = F_a \circ O_\xi$, and so do ψ_{out} and ψ_{in}, we have that $O_\xi^* \sigma_a = \sigma_a$. Let ϕ_a be the integrating map given by the Measurable Riemann Mapping Theorem (see [1, p. 57] or [2, Theorem 1.28]) which fixes $z = 0$ and $z = \infty$ and is tangent to the identity at $z = \infty$ (notice that ϕ_a is holomorphic in a neighbourhood of $z = \infty$). Then, $\phi_a^* \sigma_0 = \sigma_a$. It follows from the unicity of the integrating map modulus post-composition with conformal automorphisms of $\widehat{\mathbb{C}}$ that $\phi_a = O_\xi^{-1} \circ \phi_a \circ O_\xi$ since $O_\xi^{-1} \circ \phi_a \circ O_\xi$ would satisfy the same normalizations and $(O_\xi^{-1} \circ \phi_a \circ O_\xi)^* \sigma_0 = O_\xi^* \sigma_a = \sigma_a$.

Finally, define $G_a = \phi_a \circ F_a \circ \phi_a^{-1}$. By construction, G_a is a rational map of degree 6. Given any third root of the unity ξ, the map G_a satisfies $\xi \cdot G_a(z) = G_a(\xi \cdot z)$ since both F_a and ϕ_a satisfy the same condition. By construction, G_a maps $z = 0$ to $z = \infty$ with local degree 2, the point $z = \infty$ is super-attracting of local degree 4 and G_a has 6 critical points which are mapped onto exactly 3 critical values

(compare Remark 5.2). By Proposition 2.1 we conclude that G_a is conjugated to the map

$$M_{\lambda(a)}(z) = z^4 + \frac{\lambda(a)}{z^2},$$

under a linear map L_a. To finish the proof it is enough to take $\varphi_a = L_a \circ \phi_a$. Notice that, by construction, $Ext(\beta_2^{out})$ and $Int(\beta_2^{int})$ belong to the basin of attraction of $z = \infty$ under F_a. Therefore, $\varphi_a^{-1}(\mathcal{J}(M_{\lambda(a)})) \subset A(\beta_2^{in}(a), \beta_2^{out}(a))$ $\square$

Remark 5.4. The parameter $\lambda(a)$ depends in a as well as in the level l_0 of the equipotentials chosen to define γ_0 (Definition 3.3).

6. FURTHER RESULTS

In Theorem A we relate the dynamics of the map R_a^2 with the ones of a map $M_{\lambda(a)}$. The Escape Trichotomy Theorem (see section 2) states that if all critical orbits of M_λ escape to ∞, then $\mathcal{J}(M_\lambda)$ is either a Cantor set, a Cantor set of circles or a Sierpinski carpet. In this section we study the Julia set of the maps R_a, justify that these three cases are achieved by the map $M_{\lambda(a)}$ for different values of a (compare Figure 3). The next two results help us to understand how the Julia set moves for $|a|$ small.

Let $\delta_0(0) := \partial A_0^*(0)$ and $\delta_\infty(0) := \partial A_0^*(\infty)$ be the boundaries of the immediate basin of attraction of 0 and ∞ under R_0^2. By Lemma 3.6, $\delta_0(0)$ and $\delta_\infty(0)$ are quasicircles. The next lemma states that there is a holomorphic motion of these curves in a small neighbourhood of $a = 0$.

Lemma 6.1. *There is a holomorphic motion $H(a, \cdot)$ of $\delta_0(0) \cup \delta_\infty(0)$ parameterized by a simply connected domain $\tilde{\Lambda} \subset \Lambda$ that is a neighbourhood of 0. In particular, for all $a \in \tilde{\Lambda}$ the curves $\delta_0(a) = H(a, \delta_0(0))$ and $\delta_\infty(a) = H(a, \delta_\infty(0))$ are quasicircles.*

Proof. The sets $\delta_0(0) = \partial A_0^*(0)$ and $\delta_\infty(0) = \partial A_0^*(\infty)$ are Jordan curves. Periodic points are dense in those curves because they correspond to the rational angles in the Böttcher parametrization. For $a = 0$ there is no parabolic points so that all the aforementioned periodic points are repelling. We will prove that they stay repelling for $|a|$ small enough. Assuming this property, we get a common neighbourhood of $a = 0$ on which we can follow each repelling periodic point (by implicit function theorem). This defines a holomorphic motion of the set of periodic point in the given curves. Note that the neighbourhood can be chosen simply connected. So, it then follows from the λ-Lemma (see [9]) that $\delta^0(0)$ and $\delta^\infty(0)$ admit a holomorphic motion on this neighbourhood so that they are quasicircles through the motion.

