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PHASE PORTRAITS OF CUBIC POLYNOMIAL KOLMOGOROV DIFFERENTIAL SYSTEMS HAVING A RATIONAL FIRST INTEGRAL OF DEGREE THREE

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Abstract. We classify all global phase portraits in the Poincaré disc of the cubic polynomial Kolmogorov differential systems having a well-defined rational first integral of degree three at the origin. For such differential systems there are exactly two different global phase portraits up to a reversal of the sense of their orbits.

1. Introduction and statement of the main results

Let $P(x, y)$ and $Q(x, y)$ be two real polynomials in the variables x and y. Consider the following *planar polynomial differential system*:

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x.y), \tag{1}$$

where d/dt denotes derivative with respect to the independent real variable (or *time*) t. We say system (1) is a *cubic polynomial differential system* if the maximum of the degrees of the polynomials $P(x, y)$ and $Q(x, y)$ is 3. And we say that system (1) is of *Kolmogorov* type if x is a factor of $P(x, y)$ and y is a factor of $Q(x, y)$.

There is no need to further emphasize the importance of Kolmogorov systems in describing many natural phenomena, such as population dynamics and chemical reactions (see [2, 7, 11] for more information). In the field of qualitative theory of differential systems, many mathematicians have studied the periodic orbits of Kolmogorov systems (see [6, 8, 10]), as well as their integrability (see [3, 4, 9]).

In this paper we are interested in studying the integrability, specifically focusing on the topologically different global phase portraits that may exist when the Kolmogorov system possesses a *well-defined* rational first integral $H(x, y) = f(x, y)/g(x, y)$ at origin, where $f(x, y)$ and $g(x, y)$ are real polynomials. We say that the rational first integral $H(x, y) = f(x, y)/g(x, y)$ is *well-defined at the origin* if at least one of $f(0, 0)$ and $g(0, 0)$ is non-zero. And the degree of the first integral $H(x, y)$ is the maximum of the degree of $f(x, y)$ and $g(x, y)$.

Cairó and Llibre [3] in 2007 studied the cubic Kolmogorov systems having a rational first integral of degree 2 and obtained 28 different global phase portraits. In this paper we study the global phase portraits that may exist for a cubic Kolmogorov system having a well-defined rational first integral of degree 3 at origin.

The following is the main result of this paper:

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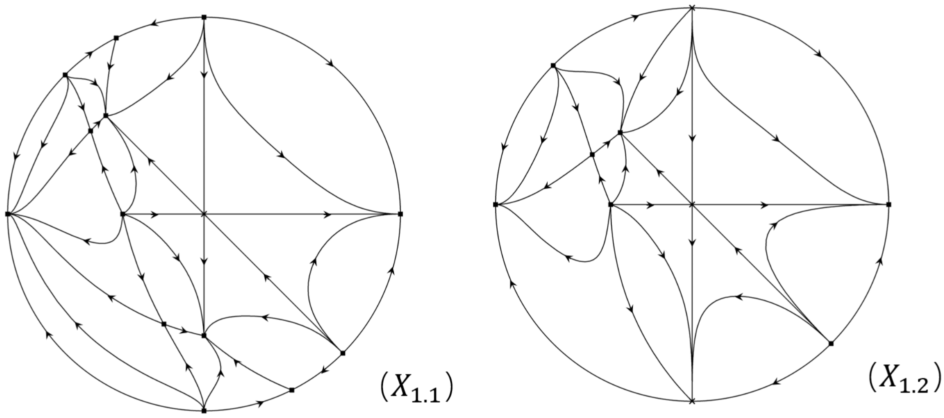

FIGURE 1. The two non-topologically equivalent phase portraits in the Poincaré disc of a cubic polynomial Kolmogorov differential system with a well-defined rational first integral of degree 3 at the origin.

Theorem 1. *The phase portrait of a cubic polynomial Kolmogorov differential system* (1) *having a well-defined rational first integral of degree* 3 *at the origin and with* $P(x, y)$ *and* $Q(x, y)$ *coprime, or the phase portrait with the sense of all orbits reversed, is topologically equivalent to one of the two phase portraits described in Figure* 1.

The paper is organized as follows: In section 2 we characterize the cubic polynomial Kolmogorov differential systems having a well-defined rational first integral of degree 3 at the origin. In section 3 we present the basic results on singular points, Poincaré compactification and topologically equivalence of two differential systems that we shall need. The rest of the section are dedicated to prove Theorem 1.

2. Characterization of cubic polynomial Kolmogorov differential systems

The cubic polynomial Kolmogorov differential systems having a well-defined rational first integral of degree 3 at the origin which is not polynomial are characterized in the next result.

Proposition 2. *A cubic polynomial Kolmogorov differential system* $dx/dt = P(x, y)$, $dy/dt = Q(x, y)$, *having a well-defined rational first integral of degree* 3 *at the origin which is not polynomial can be written as follows,*

$$\begin{aligned}\frac{dx}{dt} &= P(x,y) = x(ax + 2ady + bx^2 + 2bdxy + cdy^2),\\ \frac{dy}{dt} &= Q(x,y) = y(-2ax - ady - bx^2 - 2cxy - cdy^2),\end{aligned} \tag{2}$$

where $a \neq 0$, *and its first integral is*

$$H(x, y) = \frac{a + bx + cy}{(x + dy)xy}. \tag{3}$$

Proof. Assume that $\overline{H} = f/g$ is a well-defined rational first integral of degree 3 at origin for a cubic Kolmogorov differential system without polynomial first integrals, where f and g are coprime, having maximum degree 3, and at least one of $f(0,0)$ and $g(0,0)$ is non-zero. Without loss of generality, we can assume that $g(0,0) \neq 0$. Since first integral $\overline{H}$ is well-defined at the origin, all orbits of the cubic Kolmogorov system having $\overline{H}$ as a first integral passing through the origin are contained in the algebraic curve $f(x,y) - hg(x,y) = 0$, where $h = \overline{H}(0,0)$. Note that the straight lines $x = 0$ and $y = 0$ are two invariant straight lines passing through the origin of the differential system. Therefore, we have two cases: $f(x,y) - hg(x,y) = (Ax + By)xy$, where $A^2 + B^2 \neq 0$ or $f(x,y) - hg(x,y) = Cxy$, where $C \neq 0$.