We now prove the claim that there exists a neighbourhood of $a = 0$ on which the periodic points of $\partial A_0^*(0)$ stay repelling for $|a|$ small enough (the proof is analogous for $\partial A_0^*(\infty)$). The idea is to perform a surgery which will eliminate all free critical points and keep the dynamics of all periodic points coming from $\partial A_0^*(0)$. Since the surgery construction is analogous to the classical one proposed by Douady and Hubbard [7] for polynomial-like mappings, we only explain on which curves the cut and paste is done (see also [2, Theorem 7.4]). As in Theorem 5.3, we consider the map R_a^2. For $a = 0$, the point $z = 0$ is super-attracting of local degree 4. Let ς be a geodesic at $A_0^*(0)$, defined with the Böttcher coordinate. Let ς_0^{-1} be the preimage of ς in $A_0^*(0)$ under R_0^2. Then ς_0^{-1} is also a geodesic and is mapped 4 to 1 onto ς. Since R_a converges uniformly on compact sets of $\mathbb{C}$ to R_0, if $|a|$ is small enough then we can pick a connected component ς_a^{-1} of $R_a^{-2}(\varsigma)$ which is a continuation of

ς_0^{-1}. Then, we can use ς_a^{-1} and ς and the curves $\beta_1^{out}(a)$ and $\beta_0^{out}(a)$ to perform a cut and paste surgery in which we glue the dynamics of z^4 near 0 and ∞ erasing all free critical points while keeping the dynamics of R_a^2 in the annulus bounded by $\beta_1^{out}(a)$ and ς_a^{-1}. The resulting map is quasiconformally conjugate to z^4. Since the continuations of all periodic points of $\partial A_0^*(0)$ and their orbits are contained in the annulus bounded by $\beta_1^{out}(a)$ and ς_a^{-1}, this surgery keeps their dynamics. Moreover, since the resulting map does not have free critical points, we conclude that these periodic points are repelling. $\qquad\square$

The next lemma tells us that the previous holomorphic motion can actually be extended to the closure of the union of the basins of attraction of the roots under R_0, $\overline{A_0(1)} \cup \overline{A_0(\zeta)} \cup \overline{A_0(\zeta^2)}$, for all $a \in \tilde{\Lambda}$. This explains why for $|a|$ small we can see copies of these basins of attraction on the dynamical plane of R_a (see Figure 1).

Lemma 6.2. *There exists a holomorphic motion of $\overline{A_0(1)} \cup \overline{A_0(\zeta)} \cup \overline{A_0(\zeta^2)}$ which is parametrized by $\tilde{\Lambda}$ with $H(a, \overline{A_0(\zeta^k)}) \subset \overline{A_a(\zeta^k)}$ for $a \in \tilde{\Lambda}$ and $k = 0, 1, 2$.*

Proof. Let ϕ_a be the Böttcher map of R_a around the super-attracting fixed point 1. It is well defined for $a \in \tilde{\Lambda}$ on $A_a^*(1)$, the whole immediate basin of attraction of 1, since $A_a^*(1)$ is contained in the annulus A_{δ_a} bounded by $\delta_0(a)$ and $\delta_\infty(a)$, which contains no free critical point (by Lemma 6.1). The map $H(a, z) = \phi_a(\phi_0^{-1}(z))$ is a holomorphic motion of the immediate basin of 1, $A_0^*(1)$. It can be pulled back to every connected component of the basin of attraction of 1 whose orbit never exits the annulus A_{δ_a} since A_{δ_a} does not contain any free critical point. As a consequence, the holomorphic motion H extends to the closure $\overline{A_0(1)}$ and one easily sees that $H(a, \overline{A_0(1)}) \subset \overline{A_a(1)}$. The argument is exactly the same for $\overline{A_0(\zeta)}$ and $\overline{A_0(\zeta^2)}$. $\quad\square$

After some Lemmas in order to understand how the Julia set moves with a, we show next that all cases of the Escape Trichotomy Theorem can be achieved and correspond to maps $M_{\lambda(a)}$. First we introduce a technical lemma which will be useful to study the case of the Cantor set of quasicircles. Its proof is analogous to Lemma 4.1.

Lemma 6.3. *Let $\mathcal{C} > 0$. There exists a $\mathcal{C}' > 0$ such that if $|a|$ is small enough, $a \neq 0$, then $|R_a^2(z)| > \mathcal{C}' \frac{1}{|a|^{1/3}}$ for all z such that $|z| < \mathcal{C}|a|^{2/3}$.*

In the next proposition we study the case of Cantor set of quasicircles.

Proposition 6.4. *If $|a|$ is small enough, $a \neq 0$, then $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of quasicircles.*

Proof. By Proposition 5.1 $iv)$ and the Riemann-Hurwitz formula, we know that the annulus $A(\beta_2^{in}(a), \beta_2^{out}(a))$ contains 6 critical points and 6 zeros of R_a^2. Recall that, for $|a|$ small enough, the annulus $\mathbb{A}\left(\frac{1}{|a|^{1/3}}, \frac{3}{|a|^{1/3}}\right)$ contains the 3 free critical points of R_a together with the 3 zeros that appear after the singular perturbation. Recall also that, by definition, $\beta_2^{out}(a) = \gamma_4(a)$, $\beta_1^{out}(a) = \gamma_2(a)$, and that $\beta_0^{out}(a) = \gamma_0(a)$.

Furthermore, there exists a connected component $\mathcal{A}_0(a)$ of $R_a^{-1}\left(\mathbb{A}\left(\frac{1}{|a|^{1/3}}, \frac{3}{|a|^{1/3}}\right)\right)$ which is a doubly connected set contained in $Int(\gamma_2(a))$ that is mapped with degree 2 onto $\mathbb{A}\left(\frac{1}{|a|^{1/3}}, \frac{3}{|a|^{1/3}}\right)$ under R_a, by Proposition 4.6 $i)$. It follows that $\mathcal{A}_0(a)$ contains 6 critical points and 6 zeros of R_a^2, which correspond precisely to the 6 critical

points and 6 zeros of R_a^2 in $A(\gamma_0''(a), \gamma_2(a))$ (notice that, by Proposition 4.6 i), there is no other preimage of critical points of R_a in $A(\beta_2^{in}(a), \beta_2^{out}(a))$).

In application of Lemma 4.1, there exists a $C_1 > 0$ such that for $|a|$ small enough the set $R_a^2(\mathcal{A}_0(a))$ is contained in a disk of radius $\mathcal{C}_1|a|^{2/3}$. Since $A(\beta_1^{in}(a), \beta_2^{in}(a))$ is mapped onto $A(\beta_1^{out}(a), \beta_0^{out}(a))$ under R_a^2, it follows that $\mathcal{A}_0(a) \subset A(\beta^{in}(a), \delta_0(a))$. Notice that, for $|a|$ small enough, the disk of radius $\mathcal{C}_1|a|^{2/3}$ is contained in the region bounded by $\delta_0(a)$. Moreover, by Lemma 6.3, for $|a|$ small enough the set $\mathcal{A}_0$ is mapped under 2 iterates of R_a^2 onto $Ext(\gamma_0(a))$.

We can conclude that the critical points of $M_{\lambda(a)}$ are mapped under exactly 2 iterates of $M_{\lambda(a)}$ onto $A^*_{M_{\lambda(a)}}(\infty)$. It follows from the Escape Trichotomy that $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of quasicircles.