For the case of $f(x,y) - hg(x,y) = (Ax + By)xy$, we consider the first integral

$$\overline{H} - h = \frac{f(x,y)}{g(x,y)} - h = \frac{f(x,y) - hg(x,y)}{g(x,y)} = \frac{(Ax+By)xy}{g(x,y)}.$$

Note that $1/(\overline{H} - h)$ is also a first integral. Therefore, we consider

$$\begin{aligned} H := &\frac{g(x,y)}{(Ax+By)xy} \\ = &\frac{a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + kxy^2 + my^3}{(Ax+By)xy} \end{aligned} \tag{4}$$

as the first integral, where $a(A^2 + B^2) \neq 0$. Clearly all differential systems having H as a first integral are of the form

$$\frac{dx}{dt} = -F(x,y)\frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = F(x,y)\frac{\partial H}{\partial x}, \tag{5}$$

where $F(x,y)$ is a function. In order that the previous differential system be polynomial of degree 3, we take $F(x,y) = (Ax + By)^2x^2y^2$ and obtain

$$\begin{aligned} \frac{dx}{dt} &= x(Aax + 2Bay + Abx^2 + 2Bbxy + Bcy^2 + Adx^3 + 2Bdx^2y - Afxy^2 \\ &\quad + Bexy^2 + Agx^4 + 2Bgx^3y - Akx^2y^2 - 2Amxy^3 + Bhx^2y^2 - Bmy^4), \\ \frac{dx}{dt} &= y(-2Aax - Bay - Abx^2 - 2Acxy - Bcy^2 - Aex^2y + Bdx^2y - 2Afxy^2 \\ &\quad - Bfy^3 + Agx^4 + 2Bgx^3y - Akx^2y^2 + Bhx^2y^2 - 2Amxy^3 - Bmy^4). \end{aligned}$$

To ensure that the highest degree of the above system is 3, we must take $Ad = Ae = Af = Ag = Ak = Am = Bd = Be = Bf = Bh = Bg = Bm = 0$, leading to three cases based on $A^2 + B^2 \neq 0$.

Case 1*: $A \neq 0$ and $B \neq 0$.* In this case we have $d = e = f = g = h = k = m = 0$. Substituting these values into (4), we get $H = (a + bx + cy)/((Ax + By)xy)$.

Case 2*: $A = 0$ and $B \neq 0$.* In this case we have $d = e = f = g = h = m = A = 0$. Substituting these values into (4), we get $H = (a + bx + cy)/(Bxy^2)$.

Case 3*: $A \neq 0$ and $B = 0$.* In this case we have $d = e = f = g = k = m = B = 0$. Substituting these values into (4), we get $H = (a + bx + cy)/(Ax^2y)$.

For the case of $f(x,y) - hg(x,y) = Cxy$, we consider the first integral

$$\frac{\overline{H} - h}{C} = \frac{f(x,y) - hg(x,y)}{Cg(x,y)} = \frac{xy}{g(x,y)}.$$

Since $\frac{C}{H-h} - e$ is also a first integral, we consider the following first integral:

$$H := \frac{g(x,y)}{xy} - e = \frac{a + bx + cy + dx^2 + fy^2 + gx^3 + hx^2y + kxy^2 + my^3}{xy} \tag{6}$$

where $a \neq 0$. Substituting (6) into system (5) and taking $F(x,y) = x^2y^2$, we have

$$\begin{aligned}\frac{dx}{dt} &= x(a + bx + dx^2 - fy^2 + gx^3 - kxy^2 - 2my^3),\\ \frac{dx}{dt} &= -y(a + cy - dx^2 + fy^2 - 2gx^3 - hx^2y + my^3).\end{aligned}$$

To ensure that the highest degree of the above system is 3, we must take $g = h = k = m = 0$. However, this contradicts the degree of the rational first integral (6) being 3. So, this case does not happen.

In conclusion, the well-defined rational first integral of degree 3 at the origin of a cubic Kolmogorov differential systems is in the form $H = (a+bx+cy)/((Ax+By)xy)$, where $a(A^2 + B^2) \neq 0$. Without loss of generality, we assume that the coefficient of x^2y in the denominator of the rational first integral H is not zero; otherwise, we do the change of variables variables $(x,y) \to (y,x)$. Thus assuming $A \neq 0$, we obtain $AH = (a + bx + cy)/((x + (B/A)y)xy)$, which is also a first integral. By setting $d := B/A$, we establish that the cubic Kolmogorov differential system has the rational first integral $(a + bx + cy)/((x + dy)xy)$.

Substituting this expression into the differential system (5) and taking $F(x,y) = (x + dy)^2x^2y^2$, we obtain system (2). This completes the proof of the proposition. □

3. Basic results

In this section we will introduce some basic definitions and concepts necessary for analyzing the local phase portraits of both finite and infinite singular points of cubic polynomial Kolmogorov differential systems. These fundamental principles can be commonly found in most textbooks on qualitative theory of differential systems (for example see [5]). We list them here for the convenience of the readers.

3.1. **Singular points.** A point $p \in \mathbb{R}^2$ is called a singular point of system (1) if it satisfies $P(p) = Q(p) = 0$. Let $DX(p)$ be the Jacobian matrix of systems (1) at point p. A singular point is *non-degenerate* if the determinant of its Jacobian matrix is non-zero. If the two eigenvalues of the Jacobian matrix $DX(p)$ have non-zero real parts, we call the singular point *hyperbolic*. Furthermore, if the two eigenvalues are real numbers with opposite signs, the point is called a *hyperbolic saddle*. If they have the same sign, the point is called a *hyperbolic node*. For a hyperbolic node, if the signs of the two eigenvalues are both negative, the node is stable; otherwise, it is unstable.

Finally, if the Jacobian matrix at the singular point p is identically zero, then we say that p is *linearly zero*. The local phase portrait of such singular points will become complex. Dealing with them requires special tools like blow-ups, for more details see [5].

3.2. **Poincaré compactification.** The Poincaré compactification, proposed by the French mathematician H. Poincaré, offers a method for studying the behavior of orbits when they go or come from the infinity in different directions on the plane.

Consider a planar polynomial differential vector field of degree n, denoted as $\boldsymbol{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$. The corresponding Poincaré compactified differential system $p(\boldsymbol{X})$ is induced on $\mathbb{S}^2$ as follows: Start with the Poincaré sphere $\mathbb{S}^2 = \{(X,Y,Z) \in \mathbb{R}^3 : X^2+Y^2+Z^2 = 1\}$ and the tangent plane $T_{(0,0,1)}\mathbb{S}^2$ to $\mathbb{S}^2$ at point $(0,0,1)$. We identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$. So there is a central projection $f : \mathbb{R}^2 \to \mathbb{S}^2$. This map sends every point $x \in \mathbb{R}^2$ to two points on $\mathbb{S}^2$, one in the northern hemisphere and the other in the southern hemisphere. Clearly the equator $\mathbb{S}^1 = \{(X,Y,Z) \in \mathbb{S}^2 : Z = 0\}$ is in bijection with the infinity of $\mathbb{R}^2$. Denote by $\boldsymbol{X}'$ the differential system $Df \circ \boldsymbol{X}$ defined on $\mathbb{S}^2/\mathbb{S}$.

Next we extend $\boldsymbol{X}'$ to a differential system on $\mathbb{S}^2$ including $\mathbb{S}^1$. Multiply $\boldsymbol{X}'$ by Z^{n-1} and we get a new differential system denoted by $p(\boldsymbol{X})$ and called the Poincaré compactification of the polynomial differential system $\boldsymbol{X}$. The behavior of $p(\boldsymbol{X})$ around $\mathbb{S}^1$ corresponds with the behavior of $\boldsymbol{X}$ in a neighbourhood of the infinity. Note that $\mathbb{S}^1$ is invariant under the flow of $p(\boldsymbol{X})$. The *Poincaré disc*, denoted by $\mathbb{D}$, is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $Z = 0$ under $(X,Y,Z) \to (X,Y)$.

Since $\mathbb{S}^2$ is a differential manifold, for working with the differential system $p(\boldsymbol{X})$ on $\mathbb{S}^2$, we can consider the four local charts $U_1 = \{(X,Y,Z) \in \mathbb{S}^2 : X > 0\}$, $U_2 = \{(X,Y,Z) \in \mathbb{S}^2 : Y > 0\}$, $V_1 = \{(X,Y,Z) \in \mathbb{S}^2 : X < 0\}$, $V_2 = \{(X,Y,Z) \in \mathbb{S}^2 : Y < 0\}$ and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$, $G_i : V_i \to \mathbb{R}^2$ for $i = 1,2$ are the inverse of the central projection from the planes tangent at the points $(1,0,0)$, $(0,1,0)$, $(-1,0,0)$ and $(0,-1,0)$ respectively. We denote by $z = (z_1, z_2)$ the value of $F_1(X,Y,Z)$ or $F_2(X,Y,Z)$, then the expressions of $p(X)$ are

$$\begin{aligned} \frac{dz_1}{dt} &= z_2^n\left(Q\left(\frac{1}{z_2},\frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right)\right), \quad \frac{dz_2}{dt} = -z_2^{n+1}P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right) \text{ in } U_1,\\ \frac{dz_2}{dt} &= z_2^n\left(P\left(\frac{z_1}{z_2},\frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right)\right), \quad \frac{dz_2}{dt} = -z_2^{n+1}Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right) \text{ in } U_2. \end{aligned} \tag{7}$$

Finally, we remark that to study the singular points at infinity, it is sufficient to study the singular points on U_1 and check if the origin on U_2 is or not a singular point.

3.3. **Topological equivalence of two polynomial differential systems.** We say that two polynomial differential systems $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ on the plane are *topologically equivalent* if there exists a homeomorphism defined on the Poincaré disc $\mathbb{D}$, preserving the infinity $\mathbb{S}^1$, and mapping all orbits of $p(\boldsymbol{X}_1)$ onto orbits of $p(\boldsymbol{X}_2)$ either preserving or reversing the orientation of all orbits.

4. Normal form

Now we will reduce the number of four parameters of the cubic polynomial Kolmogorov differential systems having a well-defined rational first integral of degree 3 at origin to at most one parameter.