$\square$

Proposition 6.4 gives us a condition so that the Julia set of $M_{\lambda(a)}$ is a Cantor set of quasicircles. We would like to understand also the structure of the Julia set of the corresponding map R_a. The connectedness of the Julia set of the maps $O_{n,\alpha}$ obtained when applying Chebyshev-Halley methods to $z^n + c$, $c \in \mathbb{C}$ is studied in [4]. It is proven that these maps cannot have Herman rings. Moreover, the following characterization for the connectivity of $\mathcal{J}(O_{n,\alpha})$ is provided. It depends on whether the immediate basin of attraction of 1, $\mathcal{A}^*_{O_{n,\alpha}}(1)$, contains extra critical points.

Theorem 6.5 ([4], Theorem 3.9). *For fixed $n \geq 2$ and $\alpha \in \mathbb{C}$, the Julia set $\mathcal{J}(O_{n,\alpha})$ is disconnected if and only if $\mathcal{A}^*_{O_{n,\alpha}}(1)$ contains a critical point $c \neq 1$ and no preimage of $z = 1$ other than itself.*

Notice that R_a corresponds to $O_{3,\alpha}$ with $a = 5 - 4\alpha$. It is not difficult to see that if a is such that $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of quasicircles, then no free critical point of R_a can belong to the immediate basin of attraction of 1. It then follows from Theorem 6.5 that $\mathcal{J}(R_a)$ is connected. Moreover, from Proposition 6.4 we know that $\mathcal{J}(R_a^2)$ contains an invariant Cantor set of quasicircles (which separate 0 from ∞). The image under R_a of this Cantor set of quasicircles is another Cantor set of quasicircles which also separate 0 from ∞. From all the previous facts we obtain the next corollary.

Corollary 6.6. *If $a \in \Lambda$ and $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of quasicircles, then $\mathcal{J}(R_a)$ is connected and contains an invariant Cantor set of quasicircles which separate 0 from ∞.*

Next we study the case of the Cantor set of points. Recall that Λ (see Definition 4.5) is defined as an open simply connected set containing $a = 0$ such that $\gamma_2(a)$ can be continued and contains no critical value (and, hence, $\gamma_4(a)$ is well defined). Since the set of parameters for which the fixed points $x_{a,j}$ does not surround $a = 0$ (see Remark 4.3) we can choose Λ to contain parameters a such that the critical values lie in $\gamma_4(a)$. It would follow directly that the critical values of $M_{\lambda(a)}$ lie in $A^*_{M_{\lambda(a)}}(\infty)$ and, by the Escape Trichotomy, $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of points. More specifically, we can prove the following.

Proposition 6.7. *Let $a \in \Lambda \setminus \{0\}$. Let $A^0 := A(\gamma_2(a), \gamma_0(a))$ and let $A^1 := A(\gamma_4(a), \gamma_2(a))$. Define recursively A^{n+1} as the connected component of $R_a^{-2}(A^n)$ which separates 0 and ∞ and shares a boundary component with A^n. Then, $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of points if, and only if, the critical values of R_a belong to $\overline{A^n}$ for some $n \geq 1$.*

Proof. First we will mention why these sets are well defined. The fact that given A^n there exists a connected component of $R_a^{-2}(A^n)$ satisfying the above conditions follows inductively from the fact that A^0 and A^1 satisfy these conditions. Notice that 0 cannot belong to any A^n since it is mapped under R_a^2 to ∞.

It is not difficult to see that the sets A^n are sent to $A^*_{M_{\lambda(a)}}(\infty)$ under the surgery construction that defines $M_{\lambda(a)}$. Moreover, the critical values of R_a coincide with the image under R_a^2 of the 6 critical points of R_a^2 which appear near 0 (and are preserved by the surgery construction). Therefore, if the critical values of R_a lie in $\overline{A^n}$ for some $n \geq 1$ we obtain that the critical values of $M_{\lambda(a)}$ lie in $A^*_{M_{\lambda(a)}}(\infty)$. By the Escape Trichotomy we can conclude that $\mathcal{J}(M_{\lambda(a)})$ is a Cantor set of points

Assume that there is no n such that the critical values of R_a belong to $\overline{A^n}$. Then, by the Riemann-Hurwitz Formula, the sets A^n are doubly connected. Moreover $\partial A^n \cap \partial A^{n+1}$ is a quasicircle and A^{n+1} lies in the bounded component of $\mathbb{C} \setminus A^n$. Notice also that for all $n > 1$ we have $A^n \subset A(\beta_2^{in}(a), \beta_2^{out}(a))$ (compare Proposition 5.1). In the limit, the sets A^n need to accumulate on an invariant curve $\widehat{\delta}_a \subset A(\beta_2^{in}(a), \beta_2^{out}(a))$ which belongs to the Julia set of R_a. A quasiconformal copy of this curve will belong to $\mathcal{J}(M_{\lambda(a)})$. Therefore, $\mathcal{J}(M_{\lambda(a)})$ cannot be a Cantor set of points.