Proposition 3. *Any cubic polynomial Kolmogorov differential system $dx/dt = P(x,y)$, $dy/dt = Q(x,y)$, having a well-defined rational first integral of degree 3 at origin with*

P and Q coprime can be written as the following differential system:

$$\begin{aligned} \frac{dx}{dt} &= P_1 = x(x + 2y + x^2 + 2xy + \lambda y^2), \\ \frac{dy}{dt} &= Q_1 = y(-2x - y - x^2 - 2\lambda xy - \lambda y^2), \end{aligned} \tag{8}$$

where $\lambda \in [0, 1)$.

Proof. Let $x = \alpha x_1$, $y = \beta y_1$, $t = \gamma t_1$. Then differential system (2) can be written as

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(a\alpha\gamma x_1 + 2ad\beta\gamma y_1 + b\alpha^2\gamma x_1^2 + 2bd\alpha\beta\gamma x_1 y_1 + cd\beta^2\gamma y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2a\alpha\gamma x_1 - ad\beta\gamma y_1 - b\alpha^2\gamma x_1^2 - 2c\alpha\beta\gamma x_1 y_1 - cd\beta^2\gamma y_1^2). \end{aligned} \tag{9}$$

We consider the following cases.

Case 1*:* $bd \neq 0$. We do the change of variables $(x_1, y_1) \to (-x_1, y_1)$, $(x_1, y_1) \to (x_1, -y_1)$ or $(x_1, y_1, t_1) \to (-x_1, -y_1, -t_1)$ to system (9) respectively, and get the following three systems

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(-a\alpha\gamma x_1 + 2ad\beta\gamma y_1 + b\alpha^2\gamma x_1^2 - 2bd\alpha\beta\gamma x_1 y_1 + cd\beta^2\gamma y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(2a\alpha\gamma x_1 - ad\beta\gamma y_1 - b\alpha^2\gamma x_1^2 + 2c\alpha\beta\gamma x_1 y_1 - cd\beta^2\gamma y_1^2), \end{aligned}$$

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(a\alpha\gamma x_1 - 2ad\beta\gamma y_1 + b\alpha^2\gamma x_1^2 - 2bd\alpha\beta\gamma x_1 y_1 + cd\beta^2\gamma y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2a\alpha\gamma x_1 + ad\beta\gamma y_1 - b\alpha^2\gamma x_1^2 + 2c\alpha\beta\gamma x_1 y_1 - cd\beta^2\gamma y_1^2), \end{aligned}$$

and

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(a\alpha\gamma x_1 + 2ad\beta\gamma y_1 - b\alpha^2\gamma x_1^2 - 2bd\alpha\beta\gamma x_1 y_1 - cd\beta^2\gamma y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2a\alpha\gamma x_1 - ad\beta\gamma y_1 + b\alpha^2\gamma x_1^2 + 2c\alpha\beta\gamma x_1 y_1 + cd\beta^2\gamma y_1^2). \end{aligned}$$

Note that the three types of variable changes mentioned above only alter the sign of one of the three coefficients $a\alpha\gamma$, $2ad\beta\gamma$ and $b\alpha^2\gamma$, without changing the signs of the other two. Therefore, without loss of generality, we can assume $a\alpha\gamma > 0$, $ad\beta\gamma > 0$ and $b\alpha^2\gamma > 0$ in equation (9). By solving equations $a\alpha\gamma = ad\beta\gamma = b\alpha^2\gamma = 1$, we get $\alpha = a/b$, $\beta = a/(bd)$ and $\gamma = b/a^2$. Substituting these values into system (9), we obtain system $dx/dt = P_1$, $dy/dt = Q_1$, where $\lambda := c/(bd)$ may be zero.

Case 2*:* $d = 0$. In the case, from (9) we obtain

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(a\alpha\gamma x_1 + b\alpha^2\gamma x_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2a\alpha\gamma x_1 - b\alpha^2\gamma x_1^2 - 2c\alpha\beta\gamma x_1 y_1). \end{aligned} \tag{10}$$

Since there is a common factor x_1 in the above system, we omit this case.

Case 3*: $d \neq 0$ and $b = 0$.* In this case, from system (9) we have

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(a\alpha\gamma x_1 + 2ad\beta\gamma y_1 + cd\beta^2\gamma y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2a\alpha\gamma x_1 - ad\beta\gamma y_1 - 2c\alpha\beta\gamma x_1 y_1 - cd\beta^2\gamma y_1^2), \end{aligned} \tag{11}$$

and from (3) we have the rational first integral $H = (a+cy)/((x+dy)xy)$. Consequently, $c \neq 0$, otherwise the differential system would have a polynomial first integral $(x+dy)xy$.

Similarly we assume that $a\alpha\gamma > 0$, $ad\beta\gamma > 0$ and $cd\beta^2\gamma > 0$ in system (11). Otherwise, we do the change of variables $(x_1, y_1) \to (-x_1, y_1)$, $(x_1, y_1) \to (x_1, -y_1)$ or $(x_1, y_1, t_1) \to (-x_1, -y_1, -t_1)$ respectively. By solving equations $a\alpha\gamma = ad\beta\gamma = cd\beta^2\gamma = 1$, we obtain $\alpha = (ad)/c$, $\beta = a/c$ and $\gamma = c/(a^2 d)$. Substituting them into system (11), we obtain

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(x_1 + 2y_1 + y_1^2), \\ \frac{dy_1}{dt_1} &= y_1(-2x_1 - y_1 - 2x_1 y_1 - y_1^2). \end{aligned}$$

Doing the change of variables $(x_1, y_1, t_1) \to (y_1, x_1, -t_1)$, we get

$$\begin{aligned} \frac{dx_1}{dt_1} &= x_1(x_1 + 2y_1 + x_1^2 + 2x_1 y_1), \\ \frac{dy_1}{dt_1} &= y_1(-2x_1 - y_1 - x_1^2). \end{aligned}$$

Note that above system is the particular case of systems $dx/dt = P_1$, $dy/dt = Q_1$ with $\lambda = 0$.

Next we explain how we restricted the range of λ. For $\lambda \neq 0$ doing the change of variables $(x, y, t) \to ((1/\lambda)y,\ (1/\lambda)x, -\lambda t)$, we obtain

$$\frac{dx}{dt} = x(x + 2y + x^2 + 2xy + \frac{y^2}{\lambda}), \ \frac{dy}{dt} = y(-2x - y - x^2 - \frac{2xy}{\lambda} - \frac{y^2}{\lambda}).$$

The difference between the above system and system $dx/dt = P_1$, $dy/dt = Q_1$ lies in replacing $1/\lambda$ with λ. Without loss of generality, we can assume that $\lambda \in [-1, 0) \cup (0, 1]$. For $\lambda \neq 1$ by doing the change of variables $(x, y, t) \to (x/(1-\lambda), (x+y)/(\lambda-1), (\lambda-1)t)$, we obtain

$$\frac{dx}{dt} = x(x + 2y + x^2 + 2xy + \frac{\lambda y^2}{\lambda - 1}), \quad \frac{dy}{dt} = y(-2x - y - x^2 - \frac{2\lambda xy}{\lambda - 1} - \frac{\lambda y^2}{\lambda - 1}).$$

Note that the difference between the above system and system $dx/dt = P_1$, $dy/dt = Q_1$ lies in replacing $\lambda/(\lambda - 1)$ with λ. So, without loss of generality, we can assume that $\lambda \in [0, 1) \cup (1, 2]$. Combining the two variable changes mentioned above, we can conclude that for $\lambda \notin \{0, 1\}$, we only need to study the case where $\lambda \in (0, 1)$.

Finally, note the following fact: there exists a change of variables such that system $dx/dt = P_1$, $dy/dt = Q_1$ with $\lambda = 1$ becomes system $dx/dt = P_1$, $dy/dt = Q_1$ with $\lambda = 0$. For system $dx/dt = P_1$, $dy/dt = Q_1$ with $\lambda = 1$, doing the change of variables $(x, y, t) \to (-y, x + y, -t)$, we obtain system $dx/dt = P_1$, $dy/dt = Q_1$ with $\lambda = 0$:

$$\frac{dx}{dt} = x(x + 2y + x^2 + 2xy), \quad \frac{dy}{dt} = y(-2x - y - x^2).$$

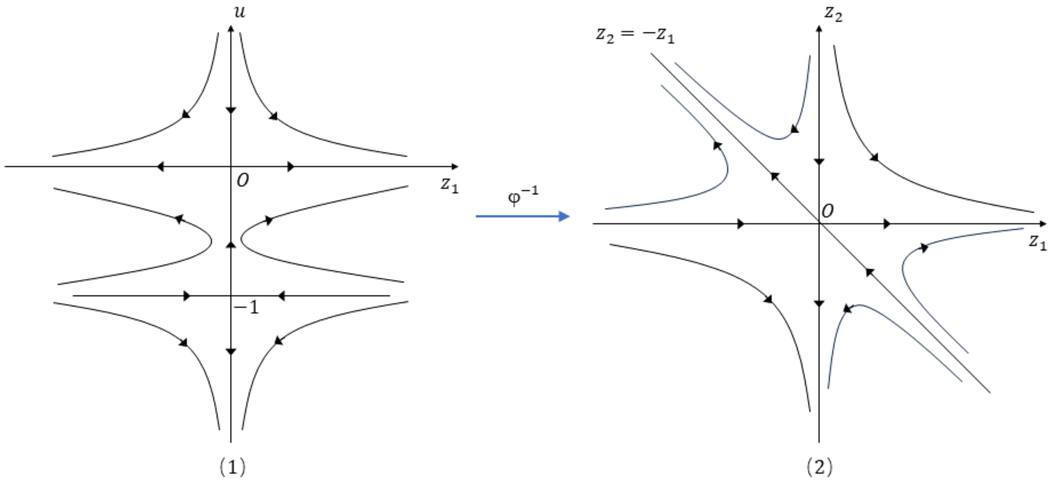

FIGURE 2. Blow-up and the local phase portrait of at the origin of system $dx/dt = P_1$, $dy/dt = Q_1$ on the chart U_2.

This ends the proof. □

5. The infinite singular points

In this section we study the infinite singular points of system $dx/dt = P_1$, $dy/dt = Q_1$.

Proposition 4. *The following statements hold.*

(*a*) *System $dx/dt = P_1$, $dy/dt = Q_1$ has three infinite singular points $(-1, 0)$, $(0, 0)$ and $(-1/\lambda, 0)$ on U_1 if $\lambda \neq 0$. Moreover, $(-1, 0)$ is a hyperbolic unstable node, $(0, 0)$ is a hyperbolic stable node and $(-1/\lambda, 0)$ is a hyperbolic saddle. This system has an infinite singular point $(0, 0)$ on U_2, which is a hyperbolic unstable node.*

(*b*) *System $dx/dt = P_1$, $dy/dt = Q_1$ has two infinite singular points $(-1, 0)$ and $(0, 0)$ on U_1 if $\lambda = 0$. Moreover, $(-1, 0)$ is a hyperbolic unstable node and $(0, 0)$ is a hyperbolic stable node. This system has an infinite singular point at $(0, 0)$ in U_2, which is linearly zero and its local phase portrait is given in Figure 2.(2).*