We would like to point out that if $\widehat{\delta}_a$ contains no critical point, then it coincides with the $\delta_0(a)$ introduced in Lemma 6.1.

$\square$

Finally, we study the case of the Sierpinski carpet case.

Proposition 6.8. *There exists* $a^* \in \Lambda \cap \mathbb{R}^-$ *such that* $\mathcal{J}(M_{\lambda(a^*)})$ *is a Sierpinski carpet.*

Proof. It follows from the Escape Trichotomy Theorem that $\mathcal{J}(M_\lambda)$ is a Sierpinski carpet if, and only if, all critical points of M_λ are mapped into $A^*_{M_\lambda}(\infty)$ in $m > 2$ iterates. Therefore, in order to prove the existence of a parameter a^* for which $\mathcal{J}(M_\lambda(a^*))$ is a Sierpinski carpet it is enough to prove that the 6 critical points of $R_{a^*}^2$ which lie in $A(\beta_2^{in}(a^*), \beta_2^{out}(a^*))$ (see Proposition 5.1 *iv*), c.f. Proposition 6.4) are mapped in exactly 2 iterates of $R_{a^*}^2$ onto $z = 0$ and thus they are mapped in 3 iterates onto $z = \infty$. It can easily be shown that in this case the origin is not contained in $A^*_{M_\lambda}(\infty)$.

Let $a \in \Lambda$. Let us denote the 6 critical points of R_a^2 which lie in $A(\beta_2^{in}(a), \beta_2^{out}(a))$ by $\tilde{c}_{a,k}$, $k = 0, \cdots, 5$. The critical points $\tilde{c}_{a,k}$ of R_a^2 are precisely the preimages in $A(\beta_2^{in}(a), \beta_2^{out}(a))$ of the critical points $c_{a,j}$, $j = 0, 1, 2$ (see (2)). In particular, the images under R_a^2 of $\tilde{c}_{a,k}$ coincides with the images under R_a of $c_{a,j}$, that is, the critical values $v_{a,j}$ (see (3)). Therefore, we need to prove that there exists $a^* \in \Lambda$ such that $R_{a^*}^2(v_{a^*,j}) = 0$.

Hereafter we restrict to real parameters $a \in (-1, 0)$. As we mention before, the 3 free critical points of R_a are denoted by $c_{a,j}$ (see (2)) and the corresponding critical values by $v_{a,j} = R_a(c_{a,j})$ (see (3)). It is easy to check that $c_{a,0}$ is real and negative and $v_{a,0}$ is real and positive for $-1 < a < 0$.

Since every attracting and parabolic cycle must contain a critical point in its immediate basins of attraction, it follows that the real map R_a, $a \in (-1, 0)$, cannot have any attracting or parabolic cycle completely contained in $\mathbb{R}^+$. Indeed, if such a cycle exists, the critical point $c_{a,0} \in \mathbb{R}^-$ would belong to the immediate basin

of attraction $\mathcal{A}^*(y)$ of an attracting or parabolic periodic point $y \in \mathbb{R}^+$. However, this is impossible since, by symmetry with respect to rotation by a third root of the unity, the critical points $c_{a,1} = \zeta c_{a,0}$ and $c_{a,2} = \zeta^2 c_{a,0}$ would belong to the immediate basins of attraction $\mathcal{A}^*(\zeta y)$ and $\mathcal{A}^*(\zeta^2 y)$, where $\zeta = e^{2\pi i/3}$. Also by symmetry, the Fatou components $\mathcal{A}^*(y)$, $\mathcal{A}^*(\zeta y)$, and $\mathcal{A}^*(\zeta^2 y)$ would have non-empty intersection, which is impossible.

If $a = 0$ the map $R_0|_{\mathbb{R}^+}$ is strictly decreasing and satisfies $\lim_{x\to 0^+} R_0(x) = +\infty$ and $\lim_{x\to +\infty} R_0(x) = 0$ (notice that $x = 1$ is super-attracting of local degree 3 and that there are no free critical points). It follows that the intersection of the immediate basin of attraction of 1, $A_0^*(1)$, with the real line consists of a period two cycle $\{q_0, q_\infty\}$ such that $0 < q_0 < 1 < q_\infty$. It is not difficult to see that q_0 is precisely the intersection of the curve $\delta_0(0) = \partial A_0^*(0)$ (compare Lemma 6.1) with $\mathbb{R}^+$. After perturbation, for $|a|$ small, there is a holomorphic motion of the periodic point $q_0(a)$ (as well as the curve $\delta_0(a)$). Moreover, since for $a \in (-1, 0)$ there can be no parabolic cycle completely contained in $\mathbb{R}^+$, the holomorphic motion of $q_0(a)$ (and $q_\infty(a)$) is well defined for all $a \in (0, 1)$ and we have $0 < q_0(a) < 1 < q_\infty(a)$. Notice that when we move a from 0 up to -1 the critical value $v_{a,0}$ moves from 0 up to $+\infty$. Therefore, we can define a_q as the parameter in $(-1, 0)$ such that $0 < v_{a,0} < q_0(a)$ if $a_q < a < 0$ and $v_{a_q,0} = q_0(a_q)$.

The periodic point $q_0(a)$ and the parameter a_q are important because they give us a dynamical condition that we can control in order to ensure that a parameter $a \in (-1, 0)$ is in the set Λ. Indeed, before perturbation the point q_0 lies between $x = 0$ and $\gamma_2 \cap \mathbb{R}^+$ (compare Figure 5). The set of parameters Λ is defined as an open simply connected set of parameters such that the holomorphic motion $\gamma_2(a)$ is well defined and $\gamma_2(a)$ contains no critical value (see Definition 4.5). Since the fixed points $x_{a,j}$ are repelling in the complement of the closed disk of centre -5 and radius 2 (see Remark 4.3), it follows that Λ can be chosen to include a neighbourhood of the interval $(a_q, 0)$ (see Figure 7).

Now we can easily prove the existence of the parameter a^*. If $a \in (-1, 0)$ the function $R_a|_{\mathbb{R}^+}$ is monotonous decreasing (it has no other critical point than $x = 1$). When x increases from 0 to 1, $R_a(x)$ decreases from $+\infty$ down to 1. When x increases from 1 to $+\infty$, $R_a(x)$ decreases from 1 down to $-\infty$. Since $v_{a,0}$ tends to 0 when a tends to 0, it is not difficult to show that $R_a(v_a, 0) \to +\infty$ and $R_a^2(v_a, 0) \to -\infty$ when a tends to 0 (compare with Lemma 6.3 and proof of Proposition 6.4). Since for a_q we have that $v_{a_q,0} = q_0(a_q)$ and, hence, $R_{a_q}^2(v_{a_q,0}) = q_0(a_q) > 0$, we conclude that there is a parameter $a^* \in (a_q, 0) \subset \Lambda$ such that $R_{a^*}^2(v_{a^*,0}) = 0$. This finishes the proof.