Proof. From (7) the Poincaré compactification of system (8) on the local chart U_1 is

$$\begin{aligned} \frac{dz_1}{dt} &= -z_1(z_1+1)(2\lambda z_1 + 3z_2 + 2), \\ \frac{dz_2}{dt} &= -z_2\left(\lambda z_1^2 + 2\left(z_2+1\right)z_1 + z_2 + 1\right). \end{aligned} \tag{12}$$

Let $D(z_1, z_2)$ be the Jacobian matrix of system (12) at point (z_1, z_2), then

$$D(z_1, z_2) = \begin{pmatrix} -6\lambda z_1^2 - 2z_1(2\lambda + 3z_2 + 2) - 3z_2 - 2, & -3z_1(z_1+1) \\ -2z_2(\lambda z_1 + z_2 + 1), & -\lambda z_1^2 - 2(2z_2+1)z_1 - 2z_2 - 1 \end{pmatrix}.$$

For $\lambda \neq 0$ (recall that $\lambda \in [0,1)$), the system has three different singular points $(-1,0)$, $(0,0)$ and $(-1/\lambda,0)$ on U_1. After simple calculations, we obtain that the eigenvalues of $D(-1,0)$ are $2(1-\lambda)>0$ and $1-\lambda>0$, the eigenvalues of $D(0,0)$ are -2 and -1, and the eigenvalues of $D(-1/\lambda,0)$ are $1/\lambda-1>0$ and $2-2/\lambda<0$. Thus the singular points $(-1,0)$, $(0,0)$ and $(-1/\lambda,0)$ on U_1 are hyperbolic unstable node, hyperbolic stable node, and hyperbolic saddle, respectively.

For $\lambda=0$ system (12) has two singular points $(-1,0)$ and $(0,0)$ on U_1. On U_1 the Jacobian matrix of singular point $(-1,0)$ has eigenvalues 2 and 1, while the Jacobian matrix of singular point $(0,0)$ has eigenvalues -2 and -1. Thus the singular points $(-1,0)$ and $(0,0)$ on U_1 are a hyperbolic unstable node and a hyperbolic stable node, respectively.

Similarly, the Poincaré compactification of system (8) on local chart U_2 is

$$\begin{aligned}\frac{dz_1}{dt}&=z_1\left(z_1+1\right)\left(2\lambda+2z_1+3z_2\right),\\ \frac{dz_2}{dt}&=z_2\left(\lambda+2z_1\left(\lambda+z_2\right)+z_1^2+z_2\right).\end{aligned}\tag{13}$$

We only need to study the point $(0,0)$ on U_2. Let $\hat{D}(z_1,z_2)$ be the Jacobian matrix of system (13) at the point (z_1,z_2), then we have

$$\hat{D}(z_1,z_2)=\begin{pmatrix}2\lambda+z_1\left(4\lambda+6z_2+4\right)+6z_1^2+3z_2 & 3z_1\left(z_1+1\right)\\ 2z_2\left(\lambda+z_1+z_2\right) & \lambda+2z_1\left(\lambda+2z_2\right)+z_1^2+2z_2\end{pmatrix}.$$

After simple calculations the singular point $(0,0)$ has two eigenvalues, 2λ and λ. Therefore the singular point $(0,0)$ in U_2 is a hyperbolic unstable node if $\lambda\neq 0$. This proves statement (a).

Noting that $\hat{D}(0,0)$ is the zero matrix if $\lambda=0$, the singular point $(0,0)$ in U_2 is linearly zero. We need to blow up the linearly zero singular point. The rescaled system $dx/dt=P_1$, $dy/dt=Q_1$ with $\lambda=0$ in the chart U_2 is given by

$$\frac{dz_1}{dt}=z_1\left(z_1+1\right)\left(2z_1+3z_2\right),\quad \frac{dz_2}{dt}=z_2\left(2z_1z_2+z_1^2+z_2\right).\tag{14}$$

By doing the change of variables $z_1\to z_1$, $z_2\to z_1u$, we obtain

$$\frac{dz_1}{dt}=(3u+2)z_1^2\left(z_1+1\right),\quad \frac{du}{dt}=-u(u+1)z_1\left(z_1+2\right).$$

Note that when we consider the behavior of orbits near $z_1=0$, the above system is equivalent to the system

$$\frac{dz_1}{dt}=(3u+2)z_1\left(z_1+1\right),\quad \frac{du}{dt}=-u(u+1)\left(z_1+2\right).\tag{15}$$

Obviously the above system has two singular point $(0,-1)$ and $(0,0)$ on $u=0$. After simple calculations, we get the eigenvalues of $(0,-1)$ are -1 and 2, while the eigenvalues of $(0,0)$ are -1 and 2. Therefore, these two singular points are saddles (see Figure 2.(1)). We denote all operations (or changes) from system (14) to system (15) as φ. Then we do the φ^{-1} which involves flipping the orbits of the second and third quadrants about the x-axis, reversing the orientation of the orbits in $x<0$, and compressing the u-axis into a single point. Finally, we obtain the local behavior of orbits near $(0,0)$ on U_2 (see Figure 2.(2)). This proves statement (b). □

6. The finite singular points

Now using the results from subsection 3.1, we study the local phase portraits of the singular points of the system $dx/dt = P_1$, $dy/dt = Q_1$. Recall that $\lambda \in [0, 1)$.

(1.1) When $\lambda \neq 0$, by solving $P_1(x, y) = Q_1(x, y) = 0$, we get that the system has six finite singular points, $(-1, 0)$, $(0, 0)$, $(\frac{1}{\lambda-1}, \frac{1}{1-\lambda})$, $(0, -\frac{1}{\lambda})$, $(\frac{\sqrt{\lambda^2-\lambda+1}-\lambda+2}{2(\lambda-1)}, -\frac{\sqrt{\lambda^2-\lambda+1}+2\lambda-1}{2(\lambda-1)\lambda})$ and $(-\frac{\sqrt{\lambda^2-\lambda+1}+\lambda-2}{2(\lambda-1)}, \frac{\sqrt{\lambda^2-\lambda+1}-2\lambda+1}{2(\lambda-1)\lambda})$. Furthermore, both eigenvalues of $(-1, 0)$ are 1, showing that it is a hyperbolic unstable node. The Jacobian matrix of $(0, 0)$ is zero, indicating it as linearly zero. Both eigenvalues of $(\frac{1}{\lambda-1}, \frac{1}{1-\lambda})$ are $\frac{1}{\lambda-1} < 0$, indicating that it is a hyperbolic stable node. Both eigenvalues of $(0, -\frac{1}{\lambda})$ are $-\frac{1}{\lambda} < 0$, indicating that it is a hyperbolic stable node. The eigenvalues of $(\frac{\sqrt{\lambda^2-\lambda+1}-\lambda+2}{2(\lambda-1)}, -\frac{\sqrt{\lambda^2-\lambda+1}+2\lambda-1}{2(\lambda-1)\lambda})$ are $\pm p$, where

$$p = \frac{\sqrt{-2\sqrt{(\lambda-1)\lambda+1}+\lambda\Big(3\sqrt{(\lambda-1)\lambda+1}+\lambda\Big(2\lambda\Big(\lambda-\sqrt{(\lambda-1)\lambda+1}-2\Big)+3\Big(\sqrt{(\lambda-1)\lambda+1}+2\Big)\Big)-4\Big)+2}}{2(1-\lambda)\lambda};$$

And the eigenvalues of $(-\frac{\sqrt{\lambda^2-\lambda+1}+\lambda-2}{2(\lambda-1)}, \frac{\sqrt{\lambda^2-\lambda+1}-2\lambda+1}{2(\lambda-1)\lambda})$ are $\pm q$, where

$$q = \frac{\sqrt{2\Big(\sqrt{(\lambda-1)\lambda+1}+1\Big)+\lambda\Big(-3\sqrt{(\lambda-1)\lambda+1}+\lambda\Big(-3\sqrt{(\lambda-1)\lambda+1}+2\lambda\Big(\lambda+\sqrt{(\lambda-1)\lambda+1}-2\Big)+6\Big)-4\Big)}}{2(1-\lambda)\lambda}.$$

In fact, it can be proven that p and q are positive real numbers, implying that the corresponding two singular points are hyperbolic saddles. The detailed proof is presented in the appendix.

(1.2) When $\lambda = 0$, the system $dx/dt = P_1$, $dy/dt = Q_1$ has four singular points, $(-3/2, 3/4)$, $(-1, 1)$, $(0, 0)$ and $(-1, 0)$. The eigenvalues of $(-3/2, 3/4)$ are $\pm 3\sqrt{3}/4$; both eigenvalues of $(-1, 1)$ are -1; the Jacobian matrix of $(0, 0)$ is zero; and the eigenvalues of $(-1, 0)$ are both 11. Thus, $(-3/2, 3/4)$ is a hyperbolic saddle; $(-1, 1)$ is a hyperbolic stable node; $(0, 0)$ is linearly zero; $(-1, 0)$ is a hyperbolic unstable node.

Note that in both cases, the origin is linearly zero, necessitating further analysis. Next, we consider the blow-ups of the origin.

By doing the change of variables $x \to x$, $y \to xu$, we obtain

$$\frac{dx}{dt} = x^2\left(\lambda u^2 x + 2u(x+1) + x + 1\right), \quad \frac{du}{dt} = -u(u+1)x(2x(\lambda u + 1) + 3).$$

Since we consider the behavior of orbits near origin, the above system is equivalent to the following system:

$$\frac{dx}{dt} = x\left(\lambda u^2 x + 2u(x+1) + x + 1\right), \quad \frac{du}{dt} = -u(u+1)(2x(\lambda u + 1) + 3).$$

There are two singular points $(0, -1)$ and $(0, 0)$ on $u = 0$. Upon simple calculations, the eigenvalues of $(0, -1)$ are -1 and 3, and the eigenvalues of $(0, 0)$ are 1 and -3. Thus, the behavior of orbits near the origin is given in Figure 2.(2).

7. Global phase portraits

Summarizing from Sections 5 and 6, we have the following result.

Proposition 5. *The phase portrait of system $dx/dt = P_1$, $dy/dt = Q_1$ in the Poincaré disc is topologically equivalent to $(X_{1.1})$ of Figure 2 if $\lambda \neq 0$, and to $(X_{1.2})$ of Figure 1 if $\lambda = 0$;*

8. Appendix

In this section we prove that p and q, as defined in Section 6, are indeed positive real numbers. Recall the expression for p:

$$p = \frac{\sqrt{-2\sqrt{(\lambda-1)\lambda+1}+\lambda\left(3\sqrt{(\lambda-1)\lambda+1}+\lambda\left(2\lambda\left(\lambda-\sqrt{(\lambda-1)\lambda+1}-2\right)+3\left(\sqrt{(\lambda-1)\lambda+1}+2\right)\right)-4\right)+2}}{2(1-\lambda)\lambda}.$$

Define the function $f(\lambda)$ as follows:

$$\begin{aligned} f(\lambda) :=& -2\sqrt{(\lambda-1)\lambda+1}+\lambda(3\sqrt{(\lambda-1)\lambda+1}+\lambda(2\lambda(\lambda-\sqrt{(\lambda-1)\lambda+1}-2) \\ &+3(\sqrt{(\lambda-1)\lambda+1}+2))-4)+2. \end{aligned}$$