$\square$

Following the proof of Proposition 6.8, in Figure 3 we show numerical examples of the three cases of the Escape Trichotomy with $a \in (-1, 0)$. First we take a negative and small enough ($a = -0.0003$) to show an example of a Cantor set of quasicircles. Then we take $a = -0.0164$, which is close to the a^*, to show the Sierpinski carpet case. Finally, we take $a = -0.028$, which is slightly smaller than the a_q, to show the Cantor set case (compare Figure 7). Notice that the parameter a_q is precisely the limit until which the holomorphic motion $\delta_0(a)$ of the immediate basin of attraction of 0 for $a = 0$ is well defined (see Lemma 6.1). Indeed, for a_q the critical value $v_{a_q,0}$ coincides with the periodic point $y_0(a_q)$ (see proof of Proposition 6.8).
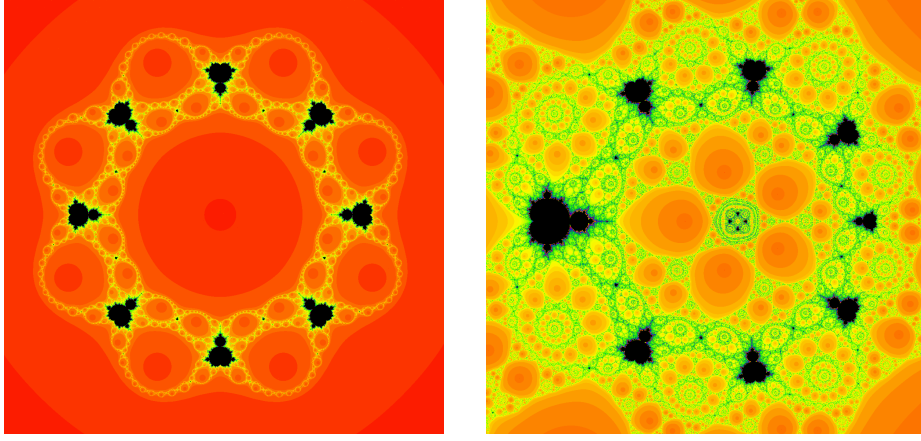
FIGURE 11. In the left side we show the parameter plane of $M_{9,3,\lambda}$ and in the right side the parameter plane of $O_{4,\alpha}$ near $\alpha = 7/6$.

6.1. **Chebyshev-Halley methods applied to $z^n - 1$.** We finish the paper with a remark. We have done the study of the Chebyshev-Halley methods applied to $z^3 - 1$. However, similar singular perturbations can be observed when these methods are applied to $z^n - 1$ with $n \geq 3$. The operator obtained when applying Chebyshev-Halley methods to $z^n - 1$ is given by the degree $2n$ rational map

$$O_{n,\alpha}(z) = z - \frac{(z^n - 1)((-1 + 2\alpha + n - 2\alpha n) + (1 - 2\alpha - 3n + 2\alpha n)z^n)}{2nz^{n-1}(\alpha(n-1)(z^n - 1) - nz^n)} =$$

$$= \frac{(1 - 2\alpha)(n - 1) + (2 - 4\alpha - 4n + 6\alpha n - 2\alpha n^2)z^n + (n - 1)(1 - 2\alpha - 2n + 2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1 - n) + (-\alpha - n + \alpha n)z^n)},$$

where $\alpha \in \mathbb{C}$. As in the degree 3 case, these maps are symmetric with respect to nth roots of the unity and have a unique free critical orbit modulo symmetry (see [3, 4] for an introduction to the dynamics of these maps). The point $z = 0$ is mapped onto ∞ with degree $n - 1$ under $O_{n,\alpha}$. If $\alpha \neq (2n - 1)/(2n - 2)$, then $z = \infty$ is a fixed point. However, if $\alpha = (2n - 1)/(2n - 2)$, then $z = \infty$ is mapped onto $z = 0$ with degree $n - 1$. As we have done for $n = 3$, this can be studied from the point of view of singular perturbations. If $\alpha = (2n - 1)/(2n - 2)$, the point $z = 0$ is a super-attracting fixed point of local degree $(n - 1)^2$ of $O_{n,\alpha}^2$. On the other hand, if $\alpha \neq (2n - 1)/(2n - 2)$ the point $z = 0$ is mapped with degree $n - 1$ onto $z = \infty$ under $O_{n,\alpha}^2$. It follows that, as we obtain for $n = 3$, the dynamics near $z = 0$ for parameters close to $\alpha = (2n - 1)/(2n - 2)$ can be related with the dynamics of the McMullen maps $M_{(n-1)^2, n-1, \lambda}(z) = z^{(n-1)^2} + \lambda/z^{n-1}$. In Figure 11 we show the parameter plane of $O_{4,\alpha}^2$ near $\alpha = 7/6$ and the parameter plane of the corresponding McMullen map $M_{9,3,\lambda}$.

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