We need to show that $f(\lambda) > 0$ for $\lambda \in (0,1)$. Note that

$$\begin{aligned} &f_1(\lambda) := \frac{f(\lambda)}{\sqrt{(\lambda-1)\lambda+1}} \\ &= (2(\sqrt{(\lambda-1)\lambda+1}-1)+\lambda(-2\sqrt{(\lambda-1)\lambda+1}+\lambda(-2\lambda+2\sqrt{(\lambda-1)\lambda+1}+3)+3)). \end{aligned}$$

Thus we must demonstrate $f_1(\lambda) > 0$. After calculations, for $\lambda \in (0,1)$ we have

$$\begin{aligned} &f_1(0) = 0, \\ &f_1^{'}(\lambda) = -3\sqrt{(\lambda-1)\lambda+1}+6\lambda\left(-\lambda+\sqrt{(\lambda-1)\lambda+1}+1\right)+3, \\ &f_1^{'}(0) = 0, \\ &f_1^{''}(\lambda) = \frac{3\left(2\sqrt{\lambda^2-\lambda+1}-2\lambda+1\right)^2}{2\sqrt{\lambda^2-\lambda+1}} > 0. \end{aligned}$$

This shows that $f(\lambda) > 0$ in $(0,1)$, implying that p is a positive real number.

Similarly, we next prove that q is also a positive real number. Recall that

$$q = \frac{\sqrt{2\left(\sqrt{(\lambda-1)\lambda+1}+1\right)+\lambda\left(-3\sqrt{(\lambda-1)\lambda+1}+\lambda\left(-3\sqrt{(\lambda-1)\lambda+1}+2\lambda\left(\lambda+\sqrt{(\lambda-1)\lambda+1}-2\right)+6\right)-4\right)}}{2(1-\lambda)\lambda}.$$

Define the function $g(\lambda)$ as:

$$\begin{aligned} g(\lambda) :=& 2\left(\sqrt{(\lambda-1)\lambda+1}+1\right)+\lambda\Big(-3\sqrt{(\lambda-1)\lambda+1} \\ &+\lambda\left(-3\sqrt{(\lambda-1)\lambda+1}+2\lambda\left(\lambda+\sqrt{(\lambda-1)\lambda+1}-2\right)+6\right)-4\Big). \end{aligned}$$

To prove q is positive for $\lambda \in (0,1)$, it suffices to show $g(\lambda) > 0$. Note that

$$\begin{aligned} &g_1(\lambda) := \frac{g(\lambda)}{\sqrt{(\lambda-1)\lambda+1}} \\ &= 2\left(\sqrt{(\lambda-1)\lambda+1}+1\right)+\lambda\left(-2\sqrt{(\lambda-1)\lambda+1}+\lambda\left(2\lambda+2\sqrt{(\lambda-1)\lambda+1}-3\right)-3\right). \end{aligned}$$

We next prove that $g_1(\lambda) > 0$. After calculations, for $\lambda \in (0,1)$ we have

$$g_1(0) = 4,\ g_1(1) = 0$$
$$g_1^{'}(\lambda) = 6\lambda\left(\lambda + \sqrt{(\lambda-1)\lambda+1} - 1\right) - 3\left(\sqrt{(\lambda-1)\lambda+1} + 1\right),$$
$$g_1^{'}(0) = -6,\ g_1^{'}(1) = 0$$
$$g_1^{''}(\lambda) = \frac{3\left(2\sqrt{\lambda^2-\lambda+1} + 2\lambda - 1\right)^2}{2\sqrt{\lambda^2-\lambda+1}} > 0.$$

This shows that $g(\lambda) > 0$ in $(0,1)$, concluding that q is a positive real number.

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References

[1] J. C. Artés, B. Grünbaum and J. Llibre, *On the number of invariant straight lines for polynomial differential systems*, Pacific. J. Math. **184** (1998) 207-230.
[2] F. H. Busse, *Transition to turbulence via the statistical limit cycle route*, Synergetics, Springer-Verlag, Berlin, 1978. P. 39
[3] L. Cairó and J. Llibre. *Phase portraits of cubic polynomial vector fields of Lotka-Volterra type having a rational first integral of degree 2*, J. Phys. A **40** (2007), 6329-6348.
[4] L. Cairó, H. Giacomini and J. Llibre. *Liouvillian first integrals for the planar Lotka-Volterra system*, Rend. Circ. Mat. Palermo 2 (5) (2003), 389-418.
[5] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative theory of planar differential systems*, New York: Springer, 2006.
[6] X. C. Huang and L. Zhu. *Limit cycles in a general Kolmogorov model*, Nonlinear Anal. **60** (2005) 1393-1414.
[7] G. Laval and R. Pellat, Plasma physics, in: Proceedings of Summer School of Theoretical Physics, Gordon and Breach, New York, 1993.
[8] C. Li and J. Llibre. *The cyclicity of period annulus of a quadratic reversible Lotka-Volterra system*, Nonlinearity **22** (2009), 2971-2979.
[9] J. Llibre and C. Valls. *Global analytic first integrals for the real planar Lotka-Volterra system*, J. Math. Phys. **48** (2007), 033507, 13pp.
[10] Z. Lu and Y. Luo. *Two limit cycles in three-dimensional Lotka-Volterra systems*, Comput. Math. Appl. **44** (2002), 51-66.
[11] R. M. May, Stability and Complexity in Model Ecosystems, Princeton, New Jersey, 1974.

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